GEOMETRICAL STRUCTURES ON QUANTUM HYPERBOLIC PLANES

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Abstract. We present several constructions of the quantum principal bundle, with its complete differential calculus and frame structure, for a large class of quantum spaces, all interpretable as a quantum version of the classical hyperbolic plane. A primary, and purely quantum, phenomenon is the appearance of an internal symmetry of the quantum hyperbolic plane, which enables us to interpret the bundle algebra, as a cross product with the circle structure group. We discuss in detail the properties of connections, their covariant derivative and curvature operators, in the light of diverse possible compatibility relations between the differential calculus on the quantum hyperbolic plane, and its internal symmetry automorphism. The standard hyperbolic quantum Hopf fibration is interpreted geometrically and discussed in detail.

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If the doors of perception were cleansed every thing would appear to man as it is, infinite.[*]

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[*]William Blake: The Marriage of Heaven and Hell.

1. Introduction

A principal objective of this paper is to introduce and study algebraic and geometric properties of a very interesting class of quantum spaces, all of which can be interpreted as quantum versions of the classical hyperbolic plane. In classical geometry, the hyperbolic plane plays a very distinctive role. Besides being a constituent of the family of the 'classical three' (together with Euclidean and elliptic planes), it provides a beautiful constructive framework for all compact Riemann surfaces of constant negative curvature. All of them can be obtained by factorizing the hyperbolic plane by an action of the appropriate infinite discrete group.

In generalizing and extending a classical theory, it is natural to expect that most interesting examples would be found amongst counterparts of objects possessing a distinctive internal harmony and simplicity, and exhibiting deep connections with other mathematical theories, objects and contexts. Therefore, it is natural to look for quantum realizations of the hyperbolic plane. As we shall see, many things from the classical Poincaré model can be incorporated into a new realm, without any essential change in music.

However, as it is natural to expect from an incomparably deeper quantum context, there appear new phenomena, completely absent in the classical world. It is always of a particular interest to study these purely quantum things. It is through them, that the quantum geometry manifests its charm.

Let us briefly walk through the contents of the paper. In the next section we present a basic cross product construction, in light of the theory of quantum principal bundles. We deal with an arbitrary automorphism at the level of function algebras and consider its extendibility to the level of differential calculus. Various degrees of compatibility are studied. This is especially important in dealing with connections, and computing their covariant derivatives and curvature forms.

As a prelude to more general cross product construction, we present the basic algebraic and geometric aspects of the Toeplitz extension algebra generated by the unilateral shift operator. There are two complementary quantum geometrical interpretations of this objects. At first, it can be viewed a 'singing quantum circle', a quantum principal circle bundle over the one-point compactication of $\mathbb{N}$. And secondly, it exhibits classical hyperbolic symmetries, which enables us to see it as the simplest of all quantum hyperbolic planes.

This more elaborated cross product construction, involving non-invertible projection symmetries and entanglement with the Toeplitz extension algebra is discussed in Section 4.

We next present in some detail, the standard example of a quantum hyperbolic Hopf fibration. We calculate completely all basic geometrical components: the bundle structure, the calculus, the algebra of horizontal forms, the Levi-Civita connection, its covariant derivative and curvature.

And in Section 4, we generalize and modify this standard example, to include a variety of spaces and bundles which all share a common background of being constructible via simple cross product extensions.

The paper ends with three complements, the extensions exhibiting an independent interest of their own. In the first Appendix, we focus on analytical aspects of the basic algebras and construct explicitly their canonical Hilbert space representations. They all fall into a classical framework of weighted shift operators, both one-sided and double-sided. There is an essential geometrical difference between the hyperbolic planes associated to one-sided and two-sided scenarios. In the first class, the spaces will be completely without points, the only classical space associated being the geometrical heaven–the circle. An important subclass is described, where the classical symmetries from $\mathrm{SU}(1,1)$ naturally act by automorphisms of the whole structure. The Toeplitz extension is obtained as the simplest concrete realization.

In the second class of two-sided shift operators, an inherent geometrical inhomogeneity appears, manifested as the presence of one classical point, the 'center' of the plane. This obviously excludes the complete group of classical hyperbolic symmetries, however it opens up a possibility for the standard quantum group of symmetries $\mathrm{S}_\mu\mathrm{U}(1,1)$ to act on the space.

In both cases, natural realizations by analytic functions are introduced, and their reproducing kernels with associated coherent states / point wave functions are computed in detail. This can be understood as the quantization of the classical Poincaré model of the hyperbolic plane. The common domain of the analytic functions is either the open unit disk $\mathbb{D}$ or $\mathbb{D}-\{0\}$, where the appropriate functions with singularity in 0 are allowed.

We also present a calculation of the metrics induced by the analytic functions Hilbert space structure. In the models where the symmetry is classical, the metrics turn out to be proportional to the classical hyperbolic metrics—as it should be.

In all cases, the space turns out to be 'infinite', and the metrics diverges when we approach the unit circle, the geometrical heaven for the space. For the two-sided shift operators model, the metrics also diverges at the unique classical point represented by 0, which therefore acts as another geometrical infinity for the space, an impossible place of infinite energy.

In the second appendix we have collected a number of interesting formulae, results and ideas which are strongly in resonance with the problematics of differential calculi on the circle. Although we are focused on the circle, several results are easily generalizable for arbitrary quantum groups, and when appropriate we have formulated them in such generality.

As we shall explain, the first order differential structures Γ on the circle are in one-one correspondence with normalized polynomials in one variable with non-zero free terms—precisely the generators of the corresponding ideals $\mathcal{R}$ in the kernel of the counit. This leads to interesting links and applications of linear algebra and rational functions, and diverse re-interpretations of the space Γ_{inv} of left-invariant one forms over $\bigcirc$.

A particular attention is given to the computation of quadratic relations space defining the universal differential envelope of a given first-order differential calculus over the circle, and to the analysis of the equilibrium condition, when the envelope and the exterior algebra coincide. These two higher-order differential structures mark the minimal and the maximal solutions respectively, for the differential forms admitting the pull-back of the product on the circle.

Both equilibrium and non-equilibrium situations are very interesting from algebraic and geometric perspectives. In particular, the non-equilibrium opens up a possibility higher-order primitive elements in the differential calculus, which in its turn allows us to generate the corresponding quantum Euler characteristic classes for the appropriate bundles.

This naturally leads to Appendix C, where a construction and basic properties of these classes are discussed, projecting the general theory of characteristic classes [11] to this particular context. The same appendix also features a general construction of the Lie algebra of infinitezimal symmetries, naturally associated to a given first-order differential calculus.

As far as our conceptual background is concerned, the basic conceptual framework is given by quantum/no-commutative geometry [2, 16, 18]. In resonance with the classical foundational treatise [14], we believe that the geometry is best described as a symbiosis of some basic space 'form', with the appropriate internal 'preintegrated' symmetries, represented by a group of transformations freely acting on a principal bundle over a given space. In other words, geometrical structures manifest themselves through principal bundles. Accordingly, in quantum geometry, the structure should best be described by appropriate quantum principal bundles, where all objects are considered as quantum—the base space, the bundle and the structure group. We shall basically follow the formalism of quantum principal bundles as developed in [4, 5, 6], and comprehensively presented in [17].

In the particular context of this paper, the bundle and the base space will be, in general, quantum, but the structure group will remain classical—it will be the circle group $\mathrm{SO}(2) \simeq \mathrm{U}(1)$. It is a natural choice for quantum Riemann surfaces. However, in spite of this, the circle will exhibit an array of quantum phenomena, primarily within the considerations of the differential calculus.

A couple of words about the notation. It is convenient to adopt here various q-expressions. They are special algebraic expressions obtained by combining a primary entity q which is assumed to be invertible and central, with integers, complex numbers and other 'classical' things.

Concretely, the q-numbers

$$(k)_q = \frac{1-q^k}{1-q} = \begin{cases} 1+q+\cdots+q^{k-1} & k>0 \\ 0 & k=0 \\ -q^{-1}-\cdots-q^{k} & k<0 \end{cases} \tag{1.1}$$

will find their ways to here. Their charming addition formula:

$$(k+j)_q = (k)_q + q^k (j)_q \qquad q^k = (k+1)_q - (k)_q. \tag{1.2}$$

We shall also meet the q-symbols

$$(z \,|\, q)_k = \prod_{j=0}^{k-1} (1 - zq^j) \tag{1.3}$$

and the infinite product version

$$(z \,|\, q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j) \tag{1.4}$$

which in order to be well-defined requires the appropriate convergence conditions. For example if q is a complex number within the open unitary disk $\mathbb{D}$, then the product is absolutely and normally convergent in z, on the whole $\mathbb{C}$.

We can express the finite symbols in terms of the infinite ones as

$$(z\,|\,q)_n = (z\,|\,q)_\infty/(zq^n\,|\,q)_\infty. \tag{1.5}$$

In such a way the definition of the symbols can be extended to all $n \in \mathbb{Z}$ and actually to all complex numbers if $q \in (0,1)$. We always have

$$(0\,|\,q)_\alpha = (z\,|\,q)_0 = 1. \tag{1.6}$$

The following reflection formula holds

$$(z\,|\,q)_{-\alpha}(q^{-\alpha}z\,|\,q)_\alpha = 1. \tag{1.7}$$

These q-expressions naturally appear in computations with quantum groups and related non-commutative structures. Their historical origins are much older than quantum theory, and are deeply intertwined with Numbers, Infinite Series and Complex Analysis.

2. Cross Product for Invertible Symmetries

2.1. **Functions Algebra Level**

Let us consider a unital *-algebra $\mathcal{V}$, of which we can think as representing a quantum space M. Let $\psi\colon \mathcal{V} \to \mathcal{V}$ be a *-automorphism of $\mathcal{V}$. The cross product construction, in its simplest form, embeds $\mathcal{V}$ into a larger *-algebra $\mathcal{B}$, by introducing a unitary $\mathfrak{u}$ which reproduces ψ via its similarity transformation. In other words $\mathfrak{u}^* = \mathfrak{u}^{-1}$ and

$$\mathfrak{u} f \mathfrak{u}^* = \psi(f) \tag{2.1}$$

for every $f \in \mathcal{V}$.

Remark 1. A basic class of examples is given by classical smooth manifolds M equipped with a diffeomorphism $T\colon M \to M$. We can define $\mathcal{V} = \mathrm{C}^\infty(M)$ with the automorphism ψ being the induced action of T on $\mathcal{V}$.

Remark 2. We shall always think of $\mathfrak{u}$ as the contextual representation of the canonical embedding U of the unitary circle $\bigcirc$ into complex numbers. The commutative *-algebra $\mathcal{A}$ generated by U, captures the group structure on the circle, via the coproduct $\phi\colon \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$, which is the pull back of the product in the circle and it is given by $\phi(U) = U \otimes U$.

The cross product algebra $\mathcal{B}$ gives us a nice quantum principal bundle [5]. The inclusion $\iota\colon \mathcal{V} \to \mathcal{B}$ can be interpreted as the projection of the corresponding quantum space P onto M. The coproduct ϕ extends to a *-homomorphism $F\colon \mathcal{B} \to \mathcal{B}\otimes\mathcal{A}$, actually a coaction of the circle group on $\mathcal{B}$, by $\mathcal{V}$-linearity. So that $\mathcal{V}$ is the F-fixed subalgebra of $\mathcal{B}$.

Lemma 1. *The constructed quantum principal bundle will be trivializable, iff ψ is an inner automorphism of $\mathcal{V}$.*

Proof. For the trivializability of P is equivalent, at the geometrical level, to the existence of a cross-section from M to P. Algebraically, there exists a unital *-homomorphism $s\colon \mathcal{B} \to \mathcal{V}$ which is a left inverse of ι. Then $\psi(f) = s\psi(f) = s(\mathfrak{u}f\mathfrak{u}^*) = s(\mathfrak{u})fs(\mathfrak{u})^*$ and thus ψ is inner. And conversely, if ψ is inner associated to a unitary W then the assignment $s(\mathfrak{u}) = W$ consistently and uniquely defines the corresponding bundle cross-section map $s\colon \mathcal{B} \to \mathcal{V}$. □

2.2. **The Simplest Differential Calculus Extension**

Let us now assume that the space M is equipped with a differential calculus, represented by a graded differential *-algebra $\Omega(M)$, built over $\mathcal{V}$. This means that $\Omega^0(M)=\mathcal{V}$, and that $\Omega(M)$, as a differential algebra, is generated by $\mathcal{V}$. We shall denote by $d_M\colon\Omega(M)\to\Omega(M)$ the corresponding differential. We shall assume that ψ extends to a differential *-automorphism of the whole $\Omega(M)$.

Remark 3. In the above mentioned classical example of smooth manifolds, this extendibility property will automatically hold, as the diffeomorphisms of the manifold always extend to differential forms.

We are going to construct a natural compatible differential calculus on the bundle P. By definition its corresponding differential *-algebra $\Omega(P)$ is given by the relations

$$\mathfrak{u}\,d\mathfrak{u}=d\mathfrak{u}\,\mathfrak{u}\qquad (d\mathfrak{u})^2=0\qquad \mathfrak{u}\lambda=\psi(\lambda)\mathfrak{u}\qquad (d\mathfrak{u})\lambda=(-)^{\partial\lambda}\psi(\lambda)(d\mathfrak{u}) \tag{2.2}$$

where $\lambda\in\Omega(M)$. We shall assume first that the calculus on the structure group $\bigcirc$ is the classical calculus, and let us denote by $\bigcirc_\wedge$ this classical calculus.

Lemma 2. *The map $F\colon\mathcal{B}\to\mathcal{B}\otimes\mathcal{A}$ is uniquely extendible to a differential *-homomorphism $\widehat{F}\colon\Omega(P)\to\Omega(P)\mathbin{\widehat{\otimes}}\bigcirc_\wedge$.*

Proof. Let us observe that the algebra $\Omega(P)$ has two components, both trivial as left or right $\Omega(M)$-modules:

$$\Omega(P)=\mathfrak{hor}(P)\oplus\mathfrak{hor}(P)(\mathfrak{u}^*d\mathfrak{u})$$

where

$$\mathfrak{hor}(P)=\Omega(M)\mathcal{B}=\mathcal{B}\Omega(M)$$

plays the role of the horizontal forms subalgebra.

Since $F\colon\mathfrak{u}\mapsto\mathfrak{u}\otimes U$, the first bimodule is clearly compatible with the definition of $\widehat{F}$. To check the compatibility of the second bimodule,

$$\begin{aligned}d\mathfrak{u}\lambda-(-)^{\partial\lambda}\psi(\lambda)d\mathfrak{u}&\rightsquigarrow(d\mathfrak{u}\lambda\otimes U+(-)^{\partial\lambda}\mathfrak{u}\lambda\otimes dU)-(-)^{\partial\lambda}\psi(\lambda)(d\mathfrak{u}\otimes U+\mathfrak{u}\otimes dU)\\&=(d\mathfrak{u}\lambda-(-)^{\partial\lambda}\psi(\lambda)d\mathfrak{u})\otimes U+(-)^{\partial\lambda}(\mathfrak{u}\lambda-\psi(\lambda)\mathfrak{u})\otimes dU\end{aligned}$$

where $\lambda\in\Omega(M)$. Thus, the relations defining $\Omega(P)$ are fully compatible. $\square$

Remark 4. In other words, we have a differential calculus on the quantum principal bundle P, as defined in [5]. The corresponding horizontal forms are given by

$$\mathfrak{hor}(P)=\widehat{F}^{-1}\Big\{\Omega(P)\otimes\mathcal{A}\Big\}$$

in resonance with the general theory. And as we are going to see, the expression $\mathfrak{u}^*d\mathfrak{u}$ defines a canonical flat connection on P.

2.3. **Connections, Covariant Derivative and Curvature**

We shall assume in this subsection that a general differential calculus on the bundle is given, such that the right action $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ extends to a differential (necessarily then grade and *-preserving) homomorphism $\widehat{F}\colon \Omega(P) \to \Omega(P) \mathbin{\widehat{\otimes}} \Gamma^{\wedge}$, where $\Gamma^{\wedge}$ is a full differential calculus for the circle built over a given bicovariant *-calculus Γ. Our aim is to see what an internal structure of the calculus looks like, if we assume the existence of at least one regular and multiplicative connection. This will provide the framework for going in the opposite direction, and constructing the calculus given the primary geometrical data on the base space.

We shall assume that the automorphism ψ is extended to the calculus $\Omega(P)$ straightforwardly, as the inner automorphism associated to $\mathfrak{u}$.

Remark 5. In general, this means that the extended ψ does not commute with the differential $d\colon \Omega(P) \to \Omega(P)$.

By setting

$$\varphi \bullet U^k = \psi^{-k}(\varphi) = \mathfrak{u}^{-k}\varphi\mathfrak{u}^k \tag{2.3}$$

we construct a right $\mathcal{A}$-module structure on $\Omega(P)$. It is easy to verify that

$$\widehat{F}(\varphi \bullet a) = [\varphi^{(0)} \bullet a^{(2)}] \otimes \kappa(a^{(1)})\varphi^{(1)}a^{(3)} \qquad [\varphi \bullet a]^* = \varphi^* \bullet \kappa(a)^*. \tag{2.4}$$

In particular, if φ is horizontal then

$$F^\wedge(\varphi \bullet a) = [\varphi^{(0)} \bullet a] \otimes \varphi^{(1)} \tag{2.5}$$

for every $a \in \mathcal{A}$. This is equivalent to saying that $\Omega(M)$ is $\bullet$-invariant.

If $\omega\colon \Gamma_{inv} \to \Omega(P)$ is an arbitrary connection on P then the horizontal obstacle to regularity ℓ_ω is composed of two generating components. The first is given by supercommutators

$$\ell_\omega(\vartheta, h) = \omega(\vartheta)h - (-)^{\partial h} h\omega(\vartheta) \tag{2.6}$$

where $h \in \Omega(M)$. Because $\bigcirc$ is Abelian, these elements are always from $\Omega(M)$. The second component is given by canonical bundle harmonics probes

$$\ell_\omega(\vartheta, \mathfrak{u}^k) = \mathfrak{u}^k \bigl[\omega(\vartheta) \bullet U^k - \omega(\vartheta \circ U^k)\bigr]. \tag{2.7}$$

We can rewrite this in the form

$$\omega(\vartheta) \bullet a - \omega(\vartheta \circ a) = \sum a_{(-)}\ell_\omega(\vartheta, a_{(+)}) \tag{2.8}$$

where $\mathcal{A} \ni a \rightsquigarrow \sum a_{(-)} \otimes a_{(+)} \in \mathcal{B} \otimes \mathcal{B}$ is the linear extension of $U^k \mapsto V^{-k} \otimes V^k$, a lifted translation map.

Lemma 3. *A necessary and sufficient pair of conditions for a connection ω to be regular is*

$$\omega(\vartheta)h = (-)^{\partial h} h\omega(\vartheta) \tag{2.9}$$

$$\omega(\vartheta \circ a) = \omega(\vartheta) \bullet a \tag{2.10}$$

for every $\vartheta \in \Gamma_{inv}$, $a \in \mathcal{A}$ and $h \in \Omega(M)$. □

Each of these two properties is worth analyzing in more detail. If the condition (2.10) holds for $a = U$ then it holds for all $a \in \mathcal{A}$. In general, the lack of $\circ-\bullet$ intertwining by ω measured in (2.8) always takes values from $\Omega(M)$. In particular, if $q(z)$ is the generating polynomial for the calculus Γ then $\omega(\vartheta) \bullet q(U) \in \Omega(M)$. On the other hand, if the $\circ-\bullet$ intertwining holds for ω and if the supercommuting (2.9) holds for $\vartheta = o_1$ then it holds for all $\vartheta \in \Gamma_{inv}$.

Lemma 4. *The assignment $\omega \rightsquigarrow \omega(o_1) = \varrho$ establishes one-one correspondence between the connections which intertwine $\circ$ and $\bullet$ and antihermitian one-forms ϱ on P satisfying*

$$\widehat{F}(\varrho) = \varrho \otimes 1 + 1 \otimes o_1 \tag{2.11}$$

$$\varrho \bullet q(U) = 0. \tag{2.12}$$

In terms of this correspondence, the connection will be regular iff ϱ supercommutes with the elements of $\Omega(M)$. □

In our geometrical setup the expressions $\varpi_k = \mathfrak{u}^{*k} d(\mathfrak{u}^k)$ for $k \in \mathbb{Z}$ exhibit a special significance. They mimic the germs $o_k = \pi(U^k)$ of the basic harmonics on the circle and satisfy

$$\begin{gathered} \widehat{F}(\varpi_k) = \varpi_k \otimes 1 + 1 \otimes o_k \\ \varpi_k^* = -\varpi_k \qquad d\varpi_k = -\varpi_k^2. \end{gathered} \tag{2.13}$$

They are all generated from ϖ_1 by iteratively applying ψ and ψ^{-1}. Explicitly $\varpi_0 = 0$ and

$$\varpi_k = \Bigl\{\sum_{j=0}^{k-1} \psi^{-j}\Bigr\}\varpi_1 \qquad \varpi_{-k} = \Bigl\{\sum_{j=1}^{k} \psi^{k}\Bigr\}\varpi_1 \tag{2.14}$$

for $k > 0$.

So if we take the first n germs $o_1, \ldots, o_n$ which span the space Γ_{inv} then for them the assignment

$$o_k \rightsquigarrow \varpi_k \tag{2.15}$$

defines a connection ϖ on P. A priori, this is not extendible to all germs, as all the ϖ_k will exhibit less linear relations than the o_k.

Remark 6. If $\sum_k c_k o_k = 0$ then and only then $\sum_k c_k \varpi_k$ will belong to the calculus on the base M.

Secondly, the elements ϖ_k are intrinsically related to the problematics of extending the symmetry ψ from $\mathcal{V}$ to the whole calculus $\Omega(M)$. By differentiating (2.3), and regrouping the terms we arrive at

$$\varpi_k h - (-)^{\partial h} h \varpi_k = \bigl\{\psi^{-k} d\psi^k - d\bigr\}h. \tag{2.16}$$

The transformed differentials $\psi^{-k} d\psi^k$ will be, in principle, all different. However, their mutual differences are always given by inner superderivations in $\Omega(P)$.

The most harmonic situation is when 'bundle germs' and group germs are completely correlated, so that the defined connection acts on all germs in the same way (2.15). This is equivalent to saying that ϖ intertwines $\circ$ and $\bullet$ so that ϖ is already 'half regular'.

Lemma 5. *In this context, ϖ is regular iff ψ is d-preserving.* □

Now, every connection on P is of the form $\omega = \varpi - \chi$ where $\chi\colon \Gamma_{inv} \to \Omega(P)$ is an antihermitian map with values in one forms on M, an appropriate 'displacement tensor'. For ω to be regular, it must supercommute with the forms on the base, and be fully $\circ{-}\bullet$ compatible. The supercommutativity with $\Omega(M)$ reads

$$\varpi_k h - (-)^{\partial h} h\varpi_k = \chi_k h - (-1)^{\partial h} h\chi_k \tag{2.17}$$

where $\chi_k = \chi(o_k) \in \Omega^1(M)$. In other words, the graded commutators with the initial connection ϖ are equivalent to internal graded commutators with the displacement. A useful special case of the identity is when $h = \chi_k$ which then reads

$$\varpi_k\chi_k + \chi_k\varpi_k = 2\chi_k^2. \tag{2.18}$$

Another important consequence of (2.17) is obtained by replacing h with dh, then differentiating it, taking into account the third equation in (2.13) the flatness property of ϖ, and doing elementary transformations:

$$\bigl[d\chi_k + \chi_k^2, h\bigr] = 0 \tag{2.19}$$

for all $k \in \mathbb{Z}$ and $h \in \Omega(M)$—an interesting centrality property.

The $\circ{-}\bullet$ intertwining property for χ says that

$$\chi_k = \Bigl\{\sum_{j=0}^{k-1} \psi^{-j}\Bigr\}\chi_1 \qquad \chi_{-k} = -\Bigl\{\sum_{j=1}^{k} \psi^k\Bigr\}\chi_1 = -\psi^k(\chi_k) \tag{2.20}$$

for all $k > 0$.

