

A Whirlwind Tour of the World of $(\infty, 1)$ -categories

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ABSTRACT. This introduction to higher category theory is intended to give the reader an intuition for what $(\infty, 1)$ -categories are, when they are an appropriate tool, how they fit into the landscape of higher category, how concepts from ordinary category theory generalize to this new setting, and what uses people have put the theory to. It is a rough guide to a vast terrain, focuses on ideas and motivation, omits almost all proofs and technical details, and provides many references.

CONTENTS

1. Introduction	2
2. The idea of higher category theory	3
2.1. The homotopy hypothesis and the problem with strictness	5
2.2. The 3-type of S^2	8
2.3. Shapes for cells	10
2.4. What does (higher) category theory do for us?	11
3. Models of $(\infty, 1)$ -categories	12
3.1. Topological or simplicial categories	12
3.2. Quasi-categories	13
3.3. Segal categories and complete Segal spaces	16
3.4. Relative categories	17
3.5. A_∞ -categories	18
3.6. Models of subclasses of $(\infty, 1)$ -categories	20
3.6.1. Model categories	20
3.6.2. Derivators	21
3.6.3. dg-categories, A_∞ -categories	22
4. The comparison problem	22
4.1. Axiomatization	24
5. Basic $(\infty, 1)$ -category theory	25
5.1. Equivalences	25
5.1.1. Further results for quasi-categories	26
5.2. Limits and colimits	26
5.3. Adjunctions, monads and comonads	28

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5.4. Less basic $(\infty, 1)$ -category theory	30
5.4.1. Fibrations and the Grothendieck construction	30
5.4.2. Stable $(\infty, 1)$ -categories	32
6. Some applications	33
6.1. Derived Algebraic Geometry	34
6.1.1. The cotangent complex	35
6.1.2. Rough sketches of the definitions	36
6.1.3. Topological modular forms	38
6.2. The cobordism hypothesis	39
References	41

1. Introduction

An $(\infty, 1)$ -category is a category-like thing that besides objects and morphisms has 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on all the way to ∞ ; but in which all k -morphisms for $k > 1$ are “invertible”, at least up to higher invertible morphisms. This is the sort of invertibility that homotopies have: the composition or concatenation of any homotopy with its reverse is not actually the identity but it is homotopic to it. So we can picture an $(\infty, 1)$ -category as a “homotopy theory”: a kind of category with objects, morphisms, homotopies between morphisms, higher homotopies between homotopies and so on.

Any context where there is a notion of homotopy, can benefit from the use of $(\infty, 1)$ -categories in place of ordinary categories. This includes homotopy theory itself, of course, but also homological algebra and more generally wherever Quillen’s version of abstract homotopy theory, the theory of model categories, has been used. Notions of homotopy are perhaps more common than one might expect since the philosophy of model categories shows that simply specifying a class of “weak equivalences” in a category, a collection of morphisms which we wish to treat as if they were isomorphisms, produces a notion of homotopy. The theory of $(\infty, 1)$ -categories plays a prominent role in derived algebraic geometry, as can be expected from the very rough description of the subject as being what is obtained by replacing the notion of commutative rings in algebraic geometry by, say, commutative differential graded algebras but only caring about them up to quasi-isomorphism.

There are now several different formalizations or models of the notion of $(\infty, 1)$ -category, detailed comparison results relating the different definitions and for one particular model of $(\infty, 1)$ -category, quasi-categories, a detailed account of how ordinary category theory generalizes to the $(\infty, 1)$ context [Joy08b], [Lur09b], [Lur12b]. (Many definitions and *statements* of results from ordinary category theory generalize straightforwardly to $(\infty, 1)$ -categories, often simply by replacing bijections of Hom-sets with weak homotopy equivalences of mapping spaces, but with current technology the traditional *proofs* do not generalize, and instead often require delicate model specific arguments: most of this work has been done using the model of quasi-categories.)

Surveying the applications of (ordinary) category theory is an impossible task: categories, and categorical constructions such as products and adjoint functors appear in very many fields of mathematics. Such a survey would turn into a survey of much of mathematics. Writing an overview of the applications of $(\infty, 1)$ -categories

could potentially be similarly doomed. This paper attempts it anyway only because $(\infty, 1)$ -categories are still relatively new and have not fully caught on yet, making it possible to list a reasonable portion of the current literature. Even so, this is just a small entry point into the world of $(\infty, 1)$ -categories and the broader context of higher category theory.

The ideal reader of this survey is someone who has heard about $(\infty, 1)$ -categories (perhaps under the name ∞ -categories), is interested in reading some work that uses them (such as the derived algebraic geometry literature), or is simply curious about them but wishes to have a better idea of what they are and how they are used before committing to read a rigorous treatment such as [Joy08b] or [Lur09b]. We will not assume any prior knowledge of $(\infty, 1)$ -categories, or even more than a cursory knowledge of 2-categories, but we will assume the reader is comfortable with notions of ordinary category theory such as limits, colimits, adjoint functors (but it's fine if the reader can't give a precise statement of Freyd's Adjoint Functor Theorem, for example). We also assume the reader is acquainted with simplicial sets; if that's not the case we recommend reading [Fri12] as a gentle introduction that gives the basic definitions and properties and focuses on conveying geometrical intuition. Anyone attempting to use the theory of $(\infty, 1)$ -categories will need to know much more about simplicial sets, and [Cur71] is a good place to continue learning about them.

We will begin by briefly exploring the landscape of higher category theory to give a context for $(\infty, 1)$ -categories and describe some basic guiding principles and requirements for the theory. Then we'll go on a quick tour of all the different models available for $(\infty, 1)$ -categories and discuss the problem of comparing different definitions; an exciting recent development is Barwick and Schommer-Pries's axiomatic characterization of higher categories [BSP12]. The next section deals with practical aspects of working with $(\infty, 1)$ -categories and describes how concepts from ordinary category theory such as isomorphisms, limits and colimits, adjunctions, monads, monoidal categories and triangulated categories generalize to the $(\infty, 1)$ setting. The final section consists of (very!) brief descriptions of some of the work that applies the theory of $(\infty, 1)$ -categories.

2. The idea of higher category theory

The first hint of higher category theory comes from the category \mathbf{Cat} of categories. It not only has objects, which are categories, and morphisms between them, functors, but there are also natural transformations between functors. Indeed, \mathbf{Cat} is the basic example of a (strict¹) 2-category, just as \mathbf{Set} is the basic example of a category. Of course, once we've imagined having another layer of things we'll call 2-morphisms connecting the morphisms (in the way natural transformations connect functors), there is no reason to stop at 2.

This gives us our first blurry picture of higher categories: an n -category will have a collection of objects, and collections of k -morphisms for $1 \leq k \leq n$ with specified identity morphisms and composition operations for morphisms satisfying appropriate associativity and unit axioms; an ∞ -infinity category will be a similar structure having k -morphisms for all $k \geq 1$.

¹We'll explain what this means and have much more to say about it in Section 2.1.

REMARK 2.1. We are being very vague and purposefully so: there is a large design space to explore. There are many possible forms for composition laws and many ways of making the axioms precise, and there are even many choices for the “shape” of morphisms, that is, choices for what data specifies the analogue of domain-and-codomain of a k -morphism. We won’t have much to say about different shapes for morphisms, so that discussion is postponed to section 2.3.

Another way to visualize this idea is also already present in the 2-category Cat : given two categories \mathcal{C} and \mathcal{D} , Cat doesn’t just have a *set* of morphisms from \mathcal{C} to \mathcal{D} , it has a whole category $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors $\mathcal{C} \rightarrow \mathcal{D}$ and whose morphisms are natural transformations. Note the funny re-indexing that takes place:

- (1) functors $\mathcal{C} \rightarrow \mathcal{D}$ are 1-morphisms in Cat but are 0-morphisms (objects) in $\text{Fun}(\mathcal{C}, \mathcal{D})$,
- (2) natural transformations are 2-morphisms in Cat but are 1-morphisms in $\text{Fun}(\mathcal{C}, \mathcal{D})$.

This gives us an alternative inductive way to think of higher categories: an n -category is like a 1-category but instead of having a *Hom-sets* between any pair of objects, its Hom-things are $(n - 1)$ -categories. Readers familiar with *enriched*² category theory will recognize that this is similar to defining an n -category as a category enriched over $(n - 1)$ -categories. That actually defines what is known as a *strict* n -category and we will argue in section 2.1 that this notion does not capture the interesting examples that one would want in higher category, so we really want some kind of category “weakly enriched” over $(n - 1)$ -categories. But before we discuss that, notice that even brushing aside the issue of strictness, this perspective does not by itself define ∞ -categories, as the inductive definition becomes circular in case $n = n - 1 = \infty$. However, if we restrict our attention to higher categories in which above a certain level the morphisms behave like homotopies, we can use the inductive perspective again.

Let’s say an (n, k) -category is an n -category in which all j -morphisms for $j \geq k + 1$ are invertible in the sense homotopies are: *not* that every j -morphism $\alpha : x \rightarrow y$ has an inverse β for which $\beta \circ \alpha$ and $\alpha \circ \beta$ are exactly *equal* to the identity j -morphisms id_x and id_y , but only that there is a β for which those composites have invertible³ $(j + 1)$ -morphisms connecting them to id_x and id_y . Of course, if $j + 1 > n$, we *do* require that $\beta \circ \alpha = \text{id}_x$ and $\alpha \circ \beta = \text{id}_y$. This convention for $j + 1 > n$ is consistent with viewing any n -category as an $(n + 1)$ -category where all the $(n + 1)$ -morphisms are identities. Finally, we can similarly talk about (∞, k) -categories (where reaching the top degree for morphisms is not an issue), and the bulk of this survey will focus on the $(\infty, 1)$ case.

REMARK 2.2. A useful metaphor has us think of an invertible morphism between two objects as a proof that they are the “same”. Just as with proofs of theorems in mathematics, sometimes one can argue that two proofs are “really the same proof”; such an argument corresponds to an invertible 2-morphism between two 1-morphisms. Then we can think of proofs establishing that two ways of showing that two proofs that two objects are the same are the same are the same, and

²The definition of enriched category is recalled in section 2.1.

³Invertible in this same sense, so this definition is recursive.

so on⁴. In other words: an $(\infty, 0)$ -category, usually called an ∞ -*groupoid*, is what a set is forced to become if we are never satisfied to just note that two things can be proven to be the same, but instead we write down the proof and contemplate the possibility that different looking proofs can be proven to be the same. This is what people mean when they say higher category theory *systematically replaces equality by isomorphism*.

For these (n, k) -categories and (∞, n) -categories, the inductive perspective says that an (n, k) -category has Hom-things which are $(n - 1, k - 1)$ -categories (which does not buy us anything new), but also that an (∞, k) -category has $(\infty, k - 1)$ -categories as Hom-things. To start picturing (∞, n) -categories, we need to know how to visualize $(\infty, 0)$ -categories, which is the next topic on our agenda.

2.1. The homotopy hypothesis and the problem with strictness. The 2-category of categories is *strict*, meaning that the composition of both its 1-morphisms and 2-morphisms is associative and has units (the identity 1- and 2-morphisms), as opposed to being just something like “associative up to homotopy”. One says that in Cat composition is *strictly* associative. As mentioned above, it is easy to define n -categories with strictly associative and unital compositions inductively using the notion of enriched category. Recall that given a monoidal category \mathcal{V} with tensor product given by a functor $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$, a category \mathcal{C} enriched over \mathcal{V} (sometimes called a \mathcal{V} -category) consists of

- a collection of objects,
- Hom-objects $\mathcal{C}(X, Y) \in \mathcal{V}$, for every pair of objects of \mathcal{C} ,
- composition morphisms $\mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ of \mathcal{V} , for every triple of objects of \mathcal{C} ,
- identities given as morphisms $I \rightarrow \mathcal{C}(X, X)$ in \mathcal{V} , for every object of \mathcal{C} (where I is the tensor unit in \mathcal{V}),

and this data is required to satisfy obvious unit and associativity axioms (whose precise statement requires using the unit and associativity constraints of \mathcal{V}). When \mathcal{V} is a category with finite products, we can take the tensor product to be the categorical product (and I to be the terminal object); when equipped with this tensor product, \mathcal{V} is said to be a *Cartesian* monoidal category. There is also a notion of \mathcal{V} -enriched functor between two categories \mathcal{C} and \mathcal{D} enriched over \mathcal{V} : a function associating to every object $X \in \mathcal{C}$ an object $FX \in \mathcal{D}$, plus a collection of morphisms of \mathcal{V} , $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$ compatible with identities and composition.

We can now give the inductive definition of strict n -categories:

DEFINITION 2.3. A *strict n -category* is a category enriched over the Cartesian monoidal category StrCat_{n-1} . The category StrCat_n whose objects are all strict n -categories and whose morphisms are StrCat_{n-1} -enriched functors is easily seen to have finite products, making the recursion well defined. The base case can be taken to be StrCat_1 , the (1-)category of categories and functors or even $\text{StrCat}_0 = \text{Set}$

The only higher category we’ve mentioned so far is Cat , and it *is* a strict 2-category, but that’s more or less it for naturally occurring examples of strict 2-categories, in the sense that almost all natural examples are based on functions and composition of functions.

⁴Limited only by the number of times we are willing to say “are the same” in a row.

EXAMPLE 2.4. A monoid M can be regarded as a category that has a single object x for which $\text{Hom}(x, x) = M$ with composition given by the monoid multiplication. In a similar way one can try to turn a monoidal category \mathcal{V} into a 2-category with one object x for which $\text{Hom}(x, x) = \mathcal{V}$ with composition given by the tensor product in \mathcal{V} . This does not produce a strict 2-category unless the tensor product is strictly associative and unital. The point of this example is that most naturally occurring monoidal categories are *not* strict. For example, the tensor product of vector spaces is only defined up to canonical isomorphism, and while $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, it is exceedingly unlikely that any actual choice of specific vector spaces for all tensor products would render both sides exactly *equal*. Similar remarks apply to products for Cartesian monoidal categories.

REMARK 2.5. There is a standard notion of non-strict 2-category: the notion of *bicategory* due to Bénabou [Bén67] (or see [Lac10]), that has a definition very similar to the usual definition of monoidal category and which reduces to it in the case of a bicategory with a single object.

While we have given what we feel are natural examples of 2-categories that fail to be strict, maybe they do not make a conclusive case for the need to weaken the associativity and unitality axioms: MacLane’s coherence theorem for monoidal categories shows that any monoidal category is (monoidally) equivalent to one where the tensor product is strictly associative. And more generally any 2-category is equivalent to a strict one⁵ (see [Lei98] for an expository account). But once we get to 3-categories, the situation is different: there are examples that cannot be made strict. We’ll give an explicit example in section 2.2, namely, the fundamental 3-groupoid of S^2 ; but first we will discuss fundamental higher groupoids and their role in higher category theory.

Higher groupoids are special cases of higher categories, namely an n -groupoid, in the terminology explained above, is an $(n, 0)$ -category and an ∞ -groupoid is a $(\infty, 0)$ -category. Before we explain what higher fundamental groupoids should be, recall that the fundamental groupoid packages the fundamental groups of a space X at all base points into a single category $\pi_{\leq 1}X$ whose objects are the points of X and whose morphisms $x \rightarrow y$ are endpoint-preserving homotopy classes of paths from x to y . Composition is given by concatenation of paths (which is not strictly associative and unital before we quotient by homotopy⁶). For a space X that has some non-zero higher homotopy groups, $\pi_{\leq 1}X$ clearly does not contain all the homotopical information of X , but for 1-*types* it does.

⁵On the other hand maybe we do have a conclusive case for considering more general notions than strict *functors*: not every functor of bicategories between strict 2-categories is equivalent to a strict 2-functor! See [Lac07, Lemma 2] for an example. Bénabou has expressed the view that the point of bicategories is not that they are non-strict themselves, but that they are the natural home for non-strict functors.

⁶We could use Moore paths, which are maps $[0, \ell] \rightarrow X$ for some $\ell \geq 0$ called the length of the path. When concatenating Moore paths, the lengths add. This operation is strictly associative and unital, but (1) the category of Moore paths is not a groupoid, since the reversal of a path only is an inverse up to homotopy, and (2) there is no analogue of Moore paths for the fundamental n -groupoid when $n > 1$.

DEFINITION 2.6. A space⁷ X is called an n -type if $\pi_k(X, x) = 0$ for all $k > n$ and all $x \in X$.

The homotopy theory of 1-types is completely captured by groupoids:

- (1) The fundamental groupoid functor induces an equivalence between (a) the homotopy category of 1-types, where the morphisms are homotopy classes of continuous functions between 1-types⁸, and (b) the homotopy category of groupoids, whose morphisms are equivalence classes of functors between groupoids, two functors being equivalent if there is a natural isomorphism between them.
- (2) The inverse of the equivalence described above can be given by a *classifying space* functor B that generalizes the well-known construction for groups and is defined before passing to homotopy categories, i.e., is a functor from the category of groupoids to the category of 1-types. Any groupoid G is equivalent to $\pi_{\leq 1}BG$, and any 1-type X is homotopy equivalent to $B\pi_{\leq 1}X$.
- (3) Given two 1-types X and Y (or more generally, an arbitrary space X and a 1-type Y), the space of maps $\text{Map}(X, Y)$ is a 1-type and its fundamental groupoid is the category of functors $\text{Fun}(\pi_{\leq 1}X, \pi_{\leq 1}Y)$ (which is automatically a groupoid too).

This means that homotopy theoretic questions about 1-types can be translated to questions about groupoids which thus provide complete algebraic models for 1-types. This is the simplest case of perhaps the main guiding principle in the search for adequate definitions in higher category theory: the *homotopy hypothesis* proposed by Grothendieck in [Gro83]. As is common now, we interpret (and phrase!) it as stating desired properties of a theory of higher categories.

The homotopy hypothesis: Any topological space should have a fundamental n -groupoid for each n (including $n = \infty$). These should furnish all examples of n -groupoids in the sense that every n -groupoid should be equivalent to the fundamental n -groupoid of some space. Furthermore, the theory of n -groupoids should be the “same” as the homotopy theory of n -types (where if $n = \infty$, “the homotopy theory of n -types” is just “homotopy theory”).

Notice that this only puts requirements on (n, k) -categories for $k = 0$, so it certainly does not tell the whole story of higher category theory, but it is enough to rule out basing the theory on strict n -categories as we’ll see in the next section. This means that we must search for definitions of higher categories that are *non-strict* or *weak*, in the same sense that monoidal categories are weak: instead of associativity meaning that given three k -morphisms f , g and h , the composites $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are *equal*, we should only require them to be linked by an invertible $(k+1)$ -morphism $(f \circ g) \circ h \rightarrow f \circ (g \circ h)$ that could be called an *associator*. The reader familiar with the definition of monoidal category will know that these associators should satisfy a condition of their own. Given four k morphisms $f, g,$

⁷For technical reasons, space here shall mean “space with the homotopy type of a CW-complex”, otherwise some of the statements need homotopy equivalences replaced by weak homotopy equivalence.

