THE MOD 2 HOMOLOGY OF FREE SPECTRAL LIE ALGEBRAS

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Abstract. The Goodwillie derivatives of the identity functor on pointed spaces form an operad $\partial_*(\mathrm{Id})$ in spectra. Adapting a definition of Behrens, we introduce mod 2 homology operations for algebras over this operad and prove these operations account for all the mod 2 homology of free algebras on suspension spectra of simply-connected spaces.

Contents

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1. Introduction

Goodwillie calculus [10] associates to appropriate functors $F : \mathrm{Top}_* \to \mathrm{Top}_*$ a tower of approximations

$$\cdots \to P_nF \to P_{n-1}F \to \cdots \to P_1F \to P_0F$$

that is analogous to the sequence of Taylor polynomials for functions of a real variable. The homotopy fibers $D_nF = \mathrm{hofib}(P_nF \to P_{n-1}F)$ are called the layers of the Goodwillie tower and are analogous to individual monomials $f^{(n)}(0)x^n/n!$ in the Taylor expansion of a function. Goodwillie proved that these layers are of the form $D_nF(X) = \Omega^\infty(\partial_nF \wedge \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$ for some sequence of spectra ∂_nF where the n-th spectrum is equipped with an action of Σ_n.

These derivatives ∂_nF are very interesting even for $F = \mathrm{Id}$ and have been much studied in that case; they can be described as the Spanier–Whitehead duals of certain finite complexes. The first such description was obtained by Johnson [13]; a second description is in terms of partitions and appears in [3]. The *partition complex* P_n is the pointed simplicial set

$$N\Pi_n \Big/ \big(N(\Pi_n \setminus \{\hat{0}\}) \cup N(\Pi_n \setminus \{\hat{1}\})\big),$$

where Π_n is the poset of partitions of a set with n elements, ordered by refinement; $\hat{0}$ and $\hat{1}$ denote its least and greatest element, respectively; and N denotes, as usual, the nerve functor. We shall regard P_n as having the action of Σ_n induced by permutations of the n-element set.

The layers of the Goodwillie tower of the identity are given by

$$D_n(\mathrm{Id})(X) = \Omega^\infty\left(\mathrm{Map}_*(P_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right),$$

where Map_* denotes the spectrum of maps from a pointed space to a spectrum. This implies that $\partial_n(\mathrm{Id})$ is $\mathrm{Map}_*(P_n, S)$, the Spanier–Whitehead dual of P_n.

In [6], Ching constructs an operad structure on $\partial_*(\mathrm{Id})$ that is easiest to describe in dual form: as a cooperad structure on P_*. That cooperad is the bar construction on the nonunital commutative operad in spectra (given by $\mathrm{Comm}_n = S$ for all $n \geq 1$), so that the operadic suspension of $\partial_*(\mathrm{Id})$ is Koszul dual to the commutative operad and we can think of $\partial_*(\mathrm{Id})$ as a shifted version of the Lie operad. Alternatively, one can also see a relation between $\partial_*(\mathrm{Id})$ and the Lie operad using what is known about the homology of the partition complex, namely that the space of n-ary operations of the Lie operad in Abelian groups is isomorphic as a $\mathbb{Z}[\Sigma_n]$-module to $\mathrm{Hom}(H_{n-2}(P_n), \mathrm{sgn})$ —where sgn is the sign representation of Σ_n.

The mod p homology of the layers,

$$\mathbb{D}_n(X) := \mathbb{D}_n(\mathrm{Id})(X) = (\partial_n(\mathrm{Id}) \wedge \Sigma^\infty X^{\wedge n})_{h\Sigma_n},$$

was studied in [3] in the case that X is a sphere. (Since the free $\partial_*(\mathrm{Id})$-algebra on a space X is given by $\bigoplus_{n\geq 0} \mathbb{D}_n(X)$, the results in that paper can be interpreted as being about the mod p homology of the free $\partial_*(\mathrm{Id})$-algebra on S^n.) In the case $p = 2$, for example, what Arone and Mahowald showed is that $H_*(\mathbb{D}_n(S^m); \mathbb{F}_p)$ is only non-zero when $n = 2^k$ is a power of 2 and in that case it is $\Sigma^{-k}CU_*$ as a module over the Steenrod algebra, where CU_* is the free graded $\mathbb{F}_p$-vector space with basis given by the "completely unadmissible" words of length k:

$$\{Q^{s_1} \cdots Q^{s_k} u : s_k \geq m, s_i > 2s_{i+1}\}$$

where u is a generator of $H_m(S^m; \mathbb{F}_p)$ and the action of the Steenrod algebra on CU_* is given by the Nishida relations.

Behrens [4] uses this computation to introduce mod 2 homology operations $\bar{Q}^j : H_d(D_i F(X)) \to H_{d+j-1}(D_{2i} F(X))$ for $j \geq d$ on the layers of the Goodwillie tower of a functor F. We adapt his definition to produce homology operations on $\partial_*(\mathrm{Id})$-algebras (see definition 5.4). Behrens shows the Arone–Mahowald computation can be interpreted as saying that the homology of $\bigoplus_{n\geq 0} \mathbb{D}_n(X)$ has an $\mathbb{F}_2$-basis consisting of completely unadmissible sequences of $\bar{Q}^j$'s with excess at least k applied to the fundamental class of S^k; furthermore, he computes the relations satisfied by the $\bar{Q}^j$'s.

In the present work we compute the mod 2 homology of the free $\partial_*(\mathrm{Id})$-algebra on a spectrum, showing that it is roughly speaking the free module over the algebra of operations $\bar{Q}^j$'s on the free Lie algebra on $H_*(X)$ (see theorem 7.1).

1.1. **Some related work.** Since the first appearance of this work as the author's PhD thesis, an analogous computation of the mod p homology for odd primes p has been carried out by Kjaer [14]. In that setting the story is less complete: due to a lack of an analogue of Priddy's trace formula [4, Lemma 1.4.3], the relations between the homology operations have yet to be fully determined.

In the upcoming paper [5], Brantner computes the $E(n)$-homology of certain free $\partial_*(\mathrm{Id})$-algebras in the category of $K(n)$-local spectra and thus computes the $E(n)$-homology operations on $K(n)$-local $\partial_*(\mathrm{Id})$-algebras. Here $E(n)$ and $K(n)$ denote Morava E-theory and Morava K-theory respectively.

Also, in the final section of the paper we will explain the relation between our computation and the E_1-page of Kuhn's spectral sequence for

Topological André–Quillen homology of an augmented E_∞-ring spectrum [15, Theorem 8.1].

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2. Homology operations on algebras for operads

We will work in a symmetric monoidal category of spectra, such as EKMM S-modules [8], taking "spectrum" to mean S-module and "E_∞-ring spectrum" to mean commutative S-algebra. This is the same framework used in [6] to put an operad structure on the derivatives of the identity.

Given an operad $\mathcal{O}$ in spectra we will denote by $F_{\mathcal{O}}$ the free $\mathcal{O}$-algebra functor. This functor is a monad, and $\mathcal{O}$-algebras are equivalently algebras for it. If E is an E_∞-ring spectrum, then there is an operad in E-module spectra we will denote by $E \wedge \mathcal{O}$, and a free $(E \wedge \mathcal{O})$-algebra functor $F_{E\wedge\mathcal{O}}$ defined on E-module spectra. The E-module of n-ary operations in $E \wedge \mathcal{O}$ is the free E-module on the spectrum $\mathcal{O}_n$: $(E \wedge \mathcal{O})_n = E \wedge \mathcal{O}_n$; we get an operad structure on $E \wedge \mathcal{O}$ induced from the operad structure on $\mathcal{O}$ because the free E-module functor is symmetric monoidal. The free algebra functors are related in the expected way: $E \wedge F_{\mathcal{O}}(X) \simeq F_{E\wedge\mathcal{O}}(E \wedge X)$. We will also make use of the functor between $\mathcal{O}$-algebras and $(E \wedge \mathcal{O})$-algebras induced by the free E-module functor, $E \wedge -$.

We will only consider cofibrant operads for which the notion of algebra is homotopy invariant, meaning that we can think of the homotopy type of the free $\mathcal{O}$-algebra as being given by

$$F_{\mathcal{O}}(X) = \bigvee_{n\geq 0} (\mathcal{O}_n \wedge X^n)_{h\Sigma_n},$$

and that we will think of an $\mathcal{O}$-algebra structure on A as providing maps $(\mathcal{O}_n \wedge A^{\wedge n})_{h\Sigma_n} \to A$.

Every class $\alpha \in E_m\left(F_{\mathcal{O}}\left(\bigvee_{i=1}^k S^{d_i}\right)\right)$ in the E-homology of the free $\mathcal{O}$-algebra on a wedge of k spheres gives a k-ary homology operation on the E-homology of any $\mathcal{O}$-algebra A, defined as follows.

Given $x_i \in E_{d_i}(A)$ $(i = 1, \ldots, k)$, we can represent each x_i by a map of spectra $S^{d_i} \to E \wedge A$, and thus the whole collection of them can be described by a single map of spectra $\bar{x} : \bigvee_{i=1}^k S^{d_i} \to E \wedge A$. Since $E \wedge A$ is an $(E \wedge \mathcal{O})$-algebra, $\bar{x}$ has an adjoint $\tilde{x}$ which is a map of $(E \wedge \mathcal{O})$-algebras to $E \wedge A$ from the free $(E \wedge \mathcal{O})$-algebra on $\bigvee_{i=1}^k S^{d_i}$, namely $F_{E\wedge\mathcal{O}}(\bigvee_{i=1}^k \Sigma^{d_i} E) = E \wedge F_{\mathcal{O}}(\bigvee_{i=1}^k S^{d_i})$.