If the equation (2.17) is satisfied for $k = 1$ then it follows that, because of (2.14) and (2.20), it holds for all $k \in \mathbb{Z}$. In summary:

Lemma 6. *The displacements for the regular connections are in one-one correspondence with antihermitian one-forms on $\Omega(M)$ whose difference with ϖ_1 supercommutes with everything in $\Omega(M)$, and whose generated cyclic $\bullet$ submodule is a projection of the cyclic $\circ$ module Γ_{inv} with its cyclic vector o_1. In this case* (2.20) *defines the full displacement map χ, the one-form is interpretable as χ_1 and* (2.17) *happily holds.* □

By combining (2.16) and (2.17) we find

$$\chi_k h - (-)^{\partial h} h\chi_k = \bigl\{\psi^{-k} d\psi^k - d\bigr\}h. \tag{2.21}$$

This brings the difference between the transformed differential and the original differential, measurable completely within $\Omega(M)$. To put it in other terms, the formula can be viewed as the definition of the extension of ψ, from its original domain $\mathcal{V} = \Omega^0(M)$ to the whole of $\Omega(M)$. To see this explicitly, rewrite (2.21) as

$$\psi^k d(h) = d\psi^k(h) + \bigl[\chi_{-k}\psi^k(h) - (-)^{\partial h}\psi^k(h)\chi_{-k}\bigr]. \tag{2.22}$$

Let us compute the curvature map $r_\omega\colon \mathcal{A} \to \mathfrak{hor}(P)$ of such a connection $\omega = \varpi - \chi$. According to the definition of this map

$$\begin{aligned} r_\omega(U^k) &= d\omega(o_k) + \omega(o_k)\omega(o_k) = d\varpi_k - d\chi_k + (\varpi_k - \chi_k)^2 = \\ &= d\varpi_k + \varpi_k^2 - d\chi_k + \chi_k^2 - \varpi_k\chi_k - \chi_k\varpi_k = -\bigl(d\chi_k + \chi_k^2\bigr). \end{aligned} \tag{2.23}$$

The curvature takes values from $\Omega^2(M)$ as it should, because $\bigcirc$ is Abelian. And, as we have already explained, the curvature is always a *central element* of $\Omega(M)$.

Let $D = D_{\omega+\xi}$ be the corresponding covariant derivative. It extends the differential $d_M\colon \Omega(M) \to \Omega(M)$ to a covariant first-order hermitian antiderivation (in the sense of satisfying the graded Leibniz rule) of $\mathfrak{hor}(P)$.

As such, it is completely determined by its values on the generator U. We have

$$D(U) = dU - U(\omega + \xi) = -U\xi. \tag{2.24}$$

And it is instructive to compare, in this very particular context, the square of the covariant derivative and the curvature. We have

$$D^2(U) = -D(U)\xi - Ud(\xi) = -Ur_{\omega+\xi} \tag{2.25}$$

in accordance with the general theory.

2.4. **Universal Solution**

Let us now consider the most general situation admitting a regular connection. We shall assume that a graded differential *-algebra $\Omega(M)$ is given, together with a graded *-automorphism $\psi\colon \Omega(M) \to \Omega(M)$. However, we shall not require any compatibility property between ψ and d_M. We shall construct the cross product algebra $\mathfrak{hor}(P)$ in the same way as before, so that

$$U\lambda = \psi(\lambda)U$$

for every $\lambda \in \Omega(M)$.

The existence of regular connections is intrinsically related to the possibility of extending the differential d_M to a covariant derivative map $D\colon \mathfrak{hor}(P) \to \mathfrak{hor}(P)$, which is a hermitian first-order antiderivation which is covariant—in the sense that the diagram

$$\begin{array}{ccc} \mathfrak{hor}(P) & \xrightarrow{F^\wedge} & \mathfrak{hor}(P)\otimes\mathcal{A} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D\otimes\mathrm{id}} \\ \mathfrak{hor}(P) & \xrightarrow[F^\wedge]{} & \mathfrak{hor}(P)\otimes\mathcal{A} \end{array} \tag{2.26}$$

is commutative.

Every such a map is completely determined by its value on U, and because of the covariance, it must be of the form

$$D(U) = -U\xi \tag{2.27}$$

where $\xi \in \Omega^1(M)$. Since D is hermitian, we have

$$0 = D(1) = D(U^*)U + U^*D(U) = -(\xi^* + \xi)$$

in other words ξ must be antihermitian.

Proposition 7. *The following identity holds:*

$$\xi\lambda - (-)^{\partial\lambda}\lambda\xi = d_M(\lambda) - \psi^{-1}d_M\psi(\lambda) \tag{2.28}$$

for every $\lambda \in \Omega(M)$. Conversely, if ξ is an antihermitian first-order element of $\Omega(M)$ satisfying this identity, then the formula (2.27) *uniquely and consistently defines a covariant hermitian first-order antiderivation $D\colon \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ extending the differential $d_M\colon \Omega(M) \to \Omega(M)$.*

Proof. By applying D on the cross product relation, we obtain

$$0 = -U^*\big\{D(U)\lambda + Ud_M(\lambda) - d_M\psi(\lambda)U - (-)^{\partial\lambda}\psi(\lambda)D(U)\big\} = \\ = \xi\lambda - d_M(\lambda) + \psi^{-1}d_M\psi(\lambda) - (-)^{\partial\lambda}\lambda\xi.$$

In other words (2.28) holds. Since the horizontal forms algebra is defined by the single cross-product relation, a necessary and sufficient condition for the consistent (and necessarily unique) extendibility of D as an antiderivation, is the compatibility with this single relation. $\square$

Remark 7. And the initial simplest situation when ψ preserves d_M, is equivalent to ξ supercommuting with all elements of $\Omega(M)$. We see that the general situation is not very different: although ψ and d_M do not commute, the difference between differentials d_M and $\psi^{-1}d_M\psi$ is, in a sense, small. It is the inner derivation associated to ξ. It is worth remembering that inner superderivations always form an ideal in the super Lie algebra of all superderivations.

Remark 8. The above formula (2.28) completely fixes the extension of the automorphism ψ, from $\mathcal{V} = \Omega^0(M)$ to the whole of $\Omega(M)$. Let us recall that we are assuming that $\Omega(M)$ is generated, as a differential algebra, by $\mathcal{V}$.

Let us calculate the curvature operator for such connections. According to the general theory, it is a map $\varrho_D\colon \mathcal{A} \to \mathfrak{hor}(P)$ uniquely defined by the formula

$$D^2(\varphi) = -\varphi^{(0)}\varrho_D(\varphi^{(1)})$$

where $F^\wedge(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)}$. In our context, since the circle group is Abelian, the curvature will be $\Omega(M)$-valued. We have

$$\varrho_D(U) = -U^*D^2(U) = U^*D(U\xi) = U^*(D(U)\xi + Ud_M(\xi)) = d_M(\xi) - \xi^2.$$

This easily extends to all powers of U:

Proposition 8. *The following identities hold*

$$\varrho_D(U^n) = \sum_{k=0}^{n-1}\psi^{-k}\big[d_M(\xi) - \xi^2\big] \qquad \varrho_D(U^{-n}) = -\sum_{k=1}^{n}\psi^{k}\big[d_M(\xi) - \xi^2\big] \tag{2.29}$$

for each $n \in \mathbb{N}$.

Proof. The square D^2 is a standard derivation. Thus,

$$\varrho_D(U^n) = -U^{-n}D^2(U^n) = -U^{-n}\sum_{k=0}^{n-1}U^{n-k-1}D^2(U)U^k = \\ = U^{-n}\sum_{k=0}^{n-1}U^{n-k-1}U[d_M(\xi) - \xi^2]U^k = \sum_{k=0}^{n-1}\psi^{-k}[d_M(\xi) - \xi^2]$$

and similarly

$$\varrho_D(U^{-n}) = -U^nD^2(U^{-n}) = -U^n\sum_{k=0}^{n-1}U^{1+k-n}D^2(U^*)U^{-k} = \\ = U^n\sum_{k=0}^{n-1}U^{1+k-n}[\xi^2 - d_M(\xi)]U^*U^{-k} = \sum_{k=1}^{n}\psi^k[\xi^2 - d_M(\xi)].$$

Above, we have used the facts that D commutes with $*$, and that the expression $d_M(\xi) - \xi^2$ is antihermitian. $\square$

From the general theory, we know that the covariant derivative of the curvature operator is zero. This is the quantum Bianchi identity for the regular connections. In our case, since the curvature takes values from $\Omega(M)$, this means that all $\varrho_D(U^m)$, for $m \in \mathbb{Z}$, must be closed. Equivalently

$$d_M \psi^m \big[d_M(\xi) - \xi^2 \big] = 0 \tag{2.30}$$

for every $m \in \mathbb{Z}$. The simplest among these Bianchi identities is the one saying that $d_M(\xi^2) = 0$. The curvature also commutes with all the elements of $\Omega(M)$. In particular, we have

$$d_M(\xi)\xi = \xi d_M(\xi) \tag{2.31}$$

which is equivalent to the above mentioned fact that ξ^2 is closed.

Remark 9. The curvature operators define, in a natural way, the minimal compatible calculus on the structure group. This calculus is based on the right ideal $\mathcal{J}$ in the kernel of the counit $\epsilon\colon \mathcal{A} \to \mathbb{C}$ consisting precisely of all the elements aniquilated by the curvature. We can also consider, instead of a single preferred connection, the appropriate affine space of them (including all possible realizations via covariant derivatives). As we can see from the above expressions for the curvature, in general, such a calculus could be arbitrarily different from the classical one. Including the universal calculus (corresponding to the trivial kernel $\mathcal{J} = \{0\}$).

Remark 10. The centrality property of the curvature is equivalent to the centrality property of the single element $d_M(\xi) - \xi^2$. The characteristic classes are all cohomology classes generated by the elementary ones $\psi^m \big[d_M(\xi) - \xi^2 \big]$. In fact, we can use the derived properties to construct the universal algebra of cross-product characteristic classes, by considering a free algebra generated by symbols ξ, $d\xi$ and U, and factorizing over the appropriate relations.

3. Singing Quantum Circle

3.1. **The Toeplitz Extension**

For non-invertible projection morphisms ψ the classical circle should be replaced by its non-commutative extension by compact operators. The *-algebra $\mathcal{T}$ is generated by a single element S satisfying

$$S^* S = 1. \tag{3.1}$$

The canonical realization is given by the unilateral shift operator

$$S|k\rangle = |k+1\rangle \qquad S^*|0\rangle = 0 \qquad S^*|k+1\rangle = |k\rangle \tag{3.2}$$

acting in the Hilbert space $\mathrm{l}^2(\mathbb{N})$. By forgetting the non-commutativity, the structure morphs into the classical $\bigcirc$, by interpreting

$$\mathcal{T}/\mathrm{M}_\infty(\mathbb{C}) \cong \mathcal{A} \tag{3.3}$$

where $\mathrm{M}_\infty(\mathbb{C})$ the *-algebra of infinite complex matrices having finitely many non-zero entries, emerges as the commutant ideal of $\mathcal{T}$. The elementary matrices are given by

$$e_{ij} = |i\rangle\langle j| = S^i(1 - SS^*)S^{*j} \tag{3.4}$$

and the elements S^iS^{*j} form a linear basis in $\mathcal{T}$, which is a multiplicative unital semigroup in $\mathcal{T}$. Explicitly

$$S^iS^{*j}\, S^kS^{*l} = \begin{cases} S^iS^{*j-k+l} & j \geq k \\ S^{i+k-j}S^{*j} & j \leq k. \end{cases} \tag{3.5}$$

The diagonal basis elements $p_k = U^kU^{*k}$ are orthogonal projectors on subspaces spanned by $|i\rangle$ where $i \geq k$.

Remark 11. The whole structure can be naturally viewed in terms of the complex geometry of the Hilbert space of square summable power series [3]. These series are all holomorphic inside of a unitary disk. The generator S is viewed as the multiplication operator by the complex variable z and the adjoint operator is charmingly $S^*\colon \psi(z) \mapsto (\psi(z) - \psi(0))/z$.

There is a canonical internal symmetry, a non-unital *-morphism $\psi\colon \mathcal{T} \to \mathcal{T}$ given, together with its left inverse, by

$$\psi(h) = ShS^* \qquad\qquad \psi^-(h) = S^*hS \tag{3.6}$$

so that

$$\psi^-\psi(h) = h \qquad \psi\psi^-(h) = \psi(1)h\psi(1). \tag{3.7}$$

Furthermore ψ^- is unital, completely positive and satisfies the following partial multiplicativity property

$$\begin{aligned} \psi^-\big[\psi(q)h\big] &= q\psi^-(h) \\ \psi^-(h)q &= \psi^-\big[h\psi(q)\big] \end{aligned} \tag{3.8}$$

for every $\lambda, h, \rho \in \mathcal{T}$. By construction $p_k = \psi^k(1)$ for all $k \in \mathbb{N}$.

By taking C*-completions, we obtain the canonical short exact sequence

$$0 \to \mathbb{K} \rightarrowtail \overline{\mathcal{T}} \twoheadrightarrow \mathrm{C}(\bigcirc) \to 0 \tag{3.9}$$

the Toeplitz extension of $\mathrm{C}(\bigcirc)$ by the compact operators $\mathbb{K}$.

The classical circle acts on $\mathcal{T}$ by rotations which defines the action $F\colon \mathcal{T} \to \mathcal{T} \otimes \mathcal{A}$ and in view of

$$F(S^iS^{*j}) = S^iS^{*j} \otimes U^{i-j} \tag{3.10}$$

the algebra $\mathcal{T}$ splits into an orthogonal sum of multiple irreducible subspaces

$$\mathcal{T} = \bigoplus_{k\in\mathbb{Z}} \mathcal{T}_k \qquad \mathcal{T}_k = \mathrm{span}\Big\{ S^iS^{*j} \;\Big|\; i - j = k \Big\}. \tag{3.11}$$

The pure terms, which are the powers of only S and S^*, together with ψ generate the ψ-invariant spaces $\mathcal{T}_k$, as

$$S^iS^{*j} = \begin{cases} \psi^j(S^{i-j}) & i \geq j \\ \psi^i(S^{*j-i}) & i \leq j. \end{cases} \tag{3.12}$$

In particular, the F-invariant part $\mathcal{T}_0$ is spanned by the projectors p_k. It is a unital commutative subalgebra algebra whose underlying space is the one-point compactification of the natural numbers $\overset{\infty}{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. In terms of the identification of the points of $\overset{\infty}{\mathbb{N}}$ with characters on $\mathcal{T}_0$, the point ∞ corresponds to a character $\varkappa_\infty$ evaluating to one in all these projectors, so it is simply the classical point $1 \in \bigcirc$. While for $i \in \mathbb{N}$ its $\varkappa_i$ evaluates to one on p_j with $j \leq i$ otherwise, its 0.

Proposition 9. *The free action subalgebra for F is spanned by the non-negative powers of U. Thus, the action F is free.*

Proof. For every a from this subalgebra we can find an array c_{ijkl} such that

$$\sum_{ijkl} c_{ijkl} S^i S^{*j}\, F(S^k S^{*l}) = 1 \otimes a.$$

Because of (3.5) and (3.10) we can remove from the sum all the terms in which the basis elements do not multiply into 1, which are the terms $i = l = k - j = 0$. $\square$

In such a way we have constructed a quantum principal $\bigcirc$-bundle over $\overset{\infty}{\mathbb{N}}$. An interesting geometrical picture emerges. The bundle unifies infinitely circular 'oscillating modes'. The limiting oscillating mode is the classical mode, corresponding to the classical part $\bigcirc$. There are inner and purely quantum 'virtual modes' with non-individualizable fibers over the classical points $i \in \mathbb{N}$ labeling them. A true quantum circle.

To further this quantum circle interpretation, there is a natural coproduct map $\phi\colon \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ which a unital *-homomorphism defined by

$$\phi(S) = S \otimes S \qquad \phi(S^*) = S^* \otimes S^*. \tag{3.13}$$

The counit is the character corresponding to the classical circle point $1 \in \bigcirc$. And the antipode $\kappa\colon \mathcal{T} \to \mathcal{T}$ is the unital antimultiplicative map specified by

$$\kappa(S) = S^* \qquad \kappa(S^*) = S. \tag{3.14}$$

So the extension becomes a (generalized) quantum group, with $\bigcirc$ interpretable as its classical part.

Another fundamental geometrical significance of this extension is that it provides the simplest quantum hyperbolic plane, possessing a full classical symmetry group $\mathrm{SU}(1,1)/\{-1,1\}$. The assignment

$$S^* \mapsto \frac{aS^* + b}{\bar{b}S^* + \bar{a}} \qquad \mathrm{SU}(1,1) = \Bigl\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \Bigm| |a|^2 - |b|^2 = 1 \Bigr\} \tag{3.15}$$

consistently and uniquely defines an action of $\mathrm{SU}(1,1)/\{1,-1\}$ by *-automorphisms of $\overline{\mathcal{T}}$. The adjoint shift operator S^* corresponds to the complex variable restricted to the unit complex disc, in the classical Poincaré model of the hyperbolic plane. The classical $\bigcirc$ is interpretable as the 'horizon heaven' for such a quantum space.

3.2. **Universal Differential Calculus**

In terms of generators and relations $\mho(\mathcal{T})$ the universal differential envelope of $\mathcal{T}$ is given by S and S^*, their unique relation defining $\mathcal{T}$ and its differential child

$$dS^*\, S + S^*\, dS = 0. \tag{3.16}$$

The universal calculus already contains a rudimentary structure for constructing a full differential calculus for the quantum circle bundle. We shall explore this structure now. The coproduct ϕ on $\mathcal{T}$ together with ϵ and κ extend from $\mathcal{T}$ to

$\mho(\mathcal{T})$ naturally. The group structure remains Abelian on $\mho(\mathcal{T})$. The antipode κ is a super antimultiplicative homomorphism on $\mho(\mathcal{T})$ satisfying

$$\begin{array}{ccc} \mho(\mathcal{T}) & \xrightarrow{\phi} & \mho(\mathcal{T}) \mathbin{\widehat{\otimes}} \mho(\mathcal{T}) \\ {\scriptstyle\kappa}\big\downarrow & & \big\downarrow{\scriptstyle\kappa\otimes\kappa} \\ \mho(\mathcal{T}) & \xrightarrow{\phi} & \mho(\mathcal{T}) \mathbin{\widehat{\otimes}} \mho(\mathcal{T}) \end{array} \qquad d\kappa = \kappa d \qquad * \kappa * = \kappa^{-1}. \tag{3.17}$$

The construction of the complete calculus for the bundle requires fixing a finitely-dimensional differential calculus Γ on $\bigcirc$ and promoting the appropriate additional relations in the envelope $\mho(\mathcal{T})$. The action map F always extends to the calculus level $\widehat{F}\colon \mho(\mathcal{T}) \to \mho(\mathcal{T}) \mathbin{\widehat{\otimes}} \Gamma^{\wedge\!\!\!\vee}$. However, only for non-universal Γ will the germ map be surjective on the free action algebra. For a first-order calculus over $\bigcirc$ being non-universal is equivalent to the finite dimensionality of Γ_{inv}.

The morphism ψ and its left inverse ψ^- directly extend from $\mathcal{T}$ to $\mho(\mathcal{T})$. It is easy to see that

$$d\psi(h) = \psi d(h) + \Bigl\{(d(S)S^*)\psi(h) + (-)^{\partial h}\psi(h)(Sd(S^*))\Bigr\} \tag{3.18}$$

$$d\psi^-(h) \;=\; \psi^- d(h) + \psi^-\Bigl\{(Sd(S^*))h + (-)^{\partial h}h(d(S)S^*)\Bigr\} \tag{3.19}$$

for every $h \in \mho(\mathcal{T})$.

Proposition 10. *Let us fix a first-order element $\theta \in \mho(\mathcal{T})$. There exists a unique homomorphism $\Pi\colon \mho(\mathcal{T}) \to \mho(\mathcal{T})$ which acts as the identity on $\mathcal{T}$ and*

$$\Pi(dS) = (1 - SS^*)dS - S\theta \qquad \Pi(dS^*) = dS^*(1 - SS^*) + \theta S^*. \tag{3.20}$$

This map is grade-preserving and the diagram

$$\begin{array}{ccc} \mho(\mathcal{T}) & \xrightarrow{\phi} & \mho(\mathcal{T}) \mathbin{\widehat{\otimes}} \mho(\mathcal{T}) \\ {\scriptstyle\Pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi\otimes\Pi} \\ \mho(\mathcal{T}) & \xrightarrow{\phi} & \mho(\mathcal{T}) \mathbin{\widehat{\otimes}} \mho(\mathcal{T}) \end{array} \tag{3.21}$$

*is commutative iff θ is primitive, that is $\phi(\theta) = 1 \otimes \theta + \theta \otimes 1$. It will be *-preserving iff $\theta^* = -\theta$.*

Proof. At first, we have to check the compatibility of the definition of Π with the defining relations (3.16). We compute

$$\begin{aligned} 0 = dS^*\, S + S^*\, dS \rightsquigarrow \bigl[dS^*(1 - SS^*) + \theta S^*\bigr]S + S^*\bigl[(1 - SS^*)dS - S\theta\bigr] &= \\ = dS^*(S - SS^*S) + (S^* - S^*SS^*)dS + \theta - \theta = 0. \end{aligned}$$

Thus Π exists and is unique. It is grade preserving because the property is trivially satisfied at the level of $\mathcal{T}$, and (3.20). Both branches of the diagram (3.21) are homomorphisms and coincide on $\mathcal{T}$ and generators dS and dS^*, if and only if θ satisfies the primitivity condition. In order for it to be a *-homomorphism, it is necessary and sufficient to check the property at the level of generators S and S^*, where because of (3.20) it means the antihermicity of θ. $\square$

The correlation between θ and Π is of mutual reciprocity:

$$\theta = \Pi(dS^*)S = -S^*\Pi(dS). \tag{3.22}$$

If $\theta = -S^*\,dS$ then and only then Π is the identity on $\mho(\mathcal{T})$. Another notable choice is $\theta = 0$. Here are some more formulae exhibiting particular usefulness in calculations:

$$(1 - SS^*)dS = d[SS^*]S \qquad dS^*(1 - SS^*) = S^*d[SS^*]$$
$$S^*d[SS^*]S = 0 \qquad d[S^*dS] + [S^*dS]^2 = S^*d(SS^*)^2S.$$

The one-form $S^*dS = -d(S^*)S$ is the simplest blueprint connection form. Its sibling $Sd(S^*) = d(SS^*) - d(S)S^*$ is almost-a-connection. The form θ is viewed as a connection displacement and then Π is interpretable as a blueprint horizontal projection. And since Π acts as identity on $\mathcal{T}$, it commutes with ψ and ψ^- on the complete calculus $\mho(\mathcal{T})$.

Lemma 11. *The map* Π *will be idempotent iff* $\Pi(\theta) = \theta$.

Proof. Calculating the square of Π we obtain

$$\Pi^2(dS) = (1 - SS^*)dS - S\Pi(\theta) \qquad \Pi^2(dS^*) = dS^*(1 - SS^*) + \Pi(\theta)S^* \tag{3.23}$$

which is the morphism associated to $\Pi(\theta)$. ∎

And if Π is a projector, then

$$\Pi\big[S^*dS + \theta\big] = 0 \tag{3.24}$$

the property interpetable as the verticality of the associated connection, corresponding to the first harmonic germ o_1 on $\bigcirc$. To produce analogs suitable for the higher harmonic germs, we should act by powers of ψ^- on this elementary connection:

$$\widehat{F}\psi^{-n}\big[S^*dS + \theta\big] = \psi^{-n}\big[S^*dS + \theta\big] \otimes 1 + 1 \otimes o_n. \tag{3.25}$$

Here, in the context of the universal envelope $\mho(\mathcal{T})$, this works only for $n > 0$ due to the crucial asymmetry between S and S^*, and the lack of the additional algebraic relations in the calculus.

An important associated map is a blueprint covariant derivative

$$D = \Pi d. \tag{3.26}$$

So that we can rewrite (3.20) as

$$D(S) = (1 - SS^*)dS - S\theta \qquad D(S^*) = dS^*(1 - SS^*) + \theta S^*. \tag{3.27}$$

These formulae completely determine D acting on $\mathcal{T}$ as a graded Π-derivation. We always have

$$D(SS^*) = d(SS^*). \tag{3.28}$$

In other words $d(SS^*)$ is Π-invariant always. Indeed,

$$\begin{aligned} \Pi[d(S)S^*] &= (1 - SS^*)d(S)S^* - S\theta S^* = d(SS^*)(SS^*) - S\theta S^* \\ \Pi[Sd(S^*)] &= S(1 - SS^*)d(S^*) + S\theta S^* = (SS^*)d(SS^*) + S\theta S^* \end{aligned} \tag{3.29}$$

and summing these two and applying the Leibniz rule we obtain the result.

Lemma 12. *The following identities hold*

$$D\psi(h) = \psi D(h) + d(SS^*)\psi(h) + (-)^{\partial h}\psi(h)d(SS^*) - \psi\Big\{\theta h - (-)^{\partial h}h\theta\Big\} \tag{3.30}$$

$$D\psi^-(h) = \psi^-\Big\{D(h) + d(SS^*)h + (-)^{\partial h}hd(SS^*) + \psi(\theta)h - (-)^{\partial h}h\psi(\theta)\Big\} \tag{3.31}$$

for Π*-invariant elements* $h \in \mho(\mathcal{T})$.

