On a theorem of Kas and Schlessinger

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Abstract. In [**KS**] A. Kas and M. Schlessinger construct a versal deformation of an analytic space which is a local complete intersection. An immediate corollary of their theorem is that a flat family of nodal curves can be locally obtained by pullback of the standard family $xy = t$. In this article, we spell out how this result follows from the theorem of Kas and Schlessinger.

This is an expository paper about Kas and Schlessinger's construction of a versal deformation space for an analytic space which is locally a complete intersection. This result has a distinct algebro-geometric flavor, but we do not assume any familiarity with concepts from algebraic geometry such as flatness or nonreducedness. In fact, we hope this paper can serve as an introduction to these ideas for geometers dealing with analytic spaces.

We extract the following definition from "Deformations of complex spaces," [**P**]. Let V_0 be an analytic space. A *deformation* of V_0 is a *flat morphism* $\pi : V \to T$ of analytic spaces such that $\pi^{-1}(0) \cong V_0$. Two deformations $\pi_1 : V_1 \to T_1$ and $\pi_2 : V_2 \to T_2$ are *isomorphic* if

- there are neighborhoods $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ of the fibers $\pi_1^{-1}(0)$ and $\pi_2^{-1}(0)$,
- there are neighborhoods $B_1 \subseteq T_1$ and $B_2 \subseteq T_2$ of the basepoints $0 \in T_1$ and $0 \in T_2$, and
- an isomorphism $U_1 \to U_2$, and an isomorphism $B_1 \to B_2$ sending $0 \mapsto 0$,

such that $\pi_i(U_i) \subseteq B_i$ and the following diagram commutes

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "\pi_1"'] & U_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$

where the induced isomorphism $\pi_1^{-1}(0) \to \pi_2^{-1}(0)$ is compatible with the isomorphisms $\pi_1^{-1}(0) \cong V_0 \cong \pi_2^{-1}(0)$. This definition is natural in the context of germs of analytic spaces as discussed in Section 2.

We say that the deformation $\pi : V \to T$ is *versal* if any other deformation $W \to S$ of V_0 is induced from π by a map $U \to T$ where $U \subseteq S$ is a neighborhood of 0, such that the deformations $W \to S$ and $V \times_T U \to U$ are isomorphic.

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Given such a V_0 which is a complete intersection in a neighborhood of an isolated singularity, Kas and Schlessinger [**KS**] construct an explicit versal deformation of a sufficiently small neighborhood of the singularity. We will explain the statement of their theorem, sketch their proof and single out the special cases which imply that flat families of nodal curves $\pi : X \to T$, can be obtained locally by pullback of the *standard family*

$$C := \{((x, y), t) \in \mathbb{C}^2 \times \mathbb{C} : xy = t\} \quad \text{and} \quad \rho : C \to \mathbb{C} \quad \text{given by} \quad \rho : ((x, y), t) \mapsto t.$$

Remark 0.1. As a note of independent interest, Theorem 2.1, asserts that a flat family of *stable curves* is locally obtained by pullback of the standard family. A stable curve is a nodal curve of finite type, admitting only finitely many automorphisms. A proper flat family of stable curves $p : X \to T$ is a flat morphism which is proper, where every fiber is a stable curve. Proper flat families of stable curves arise in certain moduli problems. In particular, let $\mathcal{M}_{g,n}$ denote the moduli space of curves of genus g with n marked points. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ is a *coarse moduli space* for the *stable curves functor* see [**HK**] (in that paper, the definition of a flat family of nodal curves is given by Corollary 3.1; in fact, this was the primary motivation for writing our paper). An independent proof of Corollary 3.1 can be found in [**ACG**, Proposition 2.1].

1. Preliminaries

1.1. A warning about analytic spaces. Many sources define analytic spaces to be reduced, but it is necessary for us to allow spaces which are not necessarily reduced since even for a morphism between reduced analytic spaces, some of the fibers can be nonreduced, as shown in the following example.

Example 1.1. Let $X = \{(x, t) \in \mathbb{C}^2 : x(x - t) = 0\}$ and let $p : X \to \mathbb{C}$ be given by $(x, t) \mapsto t$. The fiber over $t \neq 0$ consists of the points $\{0, t\}$, and is reduced. But the fiber above $t = 0$ is just the point $\{0\}$, counted with multiplicity 2; more precisely, the ring of germs of functions at 0 is $\mathcal{O}_{\mathbb{C},0}/(x^2)$.