The homology operation corresponding to α, is $\alpha_* : \bigotimes_{i=1}^k E_{d_i}(A) \to E_m(A)$ defined by setting $\alpha_*(x_1 \otimes \cdots \otimes x_k)$ to be represented by the map

$$S^m \xrightarrow{\alpha} E \wedge F_{\mathcal{O}}(\bigvee_{i=1}^k S^{d_i}) \xrightarrow{\tilde{x}} E \wedge A.$$

An analogous construction gives operations on the stable homotopy of $(E \wedge \mathcal{O})$-algebras. Given an $\mathcal{O}$-algebra A, the operations on the E-homology of A coincide with those produced on the homotopy of the $(E \wedge \mathcal{O})$-algebra $E \wedge A$.

To get a useful theory of homology operations for $\mathcal{O}$-algebras, besides computing those homology groups, the various $E_m(F_{\mathcal{O}}(\bigvee_{i=1}^k S^{d_i}))$, one must organize the operations: find a relatively small collection of operations that generate all others and find a generating set of relations for the operations. This has been carried out for $H\mathbb{F}_p$-homology of algebras for the E_n-operads, due to May in the case $n = \infty$, and due to F. Cohen in the case $1 \leq n < \infty$; see [7].

Homology operations with field coefficients are simpler to study, because of the following result:

Proposition 2.1. *Let $\mathcal{O}$ be an operad in spectra. The homology with coefficients in a field k of the free $\mathcal{O}$-algebra on a spectrum X is a functor of the homology of X.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{S}p & \xrightarrow{Hk\wedge -} & Hk\text{-Mod} & \xrightarrow{\pi} & D(k) & \xrightarrow{\cong} & \mathrm{GrVect}_k \\
\downarrow{\scriptstyle F_{\mathcal{O}}} & & \downarrow{\scriptstyle F_{Hk\wedge\mathcal{O}}} & & \downarrow{\scriptstyle \hat{F}} & & \downarrow{\scriptstyle \tilde{F}} \\
\mathcal{S}p & \xrightarrow[Hk\wedge -]{} & Hk\text{-Mod} & \xrightarrow[\pi]{} & D(k) & \xrightarrow[\cong]{} & \mathrm{GrVect}_k
\end{array}$$

Here $\mathcal{S}p$ denotes the category of spectra, Hk−Mod denotes the category of Hk-module spectra, $D(k)$ is the homotopy category of Hk−Mod or, equivalently, the unbounded derived category of vector spaces over k and GrVect_k is the category of graded vector spaces over k.

The functor π is the projection from $Hk-$Mod to its homotopy category; this functor preserves coproducts but when the characteristic of k is not 0, it does not send homotopy quotients by the action of Σ_n to quotients by the action of Σ_n, so the induced monad $\hat{F}$ is no longer the free algebra functor for an operad. Finally, when k is a field there is an equivalence $D(k) \cong \mathrm{GrVect}_k$, allowing us to define the monad $\tilde{F}$ so that the last square commutes. □

3. The spectral Lie operad and its desuspension

Recall that the suspension of an operad $\mathcal{O}$ in spectra is an operad $\mathbf{s}\mathcal{O}$ defined so that:

- $\mathbf{s}\mathcal{O}$-algebra structures on ΣA correspond to $\mathcal{O}$-algebra structures on A,
- the free algebra functors satisfy $F_{\mathbf{s}\mathcal{O}}(\Sigma X) = \Sigma F_{\mathcal{O}}(X)$, and
- as a symmetric sequence, $(\mathbf{s}\mathcal{O})_n$ is given by $(S^{-1})^{\wedge n} \wedge \Sigma\mathcal{O}_n$ with Σ_n acting diagonally, permuting the smash factors on the left and acting on $\Sigma\mathcal{O}_n$ via the suspension of the action on $\mathcal{O}_n$ (that is, it acts trivially on the suspension coordinate of $\Sigma\mathcal{O}_n$).

Following existing nomenclature, we will call the operad $\partial_*(\mathrm{Id})$ formed by the Goodwillie derivatives of the identity, the *spectral Lie operad*. Even though it is actually the desuspension $\mathbf{s}^{-1}\partial_*(\mathrm{Id})$ that is most closely analogous to the classical Lie operad and some of our formulas would be simpler for it, we will stick to the language of the $\partial_*(\mathrm{Id})$-operad and $\partial_*(\mathrm{Id})$-algebras to make using the available literature easier. As a symmetric sequence, $\mathbf{s}^{-1}\partial_*(\mathrm{Id})$ is given by the derivatives of the functor $\Omega\Sigma : \mathrm{Top}_* \to \mathrm{Top}_*$ (see [10, Section 8]).

As we said before, the easiest way to describe the operad structure of $\partial_*(\mathrm{Id})$ is to describe a cooperad structure on the bar construction of the nonunital commutative operad, and obtain the operad structure of $\partial_*(\mathrm{Id})$ by taking Spanier–Whitehead duals. For a description of Ching's cooperad structure, we refer the reader to [6, Section 4].

4. Two kinds of Lie algebras in characteristic 2

In this section we collect a few definitions about (graded) Lie algebras we will need later. We will actually need to use two different notions of Lie algebras. The usual definition of Lie algebra over a field of characteristic 0 is equivalent to being an algebra for an operad Lie in Abelian groups. One can take algebras for that operad in any category that is tensored over Abelian groups, such as the category of R-modules or of graded R-modules for a commutative ring R, and this gives one possible definition of graded Lie algebra. Since $\partial_*(\mathrm{Id})$ is the suspension of the spectral version of the Lie operad, we are also interested in algebras for the suspension sLie.

Spelling out the structure we see that a graded Lie-algebra L over a commutative ring R is a graded module equipped with a binary operation $[-,-] : L_i \otimes L_j \to L_{i+j}$ satisfying, for homogeneous elements x, y and z of degrees $|x|$, $|y|$ and $|z|$:

- anti-symmetry, $[x, y] = -(-1)^{|x||y|}[y, x]$, and
- the Jacobi identity,

$$(-1)^{|z||x|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

For sLie-algebras things are only slightly different:

- The bracket has degree -1: $[-,-] : L_i \otimes L_j \to L_{i+j-1}$.
- Anti-symmetry becomes graded commutativity:

$$[x, y] = (-1)^{|x||y|}[y, x].$$

- The Jacobi identity stays the same!

All the signs in the above formulas come from the Koszul sign rule, that is, from the signs in the symmetry isomorphism of the category of graded R-modules. Since we will work over $R = \mathbb{F}_2$ we need not worry about signs, but we mention them to point out that for an element x of even degree in a Lie-algebra (or of odd degree in a sLie-algebra), the definitions imply that $2[x, x] = 0$, but they do not actually imply $[x, x] = 0$ if 2 is not invertible in R.

If R has characteristic 2, while $[x, x]$ may not be 0, we do have that any brackets involving it are 0: by the Jacobi identity,

$$[[x, x], y] = [[x, y], x] + [[y, x], x] = 2[[x, y], x] = 0.$$

As an example showing $[x, x]$ can be nonzero, the free Lie-algebra over $\mathbb{F}_2$ on one generator x in an even degree is easily seen to have basis $\{x, [x, x]\}$.

Given a graded associative R-algebra A, the graded commutator $[x, y] = xy - (-1)^{|x||y|}yx$ gives A the structure of a Lie-algebra, but all the algebras produced this way necessarily have $[x, x] = 0$ for $|x|$ even. This means that if a Lie-algebra over an R of characteristic 2 has some nonzero $[x, x]$ with $|x|$ even, it cannot be faithfully represented by commutators, and thus does not inject into its universal enveloping algebra. This substantially changes portions of the theory of Lie algebras that require an embedding into the universal enveloping algebra and so at least one other definition of Lie algebra in characteristic 2 is sometimes used, one that forces an injection into a Lie algebra of commutators.

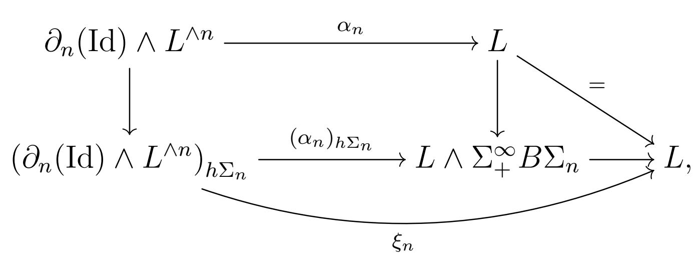
In the case of $R = \mathbb{F}_2$ this other kind of Lie algebra simply adds the requirement that $[x, x] = 0$ for all homogeneous x. We will call this kind of Lie algebra a $\mathsf{Lie}^{\mathrm{ti}}$-algebra —the ti stands for *totally isotropic*. A definition for all rings R, due to Moore, just forces the representation as a commutator Lie algebra to exist:

Definition 4.1. A graded $\mathsf{Lie}^{\mathrm{ti}}$-algebra (resp. $\mathsf{sLie}^{\mathrm{ti}}$-algebra) over R is graded R-module L with a bracket $L_i \otimes L_j \to L_{i+j}$ (resp. L_{i+j-1}) and a monomorphism $L \to A$ to some graded associative algebra so that the bracket goes to the graded commutator $xy - (-1)^{|x||y|}yx$ (resp. $xy + (-1)^{|x||y|}yx$).