Proof. Obtained by acting by Π on (3.18) and (3.19) respectively, taking (3.29) into account, all coupled with elementary properties of ψ. ☐

Let us assume that the idempotency condition holds.

Lemma 13. *We have*

$$D^2(S) = -SR \qquad RS^* = D^2(S^*) \tag{3.32}$$

with the blueprint curvature 2*-form given by*

$$R = S^*d(SS^*)^2S + D(\theta) - \theta^2. \tag{3.33}$$

Proof. A straightforward computation

$$\begin{aligned} D^2(S) = D\big[(1 - SS^*)dS - S\theta\big] &= -d(SS^*)D(S) - D(S)\Pi(\theta) - SD(\theta) = \\ &= -d(SS^*)\big[(1 - SS^*)dS - S\theta\big] - \big[(1 - SS^*)dS - S\theta\big]\theta - SD(\theta) = \\ &= SS^*d(SS^*)dS - SD(\theta) + S\theta^2 = -S\Big\{S^*d(SS^*)^2S + D(\theta) - \theta^2\Big\} \end{aligned}$$

intertwined with the above elementary identities. The second formula follows symmetrically, and for antihermitian θ is simply the conjugate of the first. ☐

Lemma 14. *If h is* Π*-invariant then*

$$D^2\psi(h) = \psi[D^2(h)] - \psi[R, h]. \tag{3.34}$$

Proof. It follows from the previous Lemma and observing that on the horizontals D acts as a graded derivation, hence D^2 is a simple derivation. ☐

3.3. **Calculus Crafting**

We shall now consider a more subtle and natural calculus for the extension algebra, obtained by introducing additional commutation relations

$$S\,dS = dS\,S \qquad S^*\,dS^* = dS^*\,S^*. \tag{3.35}$$

Remark 12. Such commutation relations can always be promoted, in the context of Hilbert spaces of holomorphic functions in a domain of $\mathbb{C}$, where the noncommutative space is built from the complex variable multiplication operator and its adjoint. In our context, it is clear that the new relations are compatible with the natural $\mathrm{SU}(1,1)$ symmetry and the above introduced quantum group structure.

Remark 13. Since we are promoting the first-order relations, the resulting complete calculus is isomorphic to the universal differential envelope of the first-order calculus Ψ.

The infinitezimal generators for the $\mathrm{SU}(1,1)$ symmetry are given by the following hermitian derivations of the basic algebra:

$$\begin{aligned} h_1(S) &= (1-S^2)/2 & h_1(S^*) &= (1-S^{*2})/2 \\ h_2(S) &= i(1+S^2)/2 & h_2(S^*) &= -i(1+S^{*2})/2 \\ h_3(S) &= iS & h_3(S^*) &= -iS^* \end{aligned}$$

They satisfy

$$[h_1, h_2] = h_3 \qquad [h_2, h_3] = -h_1 \qquad [h_3, h_1] = -h_2$$

which is the standard presentation of the Lie algebra $\mathrm{su}(1,1)$. The derivations $X \in \mathrm{su}(1,1)$ naturally incorporate into the full calculus, via the operators ι_X and l_X as discussed in Appendix C.

Remark 14. In fact, the introduced derivations cover the whole Lie algebra of the calculus, in the sense that

$$\mathrm{su}(1,1) = \mathrm{Der}(\mathcal{T} \bigm| \Psi)$$

for our context.

Our new relations imply

$$dS^* = -S^{*2}dS \qquad dS = -dS^*\, S^2. \tag{3.36}$$

At the first order level, the calculus is generated by the triplet

$$dS \qquad w = S^*\, dS = -dS^*\, S = -w^* \qquad dS^* \tag{3.37}$$

in the sense that the elements of the form

$$S^k \begin{Bmatrix} dS \\ w \\ dS^* \end{Bmatrix} S^{*l} \qquad k, l \in \mathbb{N} \tag{3.38}$$

constitute a natural linear basis for the first-order calculus.

All the second-order relations for our calculus are generated from the first-order relations. We have

$$(dS)^2 = (dS^*)^2 = dw = dS^*\, dS = 0. \tag{3.39}$$

For the forms of grade $n \geq 2$ we have the following 4-tuple linear basis

$$S^k \begin{Bmatrix} dSw^{n-2}dS^* \\ w^n \\ dS\ w^{n-1} \\ w^{n-1}dS^* \end{Bmatrix} S^{*l} \qquad k, l \in \mathbb{N} \tag{3.40}$$

and in particular we see that the differential calculus is non-trivial in all grades. This fact also follows from the primitivity formulae

$$\phi(w) = w \otimes 1 + 1 \otimes w \qquad \phi(w^2) = w^2 \otimes 1 + 1 \otimes w^2. \tag{3.41}$$

Proposition 15. *The projector $p = SS^*$ satisfies*

$$dp\, p\, dp = dS\, dS^* - Sw^2S^* + dS\, wS^* - Sw\, dS^*. \tag{3.42}$$

In particular,

$$(dp)^3 = d(Sw^2S^*). \tag{3.43}$$

Proof. The first identity follows by a direct calculus, expanding the left-hand side to the canonical basis for second order forms. It is worth noticing that the last 2 terms compactify into $d(SwS^*)$. Now by taking the differential of both sides we arrive at (3.43). □

There exists a very special second-order element which trivially contracts with all infinitezimal hyperbolic symmetries.

Proposition 16. *We have*

$$\iota_X(w^2 + dp\, p\, dp) = 0 \tag{3.44}$$

for every $X \in \mathrm{su}(1,1)$.

Proof. A direct verification with X assuming the values of the elementary basis hyperbolic derivations h_1, h_2, h_3. □

Our calculus admits further natural factorization which preserves the $\mathrm{SU}(1,1)$ symmetry, and which closes the calculus at dimension 3. Interestingly, such a closure turns out to be impossible at dimension 2. The following additional relations describe this calculus:

$$w^2 + dp\, p\, dp = 0 \tag{3.45}$$

$$dp\, dp\, dp = 0. \tag{3.46}$$

Clearly, the second relation follows by differentiating the first.

Proposition 17. *In the factor calculus the following identities hold:*

$$dS\, w^2 = Sw^3 \qquad w^2 dS^* = -w^3 S^* \tag{3.47}$$

$$dS\, wdS^* = w^3 - Sw^3 S^*. \tag{3.48}$$

Proof. Taking into account (3.43) we find $dSw^2S^* + Sw^2dS^* = 0$ and by multiplying on the left and on the right by S^* and S respectively we arrive at (3.47), mutually conjugate identities. To prove (3.48) we multiply by w the basic quadratic identity, which when expanded gives

$$0 = w^3 + dS\, dS^* w - Sw^2 S^* w + dSwS^* w - SwdS^*\, w.$$

The second and the fifth terms of the right hand side clearly vanish, while $Sw^2S^*w = Sw^2S^{*2}dS = -Sw^2dS^* = Sw^3S^*$ and $dSwS^*w = -dSwdS^*$. □

We see that 3-forms are all expressible in terms of the canonical 'volume element' w^3. It is easy to see that w^4 vanishes. Indeed $w^4 = w^3S^*\, dS = -w^2\, dS^*\, dS = 0$. This further implies $dSw^3 = w^3\, dS^* = dS\, w^2\, dS^* = 0$. In other words, all 4-forms vanish.

Remark 15. It is worth observing that such a factorized algebra is not compatible with the coproduct ϕ. In other words, the quantum group structure is lost.

4. Cross Product for General $*$-Morphisms

4.1. **General Setup**

For the sake of internal harmony & completeness, we have collected here some general definitions regarding quantum principal bundles [5]. Let us consider a quantum group G represented by a *-algebra $\mathcal{A}$ equipped with the coproduct map $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Let us assume that G acts on the right on a quantum space P via a unital *-homomorphism $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$. This means that

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, equal] & \mathcal{B}\otimes\mathcal{A} \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ \mathcal{B} \arrow[r, "\cong"'] & \mathcal{B}\otimes\mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "F"'] & \mathcal{B}\otimes\mathcal{A} \arrow[d, "\mathrm{id}\otimes\phi"] \\ \mathcal{B}\otimes\mathcal{A} \arrow[r, "F\otimes\mathrm{id}"'] & \mathcal{B}\otimes\mathcal{A}\otimes\mathcal{A} \end{tikzcd} \tag{4.1}$$

are commutative diagrams. Let $\mathcal{F}$ be the set of all elements $a \in \mathcal{A}$ for which there exists a collection of elements $b_\alpha, q_\alpha \in \mathcal{B}$ such that

$$\sum_\alpha b_\alpha F(q_\alpha) = 1 \otimes a. \tag{4.2}$$

Proposition 18. *The set $\mathcal{F}$ is a unital subalgebra of $\mathcal{A}$. It is always $*k$–invariant and moreover*

$$\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes \mathcal{A}. \tag{4.3}$$

Proof. By construction $\mathcal{F}$ is a linear space containing the scalars of $\mathcal{A}$. Another direct consequence of the definition is that $\mathcal{F}$ is closed under multiplications.

From the above definition it follows that

$$\sum_\alpha b_\alpha q_\alpha^{(0)} \otimes q_\alpha^{(1)} \otimes q_\alpha^{(2)} = 1 \otimes a^{(1)} \otimes a^{(2)}$$

which implies the right invariance of $\mathcal{F}$. On the other hand we always have

$$\sum_\alpha q_\alpha^* F(b_\alpha^*) = 1 \otimes \kappa(a)^*. \tag{4.4}$$

which shows that $\mathcal{F}$ is $*\kappa$-invariant. $\square$

It is also worth noticing that

$$\sum_\alpha b_\alpha q_\alpha = 1\epsilon(a) \tag{4.5}$$

which is another straightforward consequence of (4.2).

Definition 1. We shall call $\mathcal{F}$ the *free action algebra* for $\mathcal{A}$ & F. We shall say that F is a *free action* of G on P iff the *-algebra generated by $\mathcal{F}$ in $\mathcal{A}$ is the whole $\mathcal{A}$. Such a structuralized $P \leftrightarrow (\mathcal{B}, \iota, F)$ is called a *quantum principal G-bundle* over M.

4.2. **The Level of Bundles**

We shall assume that there is a left inverse map $\psi^-\colon \mathcal{V}\to\mathcal{V}$ satisfying

$$\psi^-\psi(f)=f \qquad \psi\psi^-(f)=\psi(1)f\psi(1) \tag{4.6}$$

for all $f\in\mathcal{V}$. The first equation above ensures that ψ is always injective. And ψ will be bijective precisely when $\psi(1)=1$. So the non-bijective ψ are possible only in infinite-dimensional contexts.

Proposition 19. *A left inverse map for ψ is unique, if it exists. It is linear, unital and *-preserving. In addition, if $\mathcal{V}$ is local then ψ^- is completely positive.*

Proof. The uniqueness of the map follows from the second equation in (4.6), coupled with the injectivity of ψ. In resonance with this

$$\begin{aligned}\psi\psi^-(\lambda f+g)=\psi(1)(\lambda f+g)\psi(1)=\lambda\psi(1)f\psi(1)+\psi(1)g\psi(1)=\\ =\lambda\psi\psi^-(f)+\psi\psi^-(g)=\psi\bigl\{\lambda\psi^-(f)+\psi^-(g)\bigr\}\end{aligned}$$

and $\psi\psi^-(1)=\psi(1)^2=\psi(1)$ as well as $\psi\psi^-(f^*)=\psi(1)f^*\psi(1)=[\psi\psi^-(f)]^*=\psi[\psi^-(f)^*]$. This proves the linearity, unitality and hermicity. If $\mathcal{V}$ is local, then the second equation of (4.6) ensures the complete positivity of $\psi\psi^-$. Since ψ is a *-monomorphism, we can safely remove it from the composition and conclude that ψ^- is completely positive as well. $\square$

For the non-bijective ψ there will be a strictly decreasing sequence of projectors

$$p_k=\psi^k(1) \tag{4.7}$$

for $k\geq 0$ with $p_0=1$.

We shall now extend the $\mathcal{V}$ to a larger *-algebra $\mathcal{B}$, by introducing a new generator $\mathfrak{u}$ satisfying

$$\mathfrak{u}^*\mathfrak{u}=1 \qquad \psi(f)=\mathfrak{u}f\mathfrak{u}^* \tag{4.8}$$

for all $f\in\mathcal{V}$. These relations are completely friendly to $\mathcal{V}$, the algebra is preserved and thus there is the canonical unital embedding $\iota\colon\mathcal{V}\rightarrowtail\mathcal{B}$. As an immediate consequence of the above generating relations, we can write

$$\mathfrak{u}f=\psi(f)\mathfrak{u} \qquad f\mathfrak{u}^*=\mathfrak{u}^*\psi(f) \qquad \psi^-(f)=\mathfrak{u}^*f\mathfrak{u}. \tag{4.9}$$

The assignments $f\rightsquigarrow f\otimes 1$ and $\mathfrak{u}\rightsquigarrow\mathfrak{u}\otimes U$ consistently and uniquely define a unital *-homomorphism $F\colon\mathcal{B}\to\mathcal{B}\otimes\mathcal{A}$, which is the action of $\bigcirc$ on P. Thus, $\mathcal{B}$ decomposes into a direct sum of the multiple irreducible spaces $\mathcal{B}_k$, where $k\in\mathbb{Z}$.

It is a fun to calculate in $\mathcal{B}$. By recursively applying these no-commutation relations it follows that

Proposition 20. *If $k>0$ then every element of $\mathcal{B}_k$ is of the form $f\mathfrak{u}^k$ and every element of $\mathcal{B}_{-k}$ is of the form $\mathfrak{u}^{*k}f$ where $f\in\mathcal{V}$. And $\mathcal{V}=\mathcal{B}_0$ is the fixed-point algebra for F. The triplet (P,ι,F) is a quantum principal $\bigcirc$-bundle over M.* $\square$

5. Standard Hyperbolic Hopf Fibration

Let us recall that the classical group $\mathrm{U}(1,1)$ is formed by complex 2×2 matrices T satisfying

$$T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}T^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}T.$$

The unitarity condition for the signature $(1,1)$ semidefinite scalar product. Its unimodular subgroup can be alternatively described as

$$\mathrm{SU}(1,1) = \left\{ T = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\gamma|^2 = 1 \right\}.$$

It is important to observe that if we assume the above form of a matrix

$$T = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$$

then the unitarity condition is equivalent to the determinant equation

$$|\alpha|^2 - |\gamma|^2 = \det(T) = 1.$$

This is the starting point for constructing the quantum version of $\mathrm{SU}(1,1)$ by Woronowicz. It is based on a *-algebra $\mathfrak{B}$ generated by the entries of a matrix

$$u = \begin{pmatrix} \alpha & \mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

where we postulate the same signature $(1,1)$ unitarity property for u. Here $\mu \in [-1,1]\setminus\{0\}$. This implies the following commutation relations for α and γ:

$$\begin{gathered} \alpha\alpha^* - \mu^2\gamma\gamma^* = 1, \quad \alpha^*\alpha - \gamma^*\gamma = 1, \\ \gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma^* = \mu\gamma^*\alpha, \quad \alpha\gamma = \mu\gamma\alpha. \end{gathered} \tag{5.1}$$

In such a way we obtain a Hopf *-algebra, with the coproduct $\phi : \mathfrak{B} \to \mathfrak{B}\otimes\mathfrak{B}$, the counit $\epsilon : \mathfrak{B} \to \mathbb{C}$ and the antipode $\kappa : \mathfrak{B} \to \mathfrak{B}$ given by

$$\phi(u_{ij}) = \sum_{k=1}^{2} u_{ik}\otimes u_{kj} \qquad \epsilon(u_{ij}) = \delta_{ij} \tag{5.2}$$

$$\kappa(u_{ij}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}_{ij} \tag{5.3}$$

or explicitly

$$\phi(\alpha) = \alpha\otimes\alpha + \mu\gamma^*\otimes\gamma \qquad \phi(\alpha^*) = \alpha^*\otimes\alpha^* + \mu\gamma\otimes\gamma^* \tag{5.4}$$

$$\phi(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma \qquad \phi(\gamma^*) = \gamma^*\otimes\alpha^* + \alpha\otimes\gamma^* \tag{5.5}$$

$$\epsilon(\alpha) = \epsilon(\alpha^*) = 1 \qquad \epsilon(\gamma) = \epsilon(\gamma^*) = 0 \tag{5.6}$$

$$\kappa(\alpha) = \alpha^* \qquad \kappa(\alpha^*) = \alpha \qquad \kappa(\gamma) = -\mu\gamma \qquad \mu\kappa(\gamma^*) = -\gamma^*. \tag{5.7}$$

We shall now introduce different coordinates, which classically correspond to the hyperbolic plane complex coordinates (in unit disc, the Poincaré model) and a unitary transformation–the pure rotation around the origin. And as it will become transparent, this will enable us to interpret the entire algebra as a cross product between the hyperbolic plane and the circle.

From now on, we shall assume that the element α is invertible. In addition, we shall assume that an elementary functional calculus is allowed in $\mathcal{B}$. Let us define

$$z = \alpha^{*-1}\gamma^* \qquad w = \frac{1}{\sqrt{\alpha\alpha^*}}\alpha. \tag{5.8}$$

It is clear that w is unitary, that is $w^* = w^{-1}$. On the other hand, we have

$$\alpha\alpha^* = \frac{1}{1 - zz^*} \qquad \alpha^*\alpha = \frac{1}{1 - z^*z}. \tag{5.9}$$

The elements zz^* and z^*z are mutually related in the following way:

$$zz^* = \frac{z^*z}{\mu^{-2} + (1 - \mu^{-2})z^*z}. \tag{5.10}$$

Let $\mathcal{V}$ be the *-subalgebra of $\mathcal{B}$ generated by z and z^*. This subalgebra can be defined as the invariant subalgebra under a natural free action of the circle group $\mathcal{B}$. Let $\mathcal{A}$ be the Hopf *-algebra corresponding to the circle group, with its canonical generator U. There is a natural Hopf *-epimorphism $\Pi\colon \mathcal{B} \to \mathcal{A}$ defined by $\Pi(\alpha) = U$, $\Pi(\alpha^*) = U^*$ and $\Pi(\gamma) = \Pi(\gamma^*) = 0$.

Remark 16. In fact, this enables us to see, for $\mu \neq 1$ the circle group as precisely the *classical part* of the quantum $\mathrm{SU}(1,1)$.

If we define $F\colon \mathcal{B} \to \mathcal{B}\otimes\mathcal{A}$ as $F = (\mathrm{id}\otimes\Pi)\phi$, then it is easy to see that the triplet $P = (\mathcal{B}, \iota, F)$, with $\iota\colon \mathcal{V} \to \mathcal{B}$ the inclusion map, is a quantum principal bundle [5]. Moreover, a direct calculation reveals that

$$F(w) = w \otimes U \qquad zz^* = w\{z^*z\}w^*. \tag{5.11}$$

Let $\psi\colon \mathcal{B} \to \mathcal{B}$ be the inner *-automorphism generated by w. In other words $\psi(b) = wbw^*$ for every $b \in \mathcal{B}$. It follows immediately that ψ commutes with the action F, and in particular $\mathcal{V}$ is ψ-invariant.

Let $\mathcal{R}$ be the *-subalgebra of $\mathcal{V}$ generated by zz^*, or equivalently z^*z. The above equation (5.10) implies that $\mathcal{R}$ is *commutative*. The same equation can be rewritten in yet another form

$$zz^* = \psi(z^*z) \tag{5.12}$$

and we see that $\mathcal{R}$ is ψ-invariant.

The algebra $\mathcal{B}$ can be viewed as the cross product between $\mathcal{V}$, equipped with the restricted automorphism $\psi\colon \mathcal{V} \to \mathcal{V}$, and the circle group. The original generators α and γ are recovered as

$$\alpha = \frac{1}{\sqrt{1 - zz^*}}w \qquad \gamma = \frac{1}{\sqrt{1 - z^*z}}z^*w.$$

Let us now focus on the differential calculus, and assume that the quantum $\mathrm{SU}(1,1)$ is equipped with the 3D calculus of Woronowicz. By definition, this calculus Ψ is a left-covariant *-calculus, defined by the following right $\mathcal{B}$-ideal:

$$\mathcal{J} = \left\langle \alpha^* + \mu^2\alpha - (1+\mu^2)1, \gamma^2, \gamma^*\gamma, \gamma^{*2}, (\alpha - 1)\gamma, (\alpha - 1)\gamma^* \right\rangle. \tag{5.13}$$

The space Ψ_{inv} of left invariant elements is 3-dimensional, and there is a canonical basis in this space, given by

$$\eta_+ = \pi(\gamma) \qquad \eta_- = \mu\pi(\gamma^*) \qquad \eta_3 = \pi(\alpha - \alpha^*) \tag{5.14}$$

where $\pi\colon \mathcal{B}\to\Psi_{inv}$ is the associated quantum germs map $\pi(b)=\kappa(b^{(1)})db^{(2)}$. The *-structure is given by

$$\eta_3^*=-\eta_3 \qquad \eta_\pm^*=\eta_\mp. \tag{5.15}$$

The canonical $\circ$-structure is given by

$$\begin{aligned} \eta_\pm\circ\alpha&=\mu^{-1}\eta_\pm & \eta_3\circ\alpha&=\mu^{-2}\eta_3\\ \eta_\pm\circ\alpha^*&=\mu\eta_\pm & \eta_3\circ\alpha^*&=\mu^2\eta_3\\ \eta_\pm\circ\{\gamma,\gamma^*\}&=0 & \eta_3\circ\{\gamma,\gamma^*\}&=0. \end{aligned}\tag{5.16}$$

Therefore, the following commutation relations hold

$$\eta_3\alpha=\mu^{-2}\alpha\eta_3 \qquad \eta_3\alpha^*=\mu^2\alpha^*\eta_3 \qquad \eta_3\gamma=\mu^{-2}\gamma\eta_3 \qquad \eta_3\gamma^*=\mu^2\gamma^*\eta_3 \tag{5.17}$$

$$\eta_\pm\alpha=\mu^{-1}\alpha\eta_\pm \quad \eta_\pm\alpha^*=\mu\alpha^*\eta_\pm \qquad \eta_\pm\gamma=\mu^{-1}\gamma\eta_\pm \quad \eta_\pm\gamma^*=\mu\gamma^*\eta_\pm \tag{5.18}$$

$$\eta_3z=z\eta_3 \qquad \eta_\pm z=z\eta_\pm \qquad \eta_3w=\mu^{-2}w\eta_3 \qquad \eta_\pm w=\mu^{-1}w\eta_\pm. \tag{5.19}$$

The calculus on the base is generated by $\mathcal{V}$. The following defines the differential on complex coordinates:

$$dz\ =\sqrt{1-zz^*}\,w\eta_-w\,\sqrt{1-z^*z}=\mu^{-1}\sqrt{1-zz^*}\,w^2\,\sqrt{1-z^*z}\eta_- \tag{5.20}$$

$$dz^*=\sqrt{1-z^*z}\,w^*\eta_+w^*\,\sqrt{1-zz^*}=\mu\sqrt{1-z^*z}w^{*2}\sqrt{1-zz^*}\eta_+. \tag{5.21}$$

This immediately tells us what are the relations between the coordinates z and z^* and their differentials:

$$\begin{gathered} dz^*\ z=\frac{\mu^2}{\mu^4+(1-\mu^4)zz^*}zdz^* \qquad dz\ z^*=\frac{\mu^{-2}}{\mu^{-4}+(1-\mu^{-4})z^*z}z^*dz\\ dz\ z=\mu^2z\ dz \qquad dz^*\ z^*=\mu^{-2}z^*dz\\ dz^*\ (zz^*)=\frac{zz^*}{\mu^4+(1-\mu^4)zz^*}dz^* \qquad dz\ (z^*z)=\frac{z^*z}{\mu^{-4}+(1-\mu^{-4})z^*z}dz. \end{gathered}$$

It is also worth observing that, in our context

$$\frac{z^*z}{\mu^{-4}+(1-\mu^{-4})z^*z}=\psi^2(z^*z)$$

which implies the following commutation relations between the coordinate differentials and the elements of $\mathcal{R}$:

$$dz\,h=\psi^2(h)dz \qquad dz^*h=\psi^{-2}(h)dz^*.$$

for every $h\in\mathcal{R}$.