⁸This definition of the category is correct because we took 1-types to have the homotopy type of a CW-complex; we could instead consider the category obtained from 1-types by inverting weak homotopy equivalences.

h and k , we can relate the composites $((fg)h)k$ and $f(g(hk))$ in two different ways (we've dropped the \circ for brevity):

$$\begin{array}{ccc}
 & (f(gh))k & \\
 & \nearrow & \searrow \\
 ((fg)h)k & & f((gh)k) \\
 \searrow & & \swarrow \\
 (fg)(hk) & \longrightarrow & f(g(hk))
 \end{array}$$

For the case of monoidal categories (where f , g , h and k are objects, $\circ = \otimes$, and the associator is a 1-morphism) we've reached the top level already and we require this diagram to commute; but in a higher category we can instead require this to commute only up to an invertible $(k+2)$ -morphism we could call a *pentagonator*. This *pentagonator* must satisfy its own condition, but only up to higher morphisms and so on. This kind of data —the associators, *pentagonators*, etc.— are what is meant to exist when saying an operation is associative up to *coherent homotopy*.

Clearly, drawing these diagrams gets complicated very quickly and indeed, definitions of n -categories along these lines have only been written down for n up to 4 —for a definition of tricategories see [GPS95] or [Gur06], for tetracategories see [Tri06] or [Hof11]. Instead people find ways of implicitly providing all these higher homotopies in a clever roundabout way. We'll see some examples in the section on models of $(\infty, 1)$ -categories.

2.2. The 3-type of S^2 . We will show that the fundamental 3-groupoid of S^2 is not equivalent to a strict 3-groupoid, or, in other words, that there is no strict 3-groupoid that models the 3-type of S^2 , which is commonly denoted P_3S^2 in the theory of Postnikov towers.⁹ What we mean by “models” is that we assume the existence of classifying space functors (with certain properties we'll spell out later that *are* satisfied for the “standard realization functors”, see the discussion after Theorem 2.4.2 of [Sim12]) that produce an n -type BG for a strict n -groupoid, and we say G models a space X if BG is homotopy equivalent to X . The argument shows, more generally, that if X is a simply connected n -type modeled by a strict n -groupoid G , X is in fact an infinite loop space and even a product of Eilenberg-MacLane spaces.

Let's investigate when we can *deloop* a given strict n -groupoid G , i.e., when G can be realized as $\text{Hom}_H(x, x)$ for some strict $(n+1)$ -groupoid H with a single object x . That's easy enough: if there exists such an H , G inherits from it a composition $\mu : G \times G \rightarrow G$ which makes it into a monoid object in StrCat_n , and clearly for each such monoid structure we can form a delooping H . If we want to deloop more than once, we need a monoid structure on H . And here something remarkable happens: a monoid structure $\nu : H \times H \rightarrow H$, in particular restricts to a new monoid structure $\nu_G : G \times G \rightarrow G$ on $G = \text{Hom}_H(x, x)$, and, since ν is a StrCat_n -enriched functor, this ν_G must be compatible with composition in H , that is, with μ . The end result

⁹Recall that the 3-type of S^2 can be obtained, say, by building S^2 as a CW-complex and then inductively attaching cells of larger and larger dimension to kill all homotopy groups π_i for $i \geq 4$.

is that G has *two* monoid structures one of which is homomorphism for the other. The classical Eckmann-Hilton argument¹⁰ implies that $\nu_G = \mu$ and that they are commutative. Also, conversely, if μ is a commutative monoid structure for G , H does in fact become a commutative monoid under $\nu(x) = x$, $\nu_G = \mu$.

This means that in the world of strict n -groupoids, being able to deloop twice already implies that you can deloop arbitrarily many times! Using this it is easy to see why you can't find a strict 3-groupoid that models P_3S^2 . If there were such a 3-groupoid G , without loss of generality we could assume G had a single object x and a single 1-morphism id_x : otherwise just take the sub-strict-3-groupoid consisting of x , id_x and the groupoid $\text{Hom}_G(\text{id}_x, \text{id}_x)$. But then G is the second delooping of $\text{Hom}_G(\text{id}_x, \text{id}_x)$, which shows that G in turn can be delooped arbitrarily many times. If we had classifying spaces for groupoids that were compatible with looping (by which we mean we had an n -type BG for each strict n -groupoid G such that if G has a single object x , ΩBG is weakly homotopy equivalent to $B(\text{Hom}_G(x, x))$), it would follow that P_3S^2 is an infinite loop space, which it is not. In fact, if classifying spaces preserved products (i.e., $B(G \times H) \cong BG \times BH$), we'd have that P_3S^2 would be a topological abelian monoid and thus homotopy equivalent to a product of Eilenberg-MacLane spaces. It would then have to be $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$, but it is not, since, for example, the Whitehead product $\pi_2S^2 \times \pi_2S^2 \rightarrow \pi_3S^2$ is non-zero.

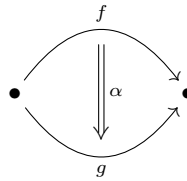
REMARK 2.7. Vanishing of the Whitehead product $\pi_2 \times \pi_2 \rightarrow \pi_3$ does not guarantee that a 3-type can be modeled by a strict 3-groupoid. Consider the space $X = P_3QS^2 = P_3 \text{colim } \Omega^n \Sigma^n S^2$. That Whitehead product for X agrees with the corresponding product for QS^2 but since QS^2 is an infinite loop space its Whitehead products of all degrees vanish. We have that $\pi_2X = \mathbb{Z}$ and $\pi_3X = \mathbb{Z}/2$, the first stable homotopy group of spheres, but one can see X is not homotopy equivalent to $K(\mathbb{Z}, 2) \times K(\mathbb{Z}/2, 3)$ by looking at the operation $\pi_2W \rightarrow \pi_3W$ given by composing (maps representing homotopy classes) with the generator of $\pi_3(S^2)$: this operation is non-zero for $W = X$, but is zero for the product of Eilenberg-MacLane spaces. By the argument above, X is not modeled by a strict 3-groupoid.

FURTHER READING. Carlos Simpson [Sim12, Section 2.7] proved that there is no classifying space functor for strict 3-groupoids such that BG is homotopy equivalent to P_3S^2 under weaker assumptions than we sketched above: he does not assume that classifying spaces are compatible with looping, in fact, he does not require there to be a family of classifying space functors for strict n -groupoids for all n at all; just a single functor for $n = 3$ satisfying the minimal requirements that BG be a 3-type and that the homotopy groups of BG are functorially isomorphic to algebraically defined ones for G . The simpler argument we sketched (under the stronger assumption of compatibility with looping) can be found in [Sim12, Section 2.6]. Clemens Berger proved a stronger result characterizing all connected 3-types (not necessarily simply connected) that can be modeled by strict 3-groupoids [Ber99, Corollary 3.4].

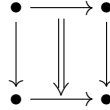
¹⁰This says that if a set M has two different monoid structures given by products \cdot and $*$, and we have $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$ —which says that $*$: $(M, \cdot) \times (M, \cdot) \rightarrow (M, \cdot)$ is a monoid homomorphism—then $\cdot = *$ and M is commutative. Here we are actually using an extension to strict n -categories instead of sets, which is essentially obtained by applying the classical statement to each degree of morphism separately.

2.3. Shapes for cells. There are many possibilities for the shapes of the morphisms in an n -category, of which we'll give a brief representative list. Here we will call the morphisms *cells*, since the word “morphism” is a little awkward when a k -morphism does not simply go from one $(k - 1)$ -morphism to another.

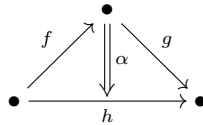
In the case of the 2-category of categories, the 2-morphisms, which are natural transformations, go between two 1-morphisms (functors) that are parallel, i.e., that share their domain and share their codomain. This pattern can be generalized for higher morphisms and is called *globular*, because drawings of such morphisms looks like topological balls, or more precisely like one of the usual CW-complex structures on disks: the one in which the boundary of the disk is divided into hemispheres meeting along a sphere, which is also divided into hemispheres and so on. Here's a picture of a globular 2-cell:



For another example of a shape for morphisms think of homotopies, homotopies between homotopies, and so on. As we mentioned in the introduction, this is one of the examples we are trying to capture. Such higher homotopies are maps $X \times [0, 1]^n \rightarrow Y$, and so naturally have a *cubical* shape. A cubical 2-morphism looks like a square, and its analogue of domain-and-codomain, the boundary of the square, has four objects and four 1-morphisms:



When we get to discussing models for $(\infty, 1)$ -categories, and specifically the model of quasi-categories (which are simplicial sets satisfying some condition), we will encounter another shape for morphisms: *simplicial*. A 2-morphism is shaped like a triangle:



We can interpret α as a homotopy between $g \circ f$ and h (or alternatively we can interpret composition as multivalued, in which case, h is some composite of g and f , and α is a witness to that fact). Similarly we can think of a higher dimensional simplex as being a coherent collection of homotopies between composites of a string of 1-morphisms. See section 3.2.

There are more elaborate cell shapes as well, such as *opetopes*, introduced by John Baez and James Dolan [BD98] (see also [Bae97] which besides describing opetopes and the proposed definition of higher category based on them, is a nice

introduction to n -categories generally). These can be interpreted as being a homotopy between the result of evaluating a *pasting diagram* and a specified target morphism. This is analogous to the above interpretation of simplices, but allowing for more general pasting diagrams than those given by strings of 1-morphisms.

FURTHER READING. To look at pictures of the zoo of higher categories, we recommend the illustrated guidebook by Eugenia Cheng and Aaron Lauda [CL04]. (In particular, the above description of opetopes is meaningless without pictures, which can be seen there.) For a concise list of many of the available definitions for n -category and ∞ -category see [Lei02]. See also the book [Lei04], particularly Chapter 10.

Those sources concentrate on definitions attempting to capture n -categories without any requirement of invertibility of morphisms. Thanks to the homotopy hypothesis and to the availability of topological spaces, simplicial sets and homotopy theory, it has turned out somewhat easier in practice to work with notions of (∞, n) -categories (which of course include n -categories as a special case). As John Baez said about climbing up the categorical ladder from 1-groupoids to ∞ -groupoids [Bae05]¹¹:

[...] the n -category theorists meet up with the topologists — and find that the topologists have already done everything there is to do with ∞ -groupoids. . . but usually by thinking of them as spaces, rather than ∞ -groupoids!

It's sort of like climbing a mountain, surmounting steep cliffs with the help of ropes and other equipment, and then finding a Holiday Inn on top and realizing there was a 4-lane highway going up the other side.

For the homotopical perspective and a focus on (∞, n) -categories see [Sim12]. The rest of this survey will mostly focus on $(\infty, 1)$ -categories.

2.4. What does (higher) category theory do for us? The reader might be asking now how exactly higher category theory is useful in mathematics. Here is one possible answer, a purely subjective and personal answer, and should be disregarded if the reader does not find it convincing. It is now widely recognized that category theory is a highly versatile and profitable organizing language for mathematics. Many fields of mathematics have objects of interest and distinguished maps between them that form categories, many comparison procedures between different kinds of objects can be represented as functors and, perhaps, most importantly, basic notions from category theory such as products, coproducts (or more general limits and colimits) and adjoint functors turn out to be well-known important constructions in the specific categories studied in many fields. While it is not reasonable to expect that category theory will swoop in and solve problems from other fields of mathematics, phrasing things categorically does help spot analogies between different fields and to pinpoint where the hard work needs to happen: often arguments are a mix of “formal” parts, which depend very little on the detailed structure of the objects being studied, and “specific” parts which involve understanding their distinguishing properties; categorical language makes short work of many formal arguments, thus highlighting the remainder, the “essential mathematical content”

¹¹John Baez's web column *This Week's Finds* is a highly recommended source for intuition about higher categories.

of an argument. Higher category theory promises to extend the scope of such formal methods to encompass situations where we wish to consider objects up to a weaker notion of equivalence than isomorphism; for example, we almost always wish to consider categories up to equivalence, in homological algebra we consider chain complexes up to quasi-isomorphism, and in homotopy theory we consider space up to homotopy equivalence or weak homotopy equivalence.

3. Models of $(\infty, 1)$ -categories

This sections gives a list of the main models of $(\infty, 1)$ -categories and attempts to motivate each definition. We spend more time discussing quasi-categories than the other models, and in later sections we'll mostly just use quasi-categories whenever we need particular models. The reader will notice an abundance of simplicial sets appearing in the definitions, and is warned again that some basic knowledge of them will be required.

Ideally we would describe for each model, say,

- the definition of $(\infty, 1)$ -category,
- the corresponding notion of functor and natural transformation or, even better, given two $(\infty, 1)$ -categories \mathcal{C} and \mathcal{D} , the $(\infty, 1)$ -category $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors and whose 1-morphisms are natural transformations,
- how to retrieve the Hom- ∞ -groupoid, or *mapping space* $\text{Map}_{\mathcal{C}}(X, Y)$ between two objects of a given $(\infty, 1)$ -category,
- the homotopy category $\text{ho}\mathcal{C}$ of a given $(\infty, 1)$ -category, which is the ordinary category with the same objects as \mathcal{C} and whose morphisms correspond to homotopy¹² classes of morphisms in \mathcal{C} .

Sadly, for reasons of space we will not do all of those for each model, but we hope to mention enough of these to give an idea of how the story goes.

One excellent feature of the $(\infty, 1)$ portion of higher category theory is that the problem of relating different definitions has a satisfactory answer which will be described in the following section.

FURTHER READING. For a more detailed introduction to the different models and the comparison problem, we recommend [Ber10], [JT06] or [Por04].

3.1. Topological or simplicial categories. As we mentioned above, we can think of an $(\infty, 1)$ -category as a category weakly enriched in ∞ -groupoids, and to satisfy the homotopy hypothesis we could “cheat” and define ∞ -groupoids as topological spaces or simplicial sets (whose homotopy theory is well-known to be equivalent to that of topological spaces). It turns out that one can always “strictify” the enrichment in ∞ -groupoids, meaning that we can model $(\infty, 1)$ -categories using:

DEFINITION 3.1. A *topological category* is a category enriched over the category of topological spaces¹³. A *simplicial category* is similarly a category enriched over the category of simplicial sets.

¹²Recall that since in an $(\infty, 1)$ -category 2-morphisms and higher are invertible, we tend to think of them as homotopies

¹³Instead of the category of all topological spaces it is better to use a so-called “convenient category of spaces” [Ste67], such as compactly generated weakly Hausdorff spaces (see,

These models of $(\infty, 1)$ -categories are perhaps the easiest to visualize and are a great psychological aid but are inconvenient to work with in practice because, among other problems, enriched functors do not furnish all homotopy classes of functors between the $(\infty, 1)$ -categories being modeled, unless the domain and codomain satisfy appropriate conditions¹⁴. And even when those conditions are met, the category of enriched functors might not correctly model the $(\infty, 1)$ -functor category, essentially because enriched functors correspond to functors that preserve composition strictly while $(\infty, 1)$ -functors are allowed to preserve it only up to coherent homotopy.

Finally, notice that although we “cheated” by putting the homotopy hypothesis into the definition, there is a sense in which we don’t trivially get it back out! We obtained a definition of $(\infty, 1)$ -category through enrichment from a definition of ∞ -groupoid, but having done so we now have a second definition of ∞ -groupoid: an $(\infty, 1)$ -category in which all 1-morphisms are invertible (up to higher morphisms, as always). In terms of the homotopy category $\text{ho}\mathcal{C}$, this definition of ∞ -groupoid that accompanies any notion of $(\infty, 1)$ -category is simply: an $(\infty, 1)$ -category \mathcal{C} for which $\text{ho}\mathcal{C}$ is a groupoid.

For topological or simplicial categories it is easy to construct $\text{ho}\mathcal{C}$: take the set of morphisms between X and Y in $\text{ho}\mathcal{C}$ to be $\pi_0(\mathcal{C}(X, Y))$; since π_0 preserves products, the composition law in \mathcal{C} induces a composition for $\text{ho}\mathcal{C}$. Now, given a topological¹⁵ category \mathcal{C} for which $\text{ho}\mathcal{C}$ is a groupoid, what is the space C that this ∞ -groupoid is supposed to correspond to? Think first of the case when \mathcal{C} has a single object X . Then $M := \mathcal{C}(X, X)$ is a topological monoid and $\text{ho}\mathcal{C}$ being a groupoid just says that $\pi_0(M)$ is a group under the operation induced by the multiplication in M . The topological category \mathcal{C} is a delooping¹⁶ of M , so we should have $\Omega C \cong M$, and there is such a space: the classifying space $C = BM$ of M ; when $\pi_0(M)$ is a group, the unit map $M \rightarrow \Omega BM$ is a weak homotopy equivalence. For general groupoids $\text{ho}\mathcal{C}$, the space C corresponding to \mathcal{C} will be a disjoint union of classifying spaces of the monoid of endomorphisms of an object chosen from each component of $\text{ho}\mathcal{C}$.

3.2. Quasi-categories. There are two classes of examples we certainly wish to have in any theory of $(\infty, 1)$ -categories: (a) ordinary categories (just add identity morphisms in all higher degrees), and (b) ∞ -groupoids, which by the homotopy hypothesis we can take to be anything modeling all homotopy types of spaces. After spaces themselves, the best known models for homotopy types are *Kan complexes*, simplicial sets X that satisfy the *horn filler* condition: that every map $\Lambda_k^n \rightarrow X$ extends to a map $\Delta^n \rightarrow X$. (Recall that Λ_k^n is obtained from the boundary $\partial\Delta^n$ of Δ^n by removing the k -th face.) Also, every category \mathcal{C} has a nerve which is a simplicial set whose n -simplices are indexed by strings of n composable morphisms of \mathcal{C} ; and the nerve functor $N : \text{Cat} \rightarrow \text{sSet}$ is fully faithful. So inside the category

for instance, **[Str]**). This is to make the comparison with other models smoother and is a technical point the reader can safely ignore. One should be aware though, that when using a different category of spaces, the product —which appears in the domain of the composition map $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ — does not carry the usual product topology.

¹⁴Namely, that the domain be cofibrant and the codomain be fibrant in the model structures discussed in section 4.

¹⁵The simplicial case is analogous.

¹⁶See section 2.2.

Set of simplicial sets we find both ordinary categories and Kan complexes and so we might expect to find a good definition of an $(\infty, 1)$ -category as a special kind of simplicial set. The following easy characterization of those simplicial sets which arise as nerves of categories shows what to do:

PROPOSITION 3.2. *A simplicial set X is isomorphic to the nerve of some category if and only if every map $\Lambda_k^n \rightarrow X$ with $0 < k < n$ extends uniquely to a map $\Delta^n \rightarrow X$.*

The least common generalization of the condition above and the definition of Kan complex is:

DEFINITION 3.3. *A quasi-category is a simplicial set in which all inner horns can be filled, that is, in which every map $\Lambda_k^n \rightarrow X$ with $0 < k < n$ extends to a map $\Delta^n \rightarrow X$.*

Probably the greatest advantage of quasi-categories over other models for $(\infty, 1)$ -categories is how straightforward it is to deal with functors. A functor $\mathcal{C} \rightarrow \mathcal{D}$ between two quasi-categories is simply a map of simplicial sets: the structure of the quasi-categories makes any such maps behave like a functor. (This is related to the nerve functor being fully faithful.) Moreover, there is a simple way to obtain the $(\infty, 1)$ -category of functors between two quasi-categories: it is just the simplicial mapping space¹⁷ $\mathcal{D}^{\mathcal{C}}$, which is automatically a quasi-category whenever \mathcal{C} and \mathcal{D} are. In fact, more generally, given a quasi-category \mathcal{C} , and an arbitrary simplicial set X , \mathcal{C}^X is a quasi-category which we think of as the category of X -shaped diagrams in \mathcal{C} .