Our main interest in these algebras is that the *basic products* appearing in Hilton's theorem about the loop space of a wedge of spheres [12] form a basis (called a Hall basis) for a totally isotropic Lie algebra, see the discussion in section 7.

5. Homology operations for spectral Lie algebras

Throughout this section L will denote an algebra for the operad $\partial_*(\mathrm{Id})$. So in particular, L is a spectrum equipped with structure maps $\xi_n : \mathbb{D}_n(L) \to L$ where $\mathbb{D}_n(L) = (\partial_n(\mathrm{Id}) \wedge L^{\wedge n})_{h\Sigma_n}$. There is a more traditional way to describe the structure of an algebra for an operad: by giving maps $\alpha_n : \partial_n(\mathrm{Id}) \wedge L^{\wedge n} \to L$ that are Σ_n-equivariant for the trivial action on the codomain and the diagonal action on the domain. The relation between these two styles of definition is captured in the following commutative diagram:



where the vertical maps are the canonical maps $Y^{\wedge n} \to Y^{\wedge n}_{h\Sigma_n}$ and the unlabeled horizontal map is $L \wedge \Sigma^\infty_+(-)$ applied to $B\Sigma_n \to *$.

In this section we will describe some operations on $H_*(L; \mathbb{F}_2)$ that will turn out to generate all others, and whose definition will only require the map

$$\xi = \xi_2 : \left(\partial_2(\mathrm{Id}) \wedge L^{\wedge 2}\right)_{h\Sigma_2} \to L.$$

Recall that $\partial_2(\mathrm{Id})$ is the Spanier–Whitehead dual of the partition complex P_2. A set with two elements has only two partitions, both fixed by Σ_2, so that $P_2 \simeq S^1$ and $\partial_2(\mathrm{Id}) \simeq S^{-1}$, both with trivial Σ_2-action. This implies that $(\partial_2(\mathrm{Id}) \wedge L^{\wedge 2})_{h\Sigma_2} \simeq \Sigma^{-1} L^{\wedge 2}_{h\Sigma_2}$.

5.1. **The shifted Lie bracket.** We will start by describing the Lie bracket. Here we remind the reader that $\partial_*(\mathrm{Id})$ is not really analogous to the Lie operad, but rather is analogous to its operadic suspension.

Definition 5.1. The *shifted Lie bracket* on the homology of an $\partial_*(\mathrm{Id})$-algebra L is the map $[\cdot,\cdot] : H_i(L) \otimes H_j(L) \to H_{i+j-1}(L)$ given by the identification $S^{-1} \to \partial_2(\mathrm{Id})$, that is, it is the map induced on homology by the suspension of the structure map $\alpha_2 : \Sigma^{-1} L^{\wedge 2} \to L$.

This operation really gives a sLie-algebra:

Proposition 5.2. *Given any $\partial_*(\mathrm{Id})$-algebra L, the shifted Lie bracket on $H_*(L)$ gives $H_*(L)$ the structure of a sLie-algebra.*

Remark 5.3. The proof of this proposition presented below is joint work with Lukas Brantner.

Proof. This is not really a result about $\mathbb{F}_2$-homology: we will show that the Lie algebra structure is present already at the level of spectra. We have already proved symmetry, when we computed $\partial_2(\mathrm{Id})$ and saw it had the trivial Σ_2-action.

Now we must prove the Jacobi identity. Consider three elements of the homology of L, say $x \in H_iL$, $y \in H_jL$ and $z \in H_kL$. Just as we defined the bracket $[x,y]$ as coming from a particular map $S^{-1} \to \partial_2(\mathrm{Id})$, the bracket $[x,[y,z]] \in H_{i+j+k-2}L$ can be obtained from a particular map $\nu : S^{-2} \to \partial_3(\mathrm{Id})$ as the effect on homology of the double suspension of the composite

$$S^{-2} \wedge L^{\wedge 3} \xrightarrow{\nu \wedge \mathrm{id}} \partial_3(\mathrm{Id}) \wedge L^{\wedge 3} \xrightarrow{\alpha_3} L.$$

The map ν simply corresponds to one of the structure maps of the $\partial_*(\mathrm{Id})$-operad under the identifications $S \simeq \partial_1(\mathrm{Id})$, $S^{-1} \simeq \partial_2(\mathrm{Id})$, namely, it is

$$S^{-1} \wedge (S \wedge S^{-1}) \simeq \partial_2(\mathrm{Id}) \wedge (\partial_1(\mathrm{Id}) \wedge \partial_2(\mathrm{Id})) \to \partial_{1+2}(\mathrm{Id}).$$

The other terms in the Jacobi identity are obtained from $[x,[y,z]]$ by cyclically permuting the variables, so, letting $\sigma = (123) \in \Sigma_3$ and using σ_* to denote the induced action of σ on $\partial_3(\mathrm{Id})$, to prove the Jacobi identity we must show $\nu + \sigma_* \circ \nu + \sigma_*^2 \circ \nu : S^{-2} \to \partial_3(\mathrm{Id})$ is null-homotopic.

We might as well work with $\Sigma^\infty P_3$, before taking Spanier–Whitehead duals, and show that $1 + \bar\sigma + \bar\sigma^2 : \Sigma^\infty P_3 \to \Sigma^\infty P_3$ is null-homotopic, where $\bar\sigma$ denotes the action of the 3-cycle σ on $\Sigma^\infty P_3$.

Now, P_3 consists of:

- a 1-simplex, corresponding to the chain $\hat{0} < \hat{1}$, connecting the basepoint $\hat{0} = \hat{1}$ with itself, and
- three 2-simplices, say τ_1, τ_2, τ_3, each filling in the above circle, corresponding to the three chains $\hat{0} < (23|1) < \hat{1}$, $\hat{0} < (13|2) < \hat{1}$, and $\hat{0} < (12|3) < \hat{1}$, respectively.

The 3-cycle σ permutes those three 2-simplices cyclically. We can compute $1 + \bar{\sigma} + \bar{\sigma}^2 : \Sigma^\infty P_3 \to \Sigma^\infty P_3$ as the composite:

$$\Sigma^\infty P_3 \xrightarrow{\Delta} \bigvee^3 \Sigma^\infty P_3 \xrightarrow{1\vee\bar{\sigma}\vee\bar{\sigma}^2} \bigvee^3 \Sigma^\infty P_3 \xrightarrow{\nabla} \Sigma^\infty P_3.$$

Non-equivariantly we have an equivalence $S^2 \vee S^2 \xrightarrow{\simeq} P_3$, where we will think of the first S^2 as mapping to P_3 by sending the northern hemisphere to τ_1, and the southern hemisphere to τ_2; we will abbreviate this map $S^2 \to P_3$ as τ_{12} and use similar notation for other maps. We will think of the second wedge summand S^2 as corresponding to the map τ_{23}.

We can think of a map $\bigvee^n \Sigma^\infty P_3 \to \bigvee^m \Sigma^\infty P_3$ as given by an $m \times n$ matrix of maps $\Sigma^\infty P_3 \to \Sigma^\infty P_3$, and each such map as given by a 2×2 matrix of maps $\Sigma^\infty S^2 \to \Sigma^\infty S^2$.

The matrices of Δ and ∇ are just the 3×1 and 1×3 matrices each of whose entries is I, the 2×2 identity matrix. Once we have the 2×2 matrix A representing $\sigma : \Sigma^\infty P_3 \to \Sigma^\infty P_3$, the matrix of $1 \vee \bar{\sigma} \vee \bar{\sigma}^2$ is given by the 3×3 diagonal matrix with I, A, A^2 along the diagonal.

To compute the matrix A, notice that $\tilde{\sigma} \circ \tau_{12} = \tau_{23}$ and $\tilde{\sigma} \circ \tau_{23} = \tau_{31}$ (where $\tilde{\sigma}$ denotes the action of σ on P_3, so that $\Sigma^\infty \tilde{\sigma} = \bar{\sigma}$). The map $\Sigma^\infty \tau_{13}$ is given by $\Sigma^\infty \tau_{12} + \Sigma^\infty \tau_{23}$, and τ_{31} differs from τ_{13} by the reflection swapping the hemispheres of S^2, which has degree -1. So,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

This means the composite map $1 + \bar{\sigma} + \bar{\sigma}^2$ has matrix:

$$\begin{pmatrix} I & I & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A^2 \end{pmatrix} \begin{pmatrix} I \\ I \\ I \end{pmatrix} = I + A + A^2,$$

which is readily computed to be 0. □

5.2. **Behrens's unary Dyer-Lashof-like operations.** In [4, Chapter 1], Behrens interprets Arone and Mahowald's calculation [3] of $H_*(\mathbb{D}_n(X))$ for a sphere X in the case $p = 2$ in terms of unary homology operations for the layer of the Goodwillie tower of a reduced finitary homotopy functor $F : \mathrm{Top}_* \to \mathrm{Top}_*$. The Arone-Ching chain rule [1] gives the symmetric sequence of derivatives of F, $\partial_*(F)$, the structure of a bimodule for $\partial_*(\mathrm{Id})$. Behrens's operations only use the left module structure and could be defined on the mod 2 homology of any symmetric sequence which is a left module over $\partial_*(\mathrm{Id})$. In particular, regarding an $\partial_*(\mathrm{Id})$-algebra as a symmetric sequence concentrated in degree 0, we get unary operations on the mod 2 homology of an $\partial_*(\mathrm{Id})$-algebra:

Definition 5.4 (adapted from [4, Section 1.5])**.** Let L be a spectrum equipped with the structure of an $\partial_*(\mathrm{Id})$-algebra. We define homology operations

$$\bar{Q}^j : H_d(L) \to H_{d+j-1}(L),$$

as follows: for $x \in H_d(L)$, we set $\bar{Q}^j x := \xi_* \sigma^{-1} Q^j x$ where

- $\xi : \Sigma^{-1} L^{\wedge 2}_{h\Sigma_2} \xrightarrow{\simeq} \mathbb{D}_2(L) \to L$ is part of the $\partial_*(\mathrm{Id})$-algebra structure of L,
- $\sigma^{-1} : H_{d+j}(L^{\wedge 2}_{h\Sigma_2}) \to H_{d+j-1}(\mathbb{D}_2(L))$ is the (de)suspension isomorphism, and
- $Q^j : H_d(L) \to H_{d+j}(L^{\wedge 2}_{h\Sigma_2})$ is a Dyer-Lashof operation.