The higher-order relations between the coordinate forms are given by

$$\eta_+\eta_-=-\mu^2\eta_-\eta_+ \quad \eta_+\eta_3=-\mu^4\eta_3\eta_+ \quad \eta_3\eta_-=-\mu^4\eta_-\eta_3 \tag{5.22}$$

$$\eta_+^2=\eta_-^2=\eta_3^2=0. \tag{5.23}$$

Here we used the universal differential envelope [4] of the first-order calculus, as the complete calculus on the bundle. By differentiating the germs-coordinate forms, we obtain

$$d\eta_3=-(1+\mu^2)\eta_-\eta_+ \qquad d\eta_+=\mu^2\eta_3\eta_+ \qquad d\eta_-=\mu^2\eta_-\eta_3. \tag{5.24}$$

The map $F\colon\mathcal{B}\to\mathcal{B}\otimes\mathcal{A}$ extends to a morphism $\widehat{F}\colon\Psi^\wedge\to\Psi^\wedge\mathbin{\widehat{\otimes}}\Gamma^\wedge$ of graded differential *-algebras. Here Γ is the projected calculus on the circle, via the differential extension $\Pi^\wedge\colon\Psi^\wedge\to\Gamma^\wedge$ of Π. In such a way, the circle acquires a nice quantum differential calculus: dU and U do not commute. This calculus is given by a unique generator χ, such that

$$dU = \frac{1}{1+\mu^2} U\chi.$$

The extended action is completely determined by

$$\widehat{F}(\eta_3) = \eta_3 \otimes 1 + 1 \otimes \chi \qquad \widehat{F}(\eta_+) = \eta_+ \otimes U^2 \qquad \widehat{F}(\eta_-) = \eta_- \otimes U^{-2}. \tag{5.25}$$

We see that the algebra $\mathfrak{hor}(P)$ of horizontal forms on the bundle, is generated by $\mathcal{B} = \mathfrak{hor}^0(P)$ and $\eta_\pm$. It terminates with the second-order forms. The calculus on the base is generated by $\mathcal{V} = \Omega^0(M)$, then $w^2\eta_-$ and $\eta_+ w^{*2}$ which span $\Omega^1(M)$, and $\eta_+\eta_-$, interpretable as the surface form on M.

There is a canonical connection $\omega : \Gamma_{inv} \to \Omega(P)$ on this bundle, defined by

$$\omega(\chi) = \eta_3.$$

This connection turns out to be regular. Its curvature $R_\omega \colon \Gamma_{inv} \to \mathfrak{hor}^2(P)$ turns out to be equal

$$R_\omega(\chi) = -(1+\mu^2)\eta_-\eta_+. \tag{5.26}$$

The corresponding covariant derivative $D = D_\omega$ is a hermitian antiderivation on $\mathfrak{hor}(P)$. It extends the differential on $\Omega(M)$, and possesses the property

$$D(\eta_+) = D(\eta_-) = 0. \tag{5.27}$$

Remark 17. The forms $\eta_\pm$ can be interpreted as canonical *coordinate forms* on M. For an arbitrary connection, their covariant derivative is interpretable as the *torsion.* Our canonical connection, in fact, generalizes the Levi-Civita connection on the hyperbolic plane.

6. Basic Non-commutative Hyperbolic Plane

Let $\mathcal{R}$ be a commutative *-algebra, generated by a single positive element $\varrho = \varrho^*$ and equipped by a *-automorphism $\psi \colon \mathcal{R} \to \mathcal{R}$. We shall think of ϱ as an abstract 'radial coordinate' and assume that its spectrum $\sigma(\varrho)$ is within $[0,1]$. We shall also assume that $\mathcal{R}$ admits an elementary functional calculus, as it will be clear from the context of our calculations. This means, in particular, that all the elements of $\mathcal{R}$ are interpretable as certain functions of ϱ, defined on its spectrum $\sigma(\varrho)$.

Let $\mathcal{U}$ be the cross product of $\mathcal{R}$ and the circle algebra, relative to ψ. In other words, we extend $\mathcal{U}$ by introducing an abstract unitary element ϖ, such that

$$\varpi h \varpi^* = \psi(h) \tag{6.1}$$

for every $h \in \mathcal{U}$. Clearly, it is sufficient to postulate this for the single generator $h = \varrho$. We can clearly extend ψ to the whole of $\mathcal{U}$.

Let us consider the following new 'complex coordinates'

$$z = \varrho\varpi \qquad z^* = \varpi^*\varrho. \tag{6.2}$$

It is clear that

$$zz^* = \varrho^2 = \psi(z^*z) = f(z^*z) \tag{6.3}$$

where f is a real function acting in $[0,1]$. From now on, we shall assume that this function is a homeomorphism of $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$. In addition, we shall assume that there exists a (necessarily unique) continuous positive function $q\colon [0,1] \to \mathbb{R}$ satisfying

$$f(t) = tq(t)^2 \qquad \forall t \in [0,1]. \tag{6.4}$$

Remark 18. This will be the case, of course, if the function f is analytic, or more generally, if it is differentiable at $t = 0$.

Let $\mathcal{V}$ be the unital *-subalgebra of $\mathcal{U}$ generated by z and z^*. Assuming the same stability under functional calculus as $\mathcal{R}$, it follows that $\mathcal{R} \subseteq \mathcal{V}$. We can equivalently define $\mathcal{V}$ as the minimal subalgebra containing $\mathcal{R}$ and coordinates z and z^*.

Lemma 21. *We have $\psi(\mathcal{V}) = \mathcal{V}$. Explicitly,*

$$\psi(z) = q(zz^*)z = zq(z^*z) \qquad \psi(z^*) = q(z^*z)z^* = z^*q(zz^*). \tag{6.5}$$

Proof. By definition of the coordinate z, we have $\psi(z) = \psi(\varrho\varpi) = \psi(\varrho)\varpi$. On the other hand

$$\varrho^2 = f(z^*z) = z^*zq^2(z^*z) = z^*q^2(zz^*)z = \varpi^*\varrho q^2(\varrho^2)\varrho\varpi = \psi^{-1}[f(\varrho^2)]$$

which implies, after applying ψ and taking square root $\psi(\varrho) = \varrho q(\varrho^2)$. So, we conclude that $\psi(z) = q(\varrho^2)\varrho\varpi = q(zz^*)z$. □

In fact, we can construct an entire group of symmetries, by slightly generalizing the above procedure and considering homeomorphisms $g\colon [0,1] \to [0,1]$ which commute with the initial f. By definition, each one of them induces a *-automorphism ψ_g of $\mathcal{U}$, by trivially acting on ϖ (so that $\psi = \psi_f$). Assuming the same regularity condition $g(t) = ts(t)^2$ for g, we obtain $\psi_g(\varrho) = (\psi_g(\varrho^2))^{1/2} = (g(\varrho^2))^{1/2} = \varrho s(\varrho^2)$ and it follows that $\psi_g(z) = \psi_g(\varrho)\varpi = s(\varrho^2)z = s(zz^*)z$.

We can now construct the differential calculus, by following the general procedure outlined in Section 2. The simplest configuration is to start with the classical calculus on $\mathcal{R}$. Let $\mathcal{R}_\wedge$ be this classical calculus. It is defined by a single relation $\varrho d(\varrho) = d(\varrho)\varrho$. Our automorphism extends to $\psi\colon \mathcal{R}_\wedge \to \mathcal{R}_\wedge$. From the differential cross product construction, we obtain a graded-differential *-algebra $\mathcal{U}_\wedge$. It is given by the relations:

$$\varpi d(\varpi) = d(\varpi)\varpi \qquad \varpi\lambda = \psi(\lambda)\varpi \qquad d(\varpi)\lambda = (-)^{\partial\lambda}\psi(\lambda)d(\varpi) \tag{6.6}$$

where $\lambda \in \mathcal{R}_\wedge$. We can now define $\Omega(M)$ to be the differential *-subalgebra generated by z and z^*. By definition $\mathcal{V} = \Omega^0(M)$ and the extended automorphism preserves $\Omega(M)$.

Lemma 22. *The following commutation relations hold:*

$$\frac{1}{z}d_M(z) = \psi^{-1}\big\{d_M(z)\frac{1}{z}\big\} \qquad z^*d_M(z) = \psi^{-1}\big\{d_M(z)z^*\big\} \tag{6.7}$$

$$\frac{1}{z^*}d_M(z^*) = \psi\big\{d_M(z^*)\frac{1}{z^*}\big\} \qquad zd_M(z^*) = \psi\big\{d_M(z^*)z\big\}. \tag{6.8}$$

Proof. It is worth remembering that, by construction, $\varpi^*d\varpi$ supercommutes with everything. We have

$$\frac{1}{z}dz = \varpi^*\frac{1}{\varrho}(d\varrho\,\varpi + \varrho\,d\varpi) = \psi^{-1}\big(\frac{1}{\varrho}d\varrho\big) + \varpi^*d\varpi = \psi^{-1}\big(\frac{1}{\varrho}d\varrho + \varpi^*d\varpi\big) =$$

$$= \psi^{-1}\big(d\varrho\frac{1}{\varrho} + d\varpi\varpi^*\big) = \psi^{-1}\big(dz\frac{1}{z}\big).$$

Quite similarly,

$$z^*dz = \varpi^*\varrho(d\varrho\,\varpi + \varrho\,d\varpi) = \psi^{-1}\bigl(\varrho d\varrho\bigr) + \varrho^2\varpi^*d\varpi = \psi^{-1}\bigl(\varrho d\varrho + \varpi^*d\varpi\bigr) =$$
$$= \psi^{-1}\bigl(d\varrho\,\varrho + \varrho d\varpi\,\varpi^*\varrho\bigr) = \psi^{-1}\bigl(dz\,z^*\bigr).$$

This proves the first pair of relations. The second one is simply the conjugate of the first. □

Appendix A. Canonical Hilbert Space Realizations

A.1. **Context For Automorphisms**

We shall consider here the basic Hilbert space realizations for the C*-algebra generated by the functional relation of the form

$$\mathfrak{z}^*\mathfrak{z} = f(\mathfrak{z}\mathfrak{z}^*),$$

where $f\colon [0,1] \to [0,1]$ is an analytical bijection such that $f(0) = 0$ and $f(1) = 1$. It is worth recalling that our main example for this is the hyperbolic quantum Hopf fibration of $\mathrm{S}_\mu\mathrm{U}(1,1)$ over the quantum hyperbolic plane, the function f is a hyperbolic Möbius tranformation having 0 and 1 as fixed points and preserving the interval $(0,1)$. It is given by

$$f(z) = \frac{z}{q + (1-q)z} \qquad q = \mu^2 \tag{A.1}$$

which can be rewritten as

$$\frac{1}{1 - f(z)} = \frac{1}{q}\frac{1}{1-z} + 1 - \frac{1}{q}. \tag{A.2}$$

A primary class of representations is given by considering $\mathrm{l}^2(\mathbb{Z})$ with its canonical orthonormal basis $\{\cdots|-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle\cdots\}$ and postulating the action of the form

$$\mathfrak{z}|n\rangle = \lambda_n^{1/2}|n+1\rangle \qquad \mathfrak{z}^*|n\rangle = \lambda_{n-1}^{1/2}|n-1\rangle$$

in such a way that the basic functional relation be satisfied. We shall assume here that these numbers are non-negative, this in fact can be understood as a part of the definition of the kets. Our operators are thus a special case of *double sided weighted shift operators*. Let us note that

$$\mathfrak{z}^*\mathfrak{z}|n\rangle = \lambda_n|n\rangle \qquad \mathfrak{z}\mathfrak{z}^*|n\rangle = \lambda_{n-1}|n\rangle$$

holds for every $n \in \mathbb{Z}$ and because of the basic functional relation we have

$$\lambda_{n+1} = f(\lambda_n).$$

In other words the numbers $\{\cdots\lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2\cdots\}$ are one *entire orbit* of the action of f in $[0,1]$. And since 0 and 1 are fixed points for f, it is wise to exclude them so we can assume that $\lambda_n \in (0,1)$ for every $n \in \mathbb{Z}$.

The map f could have some fixed points inside $(0,1)$. However, if $t \in (0,1)$ is not a fixed point, then all the elements $f^n(t)$ for $n \in \mathbb{Z}$ are mutually different and form a monotonic sequence. Indeed, if $t < f(t)$ then $f^n(t) < f^{n+1}(t)$ and the sequence is increasing, and similarly if $t > f(t)$ then $f^n(t) > f^{n+1}(t)$ and the sequence will be decreasing. In both cases the limiting points for $f^n(t)$ when $n \to \pm\infty$ will be fixed points for f. Interchanging f and f^{-1} switches between decreasing and increasing modes.

And no orbit for f can jump over a fixed point. In particular, when 0 and 1 are the unique fixed points for f then every orbit in $(0,1)$ is infinite with 0 and 1 as its limit points. Furthermore, all orbits are either increasing or decreasing.

If we fix $|0\rangle$ as a reference vector then successive actions of $\mathfrak{z}$ and $\mathfrak{z}^*$ generate the whole basis:

$$|k\rangle = \frac{\mathfrak{z}^k|0\rangle}{\sqrt{\lambda_0\lambda_1\cdots\lambda_{k-1}}} \qquad |-k\rangle = \frac{\mathfrak{z}^{*k}|0\rangle}{\sqrt{\lambda_{-1}\cdots\lambda_{-k}}}$$

where $k\in\mathbb{N}$.

In such a way every infinite orbit of f, is a natural birthplace for an infinite-dimensional representation of the quantum hyperbolic plane algebra.

Proposition 23. *For the infinite orbits, the constructed infinite-dimensional representation is irreducible.* □

Our next step is to construct a realization of the representation Hilbert spaces by analytic functions. Let us focus on the simplest scenario, when the transformation f is such that 0 and 1 are its unique fixed points, and the orbits are increasing. In this case, as we shall now explain, the natural common domain for the corresponding analytic functions is given by the interior of the unit disc without 0.

Proposition 24. *The identifications*

$$|k\rangle \leftrightsquigarrow \frac{z^k}{\sqrt{\lambda_0\lambda_1\cdots\lambda_{k-1}}} \qquad |-k\rangle \leftrightsquigarrow \frac{\sqrt{\lambda_{-1}\cdots\lambda_{-k}}}{z^k} \tag{A.3}$$

where $k\in\mathbb{N}$ induce the realization of the whole Hilbert space $\mathcal{H}$ by analytic functions in $\mathbb{D}-\{0\}$. In terms of this realization, the quantum complex variable $\mathfrak{z}$ acts simply as the multiplication operator by the coordinate z.

Proof. The elements of the Hilbert space are formally representable as

$$c_0+\sum_{k>0}\Bigl\{\frac{c_kz^k}{\sqrt{\lambda_0\lambda_1\cdots\lambda_{k-1}}}+c_{-k}\sqrt{\lambda_{-1}\cdots\lambda_{-k}}\frac{1}{z^k}\Bigr\} \qquad \sum_{k\in\mathbb{Z}}|c_k|^2<\infty.$$

The Cauchy-Hadamard formula gives for the outer and inner radii of convergence of this Laurent series

$$\frac{1}{R}=\overline{\lim}\,\Bigl(|c_k|/\sqrt{\prod_{j=0}^{k-1}\lambda_j}\Bigr)^{1/k}=\overline{\lim}\,|c_k|^{1/k}/\Bigl(\sqrt{\prod_{j=0}^{k-1}\lambda_j}\Bigr)^{1/k}=\overline{\lim}\,|c_k|^{1/k}\leq 1$$

$$\frac{1}{r}=\overline{\lim}\,\bigl(|c_{-k}|\sqrt{\lambda_{-1}\cdots\lambda_{-k}}\bigr)^{1/k}=\overline{\lim}\,|c_{-k}|^{1/k}\,\lim\bigl(\sqrt{\lambda_{-1}\cdots\lambda_{-k}}\bigr)^{1/k}=0$$

which shows that all the elements of $\mathcal{H}$, viewed as Laurent series, converge normally in their common domain $\mathbb{D}-\{0\}$. □

We see that $|0\rangle \leftrightsquigarrow 1$. Let us also define the numbers

$$\Lambda_n=\prod_{k\geq n}\lambda_k. \tag{A.4}$$

Then (A.3) simplifies into

$$|n\rangle \leftrightsquigarrow \sqrt{\Lambda_n/\Lambda_0}\,z^n \tag{A.5}$$

the expression now valid for all $n \in \mathbb{Z}$. The numbers Λ_n form a monotonic sequence on $\mathbb{Z}$ with 0 and 1 as the limiting points for the sequence. Clearly $\lambda_n = \Lambda_n : \Lambda_{n+1}$. The powers of z are mutually orthogonal with

$$\langle z^n | z^n \rangle = \Lambda_0 / \Lambda_n. \tag{A.6}$$

It follows that the reproducing kernel for this Hilbert space is given by

$$K(\bar{w}, z) = \Lambda_0^{-1} \sum_{k \in \mathbb{Z}} \Lambda_k (\bar{w} z)^k. \tag{A.7}$$

Remark 19. This is a special case of the situation when the kernel is obtained by composing the product $\bar{w}z$ with a single-argument holomorphic function:

$$K(\bar{w}, z) = \Bbbk(\bar{w} z). \tag{A.8}$$

We shall call $\Bbbk$ the *kernel generator*. All our examples will be of this form. Another important class of spaces, arising when we switch from $\mathbb{D}$ to the upper half-plane $\Im(z) > 0$ is of a similar form with $\bar{w}z$ replaced by $\bar{w} - z$. For example

$$K(\bar{w}, z) = \frac{\sin(a\bar{w} - az)}{\pi(\bar{w} - z)} \qquad a > 0$$

for classical Paley-Wiener spaces. The canonical Möbius transformation

$$z \mapsto i \frac{i + z}{i - z} \qquad\qquad \infty \mapsto -i \mapsto 0 \mapsto i \mapsto \infty$$

establishing a holomorphic equivalence between the upper half plane and the open unit disk, subtly intertwines these two types.

By fixing w and letting z free, we obtain the coherent wave functions $|w\rangle$ associated to w. In terms of the initial kets

$$|w\rangle = \Lambda_0^{-1/2} \sum_{k \in \mathbb{Z}} \sqrt{\Lambda_k}\, \bar{w}^k |k\rangle. \tag{A.9}$$

These vectors span a dense lineal in the space $\mathcal{H}$ and satisfy

$$\mathfrak{z}^* |w\rangle = \bar{w} |w\rangle. \tag{A.10}$$

In particular we see that the spectra of $\mathfrak{z}$ and $\mathfrak{z}^*$ fill the whole $\overline{\mathbb{D}}$ – as they should. It is worth recalling that

$$\langle u | v \rangle = K(\bar{v}, u)$$

always holds, this is a general identity for Hilbert spaces of analytic functions.

Remark 20. About the notation. We put numbers as kets identificators in two different contexts here. To define the initial orthonormal basis in $\mathcal{H} \cong \mathrm{l}^2(\mathbb{Z})$ and to label the coherent wave functions associated to the points $w \in \mathbb{D} - \{0\}$. Luckily, there is no ambiguity: these two labeling sets are disjoint.

Whenever we have a Hilbert space of analytic functions over some domain Ω, we can naturally associate the coherent states to the points of Ω.

$$w \leftrightsquigarrow \frac{|w\rangle\langle w|}{\langle w | w \rangle}. \tag{A.11}$$

The coherent states are viewed as 'quantum stochastic points' in this interpretation. The metrics of the Hilbert-Schmidt operators in $\mathcal{H}$ induces a metrics on Ω.

Proposition 25. *The induced metrics is given by*

$$ds^2 = \Big\{ \frac{\partial^2}{\partial \bar{w}\, \partial w} \log K(\bar{w}, w) \Big\} \delta\bar{w}\, \delta w. \tag{A.12}$$

Proof. The distance, as a quadratic form over the infinitezimal displacements, is

$$\mathrm{Tr}\Big\{ \Big(\frac{|w+\delta w\rangle\langle w+\delta w|}{\langle w+\delta w|w+\delta w\rangle} - \frac{|w\rangle\langle w|}{\langle w|w\rangle} \Big)^2 \Big\} \sim ds^2 =$$

$$= \Big\{ K(\bar{w},w) \frac{\partial^2}{\partial\bar{w}\,\partial w} K(\bar{w},w) - \frac{\partial}{\partial\bar{w}} K(\bar{w},w)\, \frac{\partial}{\partial w} K(\bar{w},w) \Big\} \frac{\delta\bar{w}\,\delta w}{K(\bar{w},w)^2}$$

where we have discregarded the higher-order terms in $\delta w, \delta\bar{w}$. This is elementary transformed into (A.12). □

Remark 21. This is a universal formula, valid for arbitrary Hilbert spaces of analytic functions. The quantum mechanical interpretation is that the square of the distance element is proportional to the probability of non-transitioning from one coherent state to another. Indeed if $\varphi, \chi \in \mathcal{H}$ are unit vectors then

$$\mathrm{Tr}\Big\{ \big(|\varphi\rangle\langle\varphi| - |\chi\rangle\langle\chi| \big)^2 \Big\} = 2(1 - |\langle\varphi|\chi\rangle|^2)$$

as a simple play with diads reveals.

Let us illustrate all this by calculating the kernel and the induced quantum metrics for the basic model (A.1). If we fix λ_0 as an origin point, its orbit is

$$\lambda_k = \frac{\lambda_0}{(1-q^k)\lambda_0 + q^k} = \frac{1}{1-q^k\varkappa} \qquad \varkappa = 1 - \frac{1}{\lambda_0}$$

where $k \in \mathbb{Z}$. So that

$$\prod_{k=0}^{n-1} \frac{1}{\lambda_k} = \prod_{k=0}^{n-1} (1-q^k\varkappa) = (\varkappa\,|\,q)_n \quad (\varkappa\,|\,q)_{-n} = \frac{1}{(q^{-n}\varkappa\,|\,q)_n} = \prod_{k=1}^{n} \frac{1}{1-q^{-k}\varkappa} = \prod_{k=1}^{n} \lambda_{-k}$$

a simple unifying expression for the positive and negative products figuring in the construction of the Hilbert space. So the Laurent series for the kernel generator simplifies into

$$\Bbbk(z) = \sum_{n\in\mathbb{Z}} (\varkappa\,|\,q)_n z^n. \tag{A.13}$$

Let us recall a deep cosmical truth:

$$\sum_{n\in\mathbb{Z}} \frac{(a\,|\,q)_n}{(b\,|\,q)_n} z^n = \frac{(az\,|\,q)_\infty (q/az\,|\,q)_\infty (q\,|\,q)_\infty (b/a\,|\,q)_\infty}{(z\,|\,q)_\infty (b/az\,|\,q)_\infty (b\,|\,q)_\infty (q/a\,|\,q)_\infty}. \tag{A.14}$$

This is the Ramanujan summation formula [1] for doubly infinite series. The convergency requirements are

$$|b/a| < |z| < 1.$$

And particularly interesting for us here is

$$\sum_{n\in\mathbb{Z}} (a\,|\,q)_n z^n = \frac{(az\,|\,q)_\infty (q/az\,|\,q)_\infty (q\,|\,q)_\infty}{(z\,|\,q)_\infty (q/a\,|\,q)_\infty} \tag{A.15}$$

a special case obtained by setting $b = 0$.

Proposition 26. *The Laurent series for the kernel generator sums into*

$$\mathbb{k}(z) = \frac{(\varkappa z\,|\,q)_\infty (q/\varkappa z\,|\,q)_\infty (q\,|\,q)_\infty}{(z\,|\,q)_\infty (q/\varkappa\,|\,q)_\infty} \tag{A.16}$$

Proof. By substituting $a \rightsquigarrow \varkappa$ and $z \rightsquigarrow \bar{w}z$ in (A.15) we obtain the closed expression for the doubly infinite sum (A.13). ☐

Proposition 27. *The induced quantum metric on* $\mathbb{D} - \{0\}$ *is given by*

$$ds^2 = \delta w\,\delta\bar{w}\Bigl\{\sum_{k\geq 0}\frac{q^k}{(1-q^k|w|^2)^2} - \varkappa\sum_{n\in\mathbb{Z}}\frac{q^n}{(1-\varkappa q^n|w|^2)^2}\Bigr\}. \tag{A.17}$$

Proof. The formula follows from the general expression (A.12) applied to (A.16). The logarithm transforms the infinite product into a simple happily convergent infinite sum, then $\partial^2/\partial w\partial\bar{w}$ term by term. ☐

Remark 22. The metrics exhibits two important divergencies. The first is the heaven horizon divergency occuring at $|w| = 1$ just as in the classical Poincaré model, related to the infinity of the space. The second, totally quantum, divergency occurs at $w = 0$ and is promoted by the negative part of the second sum. The space becomes more and more abundant when we approach the unique classical point, which behaves as the source of infinite energy.