This is an important example as it shows that the local rings of functions (opposed to just the sets of points involved) play a crucial role in the definition of fibers and pullbacks for analytic spaces.

We now recall the construction of pullbacks in the category of analytic spaces. Given analytic spaces X, Y and S with maps $f : X \to S$ and $g : Y \to S$, the pullback, denoted by $X \times_S Y$ has as its underlying set of points the expected $\{(x, y) \in X \times Y : f(x) = g(y)\}$ and the local ring of germs at a point (x, y) with $f(x) = g(y) =: s$ is given by

$$\mathcal{O}_{X \times_S Y,(x,y)} = \mathcal{O}_{X,x} \hat{\otimes}_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}.$$

Here the symbol $\hat{\otimes}$ denotes the analytic tensor product, a variant of the tensor product better suited for the kinds of local rings arising in the study of analytic spaces: *local analytic* $\mathbb{C}$*-algebras*, which are defined to be local rings isomorphic to quotients of some $\mathcal{O}_{\mathbb{C}^n,0}$ by a finitely generated ideal. The analytic tensor product is the categorical pushout in the category of local analytic $\mathbb{C}$-algebras, just as the ordinary tensor product is the pushout in the category of commutative rings. To compute analytic tensor products of local analytic algebras of the special form $\mathcal{O}_{\mathbb{C}^n,0}/I$ where the ideal I is generated by finitely many *polynomials*, one can simply pretend all holomorphic functions are polynomials, do the computation using the ordinary tensor product and then unpretend. More precisely, replace the algebras by the corresponding ones of the form $(\mathbb{C}[z_1,\ldots,z_n]/I)_{\mathfrak{m}}$, compute the ordinary tensor product and perform the inverse replacement on the result. See [**A1**] for more information on analytic tensor products and local analytic algebras.

Finally, recall that by the fiber of a map $p : X \to Y$ of analytic spaces at a point $y \in Y$ we mean the analytic space given by the pullback $X \times_Y \{y\}$. So, for example, the local ring of the fiber at 0 in Example 1.1 is computed as follows:

$$\begin{aligned}
\mathcal{O}_{p^{-1}(0),0} &= \mathcal{O}_{X\times_{\mathbb{C}}\{0\},((0,0),0)} \\
&= \mathcal{O}_{X,(0,0)}\hat{\otimes}_{\mathcal{O}_{\mathbb{C},0}}\mathcal{O}_{\{0\},0} \\
&= \mathcal{O}_{\mathbb{C}^2,(0,0)}/(x(x-t))\hat{\otimes}_{\mathcal{O}_{\mathbb{C},0}}\mathbb{C} \\
&= \mathcal{O}_{\mathbb{C}^2,(0,0)}/(x(x-t))\hat{\otimes}_{\mathcal{O}_{\mathbb{C},0}}\mathcal{O}_{\mathbb{C},0}/(t) \\
&= \mathcal{O}_{\mathbb{C},0}/(x^2).
\end{aligned}$$

1.2. Families of nodal curves. A *curve* X is a reduced 1-dimensional analytic space. A point $x \in X$ is a *node* if it has a neighborhood in X isomorphic to a neighborhood of the origin in the curve of equation $xy = 0$ in $\mathbb{C}^2$. A curve X is *nodal* if for all $x \in X$, x is a node or a smooth point. Note that nodal curves are not necessarily of finite type.

A family of nodal curves is a morphism $\pi : X \to T$ such that for all $t \in T$, $p^{-1}(t)$ is a curve with nodes.

Example 1.2. Let $X = \{(x,y,t) \in \mathbb{C}^3 : x^2y^3 = t\}$ and let $p : X \to \mathbb{C}$ be given by $p : (x,y,t) \to t$. The family $p : X \to \mathbb{C}$ is *not* a family of nodal curves; indeed, although the fiber over $t = 0$ is the union of the x-axis and the y-axis (as a set of points), the fiber is not reduced, so as analytic space, there is no neighborhood of the node which is isomorphic to a neighborhood of the origin in the curve $xy = 0$.

1.3. Flatness. In many geometric branches of mathematics, families of spaces parametrized by the points of another space play an important role. Often, this kind of notion involves some sort of local triviality, making all of the fibers isomorphic in an appropriate sense. But if one wants to study singular spaces it is best to allow families of spaces that degenerate in a continuous way. The concept of a flat family is one of the most useful ways of capturing this idea of fibers varying continuously in both algebraic geometry and analytic geometry.