Note that $\bar{Q}^j$ has degree $j-1$ but the notation for it uses "j" because it is named after Q^j. Also notice that if $j < d$ and $x \in H_d(L)$, we have $\bar{Q}^j x = 0$ simply because $Q^j x = 0$.

Remark 5.5. By modifying the setting of the definition of the $\bar{Q}^j$, we have introduced a potential ambiguity! For a free $\partial_*(\mathrm{Id})$-algebra $L = F_{\partial_*(\mathrm{Id})}(X)$ on some spectrum X, there are two different ways in which we could mean $\bar{Q}^j x$ for $x \in H_*(L)$: using definition 5.4, or using Behrens' original definition for the functor Id. Let us explain what that definition is and show it agrees with our definition in this case.

Given a functor $F : \mathrm{Top}_* \to \mathrm{Top}_*$, part of the left $\partial_*(\mathrm{Id})$-module structure on $\partial_*(F)$ is a $\Sigma_2 \wr \Sigma_i$-equivariant map $\partial_2(\mathrm{Id}) \wedge \partial_i(F)^{\wedge 2} \to \partial_{2i}(F)$. This induces a map

$$\begin{aligned} \psi_i : \Sigma^{-1}(\mathbb{D}_i(F)(X))^{\wedge 2}_{h\Sigma_2} &\simeq \left(\partial_2(\mathrm{Id}) \wedge \partial_i(F)^{\wedge 2} \wedge X^{\wedge 2i}\right)_{h\Sigma_2 \wr \Sigma_i} \\ &\to \left(\partial_{2i}(F) \wedge X^{\wedge 2i}\right)_{h\Sigma_{2i}} \\ &\simeq \mathbb{D}_{2i}(F)(X), \end{aligned}$$

and for $x \in H_d(\mathbb{D}_i(F)(X))$, Behrens defines $\bar{Q}^j x = (\psi_i)_* \sigma^{-1} Q^j x$.

Given $x \in H_d(\mathbb{D}_i(\mathrm{Id})(X)) \subset H_*(F_{\partial_*(\mathrm{Id})}(X))$ and a $j \geq d$, to show that the $\bar{Q}^j x$ from Definition 5.4 agrees with this original version of $\bar{Q}^j x \in H_{d+j}(\mathbb{D}_{2i}(\mathrm{Id})(X)) \subset H_*(F_{\partial_*(\mathrm{Id})}(X))$ we just need to unwind the definitions, the point being that both the left $\partial_*(\mathrm{Id})$-module structure of $\partial_*(\mathrm{Id})$ and the $\partial_*(\mathrm{Id})$-algebra structure of $F_{\partial_*(\mathrm{Id})}(X)$ come directly from the operad structure maps of $\partial_*(\mathrm{Id})$.

Definition 5.6. Let $\bar{\mathcal{R}}$ be the $\mathbb{F}_2$-algebra freely generated by symbols $\{\bar{Q}^j : j \geq 0\}$ subject to the following relations:

$$\bar{Q}^r\bar{Q}^s = \sum_{k=0}^{r-s-1} \binom{2s-r+1+2k}{k} \bar{Q}^{2s+1+k}\bar{Q}^{r-s-1-k}, \quad \text{if } s < r \leq 2s.$$

Also let $\bar{\mathcal{R}}_n$ be the quotient of $\bar{\mathcal{R}}$ obtained by imposing the additional relations $\bar{Q}^{j_1}\bar{Q}^{j_2}\cdots\bar{Q}^{j_k} = 0$ whenever $j_1 < j_2 + \cdots + j_k + n$.

The relations in $\bar{\mathcal{R}}_n$ allow one to rewrite any monomial in the $\bar{Q}^j$ into a linear combination of *completely unadmissible* or *CU* monomials, that is, monomials $\bar{Q}^J = \bar{Q}^{j_1}\bar{Q}^{j_2}\cdots\bar{Q}^{j_k}$ where $J = (j_1, \ldots, j_k)$ is a (possibly empty, corresponding to $1 \in \bar{\mathcal{R}}$) sequence of integers satisfying $j_i > 2j_{i+1}$ for $i = 1, \ldots, k-1$.

Definition 5.7. A positively graded module M over $\bar{\mathcal{R}}$ is called *allowable* if whenever $x \in M$ is homogeneous of degree n and $j_1 < j_2 + \cdots + j_k + n$, we have $\bar{Q}^{j_1}\bar{Q}^{j_2}\cdots\bar{Q}^{j_k}x = 0$.

Remark 5.8. This notion of allowable requires more operations to vanish than required by degree considerations, that is, more than required by the condition $\bar{Q}^j x = 0$ when $x \in M_n$, $j < n$. Indeed, that last condition only implies $\bar{Q}^{j_1}\bar{Q}^{j_2}\cdots\bar{Q}^{j_k}x = 0$ when $j_i < j_{i+1} + \cdots j_k + n - (k-i)$ for some i; note the extra negative term $-(k-i)$. The reason for this extra vanishing required is the isomorphism in [4, Theorem 1.5.1], that in the notation used there, sends $\sigma^k\bar{Q}^{j_1}\bar{Q}^{j_2}\cdots\bar{Q}^{j_k}\iota_n \mapsto Q^{j_1}Q^{j_2}\cdots Q^{j_k}\iota_n$. The Q^j do have that vanishing property just for degree reasons.

Proposition 5.9. *Given an* $\partial_*(\mathrm{Id})$*-algebra* L*, the action of the operations* $\bar{Q}^j$ *makes* $H_{>0}(L)$ *into an allowable* $\bar{\mathcal{R}}$*-module.*

Proof. We will deduce that the operations act allowably and satisfy the relations in the algebra $\bar{\mathcal{R}}$ from [4, Theorem 1.5.1]. That theorem states that

$$\bigoplus_{k\geq 0} H_*(\mathbb{D}_{2^k}(S^n)) = \bar{\mathcal{R}}_n\{\iota_n\},$$

where ι_n is the fundamental class of $\tilde{H}_n(S^n)$ (thought of as living in $H_n(\mathbb{D}_1(S^n)) \cong \tilde{H}_n(S^n)$), and the operations $\bar{Q}^j$ obey all the relations in the algebra $\bar{\mathcal{R}}_n$.

Given any class $x \in H_n(L)$, we can represent it by map $x : \Sigma^n \mathrm{H}\mathbb{F}_2 \to \mathrm{H}\mathbb{F}_2 \wedge L$ of $\mathrm{H}\mathbb{F}_2$-module spectra. This corresponds to a map $x^\dagger : \mathrm{H}\mathbb{F}_2 \wedge F_{\partial_*(\mathrm{Id})}(S^n) \to \mathrm{H}\mathbb{F}_2 \wedge L$ of $(\mathrm{H}\mathbb{F}_2 \wedge \partial_*(\mathrm{Id}))$-algebras. The naturality of the

$\bar{Q}^j$ operations shows that given any $R \in \bar{\mathcal{R}}$ we have $H_*(x^\dagger)R\iota_n = Rx$, so that if the relation $R\iota_n = 0$ is satisfied in $H_*(F_{\partial_*(\mathrm{Id})}(S^n))$, the relation $Rx = 0$ holds in $H_*(L)$. □

Notice that it also follows from theorem [4, Theorem 1.5.1], that the CU-monomials $\bar{Q}^J$ are linearly independent.

6. Algebraic structure of homology of spectral Lie algebras

We can now state the algebraic structure of the homology of an $\partial_*(\mathrm{Id})$-algebra:

Definition 6.1. An *allowable* $\bar{\mathcal{R}}$-sLie-*algebra* is a graded $\mathbb{F}_2$-vector space M, equipped with

- a shifted Lie bracket $[-,-] : M_i \otimes M_j \to M_{i+j-1}$, and
- the structure of an allowable $\bar{\mathcal{R}}$-module on $M_{>0}$,

such that

1. $\bar{Q}^k x = [x,x]$ if $x \in M_k$, and
2. $[x, \bar{Q}^k y] = 0$ for any $x \in M_i$, $y \in M_j$.

Remark 6.2. Notice that condition 2 only has content when $k \geq j$, since otherwise $\bar{Q}^k y = 0$.