The definition of quasi-category is very clean, but it may seem mysterious that it does not mention anything like composition of morphisms. Quasi-categories have something like a “multivalued” composition operation. Consider two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in a quasi-category \mathcal{C} —this really means that X , Y and Z , are vertices or 0-simplices in the simplicial set \mathcal{C} and that f and g are 1-simplices with the specified endpoints. The data (X, f, Y, g, Z) determines a map $\Lambda_1^2 \rightarrow \mathcal{C}$, that we display by drawing Λ_1^2 and labeling the simplices by their images in \mathcal{C} . A filler for this horn is a 2-simplex α whose third edge h gives a possible composite of g and f . The 2-simplex itself can be considered to be some sort of certificate that h is a composite of g and f . There may be more than one composite h , and for a given h there may be more than one certificate.

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & & Z \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & Y & \\ f \nearrow & \alpha & \searrow g \\ X & \Downarrow & Z \\ & h & \end{array}$$

This might seem like chaos, but homotopically composition is well-defined in a sense we’ll now make precise. The space of composable pairs of 1-simplices in \mathcal{C} is given by the simplicial mapping space $\mathcal{C}^{\Lambda_1^2}$ and the space of “certified compositions” is similarly \mathcal{C}^{Δ^2} . The set of vertices of $\mathcal{C}^{\Lambda_1^2}$ is precisely the set of pairs of composable 1-simplices, and the higher dimensional simplices capture homotopies between diagrams of composable pairs, and homotopies between those, and so on. Similar remarks apply to \mathcal{C}^{Δ^n} .

¹⁷This is the internal hom in \mathbf{sSet} , its n -simplices are simplicial maps $\mathcal{C} \times \Delta^n \rightarrow \mathcal{D}$.

PROPOSITION 3.4 (Joyal). *For a quasi-category \mathcal{C} , the map $\mathcal{C}^{\Delta^2} \rightarrow \mathcal{C}^{\Lambda_1^2}$ induced by composition with the inclusion $\Lambda_1^2 \hookrightarrow \Delta^2$ is a trivial Kan fibration, which implies in particular that its fibers are contractible Kan complexes.*

REMARK 3.5. Joyal proved the converse as well: if \mathcal{C} is a simplicial set such that $\mathcal{C}^{\Delta^2} \rightarrow \mathcal{C}^{\Lambda_1^2}$ is a trivial Kan fibration, then \mathcal{C} is a quasi-category.

We can think of the map in the proposition roughly as, given a “certified composition”, forgetting both the certificate and the composite being certified. That the fibers of this map are contractible says that to a homotopy theorist composition is uniquely defined after all.

This result can be extended to strings of n composable 1-simplices, namely, for a quasi-category \mathcal{C} the canonical map $\mathcal{C}^{\Delta^n} \rightarrow \mathcal{C}^{P_n}$ is a trivial Kan fibration. Here, $P_n = \Delta^1 \vee_{\Delta^0} \Delta^1 \vee_{\Delta^0} \cdots \vee_{\Delta^0} \Delta^1$ is the simplicial path of length n obtained by gluing n different 1-simplices end to end. The map $\mathcal{C}^{\Delta^n} \rightarrow \mathcal{C}^{P_n}$ is induced by the inclusion of P_n into Δ^n as the *spine of the simplex*, that is, the subsimplicial set generated by the 1-simplices $\{0, 1\}, \{1, 2\}, \dots, \{n-1, n\}$ in Δ^n . The case $n = 3$ can be interpreted as specifying a precise sense in which composition is associative.

We can generalize even further to say that when defining a functor from the free (ordinary) category on a directed graph X into a quasi-category \mathcal{C} , we can choose a diagram of 0-simplices and 1-simplices in \mathcal{C} of shape X arbitrarily: there will always be an extension to a functor, and moreover, the space of all such extension is contractible. Formally, we have:

PROPOSITION 3.6. *Let X be a reflexive¹⁸ directed graph which we will think of as a simplicial¹⁹ set which has no non-degenerate k -simplices for $k \geq 2$. For any quasi-category \mathcal{C} , the canonical map $\mathcal{C}^{NFX} \rightarrow \mathcal{C}^X$ is a trivial Kan fibration, where NFX is the nerve of the free category on X .*

For $X = P_n$, the free category on X is the category which objects $0, 1, \dots, n$ and a unique morphism from i to j when $i \leq j$; its nerve is the n -simplex Δ^n , so we recover the previous statement. As an example of this proposition, take X to be a single loop²⁰: an X -shaped diagram in a category is an object together with an endomorphism. The free category on X is just the monoid of natural numbers under addition regarded as a one object category, say, $FX = \mathbb{BN}$. In the world of ordinary categories, once you’ve chosen an object and an endomorphism f of it, you’ve uniquely specified a functor out of \mathbb{BN} : the functor sends k to $f^k = f \circ f \circ \cdots \circ f$. For quasi-categories, there is no canonical choice of f^k , you must make a choice for each k and then, to specify a functor out of \mathbb{BN} you need to further choose homotopies and higher homotopies showing you made compatible choices of iterates of f . The proposition says then that all these choices (of iterates and homotopies between their composites) can be made and that they are homotopically unique in the sense that the spaces of such choices is contractible.

We haven’t yet described how to get at mapping spaces in quasi-categories. One intuitive approach is to use the arrow $(\infty, 1)$ -category of \mathcal{C} , which is simply the simplicial mapping space \mathcal{C}^{Δ^1} . This has a projection π to $\mathcal{C} \times \mathcal{C} = \mathcal{C}^{\Delta^0 \sqcup \Delta^0}$ which

¹⁸Reflexive means the graph has a distinguished loop at each vertex; these will play the role of the identities in the free category on the graph.

¹⁹When thought of a simplicial set, it is understood that the degenerate 1-simplices are the distinguished loops in the graph.

²⁰Well, a single non-distinguished loop, in addition to the distinguished one.

sends each 1-simplex of \mathcal{C} to its source and target. Then, given two objects X and Y in \mathcal{C} , we can think of them as being picked out by maps $\Delta^0 \rightarrow \mathcal{C}$ and form the pullback:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & & \downarrow \pi \\ \Delta^0 & \xrightarrow{(X, Y)} & \mathcal{C}^{\Delta^0 \sqcup \Delta^0} \end{array}$$

This does work, that is, it produces a simplicial set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ with the correct homotopy type, but there are many other descriptions of the mapping spaces that are all homotopy equivalent but not isomorphic as simplicial sets. One such alternative description of the mapping spaces is given by Cordier’s *homotopy coherent nerve* [Cor82], used in [Lur09b] to compare quasi-categories with simplicial categories. Cordier’s construction not only provides models for the mapping spaces but is also a procedure for strictifying composition in a quasi-category: that is, constructing a simplicial category (where composition *is* required to be single-valued and strictly associative) that represents the same $(\infty, 1)$ -category as a given quasi-category. Dan Dugger and David Spivak in [DS11b] explain a really nice way to visualize the mapping spaces appearing in the homotopy coherent nerve through “necklaces” of simplices strung together; they also wrote a second paper giving a detailed comparison of the known constructions for mapping spaces in quasi-categories [DS11a].

3.3. Segal categories and complete Segal spaces. Segal categories are a different formalization of the idea discussed above for quasi-categories of a multivalued composition that is uniquely defined homotopically. Just as quasi-categories can be motivated by Proposition 3.2, Segal categories can be motivated by the following equally easy result:

PROPOSITION 3.7. *A simplicial set X is isomorphic to the nerve of a category if and only if for each n , the canonical map $X_n \rightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a bijection.*

This canonical map is the map $X^{\Delta^n} \rightarrow X^{P_n}$ we’ve already met in section 3.2: it sends an n -simplex to its *spine*, the string of 1-simplices connecting vertices 0 and 1, 1 and 2, \dots , $n - 1$ and n . It is tempting to try to make compositions only defined up to homotopy simply by requiring these canonical maps to be homotopy equivalences instead of bijections, but, of course, that requires working with spaces rather than sets.

DEFINITION 3.8. A Segal category is a simplicial space (or more precisely a simplicial simplicial-set), that is, a functor $\mathbf{\Delta}^{\mathrm{op}} \rightarrow \mathrm{sSet}$ such that

- (1) the space of 0-simplices X_0 is discrete, and
- (2) for each n , the canonical map $X_n \rightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a weak homotopy equivalence.

Complete Segal spaces, also called Rezk categories, were defined by Charles Rezk in [Rez01]; his purpose was explicitly to find a nice model for the “homotopy theory of homotopy theories”, i.e., the $(\infty, 1)$ -category of $(\infty, 1)$ -categories²¹. Their

²¹ $(\infty, 1)$ -categories naturally form an $(\infty, 2)$ -category, but we can discard non-invertible natural transformations to get an $(\infty, 1)$ -category.

definition is a little complicated and we'll only describe it informally, but they do have some advantages one of which was worked out by Clark Barwick in his PhD thesis [Bar05]: the construction of complete Segal spaces starting from simplicial sets as a model for ∞ -groupoids, can be iterated to provide a model for (∞, n) -categories. These are called *n-fold complete Segal spaces*, see [BSP12] or [Lur09c] for a definition, if Barwick's thesis proves too hard to get a hold of.

A *Segal space* like a Segal category, is also a simplicial space, but we do not require that the space of objects X_0 be discrete. In that case, the second condition must be modified to use homotopy pullbacks²² so that it reads: for each n , the canonical map $X_n \rightarrow X_1 \times_{X_0}^h X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1$ is a weak homotopy equivalence. The *completeness* condition has to do with the fact that having a non-discrete space of objects means we have two different notions of equivalence of objects: one is having an invertible morphism between them in the $(\infty, 1)$ -category modeled by X , the other is being in the same connected component of X_0 . Even better, there are two canonical ∞ -groupoids of objects: one is the *core* of the $(\infty, 1)$ -category modeled by X , this is the subcategory obtained by throwing away all non-invertible 1-morphisms (all higher morphisms are already invertible); the other is the ∞ -groupoid represented by X_0 . The core of X can be described as a simplicial set directly in terms of the simplicial space X ; the completeness condition then says that it and X_0 are homotopy equivalent.

3.4. Relative categories. Relative categories are based on the intuition that higher category theory is meant for situations where we want to treat objects up to a notion of equivalence that is weaker than isomorphism in the category they live in. The reader should have in mind the examples of equivalence of categories, homotopy equivalence of spaces, quasi-isomorphism of chain complexes, etc. The definition of a relative category couldn't be simpler:

DEFINITION 3.9. A *relative category* is a pair $(\mathcal{C}, \mathcal{W})$ of an ordinary category \mathcal{C} and a subcategory \mathcal{W} of \mathcal{C} required only to contain all the objects of \mathcal{C} . Morphisms in \mathcal{W} are called *weak equivalences*.

Implicit in the claim that these somehow provide a model for $(\infty, 1)$ -categories is the claim that out of just a collection of weak equivalences we get some sort of notion of homotopy between morphisms, to play the role of 2-morphisms in the $(\infty, 1)$ -category represented by a given relative category. To give the first idea of how this happens, let's describe the homotopy category of the $(\infty, 1)$ -category modeled by $(\mathcal{C}, \mathcal{W})$: it is $\mathcal{C}[\mathcal{W}^{-1}]$, the *localization* of \mathcal{C} obtained by formally adding inverses for all morphisms in \mathcal{W} . Let's see in the example $\mathcal{C} = \text{Top}$, $\mathcal{W} = \{\text{homotopy equivalences}\}$ that homotopic maps become equal²³ as morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$. First, notice that the projection $p : X \times [0, 1] \rightarrow X$ is a homotopy equivalence and thus becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$. This means that the two maps $i_0, i_1 : X \rightarrow X \times [0, 1]$ given by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ become equal in the localization because $p \circ i_0 = p \circ i_1$. Finally, two maps are homotopic when they can be written in the form $f \circ i_0$ and $f \circ i_1$ for a single map f .

²²The reader unfamiliar with homotopy limits can find a quick introduction in section 5.2.

²³One's first instinct —at least, if one hasn't localized rings which are not integral domains— might be that *adding* inverses to some morphisms shouldn't force other morphisms to become equal.

But of course, a satisfactory answer to the question of how higher morphisms appear in the $(\infty, 1)$ -category represented by $(\mathcal{C}, \mathcal{W})$ would construct the mapping space between two objects of \mathcal{C} , and this is precisely what an enhancement of localization called *simplicial localization* does. We refer the reader to the classic papers of William Dwyer and Dan Kan [DK80a], [DK80b] for details on how simplicial localizations may be constructed. The most intuitive construction is probably the hammock localization, which we'll explain by contrasting it with the (non-simplicial) localization $\mathcal{C}[\mathcal{W}^{-1}]$.

Morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ can be represented by *zig-zags*:

$$X_0 \xleftarrow{\sim} X_1 \rightarrow X_2 \xleftarrow{\sim} \cdots \rightarrow X_n$$

where arrows can go either way, but if they point to the left they are required to be in \mathcal{W} (this is typically indicated by placing a \sim on the arrow).²⁴ To form the mapping space in the hammock localization we add homotopies between zig-zags: the mapping space is constructed as a simplicial sets whose vertices are zig-zags and whose 1-simplices are weak equivalences of zig-zags, by which we mean diagrams of the form:

$$\begin{array}{ccccccc} & & X_1 & \xleftarrow{\sim} & X_2 & \longrightarrow & X_3 & \xleftarrow{\sim} & \cdots & \longrightarrow & X_{n-1} & & \\ & \nearrow & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & & & \downarrow \wr & & \searrow \\ X_0 & & Y_1 & \xleftarrow{\sim} & Y_2 & \longrightarrow & Y_3 & \xleftarrow{\sim} & \cdots & \longrightarrow & Y_{n-1} & & X_n \end{array}$$

in which all the left pointing morphisms and all the vertical ones are required to be in \mathcal{W} . Higher dimensional simplices are similar but have more rows (and look even more like hammocks than 1-simplices do).²⁵

FURTHER READING. The papers by Dwyer and Kan on simplicial localization already indicate that relative categories, bare-bones though they may be, can be used to model $(\infty, 1)$ -categories. The book [DHKS04] develops homotopy theory for relative categories (there called “homotopical categories” and required to satisfy a mild further axiom). More recently, Clark Barwick and Dan Kan, in a series of papers [BK12c], [BK12a], [BK12b], compare relative categories (which they named) to other models of $(\infty, 1)$ -categories and define a generalization of them that provides a model for (∞, n) -categories.

3.5. A_∞ -categories. A (non-symmetric) *operad* is a collection of spaces $\mathcal{O}(n)$ together with composition maps

$$\mathcal{O}(n) \times \mathcal{O}(k_1) \times \mathcal{O}(k_2) \times \cdots \times \mathcal{O}(k_n) \rightarrow \mathcal{O}(k_1 + k_2 + \cdots + k_n),$$

which are required to satisfy associativity conditions²⁶ that are easy to guess if one thinks of the process of substituting n functions, one of k_1 variables, one of k_2 variables, etc., into a function of n variables to obtain an overall function of $k_1 + k_2 + \cdots + k_n$ variables. As this suggests, the elements of $\mathcal{O}(n)$ are called n -ary

²⁴Really, the morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ are equivalence classes of zig-zags in the smallest equivalence relation preserved by the operations of (1) removing an identity morphism, (2) composing two consecutive morphisms that point the same way, and (3) canceling a pair of the form $\cdot \xrightarrow{w} \cdot \xleftarrow{w} \cdot$ or $\cdot \xleftarrow{w} \cdot \xrightarrow{w} \cdot$.

²⁵In this description some details are missing, see [DK80b].

²⁶We are also omitting a couple of conditions on $\mathcal{O}(0)$ and $\mathcal{O}(1)$.

operations. They can be used to parametrize all the homotopies required for a composition that is associative up to coherent homotopy:

DEFINITION 3.10. An A_∞ -operad is one such that all $\mathcal{O}(n)$ are contractible. Given any such operad, an A_∞ -category \mathcal{C} consists of

- (1) a collection of objects,
- (2) a space $\mathcal{C}(X, Y)$ for every pair of objects, and
- (3) composition maps

$$\mathcal{O}(n) \times \mathcal{C}(X_{n-1}, X_n) \times \mathcal{C}(X_{n-2}, X_{n-1}) \times \cdots \times \mathcal{C}(X_0, X_1) \rightarrow \mathcal{C}(X_0, X_n)$$

for every n and every sequence of objects X_0, X_1, \dots, X_n ,

which are required to be compatible with the composition operations of the operad in an obvious sense.²⁷

REMARK 3.11. This is not the most common definition of the term “ A_∞ -category”. Usually the term denotes a related concept using chain complexes instead of spaces. See section 3.6.3.

Notice that, just like topological categories, this model provides easy access to homotopy categories: since $\mathcal{O}(n)$ is contractible, applying π_0 to an A_∞ -category produces an ordinary category with Hom-sets given by $\pi_0(\mathcal{C}(X, Y))$.

In case the A_∞ -operad \mathcal{O} is the operad of Stasheff associahedra, an A_∞ -category with a single object is equivalent to the original notion of an A_∞ -space introduced by Jim Stasheff in his work on homotopy associative H -spaces [Sta63]. The main result of that work can be interpreted as proving the homotopy hypothesis for A_∞ -categories with a single object. We’ll state a less precise version informally:

PROPOSITION 3.12. *An A_∞ -space X is weak homotopy equivalent to a loop space ΩY (in such a way that composition in X corresponds to concatenation of loops) if and only if $\pi_0(X)$ is a group under the operation induced from composition in X .*

This *recognition principle* for loop spaces is part of the original motivation for Peter May’s definition of operad²⁸, which he used to prove a recognition principle in the same spirit for iterated loop spaces $\Omega^n Y$ and infinite loop spaces (which can be thought of as a sequence of spaces Y_0, Y_1, \dots each of which is equivalent to the loop spaces of the following one). See May’s book [May72].

FURTHER READING. This model doesn’t seem to get used that much in practice. The only example of a paper constructing some $(\infty, 1)$ -category as an A_∞ -category that the author is aware of is [AC12, Proposition 1.4]. Todd Trimble used A_∞ -categories as the first step in an inductive definition of (∞, n) -category, see [Che11]. A talk given in Morelia by Peter May [May05] expressed the hope that a simpler, more general version of the inductive approach would work. The author is unaware of the current status of May’s approach, but it seems that Michael Batanin pointed out some difficulty in carrying it out (this is mentioned in Eugenia Cheng’s paper just cited).