We now present the notion of flatness in both the algebraic and geometric contexts, as well as several criteria for flatness. The underlying theme is that flatness means something like "no unavoidable relations", and since the relevant relations are between equations or relations, it may be more precise to say "no unexpected syzygies".

1.3.1. *Flat modules.* Recall that M is a *flat module* over the ring A if tensoring with M (over A) preserves monomorphisms; that is, for every submodule N' of a module N, the natural map

$$M \otimes_A N' \to M \otimes_A N$$

is injective.

It is an exercise in commutative algebra to check that this is equivalent to the following definition:

An A-module M is flat if and only if for every ideal I of A, the natural map $I \otimes_A M \to IM$ is an isomorphism.

This last definition can be seen as an example of "no unavoidable syzygies". In geometric terms, A can be thought of as the ring of functions on some space X (in algebraic geometry one writes $X = \operatorname{Spec} A$) and M as (the module of sections of) of a (quasicoherent) sheaf on X. An ideal I of A cuts out a subspace Y of X and we can ask which sections of M vanish on Y. The restriction of M to $Y = \operatorname{Spec}(A/I)$ is given by $M \otimes_A A/I = M/IM$, so the sections that vanish on Y are IM. These are, of course, generated by sections of the form $f\sigma$ with $f \in I$. Some relations between these generators are unavoidable, namely, the ones coming from A-bilinearity of $(f, \sigma) \mapsto f\sigma$. So $I \otimes_A M$ represents the freest possible module of sections that vanish on X, IM is the actual module of vanishing sections, and the natural map between them being an isomorphism represents having no unavoidable relations.

In practice, to check whether a module M is flat over A, the following definition is perhaps the most useful: the A-module M is flat if and only if

$$\operatorname{Tor}_1^A(M, N) = 0, \text{ for any } A\text{-module } N.$$

To check that M is flat, it is sufficient to verify this for $N = A/I$, where I is a finitely generated ideal of A.

1.3.2. *Analytic flatness.* The following definition comes from algebraic geometry where it used for a morphism of schemes $\pi : X \to T$.

If $\pi : X \to T$ is a morphism of analytic spaces, then for all $x \in X$, π determines a map of local rings:

$$\pi^* : \mathcal{O}_{T,f(x)} \to \mathcal{O}_{X,x}$$

where $\mathcal{O}_{X,x}$ denotes the ring of germs of analytic functions in a neighborhood of x in X. The map π^* turns $\mathcal{O}_{X,x}$ into a module over $\mathcal{O}_{T,f(x)}$. The morphism $\pi : X \to T$ is *flat at* $x \in X$ if for all $x \in X$, $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{T,f(x)}$. We say that $\pi : X \to T$ is flat over a point $t \in T$ if for all $x \in \pi^{-1}(t)$, $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{T,f(x)}$. Note that this definition is *local* in T since it only depends on the local rings.

Following the standard terminology, we will sometimes refer to a flat morphism $\pi : X \to T$ as a *flat family* $\pi : X \to T$ over the base T.

This definition is a bit opaque, and the reader will be forgiven for not immediately realizing its geometric content; even David Mumford in [**M**] wrote: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers." A priori, flatness does not seem to be a very geometric concept; however, given certain constraints on the analytic spaces involved, there are some nice geometric descriptions.

Intuitively, a flat family $\pi : X \to T$ is one where the fibers vary continuously. More precisely, we have the following result from [**EH**].

Proposition 1.3 (Eisenbud-Harris)**.** *If the base T is a smooth 1-dimensional variety, and X is a closed subvariety of $T \times \mathbb{C}^n$, then the family $\pi : X \to T$ is flat if and only if for any $t \in T$ the closure (as an analytic space) of $X - \pi^{-1}(t)$ in X is equal to X.*

That is, if we remove the fiber over t and take the closure of what remains, we obtain our original space X.

Remark 1.4. Note that the closure in Proposition 1.3 is as analytic space; that is in a neighborhood of $x \in \pi^{-1}(t)$ the closure of $X - \pi^{-1}(t)$ in X is defined by the ideal of analytic functions that vanish on $X - \pi^{-1}(t)$. This is necessary because a closed analytic subspace of an analytic space X is not determined solely by a closed subspace of the underlying topological space of X. Indeed, the analytic structure would be missing: the sheaf of germs of analytic functions would not be specified.