Another interesting and natural one-parametric family of examples arrises if we assume the transformation function of the form

$$f(t) = t^q \qquad 0 < q < 1.$$

The iterations of the function are

$$f^n(t) = t^{q^n} \qquad n \in \mathbb{Z}$$

and a straightforward computation gives the following convolution combination of a classical and quantum geometric series, for the reproducing kernel generator

$$\mathbb{k}(z) = \sum_{n=-\infty}^{\infty} \varkappa^{-\frac{1-q^n}{1-q}} z^n. \tag{A.18}$$

For the sake of internal completeness & harmony, we shall now focus on some general facts about infinite products, and their transformations, charming in their own light. And the proofs of all the basic q-formulae used in the calculations of our diverse examples, including the binomial q-identities of Caychy and Euler, and the summation formula of Ramanujan, will be presented.

Let us consider the following functional equation

$$Q(z) = u(z)Q[v(z)] \tag{A.19}$$

involving holomorphic functions u, v and Q. By iterating this we obtain

$$Q(z) = Q[v^n(z)]\prod_{k=0}^{n-1} u[v^k(z)] \tag{A.20}$$

for every $n \in \mathbb{N}$.

We shall assume that all the basic compositions can be performed, and that 0 belongs to the domain of u and v, as well as

$$u(0) = 1 = Q(0) \qquad \lim_{n \rightsquigarrow \infty} v^n(z) = 0, \tag{A.21}$$

for every z in the domain. By using this and taking the limit $n \rightsquigarrow \infty$ in (A.20) we obtain the factorization of $Q(z)$ into an infinite product

$$Q(z) = \prod_{k=0}^{\infty} u[v^k(z)]. \tag{A.22}$$

True happiness can be found in calculating the coefficients of the power series expansion

$$Q(z) = \sum_{k=0}^{\infty} c_k z^k \qquad c_0 = 1. \tag{A.23}$$

As our basic illustration, let us consider a special case when $u(z)$ is a Möbius transformation and $v(z) = qz$ with $|q| < 1$. The most general Möbius transformation moving 0 into 1 is, modulo z-amptwists

$$u(z) = 1 - z \qquad\qquad u(z) = \frac{1 - \varkappa z}{1 - \ z} \tag{A.24}$$

where $\varkappa \in \mathbb{C}$. In the second fractional case, the functional equation (A.19) reduces to the recurrence formula

$$c_{k+1} = \frac{1 - \varkappa q^k}{1 - q^{k+1}} c_k. \tag{A.25}$$

for the coefficients. Which means

$$c_n = \frac{(\varkappa \,|\, q)_n}{(q \,|\, q)_n}. \tag{A.26}$$

The power series expansion for $Q(z)$ becomes the Cauchy q-binomial formula

$$\sum_{n=0}^{\infty} \frac{(\varkappa \,|\, q)_n}{(q \,|\, q)_n} z^n = \frac{(\varkappa z \,|\, q)_\infty}{(z \,|\, q)_\infty}. \tag{A.27}$$

For $\varkappa = 0$ this becomes

$$\sum_{n=0}^{\infty} \frac{z^n}{(q \,|\, q)_n} = \frac{1}{(z \,|\, q)_\infty} \tag{A.28}$$

which is the first Euler q-identity. The second Euler q-identity

$$\sum_{n=0}^{\infty} (-)^n \frac{q^{\binom{n}{2}} z^n}{(q \,|\, q)_n} = (z \,|\, q)_\infty \tag{A.29}$$

can be viewed as the first linear special case in (A.24). Indeed, assuming this for u we obtain

$$c_k = -\frac{q^{k-1}}{1 - q^k} c_{k-1} \tag{A.30}$$

from the fractional equation (A.19). That is to say

$$c_n = \frac{(-)^n q^{\binom{n}{2}}}{(q \,|\, q)_n} \tag{A.31}$$

and integrating into the power series expansion for $Q(z)$ we arrive to the second equation (A.29).

Lemma 28. *If Ω is a domain in $\mathbb{C}$ containing 1 and ϱ an analytic transformation of Ω into itself with $\varrho(1)=1$, and such that 1 is an attractive point for ϱ on the whole Ω with $|\varrho'(1)|<1$ then the infinite product*

$$Q(z)=\prod_{k\in\mathbb{N}}\varrho^k(z) \tag{A.32}$$

of the iterations of ϱ is normally and absolutely convergent in Ω. The following identity holds

$$Q(z)=\varrho(z)Q[\varrho(z)]. \tag{A.33}$$

Proof. For z from an appropriate neighbourhood of 1 we have $|\varrho(z)-1|\leq t|z-1|$ where $t\in(0,1)$. And if the neighbourhood is also ϱ-invariant then $|\varrho^k(z)-1|\leq t^k|z-1|$ for all $k\in\mathbb{N}$. This leads to the estimate

$$\sum_{k\in\mathbb{N}}|\varrho^k(z)-1|\leq t\frac{|z-1|}{1-t}$$

which ensures the absolute and normal convergence of the initial product. $\square$

A.2. Non-invertible Morphisms

Now we shall consider a seemingly innocent change from the previous setup, generating a C*-algebra from the same basic functional relation

$$\mathfrak{z}^*\mathfrak{z}=f(\mathfrak{z}\mathfrak{z}^*),$$

but now $f\colon[0,1]\to[0,1]$ is a non-surjective map such that $f(1)=1$ and therefore $f(0)\in(0,1]$.

This, in particular, implies that $\mathfrak{z}^*\mathfrak{z}$ is invertible with the spectrum

$$\sigma(\mathfrak{z}^*\mathfrak{z})\subseteq f[0,1].$$

On the other hand, we are interested in Hilbert space representations in which $\mathfrak{z}$ and hence $\mathfrak{z}^*$ are non-invertible. And since $\mathfrak{z}\mathfrak{z}^*$ and $\mathfrak{z}^*\mathfrak{z}$ have the same spectrum modulo 0, we conclude that

$$\sigma(\mathfrak{z}\mathfrak{z}^*)=\{0\}\cup\sigma(\mathfrak{z}^*\mathfrak{z}).$$

The existence of an isolated spectral point 0 means that $\ker(\mathfrak{z}\mathfrak{z}^*)=\ker(\mathfrak{z}^*)\neq\{0\}$. Let $|0\rangle$ be a normalized vector from this kernel. Recalling that $\mathfrak{z}\mathfrak{z}^*$ moves around in rhythm of f, that is $(\mathfrak{z}\mathfrak{z}^*)\mathfrak{z}=\mathfrak{z}f(\mathfrak{z}\mathfrak{z}^*)$ we conclude that $(\mathfrak{z}\mathfrak{z}^*)\mathfrak{z}|0\rangle=\mathfrak{z}f(\mathfrak{z}\mathfrak{z}^*)|0\rangle=f(0)\mathfrak{z}|0\rangle$ and

$$\begin{aligned}(\mathfrak{z}\mathfrak{z}^*)\mathfrak{z}^2|0\rangle&=\mathfrak{z}^2f^2(\mathfrak{z}\mathfrak{z}^*)|0\rangle=f^2(0)\mathfrak{z}^2|0\rangle\\&\vdots\\(\mathfrak{z}\mathfrak{z}^*)\mathfrak{z}^k|0\rangle&=\mathfrak{z}^kf^k(\mathfrak{z}\mathfrak{z}^*)|0\rangle=f^k(0)\mathfrak{z}^k|0\rangle\end{aligned}$$

by iterating. We can introduce the normalized vectors

$$|k\rangle=\frac{\mathfrak{z}^k|0\rangle}{\sqrt{f(0)\cdots f^k(0)}}$$

which form an orthogonal system, and generate the space $\mathcal{H}$ invariant under both $\mathfrak{z}$ and $\mathfrak{z}^*$. Explicitly, the representation is given by

$$\mathfrak{z}^*|0\rangle = 0 \qquad \mathfrak{z}^*|k\rangle = \sqrt{f^k(0)}|k-1\rangle \qquad \mathfrak{z}|k-1\rangle = \sqrt{f^k(0)}|k\rangle \qquad k \geq 1$$

and it follows that, modulo equivalences, there is only one irreducible realization.

Let us further assume that

$$\lim f^k(0) = 1. \tag{A.34}$$

In parallel with Proposition 24 we shall now construct a natural realization of the Hilbert space by analytic functions.

Proposition 29. *The identification*

$$|k\rangle \leftrightsquigarrow \frac{z^k}{\sqrt{f(0)\cdots f^k(0)}} \tag{A.35}$$

where $k \geq 0$ induce the realization of the whole Hilbert space $\mathcal{H}$ by some analytic functions in the open unit disk $\mathbb{D}$. In terms of this realization, the quantum complex variable $\mathfrak{z}$ acts simply as the multiplication operator by the coordinate z.

Proof. We proceed as in the proof of Proposition 24 but in a simpler 'one-sided' manner. The formal representation of the elements of $\mathcal{H}$ is

$$\sum_{k\geq 0} \frac{c_k z^k}{\sqrt{f(0)\cdots f^k(0)}} \qquad \sum_{k\geq 0} |c_k|^2 < \infty.$$

Because of $\lim c_k = 0$ and $\lim\bigl[f(0)\cdots f^k(0)\bigr]^{1/k} = 1$ which is a consequence of (A.34), the Cauchy-Hadamard formula gives for the radius of convergence of this power series

$$\frac{1}{R} = \overline{\lim}\bigl(|c_k|/\sqrt{f(0)\cdots f^k(0)}\bigr)^{1/k} = \overline{\lim}|c_k|^{1/k} \leq 1.$$

Hence all the elements of $\mathcal{H}$ are interpretable as power series converging normally in their common domain $\mathbb{D}$. □

The reproducing kernel for $\mathcal{H}$ is given by

$$K(\bar{w}, z) = \mathbb{k}(\bar{w}z) \qquad \mathbb{k}(z) = \sum_{k\geq 0} \frac{z^k}{f(0)\cdots f^k(0)} \tag{A.36}$$

and therefore the associated point wave functions are

$$|w\rangle = \sum_{k\geq 0} \frac{\bar{w}^k}{\sqrt{f(0)\cdots f^k(0)}}|k\rangle. \tag{A.37}$$

Remark 23. We have an apparent notational ambiguity here, similar to the one appearing in the previous subsection (see Remark 20). However here the number 0 is what two labeling sets share. Luckily $|0\rangle$ is one and the same in both cases.

A particularly interesting collection of models arises if we require that the classical hyperbolic plane symmetry, given by the group $\mathrm{SU}(1,1)$ be completely preserved. As explained in detail in [13] this promotes the following constraint

$$\frac{1}{1-f(z)} = \frac{1}{1-z} + \nu \tag{A.38}$$

with $\nu > 0$. In other words f is a parabolic Moebius transformation, conjugate to the Euclidean translation $t \mapsto t + \nu$ via $z \mapsto 1/(1-z)$ which cyclically rotates 0, 1 and ∞. Explicitly,

$$f(z) = \frac{(1-\nu)z + \nu}{1 + \nu - \nu z} \quad \longleftrightarrow \quad \begin{pmatrix} 1-\nu & \nu \\ -\nu & 1+\nu \end{pmatrix} \tag{A.39}$$

and (if we allow ν to take arbitrary real values) we actually deal with a one-parameter group of Möbius transformations parametrized by ν. In our context, the positivity of ν ensures that $f(0) \in (0, 1)$.

Clearly, f becomes the identity transformation in the limit $\nu = 0$ which gives the commutative algebra and the classical Poincaré model of the hyperbolic plane. In the non-commutative case $\nu > 0$ the reproducing kernel for the resulting Hilbert space is given by

$$K(\bar{w}, z) = (1 - \bar{w}z)^{-\lambda} \qquad \lambda = 1 + \frac{1}{\nu} \tag{A.40}$$

and from this and with the help of (A.12) it is straightforward to calculate the metrics

$$ds^2 = \frac{\lambda \delta w \, \delta \bar{w}}{(1 - |w|^2)^2}. \tag{A.41}$$

This is the classical hyperbolic metrics.

Remark 24. The Toeplitz extension is obtained as the limit $\nu \rightsquigarrow \infty$. The logarithm in (A.12) reduces the metrics to the one coming from the Toeplitz extension.

Both families (A.2) and (A.38) are included in one 2-parameter family of transformations

$$\frac{1}{1 - f(z)} = \frac{1}{q} \frac{1}{1-z} + \nu + 1 - \frac{1}{q} \tag{A.42}$$

characterized as those Möbius transformations that preserve 1 and map $[0, 1]$ into itself. Equivalently, after conjugating by $1/(1-z) : (0, 1, \infty) \mapsto (1, \infty, 0)$ they are interpretable as linear transformations mapping $[1, +\infty)$ into itself. We can rewrite the above formula in the first-translation-then-dilatation manner

$$\frac{1}{1 - f(z)} = \frac{1}{q} \Big\{ \frac{1}{1-z} + \nu \upsilon \Big\} \tag{A.43}$$

with the proportionality factor

$$\upsilon = q - \frac{1-q}{\nu}. \tag{A.44}$$

It is worth observing that

$$1 + \frac{1}{\nu} = \lambda = \frac{1-\upsilon}{1-q}.$$

The values of υ are from $(-\infty, q)$ assuming that $q \neq 1$.

As we shall now see, the q-binomial series and formula naturally appear in computing the reproducing kernel for the introduced Hilbert spaces of analytic functions.

Proposition 30. *The kernel generator for the class $0 < q < 1$ and $\nu > 0$ is explicitly given by*

$$\mathbb{k}(z) = \frac{(\upsilon z \,|\, q)_\infty}{(z \,|\, q)_\infty} = \prod_{k\geq 0} \frac{1-\upsilon z q^k}{1- z q^k}. \tag{A.45}$$

Proof. The n-fold iteration of f is a map of the same form, with the substitutions

$$q \rightsquigarrow q^n \qquad \nu \rightsquigarrow (1+\cdots+q^{n-1})\nu = \frac{1-q^n}{1-q}\nu. \tag{A.46}$$

It follows straightforwardly that the iterations of 0 by f are given by

$$f^n(0) = \frac{1-q^n}{1-q^{n-1}\upsilon}. \tag{A.47}$$

By taking the product of the first k iterations and accepting the help from the q-symbols (1.3) we find

$$f(0)\cdots f^k(0) = \frac{(q \,|\, q)_k}{(\upsilon \,|\, q)_k} \tag{A.48}$$

and therefore the reproducing kernel generator in (A.36) assumes the form

$$\mathbb{k}(z) = \sum_{k\geq 0} \frac{(\upsilon \,|\, q)_k}{(q \,|\, q)_k} z^k. \tag{A.49}$$

This is precisely the q-binomial series in z with υ as its q-exponent. Applying the q-binomial formula (A.27) closes the proof. □

Remark 25. The kernel (A.40) can be viewed as the limit of the above 2-parametric kernel, when $q \rightsquigarrow 1$.

Proposition 31. *The induced quantum metrics on $\mathbb{D}$ is given by*

$$ds^2 = \delta w\, \delta\bar{w} \sum_{k\geq 0} q^k \Bigl\{ \frac{1}{(1-q^k|w|^2)^2} - \frac{\upsilon}{(1-\upsilon q^k|w|^2)^2} \Bigr\}. \tag{A.50}$$

Proof. It follows by applying (A.12) to the infinite product (A.45). □

When q is close to 1, it is tempting to perturbatively expand the metrics around the classical hyperbolic metric, in order to study the q-fluctuations. We can do this by expanding each of the terms in (A.50) and performing the summation.

Proposition 32. *In such a way we arrive at*

$$ds^2 = \frac{\lambda \delta w\, \delta\bar{w}}{(1-|w|^2)^2} \sum_{n\geq 0} \delta_n \Bigl(\frac{|w|^2}{1-|w|^2}\Bigr)^n \tag{A.51}$$

where

$$\delta_n = (n+1)\sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{1+\upsilon+\cdots+\upsilon^j}{1+q+\cdots+q^j}. \tag{A.52}$$

In particular $\delta_0 = 1$ and

$$\delta_1 = \frac{2}{\nu}\frac{1-q}{1+q} \tag{A.53}$$

—this is the strength of the first correction term.

Proof. Using the classical binomial formula for exponent -2, observing that

$$\binom{-2}{n} = (-)^n(n+1)$$

and factoring out $1/(1-|w|^2)^2$ so that the power series is in terms of $|w|^2/(1-|w|^2)$, we calculate the remaining multiplicative parts of the coefficients as

$$\sum_{k\geq 0} q^k\Big\{(1-q^k)^n - \upsilon(1-\upsilon q^k)^n\Big\} = \sum_{k\geq 0}\sum_{j=0}^{n}(-)^j\binom{n}{j}q^{jk+k}(1-\upsilon^{j+1})$$
$$= \sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{1-\upsilon^{j+1}}{1-q^{j+1}} = \lambda\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{1+\upsilon+\cdots\upsilon^j}{1+q+\cdots q^j}.$$

Thus, the perturbative expansion holds as stated and all coefficients are easily obtained. Observe an interesting symmetry between q and υ. $\square$

Remark 26. The perturbative expansion is valid only around 0, for $|w| < 1/\sqrt{2}$. The only divergence in the metrics occurs at the heaven horizon limit $|w| = 1$.

Remark 27. We see that in the limit $q \rightsquigarrow 1$ all the perturbative terms vanish and we are left with the classical hyperbolic metrics (A.41) of the background parabolic model. Of course, this space is still highly quantum, all the realizations are 'pointless'. On the other hand, if we fix q and $\nu \rightsquigarrow +\infty$ then the limit is always the Toeplitz extension.

Remark 28. Another particularly interesting special case occurs when $\upsilon = 0$ which is equivalent to the linear transformation $f(z) = 1 - q + qz$. This sets the stage for the first Euler q-formula (A.30). The kernel and the metrics simplify into

$$\mathbb{k}(z) = (z\,|\,q)_\infty^{-1} \qquad ds^2 = \delta w\,\delta\bar{w}\sum_{k\geq 0}\frac{q^k}{(1-q^k|w|^2)^2}. \tag{A.54}$$

Appendix B. Differential Structures on Circle

B.1. General Setup

There is a natural isomorphism between $\mathcal{A}$ and $\mathbb{C}[z, 1/z]$ the extended polynomials in the complex variable z with all integer powers allowed. The isomorphism is given by restricting the rational functions from $\mathbb{C}[z, 1/z]$ on $\bigcirc$, where they can be interpreted as *trigonometric polynomials*. For a given element $f \in \mathcal{A}$ we shall write $f(z)$ to indicate the corresponding holomorphic extension in $\mathbb{C}[z, 1/z]$.

Every bicovariant differential calculus Γ on $\bigcirc$ brings an important sequence of distinguished vectors, the germs of the circle harmonics $o_k = \pi(U^k)$ where $k \in \mathbb{Z}$. By definition, these vectors span the space Γ_{inv}. Clearly $o_0 = 0$. The vector o_1 is always cyclical relative to the right $\mathcal{A}$-module structure $\circ$, with

$$o_k = o_1 \circ \big(1 + U + \cdots + U^{k-1}\big) \qquad o_{-k} = -o_1 \circ \big(U^{-1} + \cdots + U^{-k}\big) \tag{B.1}$$

for every $k > 0$. Among these vectors, the only other always cyclical one is $o_{-1} = -o_1 \circ U^{-1}$. Closely related are the recursive relations

$$o_k \circ U^m = o_{k+m} - o_k \tag{B.2}$$

valid for $k, m \in \mathbb{Z}$.

The whole $\circ$-structure on Γ_{inv} is determined by a single operator $\mho\colon \psi \mapsto \psi \circ U$.

Every Γ is uniquely determined by its ideal $\mathcal{R} = \ker(\pi) \cap \ker(\epsilon)$ so that $\Gamma_{inv} \leftrightarrow \ker(\epsilon)/\mathcal{R}$ and $\Gamma \leftrightarrow \mathcal{A} \otimes \Gamma_{inv}$. The two trivial ideals $\ker(\epsilon)$ and $\{0\}$ correspond respectively to the trivial calculus $\Gamma = \{0\}$ and the universal one where $\Gamma \leftrightarrow \ker(m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A})$ and $\Gamma_{inv} \leftrightarrow \ker(\epsilon)$.

Let us consider non-trivial ideals $\mathcal{R} \subset \ker(\epsilon)$. Every such $\mathcal{R}$ is generated by a single element of $\mathbb{C}[z]$, a uniquely defined polynomial of the form

$$p(z) = (z-1)q(z) \qquad q(z) = \prod_{\lambda \in \Lambda} (z-\lambda)^{m_\lambda} \tag{B.3}$$

where $\Lambda \subset \mathbb{C} \setminus \{0\}$ is the set of zeros λ of $q(z)$ with their respective multiplicities $m_\lambda \geq 1$. This in fact establishes a natural correspondence between the non-trivial ideals $\mathcal{R}$ and the above polynomials. For Γ the corresponding bicovariant calculus to $\mathcal{R}$ on $\bigcirc$, since $\Gamma_{inv} \cong \ker(\epsilon)/\mathcal{R}$ we have

$$\dim \Gamma_{inv} = \deg q[z] = n = \sum_{\lambda \in \Lambda} m_\lambda \tag{B.4}$$

and it follows that $o_1, \cdots, o_n$ form a basis in Γ_{inv}. In this basis, the operator $\mho$ has the form

$$\mho = \left(\begin{array}{cccc|c} -1 & -1 & \cdots & -1 & * \\ \hline 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{array}\right) \tag{B.5}$$

with the rightmost column expressing $o_{n+1} - o_1$. These numbers are the same as the coefficients of (the non-scalar part of) the polynomial $z^{n+1} - p(z)$.

Remark 29. The calculus will be *-covariant iff $p(z)$ is a *real polynomial*. This is equivalent to saying that $\Lambda = \overline{\Lambda}$ with $m_\lambda = m_{\bar{\lambda}}$.

Remark 30. The universal differential calculus, where $\mathcal{R} = \{0\}$, is the only infinite-dimensional calculus over $\bigcirc$.

Besides this simple basis of projected first n circle harmonics, another important basis in the space Γ_{inv} naturally emerges. It is integrated of blocks

$$\pi\bigl\{p(z)/(z-\lambda)^j\bigr\} \qquad \lambda \in \Lambda \qquad j \in \{1, \ldots, m_\lambda\}. \tag{B.6}$$

This is the Jordan basis for the basic $\circ$-multiplication operator $\mho$. In each block the operator has the standard form

$$\mho \leftrightarrow \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \tag{B.7}$$

We see that the spectrum of $\mho$ is Λ, and $q(z)$ is its characteristic polynomial.

With the help of the q-numbers (1.1) we can integrate both formulae in (B.1) and the trivial case $k = 0$ into a single expression

$$o_k = o_1 \circ (k)_U \tag{B.8}$$

now valid for all $k \in \mathbb{Z}$.

B.2. **Linear Algebra Link**

The formulae with the cyclic generator o_1 lead us to a canonical structural representation of every finite-dimensional differential calculus. It turns out that the diversity of these structures is the same as the diversity of linear symmetries acting on finite dimensional spaces and possessing cyclic vectors.

Indeed, let V be a (non-trivial) finite-dimensional vector space and $T\colon V \to V$ a linear invertible operator. This by the formula

$$\varphi \bullet U^k = T^k \varphi \tag{B.9}$$

gives life to a right $\mathcal{A}$-module structure on V. If in addition there is a distinguished cyclic vector $\psi \in V$ for T, define a map $\varpi\colon \mathcal{A} \to V$ by

$$\varpi(U^n) = \psi \bullet (n)_U = (n)_T \psi. \tag{B.10}$$

This map is surjective, due to the cyclicity of ψ for T.

Lemma 33. *Furthermore we have*

$$\varpi(ab) = \varpi(a) \bullet (b) + \epsilon(a)\varpi(b) \tag{B.11}$$

for every $a, b \in \mathcal{A}$.

Proof. It is sufficient to check it for circle harmonics, and for them it follows directly from the addition identity (1.2). ☐

Clearly $\varpi(1) = 0$. Let us consider the space $\mathcal{R} = \ker(\epsilon) \cap \ker(\varpi)$. By the above Leibniz rule, it follows that $\mathcal{R}$ is a right $\mathcal{A}$-ideal. Let Γ be the first-order calculus associated to $\mathcal{R}$. We can define a linear map $\widetilde{\varpi}\colon \Gamma_{inv} \to V$ by

$$\widetilde{\varpi}\pi = \varpi. \tag{B.12}$$

By construction, this map is bijective and intertwines o_1 and ψ as well as $\circ$ and $\bullet$. Let us summarize this as one important general conclusion.

