ON DEFINING EQUATION FOR QUANTUM HYPERBOLIC PLANES

MICHO ĐURĐEVICH & PERLA CECILIA LUCIO PEÑA

Abstract. We present detailed calculations of a one-parameter family of quantum spaces, which deform a classical Poincaré model of the hyperbolic plane, in such a way that all classical symmetries of $\mathrm{SU}(1,1)$ group are preserved. Using the theory of quantum principal bundles, it is then possible to naturally define and in-depth study quantum counterparts of all classical hyperbolic Riemann surfaces, compact and non-compact. The one-parametric family includes, as its two extreme cases, the classical hyperbolic plane as well as the quantum space of the Toeplitz extension.

1. Introduction

In this study, our main focus will be given by a class of quantum spaces based on a single quantum coordinate which relates to its conjugate via a simple basic equation of the form

$$z^*z = f(zz^*), \tag{1.1}$$

where f is an appropriate transition function. This defines our algebra at the basic C*-algebraic level.

Here z is a 'quantum complex variable'. We would like to extend all of complex analysis of the hyperbolic plane, into this non-commutative framework. The transition function allows us to easily switch between different non-commutative products between z and z^* and always reduce them into a finite or infinite linear combination of the canonical terms of the four mutually transformable basic types:

$$z^{*n}(zz^*)^m \qquad (zz^*)^m z^n \qquad z^n(z^*z)^m \qquad (z^*z)^m z^{*n}. \tag{1.2}$$

We are inspired by the classical Poincaré model, and so the basic equation should be complemented by more geometrical conditions on the coordinate z. We would like to have a geometry which retains the same symmetries as the classical hyperbolic geometry. This means that our relation (1.1) should be compatible with all Moebius transformations of the form

$$z \longmapsto \frac{az+b}{\bar{b}z+\bar{a}} \tag{1.3}$$

where $a, b \in \mathbb{C}$ are such that $|a|^2 - |b|^2 = 1$. These transformations form a simple 3-dimensional Lie group $\mathrm{H}(2) = \mathrm{SU}(1,1)$ of isometric motions of the hyperbolic plane.

We immediately see that the spectrum of z must be the whole unitary disk $\mathbb{D}$ with the boundary $\bigcirc$ included. This is because $\mathrm{H}(2)$ has 3 orbits in the Riemann sphere. These are $\bigcirc$, $\mathbb{D}$ and their complement. On the other hand, the spectrum must be invariant under all these transformations. Now, we are interested in the hyperbolic plane, so the solo circle as the spectrum is not appropriate—this would lead to a quantum version of the circle, not plane. And having in mind that the spectrum should be compact, we are left with the only possibility of the unitary disk and its boundary $\bigcirc$. This fits nicely with the quantum mechanical interpretation of the spectrum of an observable* as well as with the stochastic interpretation of points, as primary geometrical manifestations of an underlying fluctuating medium.

Furthermore it is evident that our relation (1.1) is apriori invariant under rotations around 0, since these rotations are of the form

$$z \longmapsto u^2 z \tag{1.4}$$

where u is a unitary complex number.

So in order to obtain a complete symmetry it is sufficient to ask that our algebra be compatible with the following one-parametric group of hyperbolic translations

$$z \longmapsto \frac{\cosh(t)z + \sinh(t)}{\sinh(t)z + \cosh(t)} \tag{1.5}$$

where $t \in \mathbb{R}$. And the compatibility with this one-parametric group, in our geometric context, is equivalent with the compatibility with its generating derivation D. By differentiating the above expression relative to t and taking $t = 0$ we obtain

$$D(z) = 1 - z^2 \qquad D(z^*) = 1 - z^{*2}. \tag{1.6}$$

So the hyperbolic symmetry emerges as soon as we are in harmony with the above formulae, in other words when these expression consistently extend to a derivation of our algebra.

Let us outline the content of this paper. In the next section we present detailed calculations determining the basic function f, by requiring that the above mentioned derivation is an infinitesimal symmetry realizable in the underlying Hilbert space. As we shall see, this essentially uniquely determines such a derivation, and the possible form of the function. We obtain a one-parametric family of quantum spaces, corresponding to a parabolic spectrum of Moebius transformations, with the fixed point at 1. This has a very nice geometrical relationship with the unit circle, which provides the horizon of quantum and classical hyperbolic planes. We derive a closed formula for stochastic points. In Section 3, we analyze some general algebraic properties of our defining relation.