Example 1.5. Consider the standard family $\rho : C \to \mathbb{C}$, where

$$C = \left\{((x,y),t) \in \mathbb{C}^2 \times \mathbb{C} : xy = t\right\}, \quad \text{and} \quad \rho : ((x,y),t) \mapsto t.$$

The fiber over $t \neq 0$ is the hyperbola $xy = t$ in $\mathbb{C}^2$, which degenerates to the union of the axes $x = 0$ and $y = 0$ over $t = 0$. For $t \neq 0$, the family is smooth and therefore flat. If we take the closure of the locus $\{xy - t \ : \ t \neq 0\} \subseteq A$, we clearly obtain the whole space C. By the criterion in Proposition 1.3, the family $\rho : C \to \mathbb{C}$ is therefore flat.

The hypothesis of the 1-dimensional base from Proposition 1.3 is necessary as demonstrated by the following example.

Example 1.6. Consider the family $\pi : X \to T$ where

$$X := \left\{((x,y),(s,t)) \in \mathbb{C}^2 \times \mathbb{C}^2 : sx + ty = 0\right\}, \quad T = \mathbb{C}^2 \quad \text{and}$$

$$\pi : ((x,y),(s,t)) \mapsto (s,t).$$

Consider the line $L_{a,b} = \{(s,t) \in T \mid as + bt = 0\}$ in the base T. For each $(s,t) \neq (0,0)$ in this line, the fiber $\pi^{-1}((s,t))$ is $bx - ay = 0$. That is, for $(s,t) \neq (0,0)$, the fiber is constant along $L_{a,b}$. The closure of $X - \pi^{-1}(0,0)$ must include all lines of the form $bx - ay = 0$ in $\{(0,0)\} \times \mathbb{C}^2$. Therefore this family satisfies all hypotheses of Proposition 1.3, except the base is not 1-dimensional.

The family is not flat because when restricted to any $L_{a,b} \subseteq T$, the fiber is constant if $(s,t) \neq (0,0)$; however, $\pi^{-1}(0,0) = \mathbb{C}^2$, so by Proposition 1.3, this restriction is not flat.

Remark 1.7. The restriction of a flat family $\pi : X \to T$ to a subvariety $T' \subseteq T$ is flat. This is a special case of the fact that pullbacks of flat families are flat families, since the restriction is simply (defined to be) the pullback $X \times_T T'$ along the inclusion $T' \hookrightarrow T$.

Proposition 1.3 can still be useful to determine if a family over a higher-dimensional base is flat, according to the following proposition.

Proposition 1.8 (Eisenbud-Harris)**.** *If the base T is a smooth variety, and X is a closed subvariety of $T \times \mathbb{C}^n$, then the family $\pi : X \to T$ is flat if and only if for any 1-dimensional variety T' mapping to T via a map $\iota : T' \to T$, the pullback*[1] *of the family $\pi : X \to T$ is flat; that is, $\iota^* X \to T'$ is flat.*

1.4. Relations criterion. Let $F_1(\mathbf{x},\mathbf{t}), \ldots F_k(\mathbf{x},\mathbf{t})$ be holomorphic functions in the variables $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{t} = (t_1, \ldots, t_m)$, and consider the family $\pi : X \to \mathbb{C}^m$ where

$$X := \{(\mathbf{x},\mathbf{t}) \in \mathbb{C}^n \times \mathbb{C}^m : \forall i \ F_i(\mathbf{x},\mathbf{t}) = 0\}, \text{ and } \pi \text{ is given by } \pi : (\mathbf{x},\mathbf{t}) \mapsto \mathbf{t}.$$

Proposition 1.9. *The family $\pi : X \to \mathbb{C}^m$ defined above is flat at $\mathbf{t} = \mathbf{0}$ if and only if for all relations*

$$\sum_{i=1}^{k} p_i(\mathbf{x}) F_i(\mathbf{x},\mathbf{0}) = 0 \textit{ with holomorphic coefficients } p_i,$$

[1]The pullback is taken in the analytic category, see Section 1.

there exist holomorphic functions $P_i(\mathbf{x},\mathbf{t})$, defined on $\mathbb{C}^n \times U$, where $U \subseteq \mathbb{C}^m$ is a neighborhood of $\mathbf{0}$, such that $P_i(\mathbf{x},\mathbf{0}) = p_i(\mathbf{x})$, and

$$\sum_{i=1}^{k} P_i(\mathbf{x},\mathbf{t}) F_i(\mathbf{x},\mathbf{t}) = 0.$$

Note that the fiber over $\mathbf{t} = \mathbf{0}$ is given by $\{\mathbf{x} \in \mathbb{C}^n : \forall\, i, F_i(\mathbf{x},\mathbf{0}) = 0\}$.