²⁷In both the description of operad and A_∞ -category we’ve omitted discussing identities. The reader can easily supply the missing details.

²⁸Which, we repeat, besides the data mentioned above includes actions of Σ_n on $\mathcal{O}(n)$ and requires the composition maps to be equivariant in an easily guessed sense.

3.6. Models of subclasses of $(\infty, 1)$ -categories. There are also several ways of modeling special classes of $(\infty, 1)$ -categories, which, when applicable can be simpler to calculate with. We'll mention model categories and derivators for which it is hard to say exactly which $(\infty, 1)$ -categories they can model, but which certainly can only model $(\infty, 1)$ -categories that have all small homotopy limits and colimits, and linear models, which model $(\infty, 1)$ -categories that are enriched over an $(\infty, 1)$ -categories of chain complexes.

3.6.1. *Model categories.* Quillen's model categories are the most successful setting for abstract homotopy theory. A *model category* \mathcal{C} is an ordinary category that has all small limits and colimits and is equipped with three collections of morphisms called *cofibrations*, *fibrations* and *weak equivalences* which are required to satisfy axioms that abstract properties that hold of the classes of maps of topological spaces that they are named after. We won't give a precise definition, but refer the reader to standard references such as the introduction [DS95], the books [Hov99] and [Hir03], the fast paced [Lur09b, Appendix A.2] or Quillen's original [Qui67] (but note that what we are calling model categories were called *closed* model categories there). The $(\infty, 1)$ -category modeled by a model category \mathcal{C} is just the one modeled by the relative category $(\mathcal{C}, \text{weak equivalences})$. We mention them separately from relative categories because the extra structure makes them easier to deal with than a random relative category, so they form an eminently practical way to construct particular $(\infty, 1)$ -categories. Even on the level of homotopy categories, the fibrations, cofibrations and the axioms make the localization better behaved. For example, in the homotopy category of a model category we do not need to consider zig-zags of arbitrary length, it is enough to look at zig-zags of the form $\cdot \xleftarrow{\sim} \cdot \rightarrow \cdot \xleftarrow{\sim}$.

REMARK 3.13. Small homotopy limits and colimits always exist in a model category and thus they can only model $(\infty, 1)$ -categories that are complete and cocomplete. It is not known to the author whether or not all such $(\infty, 1)$ -categories arise from model categories. There is however a result of Carlos Simpson's under further smallness assumptions, namely he showed that *combinatorial* model categories provide models precisely for the class of *locally presentable* $(\infty, 1)$ -categories. See [Sim99] (but beware that what are called cofibrantly generated model categories there are what we are calling combinatorial model categories), or [Lur09b, Section 5.5.1]. Roughly speaking, a locally presentable category is one that is cocomplete and generated under colimits by a small subcategory of objects which are small or compact in some sense. A combinatorial model category is required to be locally presentable (and to have a model structure which is cofibrantly generated, which is also a condition with the flavor of the whole being determined by a small portion). For information about Jeff Smith's notion of combinatorial model category see [Bek00], which is the first published account.

Model categories have been hugely successful in providing workable notions of homotopy theory in many topological and algebraic contexts. A wealth of model structures have been constructed and all provide examples of $(\infty, 1)$ -categories that people care about. When performing further constructions based on these $(\infty, 1)$ -categories, such as taking categories of diagrams in one of them, functors between two of them or homotopy limits or colimits of them it can be very hard to remain in the world of model categories. In those cases, using model categories to present the

inputs to these constructions but carrying them out in the world of $(\infty, 1)$ -categories is a very reasonable compromise.

We will meet model categories again in section 4 since the original comparison results between models of $(\infty, 1)$ -categories were formulated in that language.

3.6.2. Derivators. When working with an $(\infty, 1)$ -category \mathcal{C} , it might be tempting to do as much as possible in $\mathrm{ho}\mathcal{C}$, since ordinary categories are much simpler and more familiar objects. We can't get very far, though, we run into trouble as soon as we start talking about homotopy limits and colimits²⁹. Say we have a small (ordinary) category \mathcal{I} and wish to talk about homotopy limits or colimits of \mathcal{I} -shaped diagrams in \mathcal{C} . Homotopy limits should be homotopy invariant: if two diagrams $F, G : \mathcal{I} \rightarrow \mathcal{C}$ are connected by a natural isomorphism³⁰, they should have equivalent limits in \mathcal{C} . So, taking homotopy limits should induce a functor $\mathrm{ho}(\mathcal{C}^{\mathcal{I}}) \rightarrow \mathrm{ho}\mathcal{C}$. Now, this homotopy category $\mathrm{ho}(\mathcal{C}^{\mathcal{I}})$ is not something we can construct just from $\mathrm{ho}\mathcal{C}$ and \mathcal{I} , in particular it is not equivalent to $(\mathrm{ho}\mathcal{C})^{\mathcal{I}}$.

EXAMPLE 3.14. Let \mathcal{I} be $\mathbb{Z}/2$ regarded as a category with a single object and let \mathcal{C} be the $(\infty, 1)$ -category of ∞ -groupoids (or spaces). An \mathcal{I} -shaped diagram in \mathcal{C} is just a space with an action of $\mathbb{Z}/2$. Consider the diagrams given by the trivial action and the 180° rotation on S^1 . Since the 180° rotation is homotopic to the identity on S^1 , these two diagrams become equal in $(\mathrm{ho}\mathcal{C})^{\mathcal{I}}$, but are not isomorphic in $\mathrm{ho}(\mathcal{C}^{\mathcal{I}})$ since, for example, they have different homotopy colimits: since the rotation action is free, the homotopy colimit in that case is just $S^1/(\mathbb{Z}/2) \cong S^1$; for the trivial action, we get $(E(\mathbb{Z}/2) \times S^1)/(\text{diagonal action}) = B(\mathbb{Z}/2) \times S^1$.

The idea of derivators is to hold on to not just $\mathrm{ho}\mathcal{C}$, but $\mathrm{ho}(\mathcal{C}^{\mathcal{I}})$ for every small (ordinary) category \mathcal{I} as well. This at least allows one to hope to be able to discuss homotopy limits and colimits. Given an $(\infty, 1)$ -category \mathcal{C} , the construction $\mathcal{I} \rightarrow \mathrm{ho}(\mathcal{C}^{\mathcal{I}})$ provides a strict 2-functor $(\mathrm{Cat}_{\mathrm{small}})^{\mathrm{op}} \rightarrow \mathrm{Cat}$ where Cat is the strict 2-category of all not necessarily small categories and $\mathrm{Cat}_{\mathrm{small}}$ is the sub-2-category of small ones.³¹ By definition, derivators are strict 2-functors $(\mathrm{Cat}_{\mathrm{small}})^{\mathrm{op}} \rightarrow \mathrm{Cat}$ satisfying further conditions that guarantee that homotopy limits and colimits (and more generally homotopy versions of the left and right Kan extensions) exist and are well-behaved. As in the case of model categories: (1) the definition directly implies derivators can only model $(\infty, 1)$ -categories which are complete and cocomplete, (2) the author does not know if all such $(\infty, 1)$ -categories can be modeled, and (3) adding presentability on both sides of the equation balances it, see [Ren06]. The later [Ren09] deals with representing $(\infty, 1)$ -categories coming from left proper model categories by derivators.

FURTHER READING. Derivators were defined by Alexander Grothendieck (the term appears first in [Gro83], a few years later Grothendieck wrote [Gro91]) and independently by Alex Heller [Hel88] (who called them ‘‘homotopy theories’’).

²⁹See section 5.2 for a quick introduction.

³⁰A natural transformation whose components are invertible in the sense we always use for $(\infty, 1)$ -categories: invertible up to higher invertible morphisms.

³¹If the reader is not versed in the art of worrying about size issues, we advise not to start until after reading this survey. We do however caution that while it might seem like a merely technical point there is substance to it: for example, it is easy to prove that if the collection of morphisms of a category has size λ and the category has all products $\prod_{i < \lambda} X_i$ then it is a preorder.

Good introductions can be found in [Mal01], [Gro13], and the review section of [GPS12].

3.6.3. *dg-categories, A_∞ -categories.* Now we'll discuss two "linear" models (or more precisely, models based on chain complexes) for special kinds of $(\infty, 1)$ -categories that have seen much use in algebra and algebraic geometry. These are the notions of dg-categories, which are analogous to topological or simplicial categories, and A_∞ -categories, which are analogous to the identically named A_∞ -categories mentioned in the previous section. In both cases the analogy comes from replacing spaces by chain complexes. We will only discuss the case where the analogy is closest, namely, when the chain complexes are *connective*, that is, concentrated in non-negative degrees; but the definitions can easily be adapted to chain complexes with different boundedness conditions.

DEFINITION 3.15. Let R be a commutative ring. A *differential graded category* or *dg-category* over R is a category enriched in the monoidal category of connective chain complexes of R -modules.

The move from spaces to connective chain complexes can be thought of as going from ∞ -groupoids to the abelian, fully strict ∞ -groupoids. This philosophy is encoded by the Dold-Kan correspondence which establishes an equivalence of categories between simplicial abelian groups and chain complexes of abelian groups concentrated in non-negative degrees. Given a dg-category \mathcal{C} , we can form a category enriched in simplicial sets by applying the Dold-Kan correspondence to all of the hom complexes (and forgetting the abelian group structure), this gives the $(\infty, 1)$ -category \mathcal{C} as a simplicial category. There is also a direct construction of the quasi-category corresponding to \mathcal{C} , called the dg-nerve of \mathcal{C} , see [Lur12b, Construction 1.3.1.6].

A_∞ -categories can be defined using an operad in chain complexes, analogously to the definition in section 3.5, but there is a more explicit description of them in terms of one n -ary composition operation for each n . We will not repeat the definition here, since we won't have anything to say about it, but the interested reader can look it up in the references.

FURTHER READING. Good introductions to dg-categories include [Kel06] and [Toë10]. The $(\infty, 1)$ -category of all dg-categories is described as a model category in Gonçalo Tabuada's PhD thesis [Tab07]. That model structure is used in [Toë07] to develop Morita theory for dg-categories. A concise introduction to A_∞ -categories can be found in [Kel06]; a thorough reference is the book [BLM08]. Also see Maxim Kontsevich and Yan Soibelman's notes [KS09] which deal mostly with A_∞ -algebras which are A_∞ -categories with a single object (although that definition is anachronistic, of course). The most conspicuous example of an A_∞ -category is the Fukaya category of a symplectic manifold that plays a starring role in homological mirror symmetry. It was first mentioned in [Fuk93]. For comprehensive treatments see the books [FOOO09] and [Sei08], the later of which has a good introduction to A_∞ -categories in Chapter 1.

4. The comparison problem

Comparing different definitions of higher categories is harder than it might seem at first, since even what it means to show that two theories are equivalent

is not completely clear. Imagine we wish to compare two theories of n -categories, call them red categories and blue categories. Just like ordinary categories form a 2-category, red n -categories should form an $(n+1)$ -category, and we'd like to show that this $(n+1)$ -category is equivalent in an appropriate sense to the $(n+1)$ -category of blue n -categories. But what color are the $(n+1)$ -categories and the equivalence? It is reasonable to expect that red n -categories form a red $(n+1)$ -category and similarly that the blue n -categories form a blue $(n+1)$ -category. But this means we can't easily compare the $(n+1)$ -categories before solving the comparison for red and blue higher categories!

We might be able to assemble all blue n -categories into a red $(n+1)$ -category by an ad hoc construction, and show that that red $(n+1)$ -category is red-equivalent to the red $(n+1)$ -category of red n -categories. In that case we could say the red theory regards the two theories as equivalent. But a priori, if that happens, the blue theory might disagree and not consider the two theories equivalent!

The way the comparison problem was first solved for $(\infty, 1)$ -categories is as follows:

- (1) What gets compared are not the $(\infty, 2)$ -categories of all $(\infty, 1)$ -categories of particular kind, but rather $(\infty, 1)$ -categories of $(\infty, 1)$ -categories (obtained from the $(\infty, 2)$ -category by throwing away non-invertible natural transformations).
- (2) For each model M the $(\infty, 1)$ -category $\text{Cat}_{(\infty, 1)}^M$ of all M -style $(\infty, 1)$ -categories was described as a model category.³²
- (3) It was shown that these model categories are all connected by zig-zags of *Quillen equivalences*: these are equivalences of categories that preserve enough aspects of the model structure to ensure that two model categories have the same homotopy theory, i.e., model the same $(\infty, 1)$ -category.

In the non-standard terminology above, this says that the theory of model categories regards all models as equivalent. It can also be shown that any model regards all models as equivalent. For example, take quasi-categories. One can construct a quasi-category from a model category and show that two model categories connected by a zig-zag of Quillen equivalences produce quasi-categories that are equivalent (according to a particular definition of equivalence of quasi-categories). Then we get a quasi-category $\text{Cat}_{(\infty, 1)}^M$ for each of the models, and they are all equivalent.

Portions of the program outlined above showing the equivalence of five of the models we discussed (namely quasi-categories, simplicial categories, Segal categories, complete Segal spaces and relative categories) were carried out by Julie Bergner [Ber07], Clark Barwick and Dan Kan [BK12a], [BK12a], André Joyal and Miles Tierney [Joy08a], [JT06]. For a beautiful diagram showing the Quillen equivalences at a glance and further references see [BSP12, Figure 1]. For a summary of the model structures and Quillen equivalences comparing the first four models (i.e., excluding relative categories) see Julie Bergner's survey [Ber10].

³²This does not necessarily mean that a model structure was put precisely on some ordinary category category of all M -style $(\infty, 1)$ -categories; but rather on a larger category in which the objects which are both fibrant and cofibrant are the M -style $(\infty, 1)$ -categories. For example, for quasi-categories the model structure is on the category of all simplicial sets: every object is cofibrant and the fibrant ones are precisely the quasi-categories.

4.1. Axiomatization. A recent breakthrough in the theory of (∞, n) -categories is the axiomatization by Clark Barwick and Chris Schommer-Pries [BSP12] of the $(\infty, 1)$ -category $\text{Cat}_{(\infty, n)}$ of (∞, n) -categories. As in the direct comparison results we mentioned for $(\infty, 1)$ -categories, what gets axiomatized is not an $(\infty, n+1)$ -category of (∞, n) -categories but rather the $(\infty, 1)$ -category one obtains from that $(\infty, n+1)$ -category by throwing away higher non-invertible morphisms. Their work was inspired by Bertrand Toën’s influential [Toë05] that similarly axiomatizes *model categories* of $(\infty, 1)$ -categories. Toën’s axioms are closely related to Giraud’s axioms for toposes, while Barwick and Schommer-Pries’s axioms stray a bit further from them. Very roughly, Barwick and Schommer-Pries axioms are as follows:

- (1) There is an embedding in $\text{Cat}_{(\infty, n)}$ of the category of *gaunt* n -categories. A gaunt n -category is a strict n -category all of whose invertible k -morphisms are identities, for all k . The images of these gaunt n -categories are required to generate $\text{Cat}_{(\infty, n)}$ under homotopy colimits.
- (2) The embedding provides us in particular with (∞, n) -categories that are “walking k -cells”: they consist of a single k -morphism and its required (globular) boundary. Certain gaunt n -categories obtained by gluing a few cells together are required to still be obtained by the same gluing process inside $\text{Cat}_{(\infty, n)}$, that is, the embedding of gaunt n -categories is required to preserve a few colimits of diagrams of cells.
- (3) $\text{Cat}_{(\infty, n)}$ is required to have internal Homs, and more generally, slices $\text{Cat}_{(\infty, n)}/C_k$ are required to have internal Homs for all cells C_k .
- (4) The $(\infty, 1)$ -category $\text{Cat}_{(\infty, n)}$ is *universal* with the respect to the above properties, in the sense that any other $(\infty, 1)$ -category \mathcal{D} with an embedding of gaunt n -categories satisfying the first three axioms is obtained from $\text{Cat}_{(\infty, n)}$ via a localization functor $\text{Cat}_{(\infty, n)} \rightarrow \mathcal{D}$ that commutes with the embeddings.

Even more roughly: the first axiom gives us the ability to present (∞, n) -categories by means of generators and relations, and the second guarantees that at least for presentations of gaunt n -categories we get the expected answer. There are many imprecisions in our description of the axioms and we refer the reader to [BSP12] for a correct statement.

REMARK 4.1. One might think that discarding those morphisms loses too much information, but in a sense it doesn’t: the $(\infty, 1)$ -category $\text{Cat}_{(\infty, n)}$ characterized by the axioms is Cartesian closed, so for any two (∞, n) -categories \mathcal{C} and \mathcal{D} there is an (∞, n) -category $\text{Fun}(\mathcal{C}, \mathcal{D})$ that contains all the non-invertible natural transformations and higher morphisms that are not directly observable in the mapping space $\text{Map}_{\text{Cat}_{(\infty, n)}}(\mathcal{C}, \mathcal{D})$.

Toën also proved in [Toë05] that the only automorphism of the theory of $(\infty, 1)$ -categories is given by taking opposite categories. It is unique in the strong sense that not only is every automorphism equivalent to that one, but automorphisms themselves have no automorphisms, or more precisely, there is a naturally defined ∞ -groupoid of automorphisms of $\text{Cat}_{(\infty, 1)}$ and it is homotopy equivalent to the discrete group $\mathbb{Z}/2$. Barwick and Schommer-Pries prove the analogue of this result for $\text{Cat}_{(\infty, n)}$, showing that its space of automorphisms is the discrete group $(\mathbb{Z}/2)^n$ corresponding to the possibility of deciding for each degree k separately whether or not to flip the direction of the k -morphisms.

5. Basic $(\infty, 1)$ -category theory

In this section we review how the most basic concepts of category theory generalize to $(\infty, 1)$ -categories. The philosophy is that $(\infty, 1)$ -categories are much more like ordinary 1-categories than fully weak (∞, ∞) -categories where no morphisms are required to be invertible³³. A little more precisely, the intuition that $(\infty, 1)$ -categories have spaces of morphisms and that these spaces only matter up to (weak) homotopy equivalence usually leads to useful definitions and correct statements. We will also frequently point out what happens in the case of quasi-categories, which due both intrinsically to some features they possess and externally to the availability of [Joy08b], [Lur09b] and [Lur12b] are probably the most “practical” model to work with.

FURTHER READING. Besides those systematic treatises by Joyal and Lurie already mentioned, we recommend [Gro10], an excellent and well motivated summary of large chunks of [Lur09b], [Lur07a], [Lur07b] and [Lur07c] (the last three of which were reworked into the first few chapters of [Lur12b]). The availability of [Gro10] is why we feel justified in giving very little detail in this section, just giving the flavor of the topic. Also highly recommended is the book [Rie14], Part IV of which is about quasi-categories and includes, among other things, (1) a discussion of which aspects of $(\infty, 1)$ -categories are already captured by the 2-category whose objects are quasi-categories, whose morphisms are functors and whose 2-morphisms are homotopy classes of natural transformations; and (2) plenty of geometrical information about quasi-categories viewed as simplicial sets, such as how to visualize mapping spaces in quasi-categories or the homotopy coherent nerve that produces the equivalence between quasi-categories and simplicial categories.