Theorem 6.3. *Given any* $\partial_*(\mathrm{Id})$*-algebra* L*, the operations described above give its mod* 2 *homology* $H_*(L)$ *the structure of an allowable* $\bar{\mathcal{R}}$-sLie-*algebra.*

Proof. We have already shown that the bracket gives $H_*(L)$ the structure of a sLie-algebra and of an allowable $\bar{\mathcal{R}}$-module in Propositions 5.2 and 5.9. We will prove properties 1 and 2 from Definition 6.1 in Lemmas 6.4 and 6.5 below. □

It will be convenient to recall a construction of the Dyer-Lashof operation $Q^k : H_j(L) \to H_{j+k}(L^{\wedge 2}_{h\Sigma_2})$ for $k \geq j$. A class $x \in H_j(L)$ can be represented by a map $x : \Sigma^j \mathrm{H}\mathbb{F}_2 \to \mathrm{H}\mathbb{F}_2 \wedge L$ of $\mathrm{H}\mathbb{F}_2$-module spectra. Applying the second extended power functor we get a map $x^{\otimes 2}_{h\Sigma_2} : (\Sigma^j \mathrm{H}\mathbb{F}_2)^{\otimes 2}_{h\Sigma_2} \to (\mathrm{H}\mathbb{F}_2 \wedge L)^{\otimes 2}_{h\Sigma_2}$, where we have used $\otimes$ for the smash product of $\mathrm{H}\mathbb{F}_2$-module spectra. Since the free $\mathrm{H}\mathbb{F}_2$-module functor is symmetric monoidal and preserves homotopy colimits, $(\mathrm{H}\mathbb{F}_2 \wedge Y)^{\otimes 2}_{h\Sigma_2} \simeq \mathrm{H}\mathbb{F}_2 \wedge Y^{\wedge 2}_{h\Sigma_2}$; so that we can regard $x^{\otimes 2}_{h\Sigma_2}$ as being a map $(\Sigma^j \mathrm{H}\mathbb{F}_2)^{\otimes 2}_{h\Sigma_2} \to \mathrm{H}\mathbb{F}_2 \wedge L^{\wedge 2}_{h\Sigma_2}$.

Now, $(\Sigma^j \mathrm{H}\mathbb{F}_2)^{\otimes 2}$ has trivial Σ_2-action, so $(\Sigma^j \mathrm{H}\mathbb{F}_2)^{\otimes 2}_{h\Sigma_2} \simeq \Sigma^{2j} \mathrm{H}\mathbb{F}_2 \wedge \Sigma^\infty_+ B\Sigma_2$. One way to see this is to recall that the homotopy category of

$H\mathbb{F}_2$-module spectra is equivalent to the derived category of complexes of $\mathbb{F}_2$-vector spaces, and the definition of the symmetry of the tensor product has no signs in that case. Alternatively, one can think of spaces, before smashing with $H\mathbb{F}_2$: the fibration $(S^j)^{\wedge 2} \to S^{j\,\wedge 2}_{h\Sigma_2} \to B\Sigma_2$ is $\mathbb{F}_2$-orientable, and the above equivalence is an instance of the Thom isomorphism.

Let $q_{k-j} : \Sigma^{k-j} H\mathbb{F}_2 \to H\mathbb{F}_2 \wedge \Sigma^\infty_+ B\Sigma_2$ pick out the unique non-zero class of degree $k-j$ in $H_*(B\Sigma_2)$; then $Q^k x$ is represented by

$$\Sigma^{j+k} H\mathbb{F}_2 \xrightarrow{q_{k-j} \otimes \mathrm{id}_{\Sigma^{2j} H\mathbb{F}_2}} (H\mathbb{F}_2 \wedge \Sigma^\infty_+ B\Sigma_2) \otimes \Sigma^{2j} H\mathbb{F}_2$$
$$\simeq (\Sigma^j H\mathbb{F}_2)^{\otimes 2}_{h\Sigma_2} \xrightarrow{x^{\otimes 2}_{h\Sigma_2}} H\mathbb{F}_2 \wedge L^{\wedge 2}_{h\Sigma_2}.$$

Lemma 6.4. *For any $\partial_*(\mathrm{Id})$-algebra L and $x \in H_k(L)$, we have $\bar{Q}^k x = [x, x]$.*

Proof. This follows easily by unwinding the definitions: if x is represented by a map $x : \Sigma^k H\mathbb{F}_2 \to H\mathbb{F}_2 \wedge L$, both sides are represented by the desuspension of some composite

$$\Sigma^k H\mathbb{F}_2 \otimes \Sigma^k H\mathbb{F}_2 \to \left(\Sigma^k H\mathbb{F}_2\right)^{\otimes 2}_{h\Sigma_2} \xrightarrow{x^{\otimes 2}_{h\Sigma_2}} H\mathbb{F}_2 \wedge L^{\wedge 2}_{h\Sigma_2} \xrightarrow{H\mathbb{F}_2 \wedge \Sigma\xi} H\mathbb{F}_2 \wedge L,$$

where $\xi : \Sigma^{-1} L^{\wedge 2}_{h\Sigma_2} \to L$ is the structure map. For $[x, x]$ the first map is taken to be the quotient map, while for $\bar{Q}^k x$ it is $q_0 \otimes \mathrm{id}_{\Sigma^{2k} H\mathbb{F}_2}$, which agrees with the quotient map. □

Lemma 6.5. *For an $\partial_*(\mathrm{Id})$-algebra L and $x \in H_i(L)$, $y \in H_j(L)$ we have $[x, \bar{Q}^k y] = 0$.*

Proof. For $k < j$, $\bar{Q}^k y = 0$. For $k = j$, by Lemma 6.4, $[x, \bar{Q}^k y] = [x, [y, y]]$ and this is 0 as explained in section 4.

To analyze the case $k > j$, we begin by unwinding the definitions in terms of representing maps $x : \Sigma^i H\mathbb{F}_2 \to H\mathbb{F}_2 \wedge L$ and $y : \Sigma^j H\mathbb{F}_2 \to H\mathbb{F}_2 \wedge L$. To make the next diagram fit on the page, we introduce some temporary notation: $[i] := \Sigma^i H\mathbb{F}_2$, $\bar{L} := H\mathbb{F}_2 \wedge L$, $\overline{B\Sigma_2} := H\mathbb{F}_2 \wedge \Sigma^\infty_+ B\Sigma_2$ and $\partial_n := \partial_n(\mathrm{Id})$. Then $[x, \bar{Q}^k y] \in H_{i+j+k-2}(L)$ is represented by the the composite from the top left corner to the bottom right corner in the following commutative diagram:

$$\begin{array}{ccc}
[i+j+k-2] & & \\
\big\downarrow{\scriptstyle \mathrm{id}_{[i-1]}\otimes\Sigma^{-1}q_{k-j}\otimes\mathrm{id}_{[2j]}} & & \\
\Sigma^{-1}[i]\otimes\Sigma^{-1}(\overline{B\Sigma_2}\otimes[2j]) & & \\
{\scriptstyle\simeq}\big\downarrow & & \\
\partial_2\wedge(\partial_1\wedge[i])\otimes(\partial_2\wedge[j])^{\otimes 2}_{h\Sigma_2} & \xrightarrow{\theta_{[i],[j]}} & (\partial_3\wedge[i+2j])_{h(\Sigma_1\times\Sigma_2)} \\
{\scriptstyle(\partial_1\wedge x)\otimes(\partial_2\wedge y)^{\otimes 2}_{h\Sigma_2}}\big\downarrow & & \big\downarrow{\scriptstyle(\partial_3\wedge x\otimes y^{\otimes 2})_{h(\Sigma_1\times\Sigma_2)}} \\
\partial_2\wedge(\partial_1\wedge\bar{L})\otimes(\partial_2\wedge\bar{L})^{\otimes 2}_{h\Sigma_2} & \xrightarrow{\theta_{\bar{L},\bar{L}}} & (\partial_3\wedge\bar{L}^{\wedge 3})_{h(\Sigma_1\times\Sigma_2)} \\
{\scriptstyle\mathrm{id}\otimes(\mathrm{H}\mathbb{F}_2\wedge\xi)}\big\downarrow & & \big\downarrow{\scriptstyle\xi_3'} \\
\partial_2\wedge\bar{L}\otimes\bar{L} & \xrightarrow[\mathrm{H}\mathbb{F}_2\wedge\alpha_2]{} & \bar{L}.
\end{array}$$

The horizontal arrows whose labels involve θ are defined using the structure map $\theta:\partial_2\wedge\partial_1\wedge\partial_2\to\partial_3$, namely,

$$\theta_{X,Y}:\partial_2\wedge(\partial_1\wedge X)\otimes(\partial_2\wedge Y)^{\otimes 2}_{h\Sigma_2}\to(\partial_3\wedge X\otimes Y^{\otimes 2})_{h(\Sigma_1\times\Sigma_2)}$$

is given by $(\theta\wedge\mathrm{id}_{X\otimes Y^{\otimes 2}})_{h(\Sigma_1\times\Sigma_2)}$.

The arrow labeled ξ_3' is $\mathrm{H}\mathbb{F}_2$ smashed with the composite

$$\left(\partial_3\wedge L^{\wedge 3}\right)_{h(\Sigma_1\times\Sigma_2)}\xrightarrow{(\alpha_3)_{h(\Sigma_1\times\Sigma_2)}}L\wedge\Sigma^\infty_+B\Sigma_2\to L,$$

and that the bottom square commutes follows from the definition of algebra for an operad.