Proposition 1.9 essentially says that all of the relations corresponding to the fiber over $\mathbf{t} = \mathbf{0}$, $\sum p_i(\mathbf{x})F_i(\mathbf{x},\mathbf{0}) = 0$, come from relations in the nearby fibers, $\sum P_i(\mathbf{x},\mathbf{t})F_i(\mathbf{x},\mathbf{t}) = 0$, informally, the fiber only satisfies the relations it can't avoid satisfying by continuity.

The proof of the corresponding statement for schemes in algebraic geometry can be found in [**A2**, Part 1, Section 3]. The same proof works to prove the statement above with some slight modification: tensor products must be replaced by analytic tensor products, as discussed in Section 1.1, and one needs a replacement for the following statement: a finitely generated module M over a finite dimensional local $\mathbb{C}$-algebra A is flat if and only if $\mathrm{Tor}_1^A(M,\mathbb{C}) = 0$. In the setting of analytic spaces, the corresponding result has $\widehat{\mathrm{Tor}}$ (the derived functor of the analytic tensor product) in place of Tor and can be found in [**A1**, Proposition 4.4].

Remark 1.10. The base in Proposition 1.9 was taken to be $\mathbb{C}^m$; this is not necessary. There is a more general statement for an arbitrary affine variety as the base, and the proof indicated above actually gives this more general version. Since analytic spaces are locally affine varieties, the more general statement can be used to test for flatness for an arbitrary family $\pi : X \to T$.

2. The theorem of A. Kas and M. Schlessinger

This theorem is really a theorem about what Kas and Schlessinger call *local complex spaces*, which we will call *germs of analytic spaces*; our spaces will have basepoints, and two germs (X,a) and (Y,b) are isomorphic if there are neighborhoods (U_X,a) and (U_Y,b) such that there is an analytic isomorphism $(U_X,a) \to (U_Y,b)$. Because we are working locally, we may assume that our spaces are affine varieties (possibly nonreduced, see Example 1.1).

Let $V_0 \subseteq \mathbb{C}^n$ be an analytic variety of dimension $n-p$. Kas and Schlessinger [**KS**] give an analytic construction for a versal deformation of its germ at $\mathbf{0}$ for the case where

(1) V_0 has an isolated singularity at $\mathbf{0}$, and
(2) V_0 is a local complete intersection at $\mathbf{0}$; that is, in a neighborhood U of $\mathbf{0}$, we can find $p = \mathrm{codim}(V_0)$ holomorphic functions $f_1,\dots,f_p : U \to \mathbb{C}$ whose zero locus is equal to $U \cap V_0$.

We now give their construction. Let M be the submodule of $\prod_1^p \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ defined by

$$M := \left\{ \sum_{i=1}^{p} f_i \mathbf{a}_i + \sum_{j=1}^{n} g_j \left\langle \frac{\partial f_1}{\partial z_j}, \cdots, \frac{\partial f_p}{\partial z_j} \right\rangle \middle| \, \mathbf{a}_i \in \prod_1^p \mathcal{O}_{\mathbb{C}^n,\mathbf{0}},\ g_j \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} \right\}.$$

Point (1) is equivalent to the condition that

$$\dim_{\mathbb{C}} \left(\prod_1^p \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/M \right) < \infty.$$

To see this, note that the same formula defining M, but without taking germs at $\mathbf{0}$, defines a sheaf of $\mathcal{O}$-modules. This sheaf is supported at the singular locus of V_0 and has finite rank as an $\mathcal{O}$-module. If it is finite dimensional over $\mathbb{C}$, then the singular locus has no holomorphic functions on it (even locally) other than constants, so the singularity must be isolated (see [**A2**, p. 13]).

Now, suppose point (1) holds, and let $\mathbf{P}_1, \ldots, \mathbf{P}_l \in \prod_1^p \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ be representatives for a basis of the quotient. Let $V \subseteq U \times \mathbb{C}^l$ be the analytic space defined by

$$F_j(\mathbf{z}, \mathbf{t}) = f_j(\mathbf{z}) + \sum_{i=1}^{l} t_i P_{i,j}(\mathbf{z})$$

where $P_{i,j}$ is the jth coordinate of $\mathbf{P}_i$. We will use $\mathbf{F} : U \times \mathbb{C}^l \to \mathbb{C}^p$ to denote the vector of the F_j.