5.1. Equivalences. There are two things typically called *equivalences* in $(\infty, 1)$ -category theory: one generalizes isomorphisms in an ordinary category, and the other generalizes equivalences between categories. These are related in the following way: we can ignore non-invertible natural transformations to get $\text{Cat}_{(\infty, 1)}$, the $(\infty, 1)$ -category of $(\infty, 1)$ -categories, functors and invertible natural transformations. A functor is then an equivalence of $(\infty, 1)$ -categories if and only if it is an equivalence as a morphism in $\text{Cat}_{(\infty, 1)}$.

REMARK 5.1. That $(\infty, 1)$ -isomorphisms go by the name equivalences is probably due to (1) the case of the $(\infty, 1)$ -category of spaces where they are just (weak) homotopy equivalences, and (2) there being, in the model of topological categories (resp. simplicial categories), a second, stricter notion of isomorphism, coming from enriched category theory: morphisms such that composition with them induces homeomorphisms (resp. isomorphisms of simplicial sets) between mapping spaces.

DEFINITION 5.2. A morphism $f : X \rightarrow Y$ in an $(\infty, 1)$ -category is an *equivalence* if its image in $\text{ho}\mathcal{C}$ is an isomorphism, or, equivalently, if for every object $Z \in \mathcal{C}$, $f \circ - : \text{Map}_{\mathcal{C}}(Z, X) \rightarrow \text{Map}_{\mathcal{C}}(Z, Y)$ and $- \circ f : \text{Map}_{\mathcal{C}}(Y, Z) \rightarrow \text{Map}_{\mathcal{C}}(X, Z)$ are weak homotopy equivalences.

³³In fact, even 2-categories have some tricky points that do not arise when dealing with $(\infty, 1)$ -categories, such as the need to distinguish between several kinds of limits and colimits called pseudo-limits, lax-limits and colax-limits.

André Joyal proved, using the model of quasi-categories, that when a morphism f is an equivalence, one can coherently choose an inverse g for it, 2-morphisms showing f and g are inverses, 3-morphisms showing invertibility of those 2-morphisms, and so on. More precisely, such a coherent system of choices is given by a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ where \mathcal{J} is the ordinary category $0 \xrightarrow{\cong} 1$, which has two objects and a unique isomorphism between them. The precise statement then is as follows:

PROPOSITION 5.3 ([Joy02, Corollary 1.6]). *A morphism f in an $(\infty, 1)$ -category \mathcal{C} is an equivalence if and only if there is a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ with $F(0 \rightarrow 1) = f$, where \mathcal{J} is the walking isomorphism defined above.*

This result is in the same spirit as the much easier automatic homotopy coherence result in 3.6.

5.1.1. *Further results for quasi-categories.* The idea behind requiring the horn filler condition only for inner horns, Λ_k^n with $0 < k < n$ in the definition of quasi-category is that, for $n = 2$, filling a Λ_0^2 horn requires inverting the edge between vertices 0 and 1:

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \\ X & \xrightarrow{g} & Z \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & Y & \\ f \nearrow & \alpha \Downarrow & h \simeq g \circ f^{-1} \\ X & \xrightarrow{g} & Z \end{array}$$

And indeed Joyal proved that this is the only obstacle:

PROPOSITION 5.4 ([Joy02, Theorem 1.3]). *In a quasi-category \mathcal{C} , a morphism f is an equivalence if and only if there are fillers for every horn $\Lambda_0^n \rightarrow \mathcal{C}$ whose edge joining vertices 0 and 1 is f .*

There is, of course, a dual result about Λ_n^n horns. Putting the two together we get the homotopy hypothesis for quasi-categories:

COROLLARY 5.5. *Every 1-simplex of a quasi-category \mathcal{C} is an equivalence if and only if \mathcal{C} is a Kan complex.*

5.2. Limits and colimits. The notion of limit and colimit in an $(\infty, 1)$ -category should be thought of as generalizations of *homotopy* limits and colimits, and indeed, reduce to those for the $(\infty, 1)$ -category of spaces (or more generally, for an $(\infty, 1)$ -category coming from a model category; there is also a notion of homotopy limit and colimit in a model category). We begin by recalling what homotopy limits and colimits of diagrams of spaces are, by describing a construction of them that provides the correct intuition for $(\infty, 1)$ -categories. When reading the following description, the reader should think of paths as *invertible* morphisms in an $(\infty, 1)$ -categories, and homotopies as higher morphisms.

Let's consider a diagram of spaces $F : \mathcal{I} \rightarrow \text{Top}$ (where, for now, \mathcal{I} is an ordinary category). Recall that its (ordinary, non-homotopy) limit can be constructed as follows: it consists of the subspace of $\prod_{i \in \mathcal{I}} F(i)$ of points $(x_i)_{i \in \mathcal{I}}$ such that for any $\alpha : i \rightarrow j$ in \mathcal{I} , we have that $F(\alpha)(x_i) = x_j$. To construct instead the homotopy limit, following the philosophy described in remark 2.2, we replace the strict notion of equality in $F(\alpha)(x_i) = x_j$ by the corresponding homotopical notion: a path from $F(\alpha)(x_i)$ to x_j , this path is witness to the fact that “homotopically speaking, $F(\alpha)(x_i)$ and x_j are the same”. For simple diagrams, without composable arrows, this is enough. For example, the homotopy pullback of a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ can

be constructed as the subspace of $X \times Y \times Z \times Z^{[0,1]} \times Z^{[0,1]}$ consisting of 5-tuples $(x, y, z, \gamma, \sigma)$ with $\gamma(0) = f(x), \gamma(1) = z = \sigma(1), \sigma(0) = g(y)$.

For diagrams that have pairs of composable arrows, we need to ensure these paths that act as witnesses compose as well, that is, if we have $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ somewhere in \mathcal{I} , every point of the homotopy limit will have among its coordinates the data of

- (1) points $x_i \in F(i), x_j \in F(j), x_k \in F(k)$, and
- (2) paths $\gamma_{ij} : F(\alpha)(x_i) \rightarrow x_j, \gamma_{jk} : F(\beta)(x_j) \rightarrow x_k$ and $\gamma_{ik} : F(\beta \circ \alpha)(x_i) \rightarrow x_k$.

The paths γ_{ik} and $\gamma_{jk} \cdot (F(\beta) \circ \gamma_{ij})$ both witness that x_i and x_j are in the same path component of $F(k)$, but we shouldn't regard their testimonies as being independent! The points of the homotopy limit should also include a homotopy between these two paths. Clearly, this doesn't stop here: for diagrams with triples of composable arrows we should have homotopies between homotopies and so on. For any given small diagram shape \mathcal{I} it is clear which paths and homotopies are required.

Dually, for homotopy colimits, instead of taking a subspace of a large product where we require the presence of some paths and homotopies, we form a large coproduct of spaces and glue in paths and homotopies that enforce "sameness" of points in the colimits.

REMARK 5.6. There is also a general formula for the required homotopies in homotopy limits due to Pete Bousfield and Dan Kan [BK72]: for a functor $F : \mathcal{I} \rightarrow \text{Top}$, we have

$$\text{holim } F = \int_{i \in \mathcal{I}} F(i)^{|N(\mathcal{I} \downarrow i)|}.$$

Here $|N(\mathcal{I} \downarrow i)|$ is the geometric realization of the nerve of the slice category $\mathcal{I} \downarrow i$ whose objects are objects of \mathcal{I} with a map to i , and whose morphisms are commuting triangles. The \int sign denotes a type of limit called an *end*, and can be constructed as a subspace of the product of function spaces $\prod_{i \in \mathcal{I}} F(i)^{|N(\mathcal{I} \downarrow i)|}$ consisting of compatible families of functions $\gamma_i : |N(\mathcal{I} \downarrow i)| \rightarrow F(i)$: a morphism $i \xrightarrow{\alpha} j$ induces a functor $\mathcal{I} \downarrow i \rightarrow \mathcal{I} \downarrow j$ by composition, and thus gives a map $\alpha_* : |N(\mathcal{I} \downarrow i)| \rightarrow |N(\mathcal{I} \downarrow j)|$; the family $\{\gamma_i\}$ is compatible if $\gamma_j \circ \alpha_* = F(\alpha) \circ \gamma_i$. The reader can check that the homotopies mentioned above for composable pairs $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ are encoded by this formula as maps out of triangles that restrict on the boundary to the paths corresponding to the morphisms α, β and $\beta \circ \alpha$.

All limits and colimits in $(\infty, 1)$ -categories can be defined in terms of homotopy limits of spaces. Recall that for a diagram $F : \mathcal{I} \rightarrow \mathcal{C}$ in an ordinary category \mathcal{C} , the limit and colimit can be defined by requiring the canonical functions of sets

$$\text{Hom}_{\mathcal{C}}(X, \lim_{i \in \mathcal{I}} F(i)) \rightarrow \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(X, F(i))$$

and

$$\text{Hom}_{\mathcal{C}}(\text{colim}_{i \in \mathcal{I}} F(i), X) \rightarrow \lim_{i \in \mathcal{I}^{\text{op}}} \text{Hom}_{\mathcal{C}}(F(i), X)$$

to be bijections natural in X . The limits occurring in the codomain of these canonical maps are taken in the category of sets³⁴.

³⁴Note that colimits in the category of sets do not appear here.

DEFINITION 5.7. Given a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ between two $(\infty, 1)$ -categories, its *limit* and *colimit*, if they exist, are determined up to equivalence in \mathcal{C} by requiring that

$$\mathrm{Map}_{\mathcal{C}}(X, \lim_{i \in \mathcal{I}} F(i)) \rightarrow \mathrm{holim}_{i \in \mathcal{I}} \mathrm{Map}_{\mathcal{C}}(X, F(i))$$

and

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{i \in \mathcal{I}} F(i), X) \rightarrow \mathrm{holim}_{i \in \mathcal{I}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}}(F(i), X)$$

are weak equivalences of spaces, natural in X .

Note the special case of initial and terminal objects: an object X of an $(\infty, 1)$ -category \mathcal{C} is initial if $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is contractible for all objects Y , and terminal if $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is contractible for all Y . As expected initial objects are unique when they exist: more precisely, Joyal proved [Joy02, Proposition 4.4] that the full subcategory of \mathcal{C} consisting of all initial objects is a contractible ∞ -groupoid, that is, any two initial objects are equivalent, any two equivalences between initial objects are homotopic, etc.

Other constructions of limits and colimits in ordinary categories also generalize to the $(\infty, 1)$ -setting and give the same definition as above. For example, the limit of a functor F can be characterized as a terminal object in the category of cones over F . For concreteness we'll use quasi-categories to describe how this goes for $(\infty, 1)$ -categories: given a functor $F : \mathcal{I} \rightarrow \mathcal{C}$, the quasi-category of cones over F , $\mathrm{Cones}(F)$, is the simplicial set whose n -simplices are given by maps of simplicial sets $\Delta^n \star \mathcal{I} \rightarrow \mathcal{C}$ which restrict to F on \mathcal{I} . Its vertices are exactly what we'd expect, since $\Delta^0 \star \mathcal{I}$ is \mathcal{I} with a new initial object adjoined. Here $K \star L$ denotes the join of simplicial sets, a geometric operation that can be thought as providing a canonical triangulation of the union of line segments joining all pairs of a point of $|K|$ and a point of $|L|$; each k -simplex of \mathcal{I} contributes an $(n + k + 1)$ -simplex to $\Delta^n \star \mathcal{I}$.

REMARK 5.8. While it is perfectly fine to define a diagram in a category to be a functor for developing the theory, most people don't actually think of many common diagram shapes, such as diagrams for pullbacks, pushouts, infinite sequences $X_0 \rightarrow X_1 \rightarrow \dots$, etc., as being given by a category; instead these shapes are usually thought of as being given by a directed graph. Another advantage of the model of quasi-categories is that there is a very convenient generalization of directed graphs and diagrams shaped like them: simply take arbitrary simplicial sets for shapes and define a K -shaped diagram in \mathcal{C} to be a map of simplicial sets $K \rightarrow \mathcal{C}$. Both descriptions given above make sense for these more general types of diagrams. Also, this notion of K -shaped diagram can be used in other model of $(\infty, 1)$ -categories but is more cumbersome, since essentially one needs to define the free $(\infty, 1)$ -category on K .

Most classical results about limits and colimits hold for $(\infty, 1)$ -categories with appropriate definitions, and we hope that the examples shown here give a rough idea of how the definitions are generalized.

5.3. Adjunctions, monads and comonads. As for other concepts, the definition of adjunction in ordinary category theory that uses Hom-sets and bijections generalizes to $(\infty, 1)$ -categories by using mapping spaces and homotopy equivalences:

DEFINITION 5.9. Given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ between $(\infty, 1)$ -categories, an *adjunction* is specified by giving a *unit*, a natural transformation $u : \text{id}_{\mathcal{C}} \rightarrow g \circ f$ such that the composite map

$$\text{Map}_{\mathcal{D}}(F(C), D) \xrightarrow{G} \text{Map}_{\mathcal{C}}(G(F(C)), G(D)) \xrightarrow{-\circ u_C} \text{Map}_{\mathcal{C}}(C, G(D))$$

is a weak homotopy equivalence.

As in the case of ordinary categories, if F has a right adjoint, the adjoint is uniquely determined up to natural equivalence. The basic continuity properties also hold: left adjoints preserve colimits, and right adjoints preserve limits. For ordinary categories, Freyd’s adjoint functor theorem is a partial converse to this result. It says roughly that for $G : \mathcal{D} \rightarrow \mathcal{C}$ to have a left adjoint it is sufficient that \mathcal{D} is complete, G preserves limits and satisfies a further condition called the *solution set condition*. The precise form of the solution set condition will not matter for us, only that (1) it is a size condition in the sense that it requires there to exist a small set of morphisms that somehow control all morphisms of a certain form, (2) it is adapted to G : it is not just a condition on the category \mathcal{C} .

There is an adjoint functor theorem for $(\infty, 1)$ -categories due to Lurie which in some sense is less precise than Freyd’s theorem for 1-categories in that its size condition is not adapted to G , but rather is a global condition on \mathcal{C} and \mathcal{D} . In practice, this is not a problem: the conditions of Lurie’s theorem are usually met when they need to be. What takes the place of the solution set condition is the notion of “*accessibility*”; for $(\infty, 1)$ -categories it intuitively means that the $(\infty, 1)$ -category is determined by a small subcategory of objects which themselves are small in some sense, for functors between such categories it means it preserves certain colimits which are a generalization of filtered colimits. For precise definitions see [Lur09b, Chapter 5] or, for the analogous theory in the case of ordinary categories, [AR94]. Finally, an $(\infty, 1)$ -category is *presentable* if it has all small colimits and is accessible. Lurie’s version of the adjoint functor theorem is as follows:

THEOREM 5.10 ([Lur09b, Corollary 5.5.2.9]). *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between presentable $(\infty, 1)$ -categories.*

- (1) *The functor F has a right adjoint if and only if it preserves small colimits.*
- (2) *The functor F has a left adjoint if and only if it is accessible and preserves small limits.*

In classical category theory whenever $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ are adjoint functors (with F being the left adjoint), the composite $G \circ F$ is a monad and $F \circ G$ is comonad. This is also true in the $(\infty, 1)$ -categorical context, but much harder to show since, as the reader expects by now, the concept of monad requires not an associative multiplication but one that is associative up to coherent homotopy. The monad corresponding to an adjunction is constructed in [Lur12b, Section 6.2.2]; Lurie uses it to prove an $(\infty, 1)$ -analogue of the Barr-Beck theorem characterizing those adjunctions in which \mathcal{D} is equivalent to the category of $G \circ F$ -algebras in \mathcal{C} and G is equivalent to the forgetful functor. This result is very useful in the theory of descent in derived algebraic geometry, just as the classical version is useful in algebraic geometry.

Another construction, providing more explicit information, of the monad and comonad corresponding to an adjunction appears in [RV13]. There the authors

construct “the walking $(\infty, 1)$ -adjunction”: an $(\infty, 2)$ -category \mathcal{A} that has two objects 0 and 1 , and two morphisms $f : 0 \rightarrow 1$ and $g : 1 \rightarrow 0$ which are adjoint to each other and form the free adjunction in the sense that any pair of adjoint functors (F, G) between $(\infty, 1)$ -categories arise as the images of f and g under some functor from \mathcal{A} into the $(\infty, 2)$ -category of all $(\infty, 1)$ -categories. Given such a functor $H : \mathcal{A} \rightarrow \text{Cat}_{(\infty, 1)}$ (where temporarily, $\text{Cat}_{(\infty, 1)}$ is an $(\infty, 2)$ -category), the restriction of H to $\text{Hom}_{\mathcal{A}}(0, 0)$ gives the monad $G \circ F$ with its multiplication and all the higher coherence data. This $(\infty, 2)$ -category \mathcal{A} is surprisingly just a 2-category, the “walking (ordinary) adjunction” described in [SS86].

REMARK 5.11. We mentioned above that if F has a right adjoint G , then G is canonically determined by F . In fact, all of the adjunction data is determined by F in a sense similar to that in proposition 3.4: there is an $(\infty, 1)$ -category of adjunction data that has a forgetful functor to the arrow category of $\text{Cat}_{(\infty, 1)}$ which just keeps the left adjoint F ; this forgetful functor has contractible fibers. Similarly there is a contractible space of adjunction data with a given left adjoint F , right adjoint G and unit u . The results in [RV13] describe more generally which pieces of adjunction data determine the rest up to a contractible space of choices.

5.4. Less basic $(\infty, 1)$ -category theory. Much more than just the basic notions of category theory have been extended to $(\infty, 1)$ -categories. There is a large number of topics we could have included here, and we picked only two that are important in applications: the different sorts of fibrations between $(\infty, 1)$ -categories, and stable $(\infty, 1)$ -categories. The various fibrations play a larger role in actual use of $(\infty, 1)$ -categories, specially when incarnated as quasi-categories, than do the notions in ordinary category theory they generalize. Stable $(\infty, 1)$ -categories are widely used in derived algebraic geometry, since they replace triangulated categories. Stable $(\infty, 1)$ -categories are also downright pleasant to work with and we have no compunctions about advertising them.

A very important topic we’ve omitted is the study of monoidal and symmetric monoidal $(\infty, 1)$ -categories. As usual in higher category theory they’re tricky to define since one must make the tensor product associative up to coherent homotopy (or associative and commutative up to coherent homotopy in the symmetric case). We’ve decided to omit them since we feel that the coherence issues we’d point out about them would be similar to the ones we’ve mentioned already in other contexts. Also, there are gentler introductions to them than the one in [Lur12b] already available: we recommend Lurie’s old version in [Lur07b] and [Lur07c], or Groth’s nice exposition [Gro10, Sections 3 and 4].

Finally, a topic that will surely become increasingly important is the theory of enriched $(\infty, 1)$ -categories. Often this is dealt with in a somewhat ad hoc manner when it arises, but recently David Gepner and Rune Haugseng have produced a systematic treatment [GH14]. See also Haugseng’s follow up paper proving a strictification result for enriched $(\infty, 1)$ -categories [Hau14].