And in Section 4 a number of complementary remarks is made. We analyze the relationship of the quantum spaces with the canonical Toeplitz extension. We explain that at the basic C*-algebraic level, all non-commutative structures are the same. They differ in the interpretation of symmetries and stochastic points, which re-emphasizes the importance of these two fundamental conceptual ways in analyzing quantum spaces. The *Symmetry Way* (originating basically in Erlangen Program of Klein) and Aether Way, magnificently established in Quantum Geometry by Prugovečki. We further discuss similarities and differences regarding the Woronowicz approach of deforming $\mathrm{SU}(1,1)$ and constructing the resulting quantum hyperbolic plane as a hyperbolic Hopf fibration. Finally, we explain how the classical hyperbolic metric is obtained from quantum transition amplitude of the stochastic points, and derive the partition of unity formula for these coherent states.

*Although here our main observable is not at all self-adjoint!

2. Direct Calculations in Base Hilbert Space

We shall first analyze a concrete situation, where the fundamental relation (1.1) is realized in the Hilbert space $\mathcal{H} = l^2(\mathbb{N})$ where $\mathbb{N}$ are natural numbers (Including 0) and the operators S and S^* act in a very specific way

$$S|n\rangle = c_{n+1}|n+1\rangle \qquad S^*|n+1\rangle = c_{n+1}|n\rangle, \tag{2.1}$$

where we use the Dirac notation[†] for the elementary orthonormal base in $\mathcal{H}$. We can assume, without a lack of generality, that all $c_n \geq 0$ for $n \geq 1$. As we shall see, our symmetry conditions will give a definitive shape to this sequence, in particular all of them are strictly positive. The second equation in (2.1) should be complemented by

$$S^*|0\rangle = 0 \qquad c_0 = 0. \tag{2.2}$$

Our geometric interpretation is that S corresponds to an initial quantum complex variable z. And so S^* is its conjugate $\bar{z} \leftrightarrow z^*$. This will be especially fruitful in connecting the formalism with the theory of square-summable Hilbert spaces of holomorphic functions [1]. The vector $|0\rangle$ can be interpreted as a physical manifestation of the origin point 0 for a quantum Poincaré model. The other vectors $|n\rangle$ where $n \geq 1$ can be interpreted as higher harmonics around 0.

We follow here the philosophy of Prugovečki [8], where geometry is established by specifying un underlying field which gives, via its stochastic fluctuation modes, the structured collection of points, interpretable as a geometric space. This will be further refined by considering the corresponding coherent states, the eigenvectors of S^* corresponding to arbitrary points $w \in \mathbb{D}$ of the unitary disk $\mathbb{D}$. We shall also slightly simplify the notation and write

$$c_1 = c. \tag{2.3}$$

As we shall see, under the appropriate symmetry conditions, all further coefficients are determined by this initial one number c.

The 'radial fluctuating' modes diagonalized in our base, and given by

$$S^*S|n\rangle = c_{n+1}^2|n\rangle \qquad SS^*|n\rangle = c_n^2|n\rangle. \tag{2.4}$$

Now our abstract relation (1.1) becomes

$$c_{n+1}^2 = f(c_n^2). \tag{2.5}$$

Now we shall assume that the hyperbolic symmetries are realized by unitary operators in $\mathcal{H}$. In particular the principal hyperbolic derivation D should be given by the commutator with an anti-hermitian operator X acting in $\mathcal{H}$, so that

$$D(a) = [X, a] \tag{2.6}$$

for the elements $a \in A$. In particular,

$$XS - SX = 1 - S^2 \qquad XS^* - S^*X = 1 - S^{*2}. \tag{2.7}$$

If we act by the first equation on $|n\rangle$ we obtain a recursive formula for the vectors $X|n\rangle$. It reads

$$c_{n+1}X|n+1\rangle = SX|n\rangle + |n\rangle - c_{n+1}c_{n+2}|n+2\rangle. \tag{2.8}$$

[†]We shall also use the same notation for some important vectors and operators naturally appearing in the game.

On the other hand, the second formula in (2.7) applied on $|0\rangle$ gives

$$S^*X|0\rangle = -|0\rangle \tag{2.9}$$

which has as its simplest solution

$$X|0\rangle = \frac{-}{c}|1\rangle. \tag{2.10}$$

Now we can use the recursive formula (2.8) to find the remaining vectors.