Let $\pi : V \to \mathbb{C}^l$ be the projection $\pi : (\mathbf{z}, \mathbf{t}) \mapsto \mathbf{t}$. Point (2) guarantees that π is flat. Indeed, this follows from Proposition 1.9 because being a local complete intersection implies that any relation among the f_i is a linear combination of trivial relations $f_j f_i - f_i f_j = 0$, which clearly extend to relations among the F_i (see [**P**]). This statement has a short proof in terms of Koszul cohomology: having the trivial relations span all relations is equivalent to the first Koszul cohomology group vanishing, while being a local complete intersection implies all Koszul cohomology vanishes (see Theorem 9.4 of [**H**]).

Theorem 2.1 (Kas & Schlessinger, [**KS**])**.** *The family $\pi : (V, \mathbf{0}) \to (\mathbb{C}^l, \mathbf{0})$ is a versal deformation of the germ $(V_0, \mathbf{0})$; that is, any flat deformation $\phi : (W, \mathbf{0}) \to (S, \mathbf{0})$ of $(V_0, \mathbf{0})$ is induced from $\pi : (V, \mathbf{0}) \to (\mathbb{C}^l, \mathbf{0})$ by a map $\psi : (S, \mathbf{0}) \to (\mathbb{C}^l, \mathbf{0})$.*

2.1. Outlining the proof of Kas and Schlessinger. Their proof can be decomposed into three parts:

(1) setting up equations for the map ψ and the isomorphism
$$\Omega : (V, \mathbf{0}) \times_{(\mathbb{C}^l,\mathbf{0})} (S, \mathbf{0}) \cong (W, \mathbf{0}),$$
(2) constructing many solutions to the equations from (1) in the ring of formal power series, and
(3) showing that the series solution can be made convergent in a neighborhood (if appropriate choices are made in part (2)).

Step (1): We can assume that $(S, \mathbf{0}) \subseteq (\mathbb{C}^r, \mathbf{0})$. The flatness of ϕ implies that $(W, \mathbf{0})$ can be taken to be a subset of $(\mathbb{C}^n, \mathbf{0}) \times (S, \mathbf{0})$, for the same n which appears in the definition of V_0 (see [**KS**]).

We will again use flatness to show that $(W, \mathbf{0})$ defined by $p = \operatorname{codim}(V_0)$ equations of the form $G_j(\mathbf{w}, \mathbf{s}) = 0,\ j = 1, \ldots, p$ for some functions G_j such that for all j, $G_j(\mathbf{w}, \mathbf{0}) = f_j(\mathbf{w})$.

Indeed, since $\phi^{-1}(\mathbf{0}) = V_0$, for each j, we can choose a $G_j(\mathbf{w}, \mathbf{s})$ such that $G_j(\mathbf{w}, \mathbf{0}) = f_j(\mathbf{w})$. We must show that these already define $(W, \mathbf{0})$. Complete $\{G_1, \ldots, G_p\}$ to a generating set of the ideal defining $(W, \mathbf{0})$, and let G be an element of this generating set. Since $f_1, \ldots f_p$ generate the ideal defining V_0, $G(\mathbf{w}, \mathbf{0})$ must be of the form

$$G(\mathbf{w}, \mathbf{0}) = \sum_{i=1}^{p} b_i(\mathbf{w}) f_i(\mathbf{w}).$$

By Proposition 1.9, there exist B_0, and $B_1, \dots, B_p$ such that

$$B_0(\mathbf{w},\mathbf{s})G(\mathbf{w},\mathbf{s}) = \sum_{i=1}^{p} B_i(\mathbf{w},\mathbf{s})G_i(\mathbf{w},\mathbf{s}),$$

where $B_0(\mathbf{w},\mathbf{0}) = 1$ which means the germ $B_0(\mathbf{w},\mathbf{s})$ is a unit, showing that G is in the ideal generated by $\{G_1,\dots,G_p\}$.