Proposition 34. *There exists a natural one-one correspondence between the left-covariant (and hence bicovariant) differential first-order calculi* Γ *over* $\bigcirc$ *and (the classes of isomorphisms) of triplets* (V, ψ, T) *where* V *is a finite-dimensional vector space,* $T\colon V \to V$ *an automorphism of* V*, and* $\psi \in V$ *a cyclic vector for* T*. In terms of this correspondence*

$$\Gamma_{inv} \leftrightarrow V \qquad \pi \leftrightarrow \varpi \qquad \circ \leftrightarrow \bullet \qquad o_1 \leftrightarrow \psi. \tag{B.13}$$

The first $\leftrightarrow$ *is completely fixed by either the second, or the third & the fourth* $\leftrightarrow$*.* ☐

B.3. **Two Steps Beyond Exterior Algebra**

Let us now, still remaining within this linear-algebraic context, consider algebraic structures that are quadratically generated, but weaker in relations than the exterior algebra. So for a given vector space V, we focus on a non-trivial subspace of $\mathrm{S}(V \otimes V)$ representing the quadratic relations of an algebra built over its generator space V. As we shall now explain, such structures are always bounded from below, by elegant infinite-dimensional Clifford algebra type objects.

If the quadratic relations subspace Q is strictly a subspace of $\mathrm{S}(V \otimes V)$, then there will be always a vector $\psi \in V$ such that $\psi \otimes \psi \notin Q$. This is equivalent to saying that $\psi^2 \neq 0$ in the factor-algebra $V|Q$ of $V^{\otimes}$ over the relations Q.

The minimal departure from the exterior algebra occurs when the quadratic relations subspace Q, is of codimension one in $\mathrm{S}(V \otimes V)$.

Proposition 35. *In this minimal departure case, for every $\psi \in V$ whose square is non-zero in $V|Q$ there exists a unique symmetric bilinear form $g\colon V \times V \to \mathbb{C}$ such that*

$$xy + yx = 2g(x, y)\psi^2 \tag{B.14}$$

in $V|Q$ for all $x, y \in V$. In particular $g(\psi, \psi) = 1$.

Proof. The minimality condition says that $\mathrm{S}(V \otimes V)/Q$ is one-dimensional. This space lives in $V|Q$ and $\mathrm{S}(V \otimes V)/Q \cong \mathbb{C}\psi^2$. Every symmetric combination $xy + yx$ belongs to it and hence $\exists$ a unique scalar $g(x, y)$ so that (B.14) holds. The constructed map $g\colon V \times V \to \mathbb{C}$ is bilinear and symmetric. □

Remark 31. By interpreting $g \in \mathrm{S}(V \otimes V)^*$ we see that $Q = \ker(g)$.

Let W be the g-orthocomplement to ψ in V, so that we have $V \leftrightarrow \mathbb{C}\psi \oplus W$. It follows from (B.14) that ψ anticommutes in $V|Q$ with the elements of W. Hence ψ^2 commutes with all of them, and thus it is a *central element* of $V|Q$.

Let us now consider the graded tensor product $\wedge(W) \mathbin{\widehat{\otimes}} \mathbb{C}[z]$ of the exterior algebra $\wedge(W)$ and the polynomial algebra $\mathbb{C}[z]$, where the grade of z^n is n. There is a natural linear map

$$\wedge(W) \mathbin{\widehat{\otimes}} \mathbb{C}[z] \rightsquigarrow V|Q \tag{B.15}$$

obtained by realizing $\wedge(W)$ as antisymmetric tensors of $W^{\otimes}$, assigning ψ to z, viewing everything in $V^{\otimes}$ and then simply projecting to the factoralgebra.

Proposition 36. *The constructed map is bijective. In particular, the algebra $V|Q$ is always infinite-dimensional. It can be interpreted as a deformation of $\wedge(W)\widehat{\otimes}\mathbb{C}[z]$ the deformation parameter being the symmetric form itself.* □

Let us now assume that the codimension of Q in $\mathrm{S}(V \otimes V)$ is 2. In this case, certainly, $\dim(V) \geq 2$ and there exist two linearly independent vectors $u, v \in V$ such that projections of $u \otimes u$ and $v \otimes v$ form a basis in the factor space $\mathrm{S}(V \otimes V)/Q$. There is a natural bilinear form $g\colon V \times V \to \mathrm{S}(V \otimes V)/Q$ defined by

$$xy + yx = 2g(x, y). \tag{B.16}$$

These are the relations for the factor-algebra $V|Q$. In particular,

$$uv + uv = \alpha u^2 + \beta v^2 \tag{B.17}$$

for some $\alpha, \beta \in \mathbb{C}$. Let W be the orthocomplement of $\{u, v\}$ in V, relative to g. By definition the elements of W anticommute with both u and v. So the algebra $V|Q$ is completely specified by the restriction of g on W, as well as the numbers α & β.

If we perform a linear basis change

$$(u, v) \rightsquigarrow (u, v) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc \neq 0 \tag{B.18}$$

then it is straightforward to compute

$$(u^2, v^2) \rightsquigarrow (u^2, v^2)\begin{pmatrix} a^2+\alpha ac & b^2+\alpha bd \\ c^2+\beta ac & d^2+\beta bd \end{pmatrix} \tag{B.19}$$

$$\begin{vmatrix} a^2+\alpha ac & b^2+\alpha bd \\ c^2+\beta ac & d^2+\beta bd \end{vmatrix} = (ad-bc)(ad+bc+\alpha cd+\beta ab). \tag{B.20}$$

In order to produce new generators for the space $\mathrm{S}(V\otimes V)/Q$ we should assume also $0 \neq ad+bc+\alpha cd+\beta ab$. If so, the structural coefficients transform as follows:

$$\alpha \rightsquigarrow \frac{\alpha d^2+\beta b^2+2bd}{ad+bc+\alpha cd+\beta ab} \qquad \beta \rightsquigarrow \frac{\alpha c^2+\beta a^2+2ac}{ad+bc+\alpha cd+\beta ab} \tag{B.21}$$

and using this transformational liberty we can make them both 0, unless $\alpha\beta = 1$ in which case we can transform the system into $\alpha = \beta = 1$. Indeed, if α and β are not reciprocal and not 0 then

$$b = \frac{d}{\beta}\big[\sqrt{1-\alpha\beta}-1\big] \qquad c = \frac{a}{\alpha}\big[\sqrt{1-\alpha\beta}-1\big] \tag{B.22}$$

defines a transformation for the Disappearance Act.† We have

$$ad-bc = \frac{2ad\sqrt{1-\alpha\beta}}{1+\sqrt{1-\alpha\beta}} \qquad ad+bc+\alpha cd+\beta ab = \frac{2ad(1-\alpha\beta)}{1+\sqrt{1-\alpha\beta}}. \tag{B.23}$$

Proposition 37. *If Q is of codimension 2 in $\mathrm{S}(V\otimes V)$ then the generators $u, v \in V$ can be chosen in one of the following forms.*

—The anticommuting setup

$$uv+vu = 0. \tag{B.24}$$

The linear symmetries of the generating relation are given by diagonal or antidiagonal invertible matrices. As Möbius transformations, they correspond to amplitwists and complex inversion coupled with them.

—Or the squared setup

$$uv+vu = u^2+v^2 \quad \Leftrightarrow \quad (u-v)^2 = 0. \tag{B.25}$$

Its linear symmetries must preserve the subspace $\mathbb{C}(u-v)$ hence, as Möbius transformations, they are isomorphic to the Euclidean group $\mathrm{E}(2)$.

B.4. **Rational Functions Representation**

Another interesting representation of the space Γ_{inv} is provided via the purely singular rational functions of one complex variable. Let $\daleth[z]$ be the algebra of rational functions vanishing at infinity of $\mathbb{C}$. These are truly singular rational functions, and we have the following decomposition

$$\beth[z] = \mathbb{C}[z] \oplus \daleth[z] \tag{B.26}$$

in the field $\beth[z]$ of rational functions over $\mathbb{C}$. For every finite set Λ in $\mathbb{C}$, the following identity holds

$$\prod_{\lambda\in\Lambda}\frac{1}{z-\lambda} = \sum_{\lambda\in\Lambda}\frac{1}{z-\lambda}\Big\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{1}{\lambda-\mu}\Big\} \tag{B.27}$$

†If one of them is already 0, the case can be viewed as a limit of these formulae.

with the notable simplest special case

$$\frac{1}{(z-\alpha)(z-\beta)}=\frac{1}{\alpha-\beta}\Big(\frac{1}{z-\alpha}-\frac{1}{z-\beta}\Big). \tag{B.28}$$

This can be understood as the *multiplication formula* in $\daleth[z]$.

In the infinite-dimensional space $\daleth[z]$, a natural basis is given by elementary singular functions

$$z\mapsto\frac{1}{(z-\lambda)^k} \tag{B.29}$$

where $\lambda\in\mathbb{C}$ and $k\geq 1$.

With the help of (B.27) we can invert an arbitrary polynomial, possessing zeros with multiplicities specified by a function $m\colon\Lambda\to\mathbb{N}$, and develop its expansion in terms of the above basis:

$$\prod_{\lambda\in\Lambda}\frac{1}{(z-\lambda)^{m_\lambda}}=\frac{1}{m!}\frac{\partial^m}{\partial\Lambda}\sum_{\lambda\in\Lambda}\frac{1}{z-\lambda}\Big\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{1}{\lambda-\mu}\Big\}. \tag{B.30}$$

Here we define

$$m!=\prod_{\lambda\in\Lambda}m(\lambda)! \qquad \frac{\partial^m}{\partial\Lambda}=\prod_{\lambda\in\Lambda}\frac{\partial^{m(\lambda)}}{\partial\lambda^m}$$

and also $m_\lambda=m(\lambda)+1$ for every $\lambda\in\Lambda$. We see that the principal singular part is given by

$$\prod_{\lambda\in\Lambda}\frac{1}{(z-\lambda)^{m_\lambda}}\sim\sum_{\lambda\in\Lambda}\Big\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{1}{(\lambda-\mu)^{m_\mu}}\Big\}\frac{1}{(z-\lambda)^{m_\lambda}}. \tag{B.31}$$

In (B.26) we can freely move singularities from $\daleth[z]$ to $\mathbb{C}[z]$. The simplest variation is

$$\beth[z]=\mathbb{C}[z,\frac{1}{z}]\oplus{}_*\daleth[z] \tag{B.32}$$

where ${}_*\daleth[z]\subset\daleth[z]$ is the subalgebra of the truly singular functions which are *not* singular at 0.

In formula (B.32) we can factorize over $\mathbb{C}[z,1/z]$ and obtain a natural $\mathbb{C}[z,1/z]$-module structure on the factor-space ${}_*\daleth[z]$. Explicitly, the module structure reads

$$\begin{aligned} z|\lambda\rangle&=\lambda|\lambda\rangle & z|\lambda.k\rangle&=\lambda|\lambda.k\rangle+|\lambda.k-1\rangle\\ \frac{1}{z}|\lambda\rangle&=\frac{1}{\lambda}|\lambda\rangle & \frac{1}{z}|\lambda.k\rangle&=\sum_{j=0}^{k-1}\frac{(-)^j}{\lambda^{j+1}}|\lambda.k-j\rangle \end{aligned} \tag{B.33}$$

for $k\geq 2$. We have adopted the Dirac notation for the elementary singular functions (B.29), and we shall also write $|\lambda\rangle=|\lambda.1\rangle$.

In our main context of $\bigcirc$ when we identify $\mathcal{A}=\mathbb{C}[z,1/z]$, we shall think of this module structure as the right $\mathcal{A}$-module structure. It turns out that in the module ${}_*\daleth$ happily live as submodules, the left-invariant parts of all finite-dimensional Γ over $\bigcirc$, as we shall now see.

Proposition 38. *The assignation*

$$\Gamma_{inv} \ni \pi(a) \rightsquigarrow \bigl[a(z)/p(z)\bigr] \in {}_*\daleth[z] \tag{B.34}$$

where $a \in \ker(\epsilon)$, defines a natural $\mathcal{A}$-module embedding of Γ_{inv} into ${}_*\daleth[z]$. And in particular

$$o_1 \rightsquigarrow \frac{1}{q(z)} \tag{B.35}$$

the representation of the canonical cyclic vector is simply the inverse of the polynomial defining the first-order calculus Γ.

Proof. The map is well-defined since if $a \in \mathcal{R}$ then $p(z)|a(z)$ in $\mathbb{C}[z, 1/z]$. This divisibility condition is actually equivalent to saying $[a(z)/p(z)] = 0$, so the map is injective. Furthermore if $a \in \ker(\epsilon)$ then

$$\pi(a) \circ b = \pi(ab) \rightsquigarrow \bigl[\bigl(a(z)b(z)/p(z)\bigr)\bigr] = \bigl[\bigl(a(z)/p(z)\bigr)b(z)\bigr] = \bigl[a(z)/p(z)\bigr]b(z)$$

for every $b \in \mathcal{A}$ so that the constructed map is indeed a morphism of $\mathcal{A}$-modules. Finally the basic cyclic vector o_1 is represented by $a(z) = z - 1$ and (B.35) descends from (B.34). $\square$

Remark 32. In terms of this identification, the basis (B.6) is morphed into the elementary singular functions $|\lambda.k\rangle$ where $\lambda \in \Lambda$ and $k \in \{1, \dots, m_\lambda\}$.

Remark 33. Another interesting observation is that for every $\lambda \in \Lambda$ the space of proper vectors $\ker(\lambda - \mathring{U})$ is one-dimensional. And in particular if $\lambda = 1 \in \Lambda$ these vectors are characterized as the *central vectors* of Γ_{inv} in other words

$$\theta \in \ker(1 - \mathring{U}) \quad \Leftrightarrow \quad \theta a = a\theta \quad \Leftrightarrow \quad \theta \circ a = \epsilon(a)\theta \tag{B.36}$$

for all $a \in \mathcal{A}$.

B.5. **Some Special Properties**

Proposition 39. *For $m \neq 0$ we have*

$$o_m = 0 \tag{B.37}$$

iff $1 \notin \Lambda$ and all the zeros λ of $q(U)$ are simple zeros satisfying $\lambda^m = 1$.

Proof. The condition is equivalent to $U^m - 1 \in \mathcal{R}$ which means that as polynomials $q(U) \bigm| (1 + U + \cdots + U^{|m|-1})$. $\square$

If the above condition holds, then there is the first positive such an index. Let us call this number t. Then $o_m = 0$ iff $t|m$ and the sequence is periodic $o_m = o_{m+t}$ for every $m \in \mathbb{Z}$.

Remark 34. In general, the quantum connectedness components relative to Γ are described as the kernel of the differential $d\colon \mathcal{A} \to \Gamma$. This is always a unital subalgebra of $\mathcal{A}$. It will be a *-subalgebra if the calculus is *-covariant. If $\mathcal{A}$ is a Hopf *-algebra and Γ is bicovariant, then the kernel will in addition be a Hopf subalgebra of $\mathcal{A}$. So a Hopf *-subalgebra in the case of bicovariant *-covariant Γ. Geometrically, this means that the connectedness components are labeled by a quantum factor-group of G, the quantum space described by the kernel, consisting of the ‘functions’ constant along the corresponding ‘cosets’.

For $\bigcirc$ and Γ satisfying the above condition, we have

$$\ker(d\colon \mathcal{A} \to \Gamma) = \Big\{ \sum_{k\in\mathbb{Z}}^{t|k} c_k U^k \Big\}$$

geometrically corresponding to the canonical t-fold covering of $\bigcirc$ by $\bigcirc$. If the condition is not satisfied, then the sequence never repeats and the kernel of d is trivial, so that the circle appears 'connected'.

Remark 35. This also has a nice interpretation in terms of the cyclic groups $\mathbb{Z}_m$ understood as m-th roots of 1. Every calculus satisfying (B.37) projects to the cyclic group $\mathbb{Z}_m$. In such a projected form the space Γ_{inv} is preserved. In terms of the classification of first order bicovariant differential calculi over finite groups, where these structures are labeled by subsets consisting of entire non-trivial conjugacy classes the lineals over which are morphed into the spaces of left-invariant elements—the set of roots of $q(U)$ provides this defining set for (the projected) Γ_{inv}. Here, since the group is Abelian, the conjugacy classes are given by single elements of $\mathbb{Z}_m$. So in total, for a given m there will be 2^{m-1} solutions for the calculus. The empty subset being the trivial 0-calculus and the whole set of non-trivial roots corresponding to the universal differential calculus over $\mathbb{Z}_m$. As we shall explain shortly, this maximal solution is characterized, at the level of $\bigcirc$, by vanishing of the quadratic relations for the universal differential envelope $\Gamma^\wedge$.

Let us now focus on another interesting truly quantum phenomenon, in which the differential is represented as the commutator with a fixed element of the calculus.

Proposition 40. *The differential $d\colon \mathcal{A} \to \Gamma$ will be inner, in the sense of the existence of $\xi \in \Gamma_{inv}$ such that*

$$da = \xi a - a\xi \tag{B.38}$$

for all $a \in \mathcal{A}$ if and only if $1 \notin \Lambda$. Which is equivalent to saying $\mathcal{R} \nsubseteq \ker(\epsilon)^2$, in other words the calculus should be not projectable to the classical calculus on $\bigcirc$. And if this is the case, our ξ is unique and explicitly given by

$$\xi = -\pi\Big\{ \prod_{\lambda\in\Lambda} \Big[\frac{U-\lambda}{1-\lambda}\Big]^{m_\lambda} \Big\}. \tag{B.39}$$

Proof. The localized form of the identity (B.38) is

$$\pi(a) = \xi \circ a - \epsilon(a)\xi \tag{B.40}$$

and it is clear that we only have to check it for $a = U$. If $\xi = \pi(b)$ with $b \in \ker(\epsilon)$ then the above identity becomes

$$\pi[(b-1)(U-1)] = 0.$$

In other words $b - 1 = q(U)c$ with the simplest solution being

$$c = -\frac{1}{q(1)} = -\prod_{\lambda\in\Lambda}(1-\lambda)^{-m_\lambda}. \tag{B.41}$$

This is possible iff $q(1) \neq 0$. All other solutions are equivalent to this one, modulo the ideal $\mathcal{R}$. Another way to see this is to observe that for $1 \notin \Lambda$ there are no non-zero central elements of Γ_{inv}. $\square$

We can also write ξ simply as

$$\xi = (U\mkern-12mu\circ - 1)^{-1} o_1. \tag{B.42}$$

Remark 36. If $\mathcal{R} \subseteq \ker(\epsilon)^2$ which is to say that the calculus is projectable to the classical calculus on $\bigcirc$, then clearly the differential can not be inner—for otherwise the projected differential on the classical calculus would be inner too, which is impossible because of the commutativity of the calculus. The classical calculus is characterized by this property. It is the only commutative (first-order) calculus. The existence of (one-dimensional space of) central vectors and d being inner, are mutually *exclusive and complementary properties.*

Remark 37. The projectability on the classical calculus also means that such a Γ will be fully compatible with locally trivial principal $\bigcirc$-bundles over classical smooth manifolds [4]. In this context the calculus on the base manifold M is kept classical, and local trivializations of the bundle P always naturally induce the local trivializations of its calculus $\Omega(P)$. The condition is equivalent to the extensibility of all unital *-homomorphisms $g\colon \mathcal{A} \to \mathrm{C}^\infty(M)$ to $g\colon \Gamma^\wedge \to \Omega(M)$, which is further equivalent to the projectability of the convolution product $g_\sim = g^{-1} * (dg)$ to Γ_{inv} via π. These maps are classical localized derivations as

$$g_\sim(ab) = \epsilon(a) g_\sim(b) + g_\sim(a)\epsilon(b) \tag{B.43}$$

for all $a, b \in \mathcal{A}$. Thus $\ker(\epsilon)^2 \subseteq \mathcal{R}$ and Γ projects to the classical calculus on $\bigcirc$.

B.6. **Extending to Higher Orders**

Let us now focus on extending a given first-order calculus to a complete higher-orders calculus. Here the 'inferior limit' is given by the classical exterior algebra over Γ. This is because $\bigcirc$ is Abelian group, and as such, has trivial adjoint action and the twist operator σ is the standard transposition. The minimal solution $\Gamma^\vee$ is not very interesting from the differential viewpoint, as the differential d vanishes identically on $\Gamma^\vee_{inv}$. We can see this explicitly from the formula

$$d o_k = -o_k^2 \tag{B.44}$$

which is valid for every $k \in \mathbb{Z}$ and in any complete differential calculus. This follows from $\phi(U^k) = U^k \otimes U^k$ and the general germs identity $d\pi(a) = -\pi(a^{(1)})\pi(a^{(2)})$.

Let us recall that in general, the left-invariant component $\diamondsuit(\Gamma)$ of the quadratic relations generating the universal differential envelope consists precisely of the elements

$$\diamondsuit(\Gamma) = \Bigl\{ \pi(a^{(1)}) \otimes \pi(a^{(2)}) \Bigm| a \in \mathcal{R} \Bigr\}. \tag{B.45}$$

This morphing into the expressions $[\pi(r^{(1)}) \otimes \pi(r^{(2)})] \circ (a^{(1)} \otimes a^{(2)})$ by substituting $a \rightsquigarrow ra$ where $r \in \mathcal{R}$ are interpretable as generators of $\mathcal{R}$.

And for the circle there is, of course, a single generator $r = p(U)$. So the quadratic relations space in $\Gamma_{inv} \otimes \Gamma_{inv}$ is generated, as a right $\mathcal{A}$-module, by a single element $(\pi \otimes \pi)\phi[p(U)]$.

Remark 38. By Proposition 34 the quadratic relations $\diamondsuit(\Gamma)$ are in a natural way, the left-invariant part of another differential calculus over $\bigcirc$.

Remark 39. There is only one way for $\Gamma^\wedge$ to be finite-dimensional: to coincide with $\Gamma^\vee$. Indeed, as it follows from Proposition 36, any diminishment in full symmetric quadratic relations $\mathrm{S}(V \otimes V)$ of a finite-dimensional space V leads to an infinite-dimensional Clifford type structure with polynomial super-central part. So if $\diamondsuit(\Gamma)$ is strictly smaller than $\mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$ the algebra $\Gamma^\wedge$ will be infinite-dimensional.

For general calculi over $\bigcirc$ admitting d-generating ξ, we can construct a higher order calculus by postulating the extension of d as the graded commutator with ξ. In other words

$$\xi\varphi - (-)^{\partial\varphi}\varphi\xi = d\varphi \tag{B.46}$$

for every φ in the calculus. The vanishing of the square of d is equivalent to commuting of ξ^2 with all $a \in \mathcal{A}$ which in turn means $\xi^2 \circ a = (\xi \circ a^{(1)})(\xi \circ a^{(2)}) = \epsilon(a)\xi^2$. Combining with (B.40) this condition becomes

$$\xi\,\pi(a) + \pi(a)\,\xi + \pi(a^{(1)})\pi(a^{(2)}) \;=\; 0. \tag{B.47}$$

In other words, the anticommutator between ξ and the elements of Γ_{inv} must be equal to the known expression for the differential of those elements. This invites us to consider precisely the elements of the form

$$\xi{\otimes}\pi(a) + \pi(a){\otimes}\xi + \pi(a^{(1)}){\otimes}\pi(a^{(2)}) \tag{B.48}$$

where $a \in \ker(\epsilon)$ in $\Gamma_{inv} \otimes \Gamma_{inv}$ as generators of the ideal for the calculus. Clearly, and as it should be, it includes the defining relations for the universal envelope $\Gamma^\wedge$, as we can see by restricting on $a \in \mathcal{R}$.

Remark 40. The formula (B.48) is valid in the general context, for any quantum group equipped with a bicovariant differential calculus with an ad-invariant element $\xi \in \Gamma_{inv}$ satisfying (B.40). These elements are always σ-invariant, as it follows from

$$\begin{aligned} \sigma\big[\pi(a^{(1)}) \otimes \pi(a^{(2)})\big] \;&=\; \pi(a^{(1)}) \otimes \pi(a^{(2)}) - c^\top\pi(a) \\ c^\top\pi(a) = (\pi \otimes \pi)\mathrm{ad}(a) &\quad \mathrm{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)} \end{aligned} \tag{B.49}$$

the quantum Maurer-Cartan formula, coupled with the identities

$$\sigma\big[\xi \otimes \pi(a)\big] = \pi(a) \otimes \xi + c^\top\pi(a) \qquad\qquad \sigma\big[\pi(a) \otimes \xi\big] = \xi \otimes \pi(a). \tag{B.50}$$

The relations (B.48) are then extendible to any quadratic relations contained in $\ker(\mathrm{id} - \sigma)$ in $\Gamma_{inv} \otimes \Gamma_{inv}$, also producing a valid higher-order differential calculus.