5.4.1. Fibrations and the Grothendieck construction. The *Grothendieck construction* takes as input a weak 2-functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ where \mathcal{C} is an ordinary category (thought of as a 2-category with only identity 2-morphisms), and Cat is the 2-category of categories. Even though Cat is a strict 2-category, it makes sense to consider a weak functor F , that is, one that does not preserve composition on the nose, but rather comes equipped with a natural isomorphism $F(g) \circ F(f) \xrightarrow{\cong} F(f \circ g)$ satisfying a coherence condition for compositions of three morphisms. It produces

as output a category \mathcal{E} and a functor $P : \mathcal{E} \rightarrow \mathcal{C}$. The functor P produced is always what is called a *Grothendieck fibration*, and the Grothendieck construction provides an equivalence of 2-categories between $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat})$ and the 2-category of Grothendieck fibrations over \mathcal{C} . We won't give precise definitions here, but we'll illustrate in an example.

EXAMPLE 5.12. Let \mathcal{C} be the category Top of topological spaces and F the functor that assigns to each space X the category of vector bundles on X (the morphisms from a bundle $E_1 \rightarrow X$ to a bundle $E_2 \rightarrow X$ are maps $E_1 \rightarrow E_2$ which form a commuting triangle with the projections to X and which are linear on each fiber). For a continuous function $f : X \rightarrow Y$, $F(f)$ is given by pullback of vector bundles. For this functor, the Grothendieck construction produces a category \mathcal{E} whose objects are vector bundles $E \rightarrow X$ on arbitrary base spaces X and whose morphisms from a bundle $E_1 \rightarrow X_1$ to a bundle $E_2 \rightarrow X_2$ are commuting squares

$$\begin{array}{ccc} E_1 & \xrightarrow{e} & E_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{x} & X_2 \end{array}$$

such that the map e is linear on each fiber. The projection $P : \mathcal{E} \rightarrow \mathcal{C}$ simply forgets the bundles and keeps the bases. Now let's spell out what being a Grothendieck fibration means in this example. There are certain morphisms in \mathcal{E} that are distinguished: the *Cartesian* morphisms for which the above square is a pullback. Given any map $x : X_1 \rightarrow X_2$ in Top and any vector bundle E_2 on X_2 , there is always a Cartesian morphism in \mathcal{E} with codomain $E_2 \rightarrow X_2$ that P sends to x : namely, the one in which E_1 is the pullback $x^*(E_2)$. Every morphism e' in \mathcal{E} such that $P(e')$ factors through x can be factored through the morphism $e : x^*(E_2) \rightarrow E_2$.

The Grothendieck construction is very versatile. First, it clearly allows one to deal with weak 2-functors to Cat *without* leaving the world of 1-categories. It is used this way in the theory of stacks, for example. There one wants to replace the notion of a sheaf of sets with a 2-categorical version that has values in categories, or more typically, just groupoids. Such a thing can be defined in terms of weak 2-functors to groupoids, but can also be handled as Grothendieck fibration, which is the approach typically taken. See for example Angelo Vistoli's notes [Vis05]. (By the way, that also serves as a references for the details on the Grothendieck construction we skipped above.)

But the Grothendieck construction also allows one to calculate limits and colimits of functors to Cat . There is a dual version that given a covariant $F : \mathcal{C} \rightarrow \text{Cat}$ produces what is called a Grothendieck opfibration $\mathcal{E} \rightarrow \mathcal{C}$, and it's not too hard to show that:

- (1) $\lim F$ is given by the category of *coCartesian* sections of $P : \mathcal{E} \rightarrow \mathcal{C}$, that is, sections $\sigma : \mathcal{C} \rightarrow \mathcal{E}$ such that $P \circ \sigma = \text{id}_{\mathcal{C}}$ and $\sigma(f)$ is a coCartesian morphism in \mathcal{E} for every morphism f of \mathcal{C} .
- (2) $\text{colim } F$ is given by the localization $\mathcal{E}[\text{coCart}^{-1}]$ of \mathcal{E} obtained by inverting all coCartesian morphisms.

Here \lim and colim denote what are sometimes called pseudo-limits and pseudo-colimits which are the closest analogues in the world of 2-categories to homotopy limits and colimits. Indeed, the reader who knows how to perform the Grothendieck

construction should compare it with the description of homotopy limits and colimits in section 5.2.

All of this is generalized to $(\infty, 1)$ -categories in [Lur09b, Chapter 2]. (See also Moritz Groth’s notes [Gro10, Section 3]). Lurie defines *(co)Cartesian fibrations* corresponding to Grothendieck (op)fibrations, and also discusses Joyal’s notions of left and right fibrations. As in the classical case, functors $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{(\infty, 1)}$ for an $(\infty, 1)$ -category \mathcal{C} are classified by Cartesian fibrations $\mathcal{E} \rightarrow \mathcal{C}$; and right fibrations are the subclass of Cartesian fibrations classifying functors that land in the subcategory of $\text{Cat}_{(\infty, 1)}$ of ∞ -groupoids.

The constructions described above for limits and colimits of categories also generalize to $(\infty, 1)$ -categories. See [Lur09b, Proposition 3.3.3.1] for limits of $(\infty, 1)$ -categories; Corollary 3.3.3.4 specializes the previous result to left fibrations and thus provides a construction for homotopy limits of spaces. On the colimit side, there are corollaries 3.3.4.3 and 3.3.4.6 which construct colimits in $\text{Cat}_{(\infty, 1)}$ and homotopy colimits of spaces respectively.

5.4.2. *Stable $(\infty, 1)$ -categories.* Stable $(\infty, 1)$ -categories are a wonderfully practical replacement for the notion of triangulated category that fixes many of the problems in their theory. Many prominent examples of triangulated categories, are given to us almost by definition as homotopy categories of naturally occurring $(\infty, 1)$ -categories with nice properties that make them homotopical analogues of abelian categories. These $(\infty, 1)$ -categories are the *stable* ones we will shortly define. For example, the derived category of an abelian category \mathcal{A} is basically defined as the homotopy category of the relative category of chain complexes in \mathcal{A} with weak equivalences given by the quasi-isomorphisms. Let’s now give the definition.

DEFINITION 5.13. An $(\infty, 1)$ -category is *pointed* if it has an object which is both initial and terminal. Such an object is called a *zero object* and denoted 0 .

Between any two objects X and Y in a pointed $(\infty, 1)$ -category there is a unique homotopy class of *zero morphisms*, those that factor through the zero object.

DEFINITION 5.14. A *triangle* in a $(\infty, 1)$ -category is a commuting square of the form:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

That is, a triangle is a homotopy between $g \circ f$ and the zero morphism. If the square is a pushout square, g is called the *cofiber* of f ; if the square is a pullback, f is called the *fiber* of g .

Fibers and cofibers of maps play a role similar to kernels and cokernels in abelian categories and to cocones and cones in triangulated categories. In terms of these concepts the definition of stable $(\infty, 1)$ -category is easy to give:

DEFINITION 5.15. A *stable $(\infty, 1)$ -category* is a pointed $(\infty, 1)$ -category where every morphism has a fiber and a cofiber and every triangle is a pushout if and only if it is a pullback.

That’s it, that’s the whole definition. Of course, the notion of $(\infty, 1)$ -category is much more complicated than the notion of 1-category, but after making the initial

investment in $(\infty, 1)$ -categories, the definition of stable $(\infty, 1)$ -category is simpler not only than the definition of triangulated category but even than that of abelian category.

For readers familiar with the definition of triangulated category, we now explain how the homotopy category of any stable $(\infty, 1)$ -category is canonically triangulated. The triangles are defined to be to diagrams of the form

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & U \end{array}$$

where both squares are pushouts. This makes the outer rectangle a pushout too, and since for spaces homotopy pushouts of that form produce suspensions, we call U the suspension of X and write $U \simeq \Sigma X$. Suspension is the translation functor of the triangulated structure.

The advantage of stable $(\infty, 1)$ -categories over triangulated begins to be visible even from here: in a triangulated category the cone of a morphism is determined up to isomorphism but *not* up to a canonical isomorphism, and this is because the universal property we should be asking of the cone is homotopical in nature. In a stable $(\infty, 1)$ -category the cofiber is determined, as all colimits are, up to a contractible space of choices which is exactly canonical enough in the $(\infty, 1)$ -world to make the constructions functorial.

As further evidence of the simplicity of stable $(\infty, 1)$ -categories, notice that proving the octahedral axiom in the homotopy category becomes simply a matter of successively forming pushout squares, and putting down a suspension whenever we see a rectangle with zeros in the bottom left and top right corners:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & \Sigma X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & Z/Y & \longrightarrow & \Sigma Y \longrightarrow \Sigma(Y/X) \end{array}$$

(Here we've used notation like $X \rightarrow Y \rightarrow Y/X \rightarrow \Sigma X$ for the objects in a distinguished triangle.)

FURTHER READING. We refer the reader to [Lur12b, Chapter 1] for the development of the theory, or to Groth's presentation [Gro10, Section 5].

6. Some applications

This final section will briefly describe some uses of $(\infty, 1)$ -categories outside of higher category theory itself. (The characterization of (∞, n) -categories mentioned in section 4 is an excellent application of $(\infty, 1)$ -categories within higher category theory.) Each of these applications is a whole subject in itself and we cannot hope to do any of them justice here, rather, we hope merely to whet the readers' appetite and suggest further reading. Unfortunately many great topics had to be left out, such as algebraic K -theory, where the use of $(\infty, 1)$ -categories has finally allowed to produce a universal property of higher K -theory similar to the familiar one satisfied by K^0 : approximately, that K -theory is the universal invariant

satisfying Waldhausen’s additivity theorem. For precise descriptions and proofs of this universal property see [Bar12] and [BGT13]. Another proof of Waldhausen additivity in the $(\infty, 1)$ -setting can be found in [FL12] (but without the universal property), and an $(\infty, 1)$ -version of Waldhausen’s approximation theorem is proved in [Fio13].

6.1. Derived Algebraic Geometry. One motivation behind studying stacks in algebraic geometry is a desire to “fix” quotients of schemes by group actions. For example, if an algebraic group G acts freely on a variety X , sheaves on the quotient X/G are the same as G -equivariant sheaves on X . But if the action of G is not free, then the relation between G -equivariant geometry on X and geometry on the quotient X/G is complicated, aside from the difficulty in saying precisely what the quotient means in this case. Passing to stacks “fixes” the quotient, so that sheaves on the quotient stack $[X/G]$ are again G -equivariant sheaves on X .

Just as stacks can be seen as a way to improve certain colimits of schemes so that they always behave as they do in the nice cases, *derived algebraic geometry* can be regarded as fixing a dual problem: correcting certain limits so that they behave more often as they do in good cases. Namely, one motivation for derived algebraic geometry is to “improve” intersections so that they behave more like transverse intersections. Some intersections are already improved by passing from varieties to schemes, for example consider the case of two curves X and Y of degrees m and n in the complex projective plane. If they intersect transversely, according to Bézout’s theorem they always have mn points of intersection. When the intersections are not transverse there might be fewer than mn points of intersection, but as long as X and Y have no common irreducible component, there is a natural way to assign multiplicities to the points so that counted with multiplicity there are still mn points of intersection.

We can write the result in the transverse case in terms of cohomology classes as $[X] \cup [Y] = [X \cap Y]$, where $[X], [Y] \in H^2(\mathbb{C}P^2, \mathbb{Z})$ are the fundamental classes of X and Y , and $[X \cap Y] \in H^4(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}$ is the fundamental class of the 0-dimensional variety consisting of the finitely many points of intersection of X and Y . The point of this example is that when the intersection of X and Y is not transverse (but still assuming X and Y do not share a common component), the same formula is true provided we think of $X \cap Y$ as a scheme instead of a variety (and define fundamental classes in an appropriate way). The dimension over \mathbb{C} of the local ring of $X \cap Y$ at the points of intersection gives the relevant multiplicities.

By passing from varieties to schemes we reach a context where computing $[X] \cup [Y]$ does not require full knowledge of the pair (X, Y) , just knowledge of the intersection $X \cap Y$, and the intersection is again a scheme, the same kind of object as X and Y . So schemes “fix” the intersection of curves that don’t share a component, but they don’t fix intersections of curves that do share a component nor do they help intersections in higher dimensions. If X and Y are now subvarieties of $\mathbb{C}P^n$ of complementary dimension that intersect in a finite set of points, the dimension of the local ring of $X \cap Y$ at a point of intersection is no longer the correct multiplicity to make $[X] \cup [Y] = [X \cap Y]$ true; now the multiplicity is given by Serre’s formula:

$$(\text{multiplicity at } p \in X \cap Y) = \sum_{i \geq 0} \dim_{\mathbb{C}} \text{Tor}_i^{\mathcal{O}_{\mathbb{C}P^n, p}}(\mathcal{O}_{X, p}, \mathcal{O}_{Y, p}).$$

Notice that this sum *cannot* be computed from just the knowledge of $X \cap Y$ as a scheme, since that only determines the Tor_0 term, but the same formula suggests a fix: we'd be able to compute the correct multiplicities just from $X \cap Y$ if we could arrange for

$$\mathcal{O}_{X \cap Y, p} = \mathcal{O}_{X, p} \otimes_{\mathcal{O}_{\mathbb{C}P^n, p}}^{\mathbb{L}} \mathcal{O}_{Y, p},$$

where $\otimes^{\mathbb{L}}$ denotes the derived tensor product, a chain complex whose homology gives the relevant Tor groups. We can attempt, then, to define *derived schemes* by replacing commutative rings in the definition of schemes by the chain complex version of commutative algebras: commutative differential graded algebras or cdgas for short, and to compute structure sheaves of intersections using derived tensor products. This suggests jumping into $(\infty, 1)$ -waters, since the derived tensor product is only naturally defined up to quasi-isomorphism while sheaves expect to have values determined up to isomorphism.

EXAMPLE 6.1. The higher categorical point of view also suggests itself by considering the other case we mentioned that schemes don't fix: intersecting two curves with a common component. For the simplest possible example let's intersect the y -axis in the affine plane \mathbb{C}^2 with itself: the ring of functions we get is $\mathbb{C}[x, y]/(x) \otimes \mathbb{C}[x, y]/(x) = \mathbb{C}[x, y]/(x, x)$. If, as for ordinary rings we had $\mathbb{C}[x, y]/(x, x) = \mathbb{C}[x, y]/(x)$ the intersection would be 1-dimensional and we'd have no hope of a uniform intersection theory. We want instead to have that modding out by x twice is different than doing it once, and makes the quotient ring 0-dimensional again. The philosophy of higher categories suggests that instead of forcing x to be 0, we make it isomorphic to 0; that way if we mod out twice, we can add different isomorphisms each time. Let's do that using cdgas for simplicity. Let's start out with $\mathbb{C}[x, y]$ with x and y in degree zero. To compute $A := \mathbb{C}[x, y]/(x)$ (where we've used a double slash to avoid confusion with the ordinary quotient of rings) we add a generator u in degree one with $du = x$, to make x isomorphic (maybe "homologous" is a better term for isomorphism inside A) to 0. We can readily compute $H_0(A) = \mathbb{C}[x, y]/(x)$ (ordinary quotient), and $H_i(A) = 0$ for $i \neq 0$. Now, we quotient by x again, to get $B := A/(x) = \langle x_0, y_0, u_1, v_1 : du = dv = x \rangle$ (where the subindices on the generators indicate the degree). The non-zero homology is $H_i(B) = \mathbb{C}[x, y]/(x)$ for $i = 0, 1$. This computes the correct Tor terms from Serre's formula: $\mathrm{Tor}_i^{\mathbb{C}[x, y]}(\mathbb{C}[x, y]/(x), \mathbb{C}[x, y]/(x))$ is also given by $\mathbb{C}[x, y]/(x)$ for $i = 0, 1$ and is 0 otherwise. For similar but less trivial examples see the introduction of [Lur11b].

6.1.1. *The cotangent complex.* Another motivation for derived algebraic geometry comes from deformation theory. The cotangent complex \mathbb{L}_X of a scheme X over a field controls its deformation theory through various Ext groups of \mathbb{L}_X . Roughly speaking, it can be used to count how many isomorphism classes of first order deformations X has, and to count how many non-isomorphic ways each first order deformation extends to a second order deformation, and so on. But it seems initially to be a purely algebraic creature. Grothendieck asked in 1968 for a geometrical interpretation of the cotangent complex, and derived algebraic geometry provides one. We'll only sketch how the groups $\mathrm{Ext}^i(\mathbb{L}_{X, k}, k)$ are reified through the derived affine schemes.

For $i = 0$, we'll only need ordinary rings, in fact. Let $D_0 = \mathrm{Spec} k[\epsilon]/(\epsilon^2)$ be the walking tangent vector: it is a tiny scheme whose underlying topological space

has only one point, but that additionally carries a direction so that morphisms $D_0 \rightarrow Y$ correspond to tangent vectors on Y . We have that $\text{Ext}^0(\mathbb{L}_{X,x}, k) \simeq \text{Hom}_*(D_0, (X, x))$ (where Hom_* means base-point preserving morphisms), but to obtain similar representations of the higher Ext , we need to pass to derived rings.

One of the great advantages of admitting nilpotent elements in the rings used for algebraic geometry is that it allows for schemes like D_0 . A good intuitive picture to have for derived schemes is that they are to ordinary schemes as those are to reduced schemes, that is, derived schemes can have something like “higher nilpotent elements” in their structure sheaves. We are about to see an example of that now. Let $D_i = \text{Spec } k[\epsilon_i]/(\epsilon_i^2)$ where the ring is now a cdga with generator ϵ_i in degree i (thinking of the chain complex as an abelian higher groupoid, it has a single non-identity i -morphism). As is nicely explained in [Vez10], it turns out that $\text{Ext}^i(\mathbb{L}_{X,x}, k) \simeq \pi_0 \text{Map}_*(D_i, (X, x))$, where Map_* denotes a mapping space in an $(\infty, 1)$ -category of derived pointed schemes.

6.1.2. *Rough sketches of the definitions.* Now we can say approximately what derived schemes are: they are what you get by taking some definition of scheme in terms of commutative rings and performing two replacements: (1) replacing notions from ordinary category theory appearing in the definition with the corresponding definitions in $(\infty, 1)$ -category theory, and (2) replacing the category of commutative rings with an $(\infty, 1)$ -category of generalized commutative rings (that allow extraction of higher Tor groups from some tensor product operation defined for them).

REMARK 6.2. Since passing to derived schemes and passing to stacks are meant to solve independent problems in the category of schemes, it is entirely possible and even desirable to generalize schemes in both directions at once to obtain the theory of derived stacks. The references we will mention actually treat derived stacks as well. Derived stacks are also higher stacks in the sense that the groupoids appearing in the theory of ordinary stacks are replaced by ∞ -groupoids, but it also makes sense to consider *underived* higher stacks as in [Sim96].