We seek a map $\psi : (S,\mathbf{0}) \to (\mathbb{C}^l,\mathbf{0})$ together with an isomorphism

$$\Omega : (V,\mathbf{0}) \times_{(\mathbb{C}^l,\mathbf{0})} (S,\mathbf{0}) \to (W,\mathbf{0}),$$

commuting with the projections to $(S,\mathbf{0})$. It turns out to be simpler to find a map $\psi : (\mathbb{C}^r,\mathbf{0}) \to (\mathbb{C}^l,\mathbf{0})$ (which restricts to the desired map ψ); we will similarly enlarge the natural domains of the other maps in the equations.

The isomorphism Ω, must be of the form $\Omega : ((\mathbf{z},\mathbf{t}),\mathbf{s}) \mapsto (\omega(\mathbf{z},\mathbf{s}),\mathbf{s})$; so we will look for a map $\omega : (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^r,\mathbf{0}) \to (\mathbb{C}^n,\mathbf{0})$, and to guarantee that the map $((\mathbf{z},\mathbf{t}),\mathbf{s}) \mapsto (\omega(\mathbf{z},\mathbf{s}),\mathbf{s})$ restricts to a map $(V,\mathbf{0}) \times_{(\mathbb{C}^l,\mathbf{0})} (S,\mathbf{0}) \to (W,\mathbf{0})$ we also seek a map $H : (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^r,\mathbf{0}) \to (M_p(\mathbb{C}),\mathbf{O}) = \{p \times p \text{ matrices over } \mathbb{C}\}$ such that

$$\mathbf{G}(\omega(\mathbf{z},\mathbf{s}),\mathbf{s}) = (\mathrm{Id} + H(\mathbf{z},\mathbf{s}))\mathbf{F}(\mathbf{z},\psi(\mathbf{s})). \tag{2.1}$$

The unknowns should satisfy the obvious basepoint conditions:

$$\psi(\mathbf{0}) = \mathbf{0}, \quad \omega(\mathbf{z},\mathbf{0}) = \mathbf{z}, \quad \text{and} \quad H(\mathbf{z},\mathbf{0}) = \mathbf{O}. \tag{2.2}$$

We have reduced the proof of Theorem 2.1 to finding analytic solutions to these equations. Indeed, any solution will define a map ω, and hence a map Ω. It follows from the implicit function theorem that for a given $\mathbf{s}$, $\omega(\mathbf{z},\mathbf{s})$ is an isomorphism, whose inverse is analytic. Since the matrix $\mathrm{Id}+H(\mathbf{z},\mathbf{s})$ appearing in condition (2.1) is invertible, we have that $\mathbf{G}(\omega(\mathbf{z},\mathbf{s}),\mathbf{s}) = \mathbf{0}$ if and only if $\mathbf{F}(\mathbf{z},\psi(\mathbf{s})) = \mathbf{0}$ as required for Ω to be a bijection between its stated domain and codomain.

Step (2): We'll give the argument from [**KS**]. We'll consider the unknown functions $\omega(\mathbf{z},\mathbf{s})$, $\psi(\mathbf{s})$, and $\mathbf{H}(\mathbf{z},\mathbf{s})$ as power series in $\mathbf{s}$ (whose coefficients are functions of $\mathbf{z}$) and try to solve for them one degree at a time. Let $\omega_{<N}$ denote the sum of all terms in ω of degree at most $N-1$ (degree in $\mathbf{s}$); similarly for $\psi_{<N}$ and $\mathbf{H}_{<N}$. If we've already found power series solving equation 2.1 to order $N-1$, then

$$A_N := \mathbf{G}(\omega_{<N}(\mathbf{z},\mathbf{s}),\mathbf{s}) - (\mathrm{Id} + H_{<N}(\mathbf{z},\mathbf{s}))\mathbf{F}(\mathbf{z},\psi_{<N}(\mathbf{s})) = o(\|\mathbf{s}\|^N);$$

and to solve the equation to degree N, the terms of degree N, ω_N, ψ_N, and $\mathbf{H}_N$, must satisfy

$$A_{N+1} = A_N - \sum_{k=1}^{l} \mathbf{P}_k(\mathbf{z})\psi_{N,k}(\mathbf{s}) + B_N = o(\|\mathbf{s}\|^{N+1}).$$

where

$$B_N := \sum_{j=1}^{n} \omega_{N,j}(\mathbf{z},\mathbf{s}) \left\langle \frac{\partial f_1}{\partial z_j}, \cdots, \frac{\partial f_p}{\partial z_j} \right\rangle - \mathbf{H}_N(\mathbf{z},\mathbf{s}).$$

Since the $\mathbf{P}_k$, $k = 1 \dots l$, form a basis of $\prod_1^p \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/M$, the coefficients ψ_N are uniquely determined as is $B_N \in M$. However, the coefficient functions ω_N and $\mathbf{H}_N$ appearing in the formula for B_N are not unique, since the formula writes B_N in terms of the generators of M which are not independent.