Proposition 41. *If the calculus Γ admits its d-generating ξ then the quadratic relations generator can be computed as*

$$(\pi \otimes \pi)\phi[p(U)] = p(\mathring{U} \otimes \mathring{U})[\xi \otimes \xi]. \tag{B.51}$$

Proof. In fact, if $f(z)$ is any polynomial then

$$(\pi \otimes \pi)\phi[f(U)] = f(1)\xi \otimes \xi - \xi \otimes f(\mathring{U})[\xi] - f(\mathring{U})[\xi] \otimes \xi + f(\mathring{U} \otimes \mathring{U})[\xi \otimes \xi]. \tag{B.52}$$

Indeed, for $f(z) = z^k$ we have

$$\begin{gathered} (\pi \otimes \pi)\phi[f(U)] = o_k \otimes o_k = (\mathring{U}^k\xi - \xi) \otimes (\mathring{U}^k\xi - \xi) = (\mathring{U}^k \otimes \mathring{U}^k)[\xi \otimes \xi] + (\xi \otimes \xi) - \\ - (\xi \otimes \mathring{U}^k\xi) - (\mathring{U}^k\xi \otimes \xi) = f(1)\xi \otimes \xi - \xi \otimes f(\mathring{U})[\xi] - f(\mathring{U})[\xi] \otimes \xi + f(\mathring{U} \otimes \mathring{U})[\xi \otimes \xi]. \end{gathered}$$

and (B.52) follows by linearity on $f(z)$. Now $q(z)$ vanishes on $\mathring{U}$ and 1 is always a zero of $p(z)$, so (B.51) holds. $\square$

B.7. **Permutational Symmetry**

There is an elegant method to enlarge the quadratic relations of the universal differential envelope such that the resulting calculus exhibits the full symmetry of the classical permutations. As in general, such a calculus will be between the universal envelope $\Gamma^\wedge$ and the exterior algebra $\Gamma^\vee$. Our principal focus is $\bigcirc$, although many results of this subsection are valid for arbitrary Abelian quantum groups, and some extend to the general context.

Let us consider the following expressions

$$\theta_0 \otimes \pi(r^{(1)}) \otimes \theta_1 \otimes \cdots \otimes \theta_{n-1} \otimes \pi(r^{(n)}) \otimes \theta_n \tag{B.53}$$

where $r \in \mathcal{R}$ and θ_α are some tensor products of elements of Γ_{inv}, including the empty tensor products in which case the its left and right $\otimes$ merge into a single one. Here $n \geq 2$. These elements are from $\Gamma_{inv}^{\otimes}$. Clearly, the only elements of order 2 are the ones from $S_{inv}^{\wedge 2}$ the quadratic generating relations for $\Gamma^\wedge$. Let $T_{n|m}$ be the span of all above elements of total grade m. Obviously $n \leq m$ and $T_{n|m} \subseteq \Gamma_{inv}^{\otimes m}$.

Proposition 42. *The spaces $T_{n|m}$ are invariant under all permutations from S_m acting in $\Gamma_{inv}^{\otimes m}$.*

Proof. It is sufficient to check the invariance under $m-1$ elementary transpositions generating S_m. The multiple coproduct $r^{(1)} \otimes \cdots \otimes r^{(n)}$ is completely symmetric since $\bigcirc$ is Abelian. If some θ_k is empty so there is a neighbouring configuration $\pi(r^{(k)}) \otimes \pi(r^{(k+1)})$ in (B.53) then elementary transposition acting upon it will not change the expression. Otherwise, it will either transpose a part $\pi(r^{(k)})$ and rightmost / leftmost vector from a neighbour θ_{k-1} or θ_k, or act entirely within one of the θ_α. In any case, the transformed expression will be of the same type. □

Proposition 43. *We have*

$$T_{n|m} \circ a \subseteq \sum_{k \leq n} T_{k|m} \tag{B.54}$$

for every $a \in \mathcal{A}$ and $n \leq m$. In particular, the spaces $T_{2|m}$ are $\circ$-invariant.

Proof. If we $\circ$-act on (B.53) by U^j and use the localized Leibniz rule for the germs, the expression transforms into a $\pm$ sum of similar elements involving $\theta_\alpha \circ U^j$ and $\pi(U^j)$ as well as embedded $\pi(r^{(1)}U^j) \otimes \cdots \otimes \pi(r^{(k)}U^j)$ where $2 \leq k \leq n$, these corresponding to the switch $r \rightsquigarrow rU^j \in \mathcal{R}$. □

Remark 41. The proposition remains true for an arbitrary Abelian quantum group G and the calculus Γ.

Let $T_{n|m}^\wedge \subseteq \Gamma_{inv}^{\wedge m}$ be the projection of $T_{n|m}$ in the universal envelope $\Gamma^\wedge$.

Proposition 44. *We have*

$$d(T_{n|m}^\wedge) \subseteq T_{n|m+1}^\wedge + T_{n+1|m+1}^\wedge. \tag{B.55}$$

B.8. **Hierarchy of Differential Structures**

Proposition 45. *The calculus Ψ is projectable to the calculus Γ if and only if $\mathcal{S} \subseteq \mathcal{R}$. This is further equivalent to the divisibility of the generating polynomials $q(z)|v(z)$.* $\square$

Remark 42. The calculus will be *irreducible* in the sense of not admitting non-trivial projections, iff its polynomial possesses no non-trivial divisors. For complex polynomials this means being linear—as the linear polynomials are the 'prime numbers' of $\mathbb{C}[z]$. If we restrict to *-covariant calculi, then irreducibility is resolved within real polynomials $\mathbb{R}[z]$. In this context we have a collection of one-dimensional calculi where $q(z) = z - \lambda$ and $\lambda \in \mathbb{R}$ and the two dimensional calculi where $q(z) = (z - \lambda)(z - \bar{\lambda})$ and $\lambda \notin \mathbb{R}$.

Proposition 46. *If Ψ is projectable to Γ and $\Psi^\wedge = \Psi^\vee$ then $\Gamma^\wedge = \Gamma^\vee$.*

Proof. Since the projection intertwines the germ maps, it transforms the quadratic relations for $\Psi^\wedge$ in $\Psi_{inv} \otimes \Psi_{inv}$ into the quadratic relations for $\Gamma^\wedge$ in $\Gamma_{inv} \otimes \Gamma_{inv}$. On the other hand, the projection clearly maps the symmetric tensors of $\Psi_{inv} \otimes \Psi_{inv}$ onto the symmetric tensors of $\Gamma_{inv} \otimes \Gamma_{inv}$. So, if the quadratic relations for $\Psi^\wedge$ cover the whole symmetric subspace of $\Psi_{inv} \otimes \Psi_{inv}$, which is equivalent to saying $\Psi^\wedge = \Psi^\vee$ then the projected relations will cover the symmetric tensors in $\Gamma_{inv} \otimes \Gamma_{inv}$ in other words $\Gamma^\wedge = \Gamma^\vee$. $\square$

Proposition 47. *If the operator $\mho \otimes \mho$ exhibits multiplicities in at least one Jordan block of its restriction on the symmetric tensors of $\Gamma_{inv} \otimes \Gamma_{inv}$ then $\Gamma^\vee \neq \Gamma^\wedge$.* $\square$

B.9. **One,** 2, 3 & 4 **Dimensions**

The simplest class of solutions, of course, is given by the one-dimensional calculi. These solutions are *irreducible* and correspond to the primes of $[z]$—the linear polynomials $q(U) = U - \lambda$. So that $p(U) = U^2 - (1 + \lambda)U + \lambda$. The classical one-dimensional calculus on $\bigcirc$ is given by $\lambda = 1$. It is easy to verify that

$$o_k = (k)_\lambda o_1$$

and also

$$(\pi \otimes \pi)\phi[p(U)] = \lambda(1 + \lambda)\, o_1 \otimes o_1. \tag{B.56}$$

So for $\lambda = -1$ the universal envelope is *free of relations* thus coinciding with the tensor algebra, and for $\lambda \neq -1$ the higher-order components vanish and $\Gamma^\wedge = \Gamma^\vee$.

And for $\lambda \neq 1$ the canonical generator for the differential is given by

$$\xi = o_1/(\lambda - 1). \tag{B.57}$$

The second simplest class is, unsurprisingly, given by the 2-dimensional structures. So $q(z) = (z - \alpha)(z - \beta)$ and the generator of the ideal $\mathcal{R}$ is

$$p(U) = U^3 - (\alpha + \beta + 1)U^2 + (\alpha + \beta + \alpha\beta)U - \alpha\beta. \tag{B.58}$$

Let us first assume that $\alpha \neq \beta$. In terms of the realization in ${}_*\daleth$ the vectors

$$|\alpha\rangle = \pi\{(U - 1)(U - \beta)\} \qquad |\beta\rangle = \pi\{(U - 1)(U - \alpha)\} \tag{B.59}$$

form a basis in Γ_{inv} and

$$\mho|\alpha\rangle = \alpha|\alpha\rangle \qquad \mho|\beta\rangle = \beta|\beta\rangle \tag{B.60}$$

thus vectors $|\alpha\rangle$ and $|\beta\rangle$ diagonalizing the $\circ$ structure in Γ_{inv}. It is elementary to see that

$$o_1 = \frac{|\alpha\rangle - |\beta\rangle}{\alpha - \beta} \tag{B.61}$$

which by the way is (B.28) in terms of the realization of Γ_{inv} in ${}_*\daleth[z]$. Therefore by (B.8) we conclude that

$$o_k = \frac{(k)_\alpha|\alpha\rangle - (k)_\beta|\beta\rangle}{\alpha - \beta} \tag{B.62}$$

for all $k \in \mathbb{Z}$. And if $\alpha, \beta \neq 1$ the canonical generator for the differential turns out to be given by

$$\xi = \frac{1}{\alpha - \beta}\Bigl[\frac{|\alpha\rangle}{\alpha - 1} - \frac{|\beta\rangle}{\beta - 1}\Bigr]. \tag{B.63}$$

Let us analyze the structure of the universal differential envelope. As a straightforward computation from (B.58) reveals

$$\begin{aligned}(\pi\otimes\pi)\phi[p(U)] = \frac{\alpha\beta(1-\alpha\beta)}{(\alpha-\beta)^2}\Bigl[|\alpha\rangle\otimes|\beta\rangle + |\beta\rangle\otimes|\alpha\rangle\Bigr] +\\ + \frac{\alpha(1+\alpha)}{(\alpha-\beta)^2}(\alpha^2-\beta)|\alpha\rangle\otimes|\alpha\rangle + \frac{\beta(1+\beta)}{(\alpha-\beta)^2}(\beta^2-\alpha)|\beta\rangle\otimes|\beta\rangle.\end{aligned} \tag{B.64}$$

This is the primary *cyclic generator* of quadratic relations for the universal differential envelope $\Gamma^\wedge$. The whole space of quadratic relations is obtained by $\circ$-acting on it with $\circ(U^n \otimes U^n)$ where $n \in \mathbb{Z}$. Under such actions, each of the 3 summands in (B.64) will acquire the multiplicative factor of $\alpha^n\beta^n$, α^{2n} and β^{2n} respectively.

So in the appropriate *generic* case for $\{\alpha, \beta\}$ this will split the vectors $|\alpha\rangle\otimes|\alpha\rangle$, $|\beta\rangle\otimes|\beta\rangle$ and $|\alpha\rangle\otimes|\beta\rangle + |\beta\rangle\otimes|\alpha\rangle$ from the generator (B.64) making the universal envelope equal to the exterior algebra.

Lemma 48. *For equality $\Gamma^\wedge = \Gamma^\vee$ it is necessary and sufficient that (different and non-0) numbers α and β satisfy:*

$$\alpha\beta \neq 1 \qquad \alpha + \beta \neq 0 \qquad \alpha \neq \beta^2 \qquad \beta \neq \alpha^2 \qquad -1 \notin \{\alpha, \beta\}. \tag{B.65}$$

Proof. For precisely in this case the coefficients in (B.64) will be non-0 and numbers α^2, $\alpha\beta$ and β^2 will all be different. ☐

And here is the collection of eight *singular* cases, producing strictly smaller spaces of quadratic relations. The first 4 cases are concrete, and the second 4 allow a continuum (either complex one or 2-dimensional) of solutions.

Cubic Roots of One. This occurs when $\{\alpha, \beta\} = \{\exp(\pm 2\pi i/3)\}$ the non-trivial cubic roots of One. The generator in (B.64) vanishes and so the quadratic relations are *totally absent*, in other words $\Gamma^\wedge = \Gamma^\otimes$. As explained in detail below, this naturally extends to arbitrary dimensions where the vanishing of the quadratic relations is equivalent to the polynomial being *cyclic*.

Imaginary Units. If $\{\alpha, \beta\} = \{i, -i\}$ then the quadratic relations space is one-dimensional, and represented by the unique relation $|i\rangle^2 = |-i\rangle^2$.

Real Units. In other words $\{\alpha, \beta\} = \{-1, 1\}$. The relations are $|1\rangle^2 = 0$ and $|-1\rangle|1\rangle = -|1\rangle|-1\rangle$. This calculus is projectable to the classical calculus on $\bigcirc$.

Square Special. Here we have $\alpha = -1$ and $\beta = \pm i$ or viceversa $\alpha = \pm i$ and $\beta = -1$. In both cases the square terms vanish, and we are left with the one-dimensional quadratic relations space saying that $|\alpha\rangle|\beta\rangle = -|\beta\rangle|\alpha\rangle$ in $\Gamma^\wedge$.

Square Generic. This is characterized by exactly one of the relations $\alpha^2 = \beta$ or $\alpha = \beta^2$, but not both (in which case would reduce to the Cubic Roots of One). It is also assumed that $\{\alpha, \beta\} \neq \{-1, \pm i\}$. In this case the quadratic relations space is 2-dimensional: $|\alpha\rangle$ and $|\beta\rangle$ anticommute and the square of exactly one of them is 0 in $\Gamma^\wedge$.

Minus One. Either $\alpha = -1$ or $\beta = -1$ with $\{1, -1, i, -i\} \cap \{\alpha, \beta\} = \varnothing$. The same 2-dimensional quadratic relations space as in the case of Square Generic.

Reciprocity. If $\alpha\beta = 1$ and $\alpha, \beta \neq \exp(\pm 2\pi i/3)$ then the quadratic relations space is 2-dimensional, and spanned by vectors $|\alpha\rangle \otimes |\alpha\rangle$ and $|\beta\rangle \otimes |\beta\rangle$.

Andipodality. If $\alpha + \beta = 0$ and $\{\alpha, \beta\} \cap \{-1, 1, -i, i\} = \varnothing$ then the relations space is 2-dimensional and spanned by vectors

$$(1+\alpha)^2|\alpha\rangle \otimes |\alpha\rangle + (1+\beta)^2|\beta\rangle \otimes |\beta\rangle \qquad |\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle.$$

Let us now assume that the $\circ$-structure has only one eigenvalue $\alpha = \beta = \lambda$ so that $p(z) = (z-1)(z-\lambda)^2$. In the basis $o_1 = |\lambda.2\rangle$ and $|\lambda\rangle = \pi\bigl[(U-1)(U-\lambda)\bigr]$ we have

$$\mho = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \qquad \mho \otimes \mho = \begin{pmatrix} \lambda^2 & 0 & 0 & 0 \\ \lambda & \lambda^2 & 0 & 0 \\ \lambda & 0 & \lambda^2 & 0 \\ 1 & \lambda & \lambda & \lambda^2 \end{pmatrix}. \tag{B.66}$$

The operator $\mho \otimes \mho$ reduces to a single Jordan block in $\mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$ with the spectrum $\{\lambda^2\}$. Its Jordan basis is given by vectors

$$o_1 \otimes o_1 \qquad \lambda\bigl[o_1 \otimes |\lambda\rangle + |\lambda\rangle \otimes o_1\bigr] + |\lambda\rangle \otimes |\lambda\rangle \qquad 3\lambda^2|\lambda\rangle \otimes |\lambda\rangle \tag{B.67}$$

with $o_1 \otimes o_1$ being the cyclical vector.

The primary quadratic relation for the universal differential envelope $\Gamma^\wedge$ is

$$\begin{aligned}(\pi \otimes \pi)\phi[p(U)] = \lambda(2\lambda+1)\Bigl\{\lambda\bigl[o_1 \otimes |\lambda\rangle + |\lambda\rangle \otimes o_1\bigr] + |\lambda\rangle \otimes |\lambda\rangle\Bigr\} + \\ + \lambda^2(\lambda^2-1)\, o_1 \otimes o_1 + \lambda(2\lambda+1)\, |\lambda\rangle \otimes |\lambda\rangle.\end{aligned} \tag{B.68}$$

So if $\lambda \neq \pm 1$ the vector (B.68) will be also cyclical in $\mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$. In other words $\diamondsuit(\Gamma) = \mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$ and $\Gamma^\wedge = \Gamma^\vee$.

In two *singular* cases $\lambda = \pm 1$ the cyclical space generated by (B.68) will be 2-dimensional and spanned by second and the third vectors in (B.67). So the relations are given by

$$|\lambda\rangle^2 = 0 = |\lambda\rangle o_1 + o_1|\lambda\rangle \tag{B.69}$$

with o_1 being completely 'free'. Furthermore, the differential is non-trivial on $\Gamma^\wedge_{inv}$ with

$$do_1 = -o_1^2 \qquad d|\lambda\rangle = \begin{cases} 0 & \lambda = -1 \\ -2o_1^2 & \lambda = 1. \end{cases} \tag{B.70}$$

And for $\lambda \neq 1$ the magical d-creating ξ assumes the form

$$\xi = \frac{o_1}{\lambda - 1} - \frac{|\lambda\rangle}{(\lambda - 1)^2}. \tag{B.71}$$

Remark 43. The projection phenomenon of Proposition 46 is clearly illustrated in these examples. If $-1 \in \Lambda$ then the universal envelope $\Gamma^\wedge$ of the calculus is always strictly larger then the exterior algebra $\Gamma^\vee$ because then the calculus is projectable to the unique exceptional one-dimensional calculus where $q(z) = z + 1$, $\lambda = -1$.

Let us now consider the case of single spectral value of cubic multiplicity $q(z) = (z - \lambda)^3$. In this case $\mho \otimes \mho$ splits into one Jordan pentaplet and one singlet in $\mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$ and the whole $\mathrm{A}(\Gamma_{inv} \otimes \Gamma_{inv})$ is just one triplet: $3^2 = 1 \oplus 3 \oplus 5$.

Another very special situation occurs when the calculus is 4-dimensional, with two spectral values of multiplicity 2. In other words $q(z) = (z - \alpha)^2(z - \beta)^2$ and $\alpha \neq \beta$. The space Γ_{inv} is spanned by vectors $|\alpha.2\rangle$, $|\alpha\rangle$, $|\beta\rangle$, $|\beta.2\rangle$. The Jordan blocks of $\mho \otimes \mho$ relative to the spectral value $\alpha\beta$ are as follows

$$\begin{array}{ccc} |\alpha.2\rangle \otimes |\beta.2\rangle & \alpha|\alpha.2\rangle \otimes |\beta\rangle + \beta|\alpha\rangle \otimes |\beta.2\rangle + |\alpha\rangle \otimes |\beta\rangle & |\alpha\rangle \otimes |\beta\rangle \\ & \beta|\alpha\rangle \otimes |\beta.2\rangle - \alpha|\alpha.2\rangle \otimes |\beta\rangle & \\ & \alpha|\beta\rangle \otimes |\alpha.2\rangle - \beta|\beta.2\rangle \otimes |\alpha\rangle & \\ |\beta.2\rangle \otimes |\alpha.2\rangle & \beta|\beta.2\rangle \otimes |\alpha\rangle + \alpha|\beta\rangle \otimes |\alpha.2\rangle + |\beta\rangle \otimes |\alpha\rangle & |\beta\rangle \otimes |\alpha\rangle \end{array} \tag{B.72}$$

and anti/symmetrizing this we obtain one symmetrical and one antisymmetrical triplet, and one symmetrical and one antisymmetrical singlet.

Proposition 49. *Both cubic $q(z) = (z-\lambda)^3$ and tetric $q(z) = (z-\alpha)^2(z-\beta)^2$ with $\alpha \neq \beta$ examples manifest a Jordan block multiplicity for $\mho \otimes \mho$ in $\mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$. Hence $\Gamma^\wedge \neq \Gamma^\vee$ for all of them.* ☐

Let us now consider cubic polynomials of the form $q(z) = (z - \alpha)(z - \beta)^2$ where $\alpha \neq \beta$. The canonical basis vectors are

$$|\alpha\rangle = \frac{1}{z - \alpha} \qquad |\beta\rangle = \frac{1}{z - \beta} \qquad |\beta.2\rangle = \frac{1}{(z - \beta)^2}$$

and the initial germ-harmonics

$$o_1 = \frac{|\alpha\rangle - |\beta\rangle}{(\alpha - \beta)^2} - \frac{|\beta.2\rangle}{\alpha - \beta}. \tag{B.73}$$

A straightforward calculation shows the cyclicity projection of the quadratic relations generator

$$\begin{aligned}(\pi \otimes \pi)\phi[p(U)] \rightsquigarrow \alpha\beta^2(\alpha - 1)\frac{1 - \alpha\beta}{(\alpha - \beta)^3}\Big[|\alpha\rangle \otimes |\beta.2\rangle + |\beta.2\rangle \otimes |\alpha\rangle\Big] + \\ + \alpha(\alpha + 1)\frac{(\beta - \alpha^2)^2}{(\alpha - \beta)^4}\, |\alpha\rangle \otimes |\alpha\rangle + \beta^2(1 - \beta^2)\frac{\alpha - \beta^2}{(\alpha - \beta)^2}\, |\beta.2\rangle \otimes |\beta.2\rangle.\end{aligned} \tag{B.74}$$

In the linear combination above, the vectors generate one Jordan block doublet, singlet and triplet respectively, for the operator $\mho \otimes \mho$.

We see that the situation is in resonance with the 2-dimensional case of simple roots α and β. The generic configuration is, however, more restricted: For the coincidence of $\Gamma^\wedge$ and $\Gamma^\vee$, it is necessary in addition to all conditions listed in (B.65), to also have $1 \notin \{\alpha, \beta\}$.

The next level of complexity is given by a cubic $q(z)$ with all simple roots. In other words $q(z) = (z-\alpha)(z-\beta)(z-\gamma)$ where $\alpha \neq \beta \neq \gamma \neq \alpha$. In this case the key vectors for computing the quadratic relations $\diamondsuit(\Gamma)$ are

$$
\begin{aligned}
o_1 &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}|\alpha\rangle + \frac{1}{(\beta-\gamma)(\beta-\alpha)}|\beta\rangle + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}|\gamma\rangle\\
o_2 &= \frac{1+\alpha}{(\alpha-\beta)(\alpha-\gamma)}|\alpha\rangle + \frac{1+\beta}{(\beta-\gamma)(\beta-\alpha)}|\beta\rangle + \frac{1+\gamma}{(\gamma-\alpha)(\gamma-\beta)}|\gamma\rangle\\
o_3 &= \frac{1+\alpha+\alpha^2}{(\alpha-\beta)(\alpha-\gamma)}|\alpha\rangle + \frac{1+\beta+\beta^2}{(\beta-\gamma)(\beta-\alpha)}|\beta\rangle + \frac{1+\gamma+\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}|\gamma\rangle\\
o_4 &= \frac{1+\alpha+\alpha^2+\alpha^3}{(\alpha-\beta)(\alpha-\gamma)}|\alpha\rangle + \frac{1+\beta+\beta^2+\beta^3}{(\beta-\gamma)(\beta-\alpha)}|\beta\rangle + \frac{1+\gamma+\gamma^2+\gamma^3}{(\gamma-\alpha)(\gamma-\beta)}|\gamma\rangle.
\end{aligned}
\tag{B.75}
$$

And a straightforward calculation reveals the following beautiful expression for the primary generator of the quadratic relations:

$$
\begin{aligned}
(\pi\otimes\pi)\phi[p(U)] = o_4\otimes o_4 &+ (\alpha+\beta+\gamma+\alpha\beta+\beta\gamma+\gamma\alpha)\, o_2\otimes o_2\\
&- (1+\alpha+\beta+\gamma)\, o_3\otimes o_3 - (\alpha\beta+\beta\gamma+\alpha\gamma+\alpha\beta\gamma)\, o_1\otimes o_1 =
\end{aligned}
\tag{B.76}
$$

$$
\begin{gathered}
(\alpha+\alpha^2)\frac{(\beta-\alpha^2)(\gamma-\alpha^2)}{(\beta-\alpha)^2(\gamma-\alpha)^2}\,|\alpha\rangle\otimes|\alpha\rangle + \frac{\beta\gamma(1-\beta\gamma)(\alpha-\beta\gamma)}{(\beta-\gamma)^2(\gamma-\alpha)(\alpha-\beta)}\Big\{|\beta\rangle\otimes|\gamma\rangle + |\gamma\rangle\otimes|\beta\rangle\Big\}+\\
(\beta+\beta^2)\frac{(\gamma-\beta^2)(\alpha-\beta^2)}{(\gamma-\beta)^2(\alpha-\beta)^2}\,|\beta\rangle\otimes|\beta\rangle + \frac{\gamma\alpha(1-\gamma\alpha)(\beta-\gamma\alpha)}{(\gamma-\alpha)^2(\alpha-\beta)(\beta-\gamma)}\Big\{|\gamma\rangle\otimes|\alpha\rangle + |\alpha\rangle\otimes|\gamma\rangle\Big\}+\\
(\gamma+\gamma^2)\frac{(\alpha-\gamma^2)(\beta-\gamma^2)}{(\alpha-\gamma)^2(\beta-\gamma)^2}\,|\gamma\rangle\otimes|\gamma\rangle + \frac{\alpha\beta(1-\alpha\beta)(\gamma-\alpha\beta)}{(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha)}\Big\{|\alpha\rangle\otimes|\beta\rangle + |\beta\rangle\otimes|\alpha\rangle\Big\}.
\end{gathered}
$$

B.10. **Any # of Dimensions**

Proposition 50. *If $\Gamma^\wedge = \Gamma^\vee$ then the generating polynomial $q(z)$ for Γ is either with simple roots or there exists a single root of order 2 and all remaining roots if any, are simple.*

Proof. If the polynomial $q(z)$ does not satisfy the property mentioned, then it either has a divisor of the form $(z-\lambda)^3$ or of the form $(z-\alpha)^2(z-\beta)^2$ where $\alpha \neq \beta$. The calculus is then projectable to the 3 or 4-dimensional calculus, associated to the corresponding divisor. To complete the proof, apply Propositions 46 and 49. □

Let us now, as a preparation for considering the most general case, focus on two extreme scenarios. At first, we shall assume that the spectral points in Λ are all simple. If so, we can write

$$
o_1 = \sum_{\lambda\in\Lambda}\Big\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{1}{\lambda-\mu}\Big\}\,|\lambda\rangle \qquad \xi = \sum_{\lambda\in\Lambda}\Big\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{1}{\lambda-\mu}\Big\}\,\frac{|\lambda\rangle}{\lambda-1}
\tag{B.77}
$$

in accordance with (B.27) and (B.42), and in the formula for ξ of course, implicitly assuming $1 \notin \Lambda$.