There are choices for both what definition of schemes to generalize and what sort of rings to use. One can use the definition of schemes as locally ringed spaces locally isomorphic to affine schemes or the point of view of the functor of points. For rings one could use commutative differential graded algebras, simplicial commutative rings or, for application to homotopy theory, E_∞ -ring spectra. Also, given any commutative ring R , we can consider R -algebras of each of those three kinds.

Let’s talk about the choice of rings first. The three notions we listed are related by functors of $(\infty, 1)$ -categories $\text{SCR}_R \rightarrow \text{CDGA}_R \rightarrow E_\infty\text{-}R\text{-Alg}$. The situation is very simple if R is a \mathbb{Q} -algebra: the second functor is an equivalence, the first functor is fully faithful and its image consists of connective cdgas: those whose homology groups are concentrated in non-negative degrees. In general, it gets a little messy: neither functor nor their composite is fully faithful. Comparing free algebras might be illuminating: the free cdga on one generator x of degree 0, say, is just the polynomial ring $\mathbb{Z}[x]$ concentrated in degree 0. On the other hand, the free $E_\infty\text{-}\mathbb{Z}$ -algebra A on x is quite different, since the multiplication on A is not strictly commutative but only commutative up to coherent homotopy. This means for example, that permuting the n factors in the product x^n , doesn’t quite fix x^n , rather it produces automorphisms of x^n , and in particular we get a non-trivial homomorphism $\Sigma_n \rightarrow \pi_1(A, x^n)$ from the symmetric group on n letters. But in

$\mathbb{Z}[x]$ the multiplication *is* strictly commutative and this Σ_n -action is trivial. It seems fair to say that simplicial commutative rings are sufficient for applications of derived algebraic geometry to algebraic geometry itself, while E_∞ -ring spectra are mainly for applications to homotopy theory.

Now we'll deal with the choice of definition of scheme. Unlike what happens for the choice of rings, the choice of style of definition does not lead to different notions of derived scheme. Both the locally ringed space point of view and the functor of points view as well as the equivalence between them were described by Jacob Lurie in his PhD thesis or, more recently, in [Lur11b]. The latter also contains a very general definition of derived geometric objects that includes derived schemes with any of the above mentioned choices of rings, but also things like derived smooth manifolds studied by David Spivak in his PhD thesis (reworked into [Spi10]) and derived analytic spaces (further studied in [Lur11a]). The rest of the DAG series of papers, [Lur11c], [Lur11d], [Lur11a], [Lur11e], [Lur11f], [Lur11g], [Lur11h], [Lur12a] contain a wealth of information about algebraic geometry in the derived setting that we don't have any space to describe.

Bertrand Toën and Gabriele Vezzosi favor the functor of points in their work on homotopical algebraic geometry [TV05], [TV08]. Their approach is based on an idea of Grothendieck's developed in Hakim's thesis [Hak72]: it is possible to construct ordinary stacks, say, by a categorical procedure that receives very little outside input. Namely, one starts with the symmetric monoidal category of abelian groups: commutative monoids for the tensor product there are commutative rings, so we can get our hands on the category of affine schemes. Next, one needs a second piece of information: a topology on the category of affine schemes. Using this one can define purely categorically stacks for the topology. The idea of *relative algebraic geometry* is that one do the same for other symmetric monoidal categories and choices of topology. Toën and Vezzosi develop an $(\infty, 1)$ -version of this idea starting from a symmetric monoidal $(\infty, 1)$ -category [TV01].³⁵

Both approaches require a notion of $(\infty, 1)$ -version of sheaf and of $(\infty, 1)$ -topos. A sheaf with values in \mathcal{C} on an $(\infty, 1)$ -topos \mathcal{X} is just a functor $\mathcal{X}^{\text{op}} \rightarrow \mathcal{C}$ that sends colimits in \mathcal{X} to limits in \mathcal{C} . Just as in ordinary topos theory, for any topological space X there is a topos $\text{Shv}(X)$ of sheaves on X (in the 1-categorical case these are sheaves of sets, in the $(\infty, 1)$ -case they are sheaves of ∞ -groupoids), which completely determines the space X if it satisfies the mild technical condition of being sober. Sheaves on a space X are just defined to be sheaves on the $(\infty, 1)$ -topos $\text{Shv}(X)$, but the theory does not require derived schemes or stacks to have underlying topological spaces and is developed for $(\infty, 1)$ -toposes.

REMARK 6.3. The notion of $(\infty, 1)$ -topos is due to Charles Rezk who described them as model categories (and called them model toposes) in [Rez05]. Lurie developed their theory, using quasi-categories, in his book [Lur09b]. Toën and Vezzosi have also written about $(\infty, 1)$ -toposes using simplicial categories [TV05] and Segal categories [TV02]. For the reader familiar with the notion of *elementary topos* in ordinary category theory, we should point out that $(\infty, 1)$ -toposes only generalize the notion of Grothendieck toposes. A Grothendieck topos can be succinctly

³⁵They had their work cut out for them, since at the time the best available model for symmetric monoidal $(\infty, 1)$ -categories were monoidal model categories, which are very rigid: they have an actual monoidal structure, that is, one associative up to coherent *isomorphism* in the ordinary category on which the model structure is defined.

defined as a localization of a presheaf category, that is as a category \mathcal{E} that admits a functor $F : \mathcal{E} \hookrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$ for some \mathcal{C} such that F is fully faithful and has left adjoint which preserves finite limits. This definition generalizes to $(\infty, 1)$ -toposes by replacing Set with ∞ -groupoids. Like their 1-categorical counterparts, $(\infty, 1)$ -toposes can be characterized by analogues of Giraud’s axioms. There is also an $(\infty, 1)$ -version of the notion of site and sheaf on a site, and $(\infty, 1)$ -categories of sheaves on sites provide examples of $(\infty, 1)$ -toposes; but unlike the case for ordinary toposes, these examples are not all the $(\infty, 1)$ -toposes.

FURTHER READING. The idea of derived algebraic geometry can be traced back at least to the 1980s when the *hidden smoothness principle* came out of the work of Alexander Beilinson, Vladimir Drinfeld, Pierre Deligne and Maxim Kontsevich. This principle expressed the hope that many moduli spaces in algebraic geometry that are singular were really just truncations of smooth derived moduli spaces. An early example of the use of derived moduli spaces, before all the machinery was in place, can be found in [Kon95]. It was also Kontsevich who introduced dg-schemes and these were used by Ionut Ciocan-Fontanine and Mikhail Kapranov to construct derived versions of Quot schemes [CFK01] and Hilbert schemes [CFK02]. For more recent work on derived moduli spaces see [TV04].

For a through overview of higher and derived stacks see [Toë09]. More information about the deformation theory aspects of derived algebraic geometry, can be found in the already mentioned [Lur11e], but we also recommend Lurie’s earlier ICM address [Lur10] which gives an expository account. To see derived algebraic geometry in action, see, for example:

- (1) David Ben-Zvi and David Nadler’s work on derived loop spaces in algebraic geometry, (for example, [BZN12]), or
- (2) their paper [BZFN10], joint with John Francis, about the $(\infty, 1)$ -category $QC(X)$ of quasi-coherent sheaves on a derived stack X , where they show that given derived stacks X and Y over k satisfying an appropriate finiteness condition, any k -linear colimit preserving functor $F : QC(X) \rightarrow QC(Y)$ is given by a quasi-coherent sheaf $K \in QC(X \times_k Y)$ by means of an *integral transform* $F(S) := (\pi_Y)_*(K \otimes \pi_X^* S)$.

6.1.3. Topological modular forms. It would take us too far afield to describe topological modular forms in any kind of detail, but for readers interested in homotopy theory we want to at least mention this application of derived algebraic geometry.

There is a beautiful relation due to Quillen [Qui69] between generalized cohomology theories which have an analogue of the theory of Chern classes, properly called complex-oriented cohomology theories, and formal groups laws. A formal group law is³⁶ a power series $F(x, y) = x + y + (\text{higher order terms})$, with coefficients in some commutative ring A , that satisfies $F(x, F(y, z)) = F(F(x, y), z)$. Given any Lie group or algebraic group, the power series expansion of the multiplication at the origin gives a formal group law³⁷, but not every such law arises from this construction. There is a notion of isomorphism of formal group laws given by

³⁶Actually the definition we give is for *one-dimensional* formal groups laws. There are also *n-dimensional* formal group laws that are defined in the same way but with x, y, z denoting n -tuples of variables and $F(x, y)$ denoting an n -tuple of power series.

³⁷Of the same dimension as the group.

changes of coordinates $x' = g(x)$, $y' = g(y)$ where g is a power series. If A is a field of characteristic 0, all one-dimensional formal group laws are isomorphic to $F(x, y) = x + y$, and there is a classification due independently to Lazard [Laz55] and Dieudonné [Die55] of one-dimensional formal groups laws over algebraically closed fields of characteristic p .

A complex-oriented cohomology theory has an associated one-dimensional commutative group law F , that gives the formula for the first Chern class of a tensor product of line bundles in terms of the Chern classes of the individual line bundles, $c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$. For ordinary cohomology, the formal group law is the additive one, $F(x, y) = x + y$. For complex K -theory, it is the multiplicative formal group law, $F(x, y) = x + y + xy = (1 + x)(1 + y) - 1$. Both of these come from expanding the product of a one-dimensional commutative algebraic group, the additive group \mathbb{G}_a , and the multiplicative group \mathbb{G}_m , respectively. Over an algebraically closed field there is only one other kind of connected commutative one-dimensional algebraic group: elliptic curves, whose associated formal group laws correspond to *elliptic cohomology* theories. We can think of assembling all of the elliptic cohomology theories into something like a sheaf of cohomology theories on the moduli space of elliptic curves. If we had such a thing we could imagine that taking global sections of it would produce a new cohomology theory that samples from all elliptic cohomology theories at once. That cohomology theory is (almost³⁸) what tmf, the cohomology theory of topological modular forms, is supposed to be.

There are serious technical difficulties in constructing it, since the notion of “sheaf of cohomology theories” isn’t well-behaved enough to use for this purpose. Instead one can attempt to represent the cohomology theories by spectra, for which the $(\infty, 1)$ -notion of sheaf we mentioned earlier works well. Lifting this sheaf of cohomology theories to a sheaf of spectra turns out to be made easier, paradoxically, by making the problem harder and lifting instead to a sheaf of E_∞ -ring spectra, making tmf the global sections of an object in derived algebraic geometry. This lifting was first achieved (and shown to be essentially unique) by Mike Hopkins, Haynes Miller and Paul Goerss using an obstruction theory of E_∞ -ring spectra. A second construction due to Jacob Lurie uses more of the theory of derived algebraic geometry: he defines a moduli problem for (derived) elliptic curves that can only be stated in the derived setting and then uses his general representability criterion to show that the moduli problem is represented by a derived Deligne-Mumford stack, whose global sections then give tmf.

FURTHER READING. For a great introduction to the topic see Paul Goerss’s Bourbaki Seminar talk [Goe10] and the references therein. (That paper is also a good introduction to derived schemes.) Also recommended is Lurie’s [Lur09a] where he outlines the second construction mentioned above.

6.2. The cobordism hypothesis. Michael Atiyah [Ati88] proposed a mathematical definition of *topological quantum field theory* (henceforth TQFT) inspired by Graeme Segal’s axioms [Seg88]³⁹ for *conformal field theory*. The definition uses the category $\text{Bord}_{(n-1, n)}$ whose objects are closed oriented $(n - 1)$ -manifolds and

³⁸For one thing, we actually need a sheaf on the Deligne-Mumford compactification of the moduli space.

³⁹The reader might prefer the more recently published [Seg04] which is an expanded version of an unpublished paper that circulated widely for many years.

whose morphisms $M \rightarrow N$ are diffeomorphism classes of *bordisms*⁴⁰ from M to N , that is, n -manifolds W whose boundary is identified with $\overline{M} \sqcup N$. Here \overline{M} denotes M with the opposite orientation and the diffeomorphisms we consider are those fixing the boundary. This category has symmetric monoidal structure given by taking disjoint unions (both at the level of $(n-1)$ -manifolds and n -dimensional bordisms). The brunt of Atiyah's axioms is that a TQFT is a symmetric monoidal functor $F : \text{Bord}_{\langle n-1, n \rangle} \rightarrow \text{Vect}$, where Vect carries the monoidal structure given by the tensor product of vector spaces.

This definition implies that one can calculate the value of F on some bordism W by chopping it up into a composition of simpler bordisms and composing the images under F of those pieces. For example, when $n = 2$, we could chop any surface into a combination of pairs of pants, cylinders and spherical caps. This suggests that such a functor is determined by just a handful of its values. This approach becomes more and more complicated in higher dimensions because the pieces which are required to build up all bordisms increase in number and complexity. One might hope that a simpler theory could result from allowing more general ways to cut up bordisms: Atiyah's definition allows cutting along codimension 1 submanifolds, but if we allowed, say, arbitrary codimensional cuts, we could triangulate every manifold. The cobordism hypothesis doesn't make it quite that simple to cut up manifolds, but we will allow pieces of all dimensions.

John Baez and James Dolan's *cobordism hypothesis* [BD95] is about such extended TQFT's and does indeed say that they are determined by very few of their values, namely, that they are determined by where they send single points! This hypothesis has now been proved in the $n = 2$ case by Jacob Lurie and Mike Hopkins and in general by Lurie. A detailed and very readable sketch of the proof is available as [Lur09c]. It might be a little unfair to say that the cobordism hypothesis fulfills the promise of a simpler theory mentioned in the previous paragraph, since there were complicated foundational issues involved in both the precise statement in Lurie's paper and in its proof. We'll say a little more about these issues after stating a version of the cobordism hypothesis.

To allow chopping up manifolds into pieces of arbitrary dimension, we replace the domain category of a TQFT by a higher category $\text{Bord}_n^{\text{fr}}$ whose k -morphisms are n -framed⁴¹ k -manifolds with corners, and whose $(k+1)$ -morphisms are n -framed bordisms between such. (The framing is used to remove some ambiguity coming from the diffeomorphism group of \mathbb{R}^m , it is important to understand and to know what happens without it, but we leave the explanations to the references.) What we've said so far is a little sloppy. Baez and Dolan's original formulation involved an n -category, so the description of the morphisms we gave is only valid for $k < n$; for $k = n$, we must again take diffeomorphism classes of bordisms.

A \mathcal{C} -valued framed n -dimensional extended TQFT then is a symmetric monoidal functor from $\text{Bord}_n^{\text{fr}} \rightarrow \mathcal{C}$; where \mathcal{C} is an arbitrary symmetric monoidal n -category. We generalize to arbitrary symmetric monoidal n -categories partly because there is no canonical substitute for Vect in this case; categorifying Vect to make an

⁴⁰Perhaps confusingly, there is no difference between bordisms and cobordisms. We are following someone's terminology in each case. . .

⁴¹An n -framing on a k -manifold M is a trivialization of its stabilized tangent bundle, $TM \oplus \mathbb{R}^{n-k}$.

n -category is an interesting question of its own. Fortunately, we don't need any particular symmetric monoidal n -category for the statement:

Baez and Dolan's cobordism hypothesis: A framed n -dimensional extended TQFT with values in \mathcal{C} is completely determined by its value on the point. Moreover, the value on the point is always a *fully dualizable* object of \mathcal{C} and there is a bijection between isomorphism classes of such TQFTs and isomorphism classes of fully dualizable objects in \mathcal{C} .

We will now explain what is meant by fully dualizable objects. First, notice that the usual definition of adjoint functors can be phrased in terms of 1-morphisms and 2-morphisms in Cat and thus makes sense in any 2-category, and can even be used for k - and $(k+1)$ -morphisms in a higher category to define adjunctions of k -morphisms. Call a k -morphism dualizable if it has both a left and a right adjoint, and call it n -times dualizable if it is dualizable and the $(k+1)$ -morphisms that show it has adjoints are themselves $(n-1)$ -times dualizable⁴². Finally, a monoidal n -category \mathcal{C} can be thought of as an $(n+1)$ -category with a single object $B\mathcal{C}$, so we can say that an object of \mathcal{C} is m -times dualizable if as a 1-morphism in $B\mathcal{C}$, it is m -times dualizable. Then a fully dualizable object of a symmetric monoidal n -category \mathcal{C} is just an $(n-1)$ -times dualizable object. Notice that this really does depend on n : a symmetric monoidal n -category can also be regarded as a symmetric monoidal $(n+1)$ -category (with only identity $(n+1)$ -morphisms), but being $(n-1)$ -times dualizable is different from being n -times dualizable.

This n -category version of the cobordism hypothesis was proven for $n=2$ by Chris Schommer-Pries in his PhD thesis [SP09], but for higher n a lack of a solid, practical theory of n -categories impeded progress. Lurie's key insight was to prove instead a more general version using the easier theory of (∞, n) -categories and then deduce the original formulation by truncation. So Lurie replaced the n -category described above by one in which

- (1) n -morphisms are bordisms, not diffeomorphism classes of such,
- (2) $(n+1)$ -morphisms are diffeomorphisms,
- (3) $(n+2)$ -morphisms are isotopies of diffeomorphisms,
- (4) $(n+3)$ -morphisms are isotopies of isotopies, and so on.

FURTHER READING. For a more detailed introduction to the cobordism hypothesis and its applications, Dan Freed's survey [Fre13] is highly recommended, as is, of course, Lurie's [Lur09c] which is not just a detailed outline of the proof but also includes a lot of motivation (and an introduction to higher category theory!), and many other interesting versions of the homotopy hypothesis, and applications. The reader might also be interested in Julie Bergner's survey [Ber11] which focuses on models for (∞, n) -categories but does state Lurie's version of the cobordism hypothesis in some detail, describing the construction of $\text{Bord}_n^{\text{fr}}$ as an n -fold complete Segal space.

References

- [AC12] Greg Arone and Michael Ching, *A classification of Taylor towers of functors of spaces and spectra*, preprint, available as [arXiv:1209.5661](https://arxiv.org/abs/1209.5661) [math.AT], 2012.
- [AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. MR 1294136 (95j:18001)

⁴²Where, of course, "1-times dualizable" just means "dualizable".