Lemma 2.1. *Under the above assumptions we have*

$$X|n\rangle = \frac{n}{c_n}|n-1\rangle - c_{n+1}(n+\frac{1}{c^2})|n+1\rangle \tag{2.11}$$

for each $n \geq 1$. *In particular, all the coefficients* c_n *must be non-zero.*

Proof. For $n = 1$ the formula reads

$$X|1\rangle = \frac{1}{c}|0\rangle - c_2(1+\frac{1}{c^2})|2\rangle$$

which follows by substituting (2.9) in the recursive formula for $n = 1$. Let us assume that the formula holds for some n. Then (2.8) gives

$$X|n+1\rangle = \frac{1}{c_{n+1}}\Bigl\{S\Bigl[\frac{n}{c_n}|n-1\rangle - c_{n+1}(n+\frac{1}{c^2})|n+1\rangle\Bigr] + |n\rangle\Bigr\} - c_{n+2}|n+2\rangle =$$
$$= \frac{n+1}{c_{n+1}}|n\rangle - c_{n+2}(n+1+\frac{1}{c^2})|n+2\rangle$$

and so the formula holds for $n+1$, too. The statement follows by induction. □

We have derived the formula for the action of X on all the elementary basis vectors. But we have still not used the antihermicity condition for X.

Lemma 2.2. *The antihermicity condition is equivalent to*

$$c_{n+1} = \sqrt{\frac{(n+1)c^2}{1+nc^2}} \qquad \forall n \in \mathbb{N}. \tag{2.12}$$

In particular we see that all $0 < c_n < 1$ *and that our sequence is strictly increasing with* $\lim_n c_n = 1$.

Proof. We see that the matrix representing X is almost diagonal: non-zero things happen only on the first upper and lower diagonals. So the antihermicity is equivalent to

$$\langle n|X|n+1\rangle = -\langle n+1|X|n\rangle. \tag{2.13}$$

A direct calculation of the above matrix elements by using (2.11) leads to the desired formula. □

From this, we can easily find the direct link between two radial forms and derive the function f. Indeed, it follows that

$$\frac{1}{1-c_{n+1}^2} = \frac{1}{1-c_n^2} + \frac{c^2}{1-c^2}. \tag{2.14}$$

So the resulting transformation is a parabolic Moebius transformation

$$\frac{1}{1-f(t)} = \frac{1}{1-t} + \nu \qquad \nu = \frac{c^2}{1-c^2} \tag{2.15}$$

the similarity transformation of the translation $t \mapsto t+\nu$ via the inversion $t \mapsto 1 : (1-t)$. Quite explicitly, the defining function f is given by

$$f(t) = \frac{c^2 + (1-2c^2)t}{1-c^2t} = \frac{(1-\nu)t+\nu}{1+\nu-\nu t}. \tag{2.16}$$

The last form is particularly suitable for calculations as it gives the perfect matrix representation

$$f \longleftrightarrow \begin{pmatrix} 1-\nu & \nu \\ -\nu & 1+\nu \end{pmatrix} \tag{2.17}$$

of the entire parabolic spectrum of circles, as a one-parameter group of Moebius transformations.

Let us know consider the coherent states for our system. They are given by eigenvectors of S^*. Given $w \in \mathbb{D}$ we shall denote by $|w\rangle$ a natural normalized such a vector. Specifically,

$$S^*|w\rangle = \bar{w}|w\rangle \qquad \langle w|w\rangle = 1. \tag{2.18}$$

Clearly this still leaves an ambiguity of one single phase factor. Let us calculate these states. We have

$$S^*|w\rangle = \sum_{n\geq 1} w_n c_n |n-1\rangle = \sum_{n\geq 0} \bar{w} w_n |n\rangle \tag{2.19}$$

which gives a recurrent expression

$$w_{n+1} = \bar{w} w_n / c_{n+1} \tag{2.20}$$

or explicitly

$$w_n = \varkappa \bar{w}^n / \prod_{k=1}^{n} c_k \tag{2.21}$$

where $\varkappa > 0$ is the normalization factor.

In order to proceed with our calculations, let us introduce an important auxiliary parametrized function

$$\hbar(z,\nu) = \sum_{n=0}^{\infty} z^n \prod_{k=1}^{n} c_k^{-2} = \sum_{n=0}^{\infty} z^n \Big\{ \prod_{k=1}^{n} (1 + \frac{1}{k\nu}) \Big\}. \tag{2.22}$$

Lemma 2.3. *The function $\hbar(z)$ is holomorphic inside of the unit disk $\mathbb{D}$ where the above series converges absolutely and uniformly on compact sets. The following expression holds:*

$$\hbar(z,\nu) = \Big[\frac{1}{1-z}\Big]^{1+1/\nu} \tag{2.23}$$

for each $z \in \mathbb{D}$ and $\nu > 0$.