To conclude the proof, we will show that $(\partial_3\wedge[i+2j])_{h(\Sigma_1\times\Sigma_2)}$ is concentrated in degree $i+2j-2$, which means the composite from the top of the diagram to that point must be null if $k\neq j$. Now, that spectrum is equivalent to $\mathrm{H}\mathbb{F}_2\wedge\Sigma^{i+2j}(\partial_3)_{h(\Sigma_1\times\Sigma_2)}$ because the $(\Sigma_1\times\Sigma_2)$-action on $[i+2j]$ is trivial. So we need to describe ∂_3 as a $(\Sigma_1\times\Sigma_2)$-spectrum. Recall the description of P_3 from the proof of Proposition 5.2: it consists of three 2-dimensional disks with their boundaries identified, one for each of the three partitions $(12|3)$, $(13|2)$, $(23|1)$. The $(\Sigma_1\times\Sigma_2)$-action fixes one of the disks and swaps the other two, so that P_3 is equivariantly equivalent to $\Sigma^2\Sigma^\infty_+\Sigma_2$, the double suspension of the regular representation of Σ_2. Then ∂_3 is $\Sigma^{-2}\Sigma^\infty_+\Sigma_2$ and $(\partial_3)_{h(\Sigma_1\times\Sigma_2)}\simeq S^{-2}$, as required. $\square$

Remark 6.6. Lukas Brantner has written an alternative proof of this lemma that will appear in [5]. His argument analyzes the structure map θ showing it is the double desuspension of the transfer map $\Sigma^\infty_+B\Sigma_2\to$

S and thus vanishes on mod 2 homology. I am grateful to him for sharing his proof with me at a time when I was still confused about the "bottom operation" and thought this result only held for $k > j$.

7. Homology of free spectral Lie algebras on simply-connected spaces

Now we can state our main result:

Theorem 7.1. *Given a simply-connected space X, the mod 2 homology of the free $\partial_*(\mathrm{Id})$-algebra on $\Sigma^\infty X$ is the free allowable $\bar{\mathcal{R}}$-sLie-algebra $\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}_*(X))$ on the reduced homology $\tilde{H}_*(X)$.*

More precisely, the canonical map $\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}_*(X)) \to H_*(F_{\partial_*(\mathrm{Id})}(\Sigma^\infty X))$ is an isomorphism.

Remark 7.2. The restriction to suspension spectra of simply-connected spaces is a consequence of the use of the Hilton–Milnor theorem in section 7.2; it would be interesting to work out more general cases, and that seems to be the main obstacle. The restrictions to positive degrees in Definition 5.7 and Proposition 5.9 are not essential, as the homology of $\bigoplus_{k\geq 0} H_*(\mathbb{D}_{2^k}(S^n))$ can also be calculated for negative n. Indeed, since $\mathrm{H}\mathbb{F}_2 \wedge \Sigma^j S^{n\wedge m}_{h\Sigma_m} \simeq \Sigma^{jm}(\mathrm{H}\mathbb{F}_2 \wedge S^{n\wedge m}_{h\Sigma_m})$ (by the same arguments used near the end of the proof of Theorem 6.3: the symmetry of the tensor product of $\mathrm{H}\mathbb{F}_2$-spectra has no signs, or a Thom isomorphism), one can easily show deduce the homology for negative spheres from the Arone–Mahowald calculation. We have left the restrictions to positive degrees *there* for simplicity, since we do not know how to avoid them in the main result anyway.

We will prove Theorem 7.1 in special cases of increasing generality in the next few sections, but first we will give a convenient construction of the free allowable $\bar{\mathcal{R}}$-sLie-algebra. This will involve the notion of *basic products*, that we now recall:

Definition 7.3. The *basic products* on a set of *letters* $x_1, \ldots, x_n$ are defined and ordered recursively as follows:

The basic products of weight 1 are $x_1, x_2, \ldots, x_n$, ordered arbitrarily.

Suppose the basic products of weight less than k have been defined and ordered. A basic product of weight k is a formal bracket $[\![w_1, w_2]\!]$ where

- w_1 and w_2 are basic products whose weights add up to k,
- $w_1 < w_2$ in the order defined so far,
- if $w_2 = [\![w_3, w_4]\!]$ for some basic products w_3 and w_4, then we require that $w_3 \leq w_1$.

Once all the products of weight k are defined, they are ordered arbitrarily among themselves and declared to be greater that all basic products of lower weight. We will assume these choices of order are fixed once and for all.

Marshall Hall proved in [11] that the basic products form a basis for the free Lie algebra on $x_1, x_2, \ldots, x_n$. That result is for the totally isotropic, ungraded version of Lie algebra, but it clearly extends, at least for $R = \mathbb{F}_2$ where the grading does not introduce signs, to both $\mathsf{Lie}^{\mathrm{ti}}$-algebras and $\mathsf{sLie}^{\mathrm{ti}}$-algebras: if the letters have assigned degrees $|x_i|$, we assign to each basic product w with ℓ letters of total degree d, the degree $|w| = d$ in the $\mathsf{Lie}^{\mathrm{ti}}$ case and $|w| = d - \ell$ in the $\mathsf{sLie}^{\mathrm{ti}}$ case.

Proposition 7.4. *The free allowable $\bar{\mathcal{R}}$-sLie-algebra $\mathrm{s}\mathcal{L}_{\bar{\mathcal{R}}}(V)$ on a graded $\mathbb{F}_2$-vector space V is the free allowable $\bar{\mathcal{R}}$-module on the free $\mathsf{sLie}^{\mathrm{ti}}$ algebra on V, denoted by $\mathcal{A}_{\bar{\mathcal{R}}}(F_{\mathsf{sLie}^{\mathrm{ti}}}(V))$, equipped with a bracket defined as follows:*

First, fix a basis β of V and consider the basis of $\mathcal{A}_{\bar{\mathcal{R}}}(F_{\mathsf{sLie}^{\mathrm{ti}}}(V))$ consisting of all $\bar{Q}^J w$ where:

- *$J = (j_1, \ldots, j_k)$ is a CU-sequence of integers, and*
- *w is a basic product of degree at most j_k in letters from β.*

Now define the bracket on $\mathcal{A}_{\bar{\mathcal{R}}}(F_{\mathsf{sLie}^{\mathrm{ti}}}(V))$ on that basis as indicated below and extended bilinearly:

- *$[\bar{Q}^{J_1} w_1, \bar{Q}^{J_2} w_2] = 0$ if $J_1 \neq \emptyset$ or $J_2 \neq \emptyset$.*
- *The bracket $[w_1, w_2]$ of basic products is defined recursively as follows:*
  1. *If $[\![w_1, w_2]\!]$ is a basic product, we let $[w_1, w_2] = [\![w_1, w_2]\!]$.*
  2. *$[w_1, w_2] = \bar{Q}^{|w_1|} w_1$ if $w_1 = w_2$.*
  3. *$[w_1, w_2] = [w_2, w_1]$ if $w_1 > w_2$.*
  4. *$[w_1, w_2] = [w_3, [w_1, w_4]] + [w_4, [w_1, w_3]]$ if $w_1 < w_2$ and $w_2 = [w_3, w_4]$ with $w_1 < w_3$.*

Proof. In [11], Hall defines the $\mathsf{Lie}^{\mathrm{ti}}$ bracket on the linear span of the basic products as above, except that (2) is replaced with $[w_1, w_1] = 0$. He then proves that the recursion in the definition does terminate and that it produces a $\mathsf{Lie}^{\mathrm{ti}}$-algebra, that is, that the bracket is anti-symmetric, satisfies the Jacobi identity and $[x, x] = 0$ for all x. A straightforward adaptation of his proof will show that the above definition also terminates and produces an allowable $\bar{\mathcal{R}}$-sLie-algebra. But before we explain that, let us assume the bracket does define a $\bar{\mathcal{R}}$-sLie-algebra and check that it is free. Let $f : V \to E$ be a morphism of graded vector spaces where E is an allowable $\bar{\mathcal{R}}$-sLie-algebra. There is a unique bracket-preserving extension of f to the linear span of the basic products, and therefore a unique extension of f to a morphism of allowable $\bar{\mathcal{R}}$-modules $\mathcal{A}_{\bar{\mathcal{R}}}(F_{\mathsf{sLie}^{\mathrm{ti}}}(V)) \to E$. That this unique extension is also a morphism of allowable $\bar{\mathcal{R}}$-sLie-algebras is clear from the above definition of the bracket.

And now we check the bracket correctly produces an allowable $\bar{\mathcal{R}}$-sLie-algebra. First of all, notice that the degrees of the various parts of the definition are correct for a shifted bracket.

Secondly, having $[w_1, w_1] = \bar{Q}^{|w_1|}w_1$ instead of 0 does not affect termination of the recursion at all. Both 0 and $\bar{Q}^{|w_1|}w_1$ have the following properties: (1) they are expressions containing no further brackets, so if a term reduces to one of them that term requires no further reduction, and (2) if they appear *inside* a bracket, the term containing that bracket is 0. This means that the process of reducing a bracket $[x, y]$ to a linear combination of basic products by repeatedly applying the recursive definition uses exactly the same steps in both Hall's $\mathsf{Lie}^{\mathrm{ti}}$ case and in our $\bar{\mathcal{R}}$-sLie case, the only difference being that any $[w, w]$ that appear on their own (that is, not inside a bracket) will reduce to $\bar{Q}^{|w|}w$ instead of 0.