- [Ati88] Michael Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 175–186 (1989). MR 1001453 (90e:57059)
- [Bae97] John Baez, *An introduction to n -categories*, Category Theory and Computer Science, Springer, 1997, pp. 1–33.
- [Bae05] ———, *This Week's Finds in Mathematical Physics (Week 223)*, 2005, <http://math.ucr.edu/home/baez/week223.html>.
- [Bar05] Clark Barwick, *(∞, n) -Cat as a closed model category*, Ph.D. thesis, University of Pennsylvania, 2005.
- [Bar12] ———, *On the algebraic k -theory of higher categories. the universal property of Waldhausen K -theory*, submitted, preprint available as [arXiv:1204.3607](https://arxiv.org/abs/1204.3607) [math.KT], 2012.
- [BD95] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105. MR 1355899 (97f:18003)
- [BD98] ———, *Higher-dimensional algebra. III. n -categories and the algebra of opetopes*, Adv. Math. **135** (1998), no. 2, 145–206. MR 1620826 (99j:18005)
- [Bek00] Tibor Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 447–475. MR 1780498 (2001i:18015)
- [Bén67] Jean Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77. MR 0220789 (36 #3841)
- [Ber99] Clemens Berger, *Double loop spaces, braided monoidal categories and algebraic 3-type of space*, Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), Contemp. Math., vol. 227, Amer. Math. Soc., Providence, RI, 1999, pp. 49–66. MR 1665460 (2000e:55009)
- [Ber07] Julia E. Bergner, *Three models for the homotopy theory of homotopy theories*, Topology **46** (2007), no. 4, 397–436. MR 2321038 (2008e:55024)
- [Ber10] ———, *A survey of $(\infty, 1)$ -categories*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83. MR 2664620 (2011e:18001)
- [Ber11] ———, *Models for (∞, n) -categories and the cobordism hypothesis*, Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 17–30. MR 2742424 (2012j:55023)
- [BGT13] Andrew Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K -theory*, Geometry and Topology (2013), to appear, preprint available as [arXiv:1001.2282](https://arxiv.org/abs/1001.2282) [math.KT].
- [BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972. MR 0365573 (51 #1825)
- [BK12a] Clark Barwick and Daniel M. Kan, *A characterization of simplicial localization functors and a discussion of DK equivalences*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 69–79. MR 2877402
- [BK12b] ———, *n -relative categories: A model for the homotopy theory of n -fold homotopy theories*, to appear in *Homology, Homotopy, and Applications*, preprint available as [arXiv:1102.0186](https://arxiv.org/abs/1102.0186) [math.AT], 2012.
- [BK12c] ———, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 42–68. MR 2877401 (2012m:55019)
- [BLM08] Yuri Beshpalov, Volodymyr Lyubashenko, and Oleksandr Manzyuk, *Pretriangulated A_∞ -categories*, Institut matematiki NAN Ukrainy, 2008.
- [BSP12] Clark Barwick and Chris Schommer-Pries, *On the unicity of the homotopy theory of higher categories*, preprint available as [arXiv:1112.0040](https://arxiv.org/abs/1112.0040) [math.AT], 2012.
- [BZFN10] David Ben-Zvi, John Francis, and David Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966. MR 2669705 (2011j:14023)
- [BZN12] David Ben-Zvi and David Nadler, *Loop spaces and connections*, J. Topol. **5** (2012), no. 2, 377–430. MR 2928082
- [CFK01] Ionuț Ciocan-Fontanine and Mikhail Kapranov, *Derived quot schemes*, Annales Scientifiques de l'École Normale Supérieure, vol. 34, Elsevier, 2001, pp. 403–440.
- [CFK02] ———, *Derived hilbert schemes*, Journal of the American Mathematical Society **15** (2002), no. 4, 787–815.

- [Che11] Eugenia Cheng, *Comparing operadic theories of n -category*, Homology Homotopy Appl. **13** (2011), no. 2, 217–248. MR 2854336
- [CL04] Eugenia Cheng and Aaron Lauda, *Higher-Dimensional Categories: an illustrated guide book*, draft version available as <http://cheng.staff.shef.ac.uk/guidebook/guidebook-new.pdf>, 2004.
- [Cor82] Jean-Marc Cordier, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers Topologie Géom. Différentielle **23** (1982), no. 1, 93–112, Third Colloquium on Categories, Part VI (Amiens, 1980). MR 648798 (83g:18018)
- [Cur71] Edward B. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971), 107–209 (1971). MR 0279808 (43 #5529)
- [DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004. MR 2102294 (2005k:18027)
- [Die55] Jean Dieudonné, *Lie groups and Lie hyperalgebras over a field of characteristic $p > 0$. II*, Amer. J. Math. **77** (1955), 218–244. MR 0067872 (16,789f)
- [DK80a] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284. MR 579087 (81h:55018)
- [DK80b] William G. Dwyer and Daniel M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35. MR 578563 (81h:55019)
- [DS95] William G. Dwyer and Jan Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology (1995), 73–126.
- [DS11a] Daniel Dugger and David I. Spivak, *Mapping spaces in quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 263–325. MR 2764043 (2012e:55026)
- [DS11b] ———, *Rigidification of quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 225–261. MR 2764042 (2012f:55025)
- [Fio13] Thomas Fiore, *Approximation in K -theory for Waldhausen quasicategories*, preprint available as [arXiv:1303.4029](https://arxiv.org/abs/1303.4029) [math.AT], 2013.
- [FL12] Thomas Fiore and Wolfgang Lück, *Waldhausen additivity: Classical and quasicategorical*, preprint available as [arXiv:1207.6613](https://arxiv.org/abs/1207.6613) [math.AT], 2012.
- [FOOO09] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian intersection floer theory: anomaly and obstruction, part i*, American Mathematical Soc., 2009.
- [Fre13] Daniel S. Freed, *The Cobordism hypothesis*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 1, 57–92. MR 2994995
- [Fri12] Greg Friedman, *Survey article: an elementary illustrated introduction to simplicial sets*, Rocky Mountain J. Math. **42** (2012), no. 2, 353–423. MR 2915498
- [Fuk93] Kenji Fukaya, *Morse homotopy, A_∞ -categories, and floer homologies*, Proceedings of GARC workshop on Geometry and Topology (H. J. Kim, ed.), Seoul National University, 1993.
- [GH14] David Gepner and Rune Haugseng, *Enriched ∞ -categories via non-symmetric ∞ -operads*, preprint available as [arXiv:1312.3178](https://arxiv.org/abs/1312.3178) [math.AT], 2014.
- [Goe10] Paul G. Goerss, *Topological modular forms [after Hopkins, Miller and Lurie]*, Astérisque (2010), no. 332, Exp. No. 1005, viii, 221–255, Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. MR 2648680 (2011m:55003)
- [GPS95] R. Gordon, A. J. Power, and Ross Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. **117** (1995), no. 558, vi+81. MR 1261589 (96j:18002)
- [GPS12] Moritz Groth, Kate Ponto, and Michael Shulman, *The additivity of traces in monoidal derivators*, preprint, available as [arXiv:1212.3277](https://arxiv.org/abs/1212.3277) [math.AT], 2012.
- [Gro83] Alexander Grothendieck, *Pursuing stacks*, unpublished manuscript, 1983.
- [Gro91] ———, *Les Dérivateurs*, unpublished manuscript, 1991.
- [Gro10] Moritz Groth, *A short course on ∞ -categories*, preprint available from author’s website as http://www.math.ru.nl/~mgroth/preprints/groth_scinfinit.pdf.
- [Gro13] ———, *Derivators, pointed derivators, and stable derivators*, Algebraic & Geometric Topology (2013), no. 13, 313–374.
- [Gur06] Michael Nicholas Gurski, *An algebraic theory of tricategories*, Ph.D. thesis, University of Chicago, Department of Mathematics, 2006.
- [Hak72] Monique Hakim, *Topos annelés et schémas relatifs*, no. 64, Springer-Verlag, 1972.

- [Hau14] Rune Haugseng, *Rectification of enriched ∞ -categories*, preprint available as [arXiv:1312.3881](https://arxiv.org/abs/1312.3881) [math.AT], 2014.
- [Hel88] Alex Heller, *Homotopy theories*, Mem. Amer. Math. Soc. **71** (1988), no. 383, vi+78. MR 920963 (89b:55013)
- [Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR 1944041 (2003j:18018)
- [Hof11] Alexander Hoffnung, *Spans in 2-categories: A monoidal tricategory*, preprint, available as [arXiv:1112.0560](https://arxiv.org/abs/1112.0560) [math.CT], 2011.
- [Hov99] Mark Hovey, *Model categories*, no. 63, American Mathematical Society, 1999.
- [Joy02] André Joyal, *Quasi-categories and kan complexes*, Journal of Pure and Applied Algebra **175** (2002), no. 13, 207 – 222, Special Volume celebrating the 70th birthday of Professor Max Kelly.
- [Joy08a] ———, *Notes on quasi-categories*, in preparation, 2008.
- [Joy08b] André Joyal, *The theory of quasi-categories I*, 2008, in progress.
- [JT06] André Joyal and Miles Tierney, *Quasi-categories vs Segal spaces*, preprint, available as [arXiv:math/0607820](https://arxiv.org/abs/math/0607820) [math.AT], 2006.
- [Kel06] Bernhard Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190. MR 2275593 (2008g:18015)
- [Kon95] Maxim Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368. MR 1363062 (97d:14077)
- [KS09] M. Kontsevich and Y. Soibelman, *Notes on A_∞ -algebras, A_∞ -categories and non-commutative geometry*, Homological mirror symmetry, Lecture Notes in Phys., vol. 757, Springer, Berlin, 2009, pp. 153–219. MR 2596638 (2011f:53183)
- [Lac07] Stephen Lack, *Bicat is not triequivalent to Gray*, Theory and Applications of Categories **18** (2007), no. 1.
- [Lac10] ———, *A 2-categories companion*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 105–191. MR 2664622 (2011d:18012)
- [Laz55] Michel Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France **83** (1955), 251–274. MR 0073925 (17,508e)
- [Lei98] Tom Leinster, *Basic bicategories*, preprint available as [arXiv:math/9810017](https://arxiv.org/abs/math/9810017) [math.CT], 1998.
- [Lei02] ———, *A survey of definitions of n -category*, Theory and Applications of Categories **10** (2002), no. 1, 1–70.
- [Lei04] ———, *Higher operads, higher categories*, vol. 298, Cambridge University Press, 2004.
- [Lur07a] Jacob Lurie, *Derived Algebraic Geometry I: Stable ∞ -categories*, preprint available as [arXiv:math/0608228](https://arxiv.org/abs/math/0608228) [math.CT], 2007.
- [Lur07b] ———, *Derived Algebraic Geometry II: Noncommutative Algebra*, preprint available as [arXiv:math/0702299](https://arxiv.org/abs/math/0702299) [math.CT], 2007.
- [Lur07c] ———, *Derived Algebraic Geometry III: Commutative Algebra*, preprint available as [arXiv:math/0703204](https://arxiv.org/abs/math/0703204) [math.CT], 2007.
- [Lur09a] J. Lurie, *A survey of elliptic cohomology*, Algebraic topology, Abel Symp., vol. 4, Springer, Berlin, 2009, pp. 219–277. MR 2597740 (2011f:55009)
- [Lur09b] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009, also available from author’s website as <http://www.math.harvard.edu/~lurie/papers/croppedtopoi.pdf>. MR 2522659 (2010j:18001)
- [Lur09c] ———, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, also available from author’s website as <http://www.math.harvard.edu/~lurie/papers/cobordism.pdf>, pp. 129–280. MR 2555928 (2010k:57064)
- [Lur10] ———, *Moduli problems for ring spectra*, Proceedings of the International Congress of Mathematicians. Volume II (New Delhi), Hindustan Book Agency, 2010, pp. 1099–1125. MR 2827833 (2012i:55008)
- [Lur11a] ———, *Derived Algebraic Geometry IX: Closed Immersions*, preprint available from author’s website as <http://math.harvard.edu/~lurie/papers/DAG-IX.pdf>, 2011.

- [Lur11b] ———, *Derived Algebraic Geometry V: Structured Spaces*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-V.pdf>, 2011.
- [Lur11c] ———, *Derived Algebraic Geometry VII: Spectral Schemes*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-VII.pdf>, 2011.
- [Lur11d] ———, *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-VIII.pdf>, 2011.
- [Lur11e] ———, *Derived Algebraic Geometry X: Formal Moduli Problems*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-X.pdf>, 2011.
- [Lur11f] ———, *Derived Algebraic Geometry XI: Descent Theorems*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-XI.pdf>, 2011.
- [Lur11g] ———, *Derived Algebraic Geometry XII: Proper Morphisms, Completions, and the Grothendieck Existence Theorem*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-XII.pdf>, 2011.
- [Lur11h] ———, *Derived Algebraic Geometry XIII: Rational and p -adic Homotopy Theory*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-XIII.pdf>, 2011.
- [Lur12a] ———, *Derived Algebraic Geometry XIV: Representability Theorems*, preprint available from author's website as <http://math.harvard.edu/~lurie/papers/DAG-XIV.pdf>, 2012.
- [Lur12b] ———, *Higher algebra*, 2012, draft available from author's website as <http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf>.
- [Mal01] Georges Maltsiniotis, *Introduction à la théorie des dérivateurs (d'après grothendieck)*, preprint available from author's website, as <http://www.math.jussieu.fr/~maltsin/ps/m.ps>, 2001.
- [May72] J. P. May, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics, vol. Vol. 271, Springer-Verlag, Berlin, 1972, also available from the author's website as <http://www.math.uchicago.edu/~may/B00KS/gils.pdf>, pp. viii+175. MR 0420610 (54 #8623b)
- [May05] Peter J. May, *Operadic categories, A_∞ -categories and n -categories*, 2005, talk given in Morelia, México, <http://www.math.uchicago.edu/~may/NCATS/PostMexico.pdf>.
- [Por04] Timothy Porter, *S -categories, S -groupoids, Segal categories and quasi-categories*, preprint available as [arXiv:math/0401274](https://arxiv.org/abs/math/0401274) [math.AT], 2004.
- [Qui67] Dan Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, 1967.
- [Qui69] Daniel Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. MR 0253350 (40 #6565)
- [Ren06] Olivier Renaudin, *Théories homotopiques de Quillen combinatoires et dérivateurs de Grothendieck*, preprint available as [arXiv:math/0603339](https://arxiv.org/abs/math/0603339) [math.AT].
- [Ren09] ———, *Plongement de certaines théories homotopiques de Quillen dans les dérivateurs*, J. Pure Appl. Algebra **213** (2009), no. 10, 1916–1935. MR 2526867 (2010c:18016)
- [Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic). MR 1804411 (2002a:55020)
- [Rez05] ———, *Toposes and homotopy toposes*, 2005.
- [Rie14] Emily Riehl, *Categorical homotopy theory*, vol. 24, Cambridge University Press, 2014, also available from the author's website as <http://www.math.harvard.edu/~eriehl/cathtpy.pdf>.
- [RV13] Emily Riehl and Dominic Verity, *Homotopy coherent adjunctions and the formal theory of monads*, in preparation, 2013.
- [Seg88] G. B. Segal, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., Dordrecht, 1988, pp. 165–171. MR 981378 (90d:58026)
- [Seg04] Graeme Segal, *The definition of conformal field theory*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 421–577. MR 2079383 (2005h:81334)

- [Sei08] Paul Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2441780 (2009f:53143)
- [Sim96] Carlos Simpson, *Algebraic (geometric) n -stacks*, preprint available as [arXiv:alg-geom/9609014](https://arxiv.org/abs/alg-geom/9609014), 1996.
- [Sim99] Carlos Simpson, *A Giraud-type characterization of the simplicial categories associated to closed model categories as ∞ -pretopoi*, preprint available as [arXiv:math/9903167](https://arxiv.org/abs/math/9903167) [[math.AT](https://arxiv.org/abs/math)].
- [Sim12] ———, *Homotopy theory of higher categories*, New Mathematical Monographs, vol. 19, Cambridge University Press, Cambridge, 2012, draft available as [arXiv:1001.4071](https://arxiv.org/abs/1001.4071) [[math.CT](https://arxiv.org/abs/math)]. MR 2883823 (2012m:18019)
- [SP09] Christopher John Schommer-Pries, *The classification of two-dimensional extended topological field theories*, Ph.D. thesis, University of California, Berkeley, 2009, p. 254.
- [Spi10] David I. Spivak, *Derived smooth manifolds*, *Duke Math. J.* **153** (2010), no. 1, 55–128. MR 2641940 (2012a:57043)
- [SS86] Stephen Schanuel and Ross Street, *The free adjunction*, *Cahiers de topologie et géométrie différentielle catégoriques* **27** (1986), no. 1, 81–83.
- [Sta63] James Dillon Stasheff, *Homotopy associativity of H -spaces. I, II*, *Trans. Amer. Math. Soc.* **108** (1963), 275–292; *ibid.* **108** (1963), 293–312. MR 0158400 (28 #1623)
- [Ste67] Norman E. Steenrod, *A convenient category of topological spaces*, *Michigan Math. J.* **14** (1967), 133–152. MR 0210075 (35 #970)
- [Str] Neil Strickland, *The category of Compactly Generated Weakly Hausdorff spaces*, notes from a course taught by the author, available at <http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf>.
- [Tab07] Gonçalo Tabuada, *Homotopy theory of dg-categories*, Ph.D. thesis, Paris, 2007, available as <http://people.math.jussieu.fr/~keller/TabuadaThese.pdf>.
- [Toë05] Bertrand Toën, *Vers une axiomatisation de la théorie des catégories supérieures*, *K-Theory* **34** (2005), no. 3, 233–263. MR 2182378 (2006m:55041)
- [Toë07] ———, *The homotopy theory of dg-categories and derived Morita theory*, *Invent. Math.* **167** (2007), no. 3, 615–667. MR 2276263 (2008a:18006)
- [Toë09] ———, *Higher and derived stacks: a global overview*, Algebraic geometry—Seattle 2005. Part 1, *Proc. Sympos. Pure Math.*, vol. 80, Amer. Math. Soc., Providence, RI, 2009, also available as a preprint, [arXiv:math/0604504](https://arxiv.org/abs/math/0604504) [[math.AG](https://arxiv.org/abs/math)], pp. 435–487. MR 2483943 (2010h:14021)
- [Toë10] ———, *Lectures on dg-categories*, notes available from author’s website as <http://www.math.univ-toulouse.fr/~toen/swisk.pdf>, 2010.
- [Tri06] Todd Trimble, *Notes on Tetracategories*, <http://math.ucr.edu/home/baez/trimble/tetracategories.html>, October 2006.
- [TV01] Bertrand Toën and Gabriele Vezzosi, *Algebraic geometry over model categories (a general approach to derived algebraic geometry)*, preprint available as [arXiv:math/0110109](https://arxiv.org/abs/math/0110109) [[math.AG](https://arxiv.org/abs/math)], 2001.
- [TV02] Bertrand Toën and Gabriele Vezzosi, *Segal topoi and stacks over segal categories*, preprint available as [arXiv:math/0212330](https://arxiv.org/abs/math/0212330) [[math.AG](https://arxiv.org/abs/math)], 2002.
- [TV04] ———, *From HAG to DAG: derived moduli stacks*, *Axiomatic, enriched and motivic homotopy theory*, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 173–216. MR 2061855 (2005f:14001)
- [TV05] ———, *Homotopical algebraic geometry. I. Topos theory*, *Adv. Math.* **193** (2005), no. 2, 257–372. MR 2137288 (2007b:14038)
- [TV08] ———, *Homotopical algebraic geometry. II. Geometric stacks and applications*, *Mem. Amer. Math. Soc.* **193** (2008), no. 902, x+224. MR 2394633 (2009h:14004)
- [Vez10] Gabriele Vezzosi, *A note on the cotangent complex in derived algebraic geometry*, preprint available as [arXiv:1008.0601](https://arxiv.org/abs/1008.0601) [[math.AG](https://arxiv.org/abs/math)], 2010.
- [Vis05] Angelo Vistoli, *Grothendieck topologies, fibered categories and descent theory*, *Fundamental algebraic geometry*, *Math. Surveys Monogr.*, vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104. MR 2223406

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