Proposition 51. *The quadratic relations $\diamondsuit(\Gamma)$ generator when all zeros of $q(z)$ are simple, is explicitly given by*

$$(\pi\otimes\pi)\phi[p(U)] = \sum_{\lambda\in\Lambda}\lambda(\lambda+1)\Bigl\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{(\lambda^2-\mu)}{(\lambda-\mu)^2}\Bigr\}\,|\lambda\rangle\otimes|\lambda\rangle - \tag{B.78}$$
$$-\frac{1}{2}\sum_{\mu,\nu\in\Lambda}^{\mu\neq\nu}\mu\nu\frac{\mu\nu-1}{(\mu-\nu)^2}\Bigl\{\prod_{\lambda\in\Lambda}^{\lambda\neq\mu,\nu}\frac{\mu\nu-\lambda}{(\mu-\lambda)(\nu-\lambda)}\Bigr\}\Bigl\{|\mu\rangle\otimes|\nu\rangle+|\nu\rangle\otimes|\mu\rangle\Bigr\}.$$

Proof. The formula follows from (B.51) and the above expression for ξ. This is applicable when $1\notin\Lambda$ but clearly the singular/zero terms around 1 cancel out and the formula by continuity extends to cover the case $1\in\Lambda$. ☐

A complementary situation to all-zeros-simple is $q(z)=(z-\lambda)^n$, there is only one zero λ of multiplicity n. In this case simply $o_1=|\lambda.n\rangle$ and

$$\xi = -\Bigl\{\frac{|\lambda.n\rangle}{1-\lambda}+\frac{|\lambda.n-1\rangle}{(\lambda-1)^2}+\cdots+\frac{|\lambda.2\rangle}{(1-\lambda)^{n-1}}+\frac{|\lambda\rangle}{(1-\lambda)^n}\Bigr\}. \tag{B.79}$$

The principal cyclic component for the generator of $\diamondsuit(\Gamma)$ evaluates into

$$(\pi\otimes\pi)\phi[p(U)] = p(\mho\otimes\mho)[\xi\otimes\xi] \sim \lambda^n(\lambda+1)(\lambda-1)^{n-1}\,|\lambda.n\rangle\otimes|\lambda.n\rangle. \tag{B.80}$$

Proposition 52. *If $\lambda\neq-1,1$ then $\diamondsuit(\Gamma)$ has exactly $2n-1$ dimensions.* ☐

Proposition 53. *The special case $\lambda=-1$ steals one dimension, so that $\diamondsuit(\Gamma)$ is $2n-2$-dimensional. And if $\lambda=1$ then the dimension of $\diamondsuit(\Gamma)$ is n, and $\diamondsuit(\Gamma)$ is naturally isomorphic to Γ_{inv}.* ☐

Remark 44. If $q(z)=(z-1)^n$ for $n\geq 2$ then Γ consists of classical n-currents over $\bigcirc$.

We are now ready to turn to the general case. So an arbitrary multiplicity map $m\colon\Lambda\to\mathbb{N}$ is given, with actual multiplicities $m_\lambda=m(\lambda)+1$ as introduced in Subsection B.4. As it follows from (B.31) the principal singular part of ξ is:

$$\xi\sim\sum_{\lambda\in\Lambda}\Bigl\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{1}{(\lambda-\mu)^{m_\mu}}\Bigr\}\frac{|\lambda.m_\lambda\rangle}{\lambda-1}. \tag{B.81}$$

Using this and taking advantage of Property 41 we compute the highest cyclic component of the quadratic relations generator:

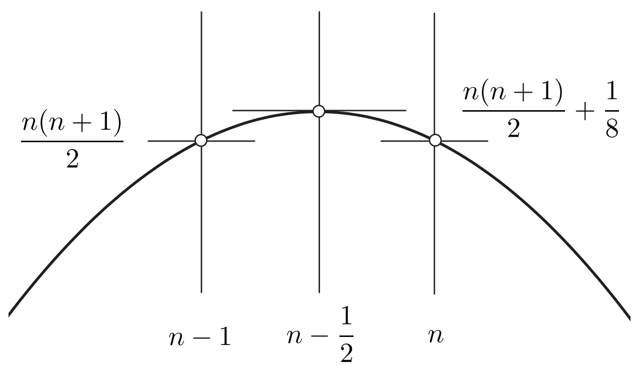
$$(\pi\otimes\pi)\phi[p(U)] = p(\mho\otimes\mho)[\xi\otimes\xi]\sim\sum_{\lambda\in\Lambda}C_\lambda\,|\lambda.m_\lambda\rangle\otimes|\lambda.m_\lambda\rangle+ \tag{B.82}$$
$$+\frac{1}{2}\sum_{\mu,\nu\in\Lambda}^{\mu\neq\nu}C_{\mu\nu}\Bigl\{|\mu.m_\mu\rangle\otimes|\nu.m_\nu\rangle+|\nu.m_\nu\rangle\otimes|\mu.m_\mu\rangle\Bigr\}$$

with easy-to-remember constants

$$C_{\mu\nu}=(\mu\nu-1)\frac{(\mu-1)^{m(\nu)}(\nu-1)^{m(\mu)}}{(\mu/\nu-1)^{m_\nu}(\nu/\mu-1)^{m_\mu}}\Bigl\{\prod_{\lambda\in\Lambda}^{\lambda\neq\mu,\nu}\Bigl[\frac{\mu\nu-\lambda}{(\mu-\lambda)(\nu-\lambda)}\Bigr]^{m_\lambda}\Bigr\}$$
$$C_\lambda=\lambda^{m_\lambda}(\lambda+1)(\lambda-1)^{m(\lambda)}\Bigl\{\prod_{\mu\in\Lambda}^{\mu\neq\lambda}\frac{(\lambda^2-\mu)^{m_\mu}}{(\lambda-\mu)^{2m_\mu}}\Bigr\}.$$

Proposition 54. *For the size of* Λ *and the dimension* n *of the calculus both fixed, the maximal dimension of the space of universal quadratic relations* $\diamondsuit(\Gamma)$ *is given by the number* $(|\Lambda|+1)(n-|\Lambda|/2)$.

Remark 45. The quadratic function $t \mapsto (t+1)(n-t/2)$ reaches its maximum value of $(n+1/2)^2/2 = n(n+1)/2 + 1/8$ at $t = n - 1/2$. The closest integer points $n-1$ and n produce the value of $n(n+1)/2$ which is the dimension of $\mathrm{S}(\Gamma_{inv} \otimes \Gamma_{inv})$. So this precisely corresponds to the equilibrium situation $\Gamma^\wedge = \Gamma^\vee$. If $|\Lambda| = n$ then all the zeros of $q(z)$ are simple, and $|\Lambda| = n-1$ means that only one zero is non-simple, and with multiplicity 2.



B.11. **Vanishing Envelope**

A very interesting special context occurs when the quadratic relations defining the universal differential envelope $\Gamma^\wedge$ trivialize, so that it becomes the whole tensor algebra $\Gamma^\otimes$.

Proposition 55. *In the general context, the quadratic relations defining the universal differential envelope* $\Gamma^\wedge$ *will trivialize iff the right ideal* $\mathcal{R}$ *defining* Γ *is also a coideal, in the sense of*

$$\phi(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{R}. \tag{B.83}$$

In this case the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi} & \Gamma_{inv} \\ \big\downarrow \phi & & -d \big\downarrow \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow[\pi \otimes \pi]{} & \Gamma_{inv} \otimes \Gamma_{inv} \end{array} \tag{B.84}$$

is commutative, in other words Γ_{inv} *possesses a natural induced coalgebra structure.*

Proof. It is clear from (B.45) that if (B.83) holds then the quadratic relations vanish. Conversely, the vanishing of the relations implies $\phi(a) = b \otimes 1 + 1 \otimes c + \sum_k R_k \otimes u_k + v_k \otimes R_k$ for every $a \in \mathcal{R}$, where $b, c \in \ker(\epsilon)$ and $R_k \in \mathcal{R}$. From the counit identity, it follows that b and c are, as a matter of fact, from $\mathcal{R}$. $\square$

Remark 46. Continuing with the general context: if $\mathcal{R}$ is generated, as a right $\mathcal{A}$ ideal, by some generators $r_1, \ldots, r_m$ satisfying the co-ideal condition $\phi(r_i) = \sum_j \big[r_j \otimes b_{ji} + c_{ij} \otimes r_j\big]$ then (B.83) holds.

Proposition 56. *In the case of $\bigcirc$ a necessary and sufficient condition for the above mentioned trivialization of the universal differential envelope $\Gamma^\wedge$ to occur, is that $p(U) = U^{n+1} - 1$. Such polynomials will be called cyclic.*

Proof. For a cyclic polynomial $p(U) = U^{n+1} - 1$ we have

$$\phi[p(U)] = p(U) \otimes 1 + U^{n+1} \otimes p(U)$$

and so the condition (B.83) holds. Conversely, if this condition holds then the set of zeros of the polynomial is closed under multiplication. This follows from the fact that the points of $\mathbb{C} \setminus \{0\}$ are in one–one correspondence with multiplicative functionals on the trigonometric polynomials $\mathcal{A}$ the correspondence being given by $\chi(U) \leftrightarrow \chi$. In terms of this correspondence the convolution product of multiplicative functionals is the product of the associated numbers.

It follows that zeros of $p(U)$ are unitary and form a cyclic group. In addition, all the zeros must be simple. $\square$

For these cyclic calculi, the element ξ generating the differential on $\mathcal{A}$ is of a particularly simple form

$$\xi = -\frac{1}{n+1} \sum_{k=1}^{n} o_k. \tag{B.85}$$

For any calculus admitting the generator ξ the relations (B.47) for the associated complete calculus can be rewritten as:

$$\xi o_k + o_k \xi = -o_k^2. \tag{B.86}$$

In the particular context of cyclic polynomials and relations-free universal differential envelopes, the constructed calculus is non-trivial and different from the exterior algebra, providing an instructive example of a truly intermediate solution for all higher dimensions n. For $n = 1$ the condition is trivially fulfilled.

And reconnecting with Remark 35, these cyclic calculi correspond to the universal differential calculus over $\mathbb{Z}_{n+1}$, effectively including the universal differential envelopes of all finite sets. The first non-trivial solution $n = 1$ corresponds to the unique singular one-dimensional case $\lambda = -1$. The universal calculus is then over the two-points set $\{-1, 1\}$. Here is the full specification of the universal calculus in terms of the characteristic functions $\circledast$ and $\ominus$ for these two points:

$$\begin{gathered} \ominus\, d\ominus = d\ominus\, \circledast \qquad \circledast\, d\circledast = d\circledast\, \ominus \qquad d\ominus + d\circledast = 0 \qquad \ominus + \circledast = 1 \\ \ominus\, d\ominus\, \ominus = 0 = \circledast\, d\circledast\, \circledast \\ \ominus\, d\ominus + \circledast\, d\circledast = 0.5\, o_1 = -\xi \qquad do_1 = 2(d\ominus)^2 = 2(d\circledast)^2 = -o_1^2. \end{gathered} \tag{B.87}$$

The spaces $\Omega^k\{-1, 1\}$ are all two-dimensional with a canonical basis given by k-forms $\ominus(d\ominus)^k$ and $\circledast(d\circledast)^k$.

Appendix C. Infinitezimal Symmetries & Primitive Elements

C.1. Infinitezimal Symmetries

Let us consider a general algebraic setup, of a first-order calculus Γ over a quantum space given by a unital *-algebra $\mathcal{A}$. Let $\mathrm{Der}(\mathcal{A} \mid \Gamma)$ be the set of all derivations $X \in \mathrm{Der}(\mathcal{A})$ for which there exists (necesarrily unique and of grade -1) antiderivation $\iota_X \colon \Gamma^\wedge \to \Gamma^\wedge$ such that

$$\iota_X(da) = X(a) \tag{C.1}$$

for every $a \in \mathcal{A}$.

Remark 47. The antiderivation requirement for ι_X when restricted to Γ says that $\iota_X \colon \Gamma \to \mathcal{A}$ is a bimodule homomorphism. This implies that X satisfies the Leibniz rule, in other words it is a derivation on $\mathcal{A}$.

Proposition 57. *A necessary and sufficient condition for $X \in \mathrm{Der}(\mathcal{A})$ to belong in $\mathrm{Der}(\mathcal{A} \mid \Gamma)$ is given by the implications*

$$\sum a\, db = 0 \quad \Rightarrow \quad \sum aX(b) = 0 \quad \& \quad \sum \{X(a)db - da\, X(b)\} = 0.$$

Proof. A direct consequence of the definition of the universal differential envelope for a first-order differential calculus. $\square$

Proposition 58. *The vector space $\mathrm{Der}(\mathcal{A} \mid \Gamma)$ is closed under commutators, in other words it is a Lie subalgebra of $\mathrm{Der}(\mathcal{A})$.*

Proof. A direct calculation, using the previous proposition criterion, shows that if $X, Y \in \mathrm{Der}(\mathcal{A} \mid \Gamma)$ then $[X, Y] = XY - YX \in \mathrm{Der}(\mathcal{A} \mid \Gamma)$. $\square$

For every $X \in \mathrm{Der}(\mathcal{A} \mid \Gamma)$ let $l_X \colon \Gamma^\wedge \to \Gamma^\wedge$ be a map given by

$$l_X = d\iota_X + \iota_X d. \tag{C.2}$$

As an anticommutator of antiderivations, it is clearly a grade-preserving derivation on $\Gamma^\wedge$ commuting with d and extending X.

Proposition 59. *The following identities hold*

$$[l_X, l_Y] = l_{[X,Y]} \qquad [l_X, \iota_Y] = \iota_{[X,Y]}$$
$$\iota_X \iota_Y + \iota_Y \iota_X = 0$$

for all $X, Y \in \mathrm{Der}(\mathcal{A} \mid \Gamma)$.

Proof. Since the commutator of derivations is a derivation, of a derivation and an antiderivation is an antiderivation, and the anticommutator of antiderivations is again an antiderivation — it is sufficient to verify all these identities on the elements of $\mathcal{A}$ and $d(\mathcal{A})$. $\square$

C.2. **Quantum Euler Classes**

Quantum Euler classes are the simplest general quantum characteristic classes, and can be defined for arbitrary quantum principal bundles. Our main class of examples are quantum Riemann surfaces obtained by appropriately projecting the quantum hyperbolic planes, so the structure group is then, of course, the classical circle.

We shall commence by considering a general compact quantum group G represented by a Hopf *-algebra $\mathcal{A}$ and equipped with a bicovariant first-order *-calculus Γ. We shall denote by $\Gamma^{\vee\!\!\!\wedge}$ any complete *-calculus built over Γ, so that the coproduct map $\phi\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ extends (necessarily uniquely) to a differential homomorphism $\widehat{\phi}\colon \Gamma^{\vee\!\!\!\wedge}\to\Gamma^{\vee\!\!\!\wedge}\mathbin{\widehat{\otimes}}\Gamma^{\vee\!\!\!\wedge}$, which is then automatically *-preserving, too. It is worth recalling that all such calculi are found between the two 'extreme' solutions. The maximal calculus $\Gamma^{\wedge}$ given by the universal differential envelope of Γ, and the minimal calculus $\Gamma^{\vee}$ given by the corresponding braided exterior algebra.

Definition 2. An element $\tau\in\Gamma^{\vee\!\!\!\wedge}$ is called primitive if

$$\widehat{\phi}(\tau)=\tau\otimes 1+1\otimes\tau. \tag{C.3}$$

Clearly, every primitive τ is left invariant, in other words $\tau\in\Gamma^{\vee\!\!\!\wedge}_{inv}$. It is also right-invariant (in this case of left invariantness we would say) ad-invariant. If $\tau\in\Gamma_{inv}$ then τ is primitive iff it is ad-invariant $\mathrm{ad}(\tau)=\tau\otimes 1$. For second-order elements of the form $\tau=\sum\theta\eta$ where $\theta,\eta\in\Gamma_{inv}$ the primitivity is equivalent to the ad-invariance coupled with $\sum\sigma(\theta\otimes\eta)=\sum\theta\otimes\eta$. Similarly, for higher order elements all the shuffling maps should vanish on them. But this, in particular, means that the braided exterior algebra has no higher-order primitive elements, so more subtle considerations are required to construct them.

Lemma 60. *If τ is primitive then $d\tau$ is primitive, too. If ρ is another primitive element then the graded commutator $\tau\rho-(-)^{\partial\tau\partial\rho}\rho\tau$ is primitive, too. In particular, if ρ is of odd degree, then ρ^2 is primitive.* □

Remark 48. It is worth observing that if τ is of odd order then $d(\tau^2)=[d\tau,\tau]$ so if τ and $d\tau$ commute then τ^2 will be *closed*. This is a nice way of producing closed primitive elements, besides taking differentials of primitive elements.

If τ is a primitive element then the ideal $I(\tau)$ generated by τ and $d\tau$ is a coideal, too. In other words

$$\widehat{\phi}[I(\tau)]\subseteq I(\tau)\otimes\Gamma^{\vee\!\!\!\wedge}+\Gamma^{\vee\!\!\!\wedge}\otimes I(\tau). \tag{C.4}$$

By definition $I(\tau)$ is d-inviariant. So we can pass to the differential factor-algebra $\Gamma^{\vee\!\!\!\wedge}/I(\tau)=\Gamma^{\tau|\vee\!\!\!\wedge}$. The first-order calculus remains intact in this factorization, and the coproduct map on $\mathcal{A}$ maintains its extensibility to the $\widehat{\phi}\colon\Gamma^{\tau|\vee\!\!\!\wedge}\to\Gamma^{\tau|\vee\!\!\!\wedge}\mathbin{\widehat{\otimes}}\Gamma^{\tau|\vee\!\!\!\wedge}$.

Remark 49. This in particular implies that $\Gamma^{\tau|\vee\!\!\!\wedge}$ still projects over the braided exterior algebra $\Gamma^{\vee}$. Hence $\Gamma^{\vee}$ is free of higher-order primitive elements. The only primitive elements of $\Gamma^{\vee}$ are the ad-invariant elements of Γ_{inv}.

We can always make primitive elements closed, by appropriately factorizing. Indeed, if τ is non-closed primitive element, project it down into $\Gamma^{d\tau|\vee\!\!\!\wedge}$ where by construction will (remain non-zero primitive and) become closed.

Closed and primitive elements are very interesting indeed. In addition they happily generate charming characteristic classes for quantum principal bundles—the quantum Euler classes, as we shall now explain.

Let us assume that a quantum principal G-bundle $P = (\mathcal{B}, \iota, F)$ over a quantum space M is given, equipped with a differential calculus $\Omega(P)$. So in particular, we have a grade preserving differential coaction *-homomorphism $\widehat{F}\colon \Omega(P) \to \Omega(P) \mathbin{\widehat{\otimes}} \Gamma^{\wedge}$ extending the coaction $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$.

Let ω be a connection on this bundle, understood in its extended form, as a unital grade and *-preserving map $\omega\colon \Gamma^{\wedge}_{inv} \to \Omega(P)$ such that the diagram

$$\begin{tikzcd} \Gamma^{\wedge}_{inv} \arrow[r, "\widehat{\phi}"] \arrow[d, "\omega"'] & \Gamma^{\wedge}_{inv} \mathbin{\widehat{\otimes}} \Gamma^{\wedge} \arrow[d, "\omega \otimes \mathrm{id}"] \\ \Omega(P) \arrow[r, "\widehat{F}"'] & \Omega(P) \mathbin{\widehat{\otimes}} \Gamma^{\wedge} \end{tikzcd} \tag{C.5}$$

is commutative.

Remark 50. As discussed in detail in [5], we can start from a pseudotensorial quantum one-form $\omega\colon \Gamma_{inv} \to \Omega(P)$ and build the extension of the connection on the whole $\Gamma^{\wedge}_{inv}$ by composing the natural (unital and multiplicative) extension $\omega^{\otimes}\colon \Gamma^{\otimes}_{inv} \to \Omega(P)$ with an appropriate embedding of $\Gamma^{\wedge}_{inv}$ into $\Gamma^{\otimes}_{inv}$. The extension will, by construction, always map into the algebra generated by the values of the original one-form map ω. A particular significance is played by *multiplicative connections* for which the extension is independent of the above mentioned embedding. Here we are not interested in these subtleties, and connections can be dealt with in a broader context, as covariant (in the sense of (C.5)) unital grade and *-preserving maps between $\Gamma^{\wedge}_{inv}$ and $\Omega(P)$. Such an interpretation breaks with Classical Origins [14] but it brings additional Flexibility via its Purity & Simplicity.

Lemma 61. *If $\tau \in \Gamma^{\wedge}_{inv}$ is primitive and closed, then $d\omega(\tau) \in \Omega(M)$. The cohomology class of this form is independent of the connection ω.*

Proof. We compute

$$\begin{aligned} \widehat{F}[d\omega(\tau)] = d\widehat{F}[\omega(\tau)] = d\bigl\{(\omega \otimes \mathrm{id})[\widehat{\phi}(\tau)]\bigr\} = d\bigl\{\omega(\tau) \otimes 1 + 1 \otimes \tau\bigr\} = \\ = d\omega(\tau) \otimes 1 + 1 \otimes d\tau = d\omega(\tau) \otimes 1 \end{aligned}$$

in other words $d\omega(\tau)$ is a form on the base M. If $\tilde{\omega}$ is another connection, then

$$\widehat{F}[\omega(\tau) - \tilde{\omega}(\tau)] = \omega(\tau) \otimes 1 + 1 \otimes \tau - \tilde{\omega}(\tau) \otimes 1 - 1 \otimes \tau = [\omega(\tau) - \tilde{\omega}(\tau)] \otimes 1$$

which means that $\omega(\tau)$ and $\tilde{\omega}(\tau)$ differ by a form on $\Omega(M)$, and this property being independent of the closeness of a primitive τ. Therefore $d\omega(\tau)$ and $d\tilde{\omega}(\tau)$ differ by an exact form in $\Omega(M)$ and hence, in the case of closed primitive τ, belong to the same cohomology class in $\Omega(M)$. ☐

Definition 3. The constructed cohomology class in $\mathrm{H}\Omega(M)$ is called the *quantum Euler class* of M associated to τ.

Remark 51. The quantum Euler classes are the simplest possible quantum characteristic classes [11], when the algebraic expression, the non-commutative polynomial involving ω and $d\omega$, reduces to just one single term, which is always precisely of the form $d\omega(\tau)$ where τ is a primitive and closed element of $\Gamma_{inv}^{\varkappa}$.

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