Remark 2.2. The nonuniqueness seen here reflects the fact that the deformation $\pi : (V,\mathbf{0}) \to (\mathbb{C}^l,\mathbf{0})$ is *versal* opposed to *universal*.

Step (3): We won't say much about this step. It follows from Artin's powerful realizability theorem [**A3**], which says that if a system of finitely many analytic equations has solutions which are formal power series with no constant term, then it also has solutions which are convergent power series.

Kas and Schelssinger include an argument for the specific system of equations (2.1) needed here. Their overall strategy is as follows: using complex analysis, Kas and Schlessinger first prove a lemma which asserts that certain bounds can be achieved for the coefficients ω_N and $\mathbf{H}_N$ chosen at each step. The rest of their argument shows that the greedy procedure of making independent choices at each step yields convergent power series.

3. Flat families of nodal curves

In this section we apply Theorem 2.1 to show that a flat family of nodal curves is locally a pullback of the standard family $\rho : C \to \mathbb{C}$ in Example 1.5.

Corollary 3.1. *Let $p : A \to B$ be a flat family of nodal curves. Then for every $a \in A$, there is a neighborhood U of a, neighborhood V of $b := p(a)$, a map $\psi : V \to \mathbb{C}$ and an open embedding $\widetilde{\psi} : U \hookrightarrow \psi^* C$ such that the diagram*

$$\begin{tikzcd} U \arrow[r, hook, "\widetilde{\psi}"] \arrow[dr, "p"'] & \psi^* C \arrow[r] \arrow[d] & C \arrow[d, "\rho"] \\ & V \arrow[r, "\psi"] & \mathbb{C} \end{tikzcd}$$

commutes.

Proof. First suppose that $a \in A$ is a smooth point; that is, a is not a node in its fiber. Then the germ $(p^{-1}(b), a)$ is isomorphic to $(\mathbb{C}, 0)$, and Theorem 2.1 proves that the family is trivial near a; that is, it is obtained by pullback via a constant map $\psi \neq 0$.

Now suppose that $a \in A$ is a singular point; that is, a is a node in its fiber. Then the germ $(p^{-1}(b), a)$ is isomorphic to $(\{(x, y) \in \mathbb{C}^2 : xy = 0\}, \mathbf{0})$. We now compute the versal deformation according to Theorem 2.1. The construction simplifies when $V_0 = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$ is defined by a single equation, so $p = 1$. Let $f(x, y) = xy$; we need a basis of

$$\mathcal{O}_{\mathbb{C},\mathbf{0}} \Big/ \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

We may take the basis $\{-1\}$, so $l = 1$ and $P_1(x, y) = -1$. The versal deformation is defined by the single equation $F((x, y), t) = f(x, y) + tP_1(x, y) = xy - t = 0$. $\square$

References

[A1] J. Adamus *Topics in complex analytic geometry, Lecture notes II*, www.math.uwo.ca/ jadamus/papers/AGII.pdf, 2008.

[ACG] E. Arbarello, M. Cornalba & P. Griffiths *Geometry of algebraic curves, volume II*, Fundamental Principles of Mathematical Sciences, 268. Springer, 2011.

[A2] M. Artin *Lectures on deformations of singularities*, The Tata Institute of Fundamental Research, 1976.

[A3] M. Artin *On the solutions of analytic equations*, Invent. Math. 5, 1968, 277–291.

[E] D. Eisenbud *Commutative algebra with a view toward algebraic geometry*, Springer, 1995.

[EH] D. Eisenbud & J. Harris *The geometry of schemes*, Springer, 2000.

[H] R. Hartshorne *Deformation theory*, Springer, 2010.

[HK] J. H. Hubbard & S. Koch *An analytic construction of the Deligne-Mumford compactification of the moduli space of curves*, submitted.

[KS] A. Kas & M. Schlessinger *On the versal deformation of a complex space with an isolated singularity*, Math. Ann. 196, 1972, 23–29.

[M] D. Mumford *Red book on varieties and schemes*, Springer, 1999.

[P] V. P. Palamodov *Deformations of complex spaces*, Rus. Math. Surveys 31:3, 1976, 129–197.

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