Next we must check that this bracket satisfies $[x, \bar{Q}^k y] = 0$, $[x, x] = \bar{Q}^{|x|}x$, symmetry and the Jacobi identity. All of these need only be checked on the given basis. Symmetry and that $[x, \bar{Q}^k y] = 0$ are directly built in to the definition, as is the fact that $[x, x] = \bar{Q}^{|x|}x$ when x is a basic product. When $x = \bar{Q}^J w$ for $J = (j_1, \ldots, j_k)$ with $k \geq 1$, we have $[x, x] = 0$ (since $J \neq \emptyset$), but we also have $|x| = j_1 + \cdots + j_k + |w| - k < j_1 + \cdots + j_k + |w|$ so that $\bar{Q}^{|x|}x = \bar{Q}^{|x|}\bar{Q}^J w = 0$ is required by allowability.

Now only the Jacobi identity remains to be checked:

$$\sum_{\text{cyclic}} \left[\bar{Q}^{J_1}w_1, [\bar{Q}^{J_2}w_2, \bar{Q}^{J_3}w_3]\right] = 0.$$

If any $J_i \neq \emptyset$, all three terms are 0, so assume all $J_i = \emptyset$. This remaining case can be proved exactly as in [11, Section 3, p. 579], with one tiny change. There is only one place in that proof where the condition $[w, w] = 0$ is used: it is at the very beginning of the argument for the Jacobi identity. The proof starts by considering the case when two of the w_i are equal, say $w_1 = w_2$. Then the terms $[w_1, [w_1, w_3]]$ and $[w_1, [w_3, w_1]]$ cancel by anti-symmetry and the remaining term is 0 since $[w_3, [w_1, w_1]] = [w_3, 0]$. In our case, that last term still vanishes: $[w_3, [w_1, w_1]] = [w_3, \bar{Q}^{|w_1|}w_1] = 0$. The rest of Hall's argument goes through verbatim. □

7.1. **The free spectral Lie algebra on a sphere.** For $X = S^n$, Theorem 7.1 is essentially a restatement of [4, Theorem 1.5.1] using

Proposition 7.4. Indeed, the free $\mathsf{sLie}^{\mathrm{ti}}$-algebra on $\tilde{H}_*(S^n) = \mathbb{F}_2\{\iota_n\}$ is just $\mathbb{F}_2\{\iota_n\}$ again, so that $\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}_*(S^n)) = \mathcal{A}_{\bar{\mathcal{R}}}(\mathbb{F}_2\{\iota_n\})$, which is what Behrens shows $H_*(F_{\partial_*(\mathrm{Id})}(S^n))$ to be.

7.2. **The free spectral Lie algebra on a finite wedge of spheres.** Now we consider the case of $X = S^{d_1} \vee S^{d_2} \vee \cdots \vee S^{d_k}$ for some integers $d_i \geq 2$; in this case $F_{\partial_*(\mathrm{Id})}(X)$ can be computed from the results of [2], which we now summarize.

Consider a bit more generally the case $X = \Sigma(X_1 \vee \cdots \vee X_k)$, where the X_i are some connected spaces. In [2] there is a computation of $\mathbb{D}_n(\mathrm{Id})(X) = \Sigma\mathbb{D}_n(\Omega\Sigma)(X_1 \vee \cdots \vee X_k)$ that "takes multi-variable Goodwillie derivatives on both sides of the Hilton–Milnor theorem".

The Hilton–Milnor theorem (see [16, Section XI.6]) gives a homotopy equivalence between $\Omega\Sigma X = \Omega\Sigma(X_1 \vee \cdots \vee X_k)$ and the weak[1] infinite product $\prod_w \Omega\Sigma Y_w(X_1, \ldots, X_k)$, where w runs over the basic products on k letters, and each Y_w is the functor obtained from the word w by interpreting the i-th letter as X_i, and the bracket as the smash product; so that $Y_w(X_1, \ldots, X_k) = X_1^{\wedge m_1(w)} \wedge \cdots \wedge X_k^{\wedge m_k(w)}$ with $m_i(w)$ counting the number of occurrences of the i-th letter in w.

Given a basic product w there is a map $h_w : Y_w(X_1, \ldots, X_n) \to \Omega\Sigma X$ obtained from w by interpreting the i-th letter as the canonical map $X_i \hookrightarrow X \to \Omega\Sigma X$ and interpreting the bracket as the Samelson product. Let $\bar{h}_w : \Omega\Sigma Y_w(X_1, \ldots, X_n) \to \Omega\Sigma X$ be the extension of h_w to a map of A_∞-spaces and for any set B of basic words let $\bar{h}_B$ be the composite $\prod_{w \in B} \Omega\Sigma Y_w(X_1, \ldots, X_n) \xrightarrow{\prod \bar{h}_w} (\Omega\Sigma X)^B \xrightarrow{\mu} \Omega\Sigma X$. Then the Hilton–Milnor theorem can be stated as saying that the colimit of $\bar{h}_B$ over all finite sets of basic products is an equivalence.

The result Arone and Kankaanrinta obtain from the Hilton–Milnor equivalence [2, Theorem 0.1] is the following equivalence of spectra:

$$\left(\partial_n(\Omega\Sigma) \wedge \Sigma^\infty \left(X_1^{\wedge n_1} \wedge \cdots \wedge X_k^{\wedge n_k}\right)\right)_{h(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k})} \simeq$$
$$\bigvee_{d | \gcd(n_1, \ldots, n_k)} \left(\bigvee_{w \in W(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} \mathbb{D}_d(\Omega\Sigma)\left(Y_w(X_1, \ldots X_k)\right) \right),$$

where $W(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$ is the set of basic products on k-letters involving the i-th letter exactly $\frac{n_i}{d}$ times.

[1]This means the homotopy colimit of the finite products, where the maps in the colimit include a product into a larger product using the basepoint on the extra factors.

We can use this to get a nice formula for $\Sigma^{-1}F_{\partial_*(\mathrm{Id})}(X)$:

$$\begin{aligned}
&F_{\partial_*(\Omega\Sigma)}(X_1 \vee \cdots \vee X_k) = \bigvee_n \mathbb{D}_n(\Omega\Sigma)(X_1 \vee \cdots \vee X_k) \\
&= \bigvee_n \left(\partial_n(\Omega\Sigma) \wedge \Sigma^\infty (X_1 \vee \cdots \vee X_k)^{\wedge n}\right)_{h\Sigma_n} \\
&= \bigvee_n \left(\partial_n(\Omega\Sigma) \wedge \bigvee_{n_1+\cdots+n_k=n} \mathrm{Ind}_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}}^{\Sigma_n} (X_1^{\wedge n_1} \wedge \cdots \wedge X_k^{\wedge n_k})\right)_{h\Sigma_n} \\
&= \bigvee_{n_1,\ldots,n_k} \left(\partial_{n_1+\cdots+n_k}(\Omega\Sigma) \wedge (X_1^{\wedge n_1} \wedge \cdots \wedge X_k^{\wedge n_k})\right)_{h(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k})} \\
&= \bigvee_{n_1,\ldots,n_k} \left(\bigvee_{d\mid\gcd(n_1,\ldots,n_k)} \left(\bigvee_{w\in W(\frac{n_1}{d},\ldots,\frac{n_k}{d})} \mathbb{D}_d(\Omega\Sigma)\left(Y_w(X_1,\ldots,X_k)\right)\right)\right) \\
&= \bigvee_{m_1,\ldots,m_k,d} \left(\bigvee_{w\in W(m_1,\ldots,m_k)} \mathbb{D}_d(\Omega\Sigma)\left(Y_w(X_1,\ldots,X_k)\right)\right) \\
&= \bigvee_{w\in W} F_{\partial_*(\Omega\Sigma)}\left(Y_w(X_1,\ldots,X_k)\right),
\end{aligned}$$

where the last wedge runs over all basic products in k letters, and the next to last step uses the change of variables $m_i = \frac{n_i}{d}$: this gives a bijection between all $(k+1)$-tuples $(n_1,\ldots,n_k,d)$ of positive integers with $d \mid \gcd(n_1,\ldots,n_k)$, and all $(k+1)$-tuples $(m_1,\ldots,m_k,d)$ of positive integers.

This in turn tells us, for $F_{\partial_*(\mathrm{Id})}$, that:

$$F_{\partial_*(\mathrm{Id})}\left(\Sigma(X_1 \vee \cdots \vee X_k)\right) = \bigvee_{w\in W} F_{\partial_*(\mathrm{Id})}\left(\Sigma Y_w(X_1,\ldots,X_k)\right).$$

Plugging in $X_i = S^{d_i-1}$, for some $d_i \geq 2$, we get that

$$F_{\partial_*(\mathrm{Id})}\left(S^{d_1} \vee \cdots \vee S^{d_k}\right) = \bigvee_{w\in W} F_{\partial_*(\mathrm{Id})}\left(S^{1+\sum_i m_i(w)(d_i-1)}\right),$$

so that Proposition 7.4 allows us to conclude Theorem 7.1 for the wedge $S^{d_1} \vee \cdots \vee S^{d_k}$ from the case of single spheres.

7.3. **The free spectral Lie algebra on a simply-connected space.** Bootstrapping from the previous cases to $F_{\partial_*(\mathrm{Id})}(X)$ for general simply-connected X is purely formal using the fact that $H_*(F_{\partial_*(\mathrm{Id})}(X))$ only depends on the homology of X, as shown in Proposition 2.1.

Let $\phi_X : \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}(X)) \to H_*(F_{\partial_*(\mathrm{Id})}(\Sigma^\infty X))$ be the canonical map coming from the universal property of the the free allowable $\bar{\mathcal{R}}$-sLie-algebra.

If X is an arbitrary wedge of spheres, each of dimension at least 2, then we can write X as a filtered colimit of finite wedges of spheres and these fall under the previous case. Since homology and the free functors we are using all commute with filtered colimits, the result also holds for such an X.

Now, for a general simply-connected X, pick an $\mathbb{F}_2$-basis $\{x_j\}$ of $\tilde{H}_*(X)$ and use it to construct an equivalence of $\mathrm{H}\mathbb{F}_2$-module spectra $f : \bigvee_j \Sigma^{|x_j|}\mathrm{H}\mathbb{F}_2 \to \mathrm{H}\mathbb{F}_2 \wedge \Sigma^\infty X$. The natural transformation ϕ is a special case of a natural transformation $\psi_V : \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(V)) \to \pi_*(F_{\mathrm{H}\mathbb{F}_2\wedge\partial_*(\mathrm{Id})}(V))$ for $\mathrm{H}\mathbb{F}_2$-module spectra V, in the sense that $\phi_X = \psi_{\mathrm{H}\mathbb{F}_2\wedge\Sigma^\infty X}$. In the naturality square

$$\begin{array}{ccc}
\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(\bigvee_j \Sigma^{|x_j|}\mathrm{H}\mathbb{F}_2)) & \xrightarrow{\psi_{(\bigvee_j \Sigma^{|x_j|}\mathrm{H}\mathbb{F}_2)}} & \pi_*(F_{\mathrm{H}\mathbb{F}_2\wedge\partial_*(\mathrm{Id})}(\bigvee_j \Sigma^{|x_j|}\mathrm{H}\mathbb{F}_2)) \\
\downarrow{\scriptstyle \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(f))} & & \downarrow{\scriptstyle \pi_*(F_{\mathrm{H}\mathbb{F}_2\wedge\partial_*(\mathrm{Id})}(f))} \\
\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(\mathrm{H}\mathbb{F}_2\wedge\Sigma^\infty X)) & \xrightarrow{\psi_{(\mathrm{H}\mathbb{F}_2\wedge\Sigma^\infty X)}} & \pi_*(F_{\mathrm{H}\mathbb{F}_2\wedge\partial_*(\mathrm{Id})}(\mathrm{H}\mathbb{F}_2\wedge\Sigma^\infty X)),
\end{array}$$

all maps are known to be isomorphisms (the vertical ones because f is an equivalence, the top one because it is $\phi_{(\bigvee_j S^{|x_j|})}$) except the bottom one, which therefore also is an isomorphism.

8. Divided power algebras and Koszul duality

In this section we review some results from [15] and explain heuristically their relation to our computation.

Let R be an augmented E_∞-ring spectrum, or more precisely, an augmented commutative S-algebra. In [15], Kuhn discusses a model for the Topological André–Quillen homology spectrum $TAQ(R)$ of R given by

$$TAQ(R) = \operatorname*{hocolim}_{n\to\infty} \Omega^n S^n \otimes R$$

where $\otimes$ denotes the tensoring of augmented commutative S-algebras over pointed spaces ([15, Definition 7.2]). As mentioned in that paper, this model was shown by Mandell to be equivalent to standard definitions of $TAQ(R)$.

Kuhn defines a filtration on $TAQ(R)$ (coming from a filtration on these tensor products) and identifies [15, Corollary 7.5] the associated graded pieces as

$$F_d TAQ(R)/F_{d+1}TAQ(R) \simeq \left(P_d \wedge (R/S)^{\wedge d}\right)_{h\Sigma_d},$$

where P_d is the partition complex and R/S denotes the cofiber of the unit $\eta : S \to R$ of the ring spectrum R.

One can use the spectral sequence associated to this filtration to compute the cohomology $H^*(TAQ(R); \mathbb{F})$ and Kuhn identifies the E_1-page [15, Theorem 8.1] for $\mathbb{F}$ of characteristic p as

$$E_1^{*,*}(TAQ(R); \mathbb{F}) = \mathcal{R}(\Sigma L_r(\Sigma^{-1}\tilde{H}^*(R; \mathbb{F}))),$$

where L_r is the free restricted Lie algebra functor and $\mathcal{R}$ is, roughly speaking, the free algebra-over-the-Dyer-Lashof-algebra functor.

Before explaining how this result is related to this paper, a quick reminder of divided power algebras is in order. Associated to an operad $\mathcal{O}$ one has the free algebra monad whose functor part is

$$X \mapsto \bigvee_{n \geq 0} \left(\mathcal{O}(n) \wedge X^{\wedge n}\right)_{h\Sigma_n},$$

but also, under certain relatively mild conditions, another monad defined using homotopy fixed points instead:

$$X \mapsto \bigvee_{n \geq 0} \left(\mathcal{O}(n) \wedge X^{\wedge n}\right)^{h\Sigma_n}.$$

The algebras for this second monad are called *divided power $\mathcal{O}$-algebras* because of the special case of the commutative operad in vector spaces over a field, where they are the traditional divided power algebras. In the case of the classical Lie operad over $\mathbb{F}_p$, the divided power algebras turn out to be restricted Lie algebras ([9, Theorem 0.1]).

Kuhn's result is related to the results in this paper through Koszul duality: the spectral Lie operad should be thought of as Koszul dual to the commutative operad in spectra, whose algebras are, of course, E_∞-ring spectra. Koszul duality of operads, roughly speaking, should result in an equivalence between the category of algebras for one operad and the category of divided power algebras for the other. In this case $D \circ TAQ$ should implement the equivalence (where D denotes Spanier–Whitehead duality), taking E_∞-ring spectra to divided power spectral Lie algebras —which might be called *spectral restricted Lie algebras*, at least when working with the version for $H\mathbb{F}_p$-module spectra.

The associated graded of Kuhn's filtration of $TAQ(R)$, namely,

$$\bigvee_{d \geq 0} \left(P_d \wedge (R/S)^{\wedge d}\right)_{h\Sigma_d}$$

looks a lot like the free $\partial_*(\mathrm{Id})$-algebra on R/S, except that it has the partition complex P_d instead of its Spanier–Whitehead dual $\partial_n(\mathrm{Id})$.

The cohomology of the associated graded is given by the homotopy groups of the mapping spectrum

$$\mathrm{Map}(\bigvee_{d\geq 0}\left(P_d\wedge (R/S)^{\wedge d}\right)_{h\Sigma_d}, H\mathbb{F})\simeq \bigvee_{d\geq 0}\left(\partial_d(\mathrm{Id})\wedge \mathrm{Map}(R/S, H\mathbb{F})\right)^{h\Sigma_d}$$

so one can think of Kuhn's formula for the E_1-page as giving the homology of the free divided power $\partial_*(\mathrm{Id})$-algebra on R/S. From this point of view it is no surprise that Kuhn's formula involves free *restricted* Lie algebras, while our formulas involve free Lie algebras.

References

[1] G. Arone and M. Ching. Operads and chain rules for the calculus of functors. *Astérisque*, (338):vi+158, 2011.

[2] G. Arone and M. Kankaanrinta. The homology of certain subgroups of the symmetric group with coefficients in $\mathcal{L}ie(n)$. *J. Pure Appl. Algebra*, 127(1):1–14, 1998.

[3] G. Arone and M. Mahowald. The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Invent. Math.*, 135(3):743–788, 1999.

[4] M. Behrens. The Goodwillie tower and the EHP sequence. *Mem. Amer. Math. Soc.*, 218(1026):xii+90, 2012.

[5] L. Brantner. The Lubin-Tate theory of spectral Lie algebras. In preparation.

[6] M. Ching. Bar constructions for topological operads and the Goodwillie derivatives of the identity. *Geom. Topol*, 9:833–933, 2005.

[7] F. R. Cohen, T. J. Lada, and J. P. May. *The homology of iterated loop spaces*, volume 533. Springer-Verlag Berlin, 1976.

[8] A. Elmendorf, I. Kriz, M. Mandell, and J. May. *Rings, modules, and algebras in stable homotopy theory.* Number 47. American Mathematical Soc., 2007.

[9] B. Fresse. On the homotopy of simplicial algebras over an operad. *Trans. Amer. Math. Soc.*, 352(9):4113–4141, 2000.

[10] T. G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.

[11] M. Hall. A basis for free Lie rings and higher commutators in free groups. *Proceedings of the American Mathematical Society*, 1(5):575–581, 1950.

[12] P. J. Hilton. On the homotopy groups of the union of spheres. *J. London Math. Soc*, 30(154-172):71, 1955.

[13] B. Johnson. The derivatives of homotopy theory. *Trans. Amer. Math. Soc.*, 347(4):1295–1321, 1995.

[14] J. J. Kjaer. On the odd primary homology of free algebras over the spectral Lie operad. `arXiv:1608.06605 [math.AT]`, 2016.

[15] N. J. Kuhn. The McCord model for the tensor product of a space and a commutative ring spectrum. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 213–236. Birkhäuser, Basel, 2004.

[16] G. Whitehead. *Elements of homotopy theory*, volume 61. Springer-Verlag New York, 1978.

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