Proof. Let us recall the binomial series expansion

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \tag{2.24}$$

which is valid for every $\alpha\in\mathbb{C}$ and $z\in\mathbb{D}$. Inside of the unitary disk $\mathbb{D}$ the series converges normally. Now we can rewrite this as

$$(1-z)^\alpha=\sum_{n=0}^\infty\Bigl\{\prod_{k=1}^n\bigl(1-\frac{\alpha+1}{k}\bigr)\Bigr\}z^n \tag{2.25}$$

which reproduces our expression for $\alpha+1=-1/\nu$. □

We can write down an explicit expression for the above normalization constant:

$$\varkappa^2=(1-|w|^2)^{1+1/\nu}. \tag{2.26}$$

It is also worth noticing that the above exponent is simply $1+1/\nu=1/c^2$. Around 0 our function f has the following power series expansion:

$$f(t)=\frac{\nu}{\nu+1}+\frac{1}{\nu(\nu+1)}\sum_{n=1}^\infty\Bigl[\frac{\nu t}{1+\nu}\Bigr]^n \tag{2.27}$$

3. Abstract Generators/Relations Approach

We shall here focus on abstract algebraic properties of the defining function f. As already mentioned, the basic functional relation allows us to 'move around' in diverse calculations within our algebra. If g is any analytic function then

$$zg(z^*z)=g(zz^*)z \qquad z^*g(zz^*)=g(z^*z)z^* \tag{3.1}$$

which shows how to move from left to right of the radial expressions—the ones involving zz^* and z^*z. The radial expressions generate a commutative subalgebra.

We can generalize the above expressions, by iterating them, to include the powers of z and z^*. In such a way we obtain

$$\begin{gathered} z^ng(z^*z)=[gf^{-n}](z^*z)z^n\\ z^{*n}g(zz^*)=[gf^n](zz^*)z^{*n}\end{gathered} \tag{3.2}$$

for every $n\in\mathbb{N}$.

Now let us assume that there is a derivation D acting on our algebra, such that (1.6) holds. If we act by it on constituting elements of the relation (1.1) we obtain

$$D(z^*z)=(1-z^{*2})z+z^*(1-z^2)=z\bigl[1-f^2(zz^*)\bigr]+\bigl[1-f^2(zz^*)\bigr]z^*,$$

and also

$$\begin{gathered}D[(zz^*)^n]=\sum_{k+l}^{n-1}(zz^*)^k\bigl[(1-z^2)z^*+z(1-z^{*2})\bigr](zz^*)^l=\sum_{k+l}^{n-1}(zz^*)^kf(zz^*)^lz^*\\ -\sum_{k+l}^{n-1}zf(zz^*)^k(zz^*)^{l+1}+\sum_{k+l}^{n-1}zf(zz^*)^k(zz^*)^l-\sum_{k+l}^{n-1}(zz^*)^{k+1}f(zz^*)^lz^*.\end{gathered}$$

By combining the terms of the positive and negative rotational symmetry, we arrive at

$$1-f^2(zz^*)=\sum_{n=1}^\infty\Bigl\{\sum_{k+l}^{n-1}\bigl[(zz^*)^k-(zz^*)^{k+1}\bigr]f(zz^*)^l\Bigr\}\o_n \tag{3.3}$$

where we have assumed that

$$f(t) = \sum_{n=0}^{\infty} \o_n t^n. \tag{3.4}$$

By further rearranging the terms we obtain the following functional equation

$$1 - f[f(t)] = (1-t) \sum_{n=1}^{\infty} \big\{\rho_n[f](t)\big\} f(t)^{n-1} \tag{3.5}$$

where ρ_n are the n-fold truncated parts of the power series:

$$\rho_n[f](t) = \sum_{k=0}^{\infty} \o_{n+k} t^k. \tag{3.6}$$

4. Additional Comments

4.1. **Comparison with the Quantum** SU(1, 1) **Group of Woronowicz**

4.2. **Relation With Toeplitz Extension**

We see that for any positive ν the operator S^*S is invertible and we have

$$\frac{1}{\sqrt{S^*S}} S\colon |n\rangle \mapsto |n+1\rangle \tag{4.1}$$

which shows that the standard shift operator S_∞ and hence the entire Toeplitz extension, is included in the C*-algebra A. And since the coefficients c_n tend to one, it is easy to see that moreover

$$S - S_\infty \in \mathbb{K} \tag{4.2}$$

which shows that the Toeplitz extension (the C*-algebra generated by S_∞ and S_∞^*f) is actually identical to the C*-algebra A. In particular we have a short exact sequence

$$0 \to \mathbb{K} \rightarrowtail A \twoheadrightarrow \mathrm{C}(\bigcirc) \to 0 \tag{4.3}$$

of C*-algebras. However, although the structure exhibits the same C*-algebra, geometrically we have *different* spaces, since the geometry is specified by

- **Symmetry Way:** The basic transformations of the quantum space. In our case the group is H(2) and it acts differently depending on the translation parameter ν.
- **Aether Way:** The collection of states that provides a physical description of stochastic points. These are different vectors and the transition probabilities between the stochastic points are different, although they all lead to the same classical hyperbolic metric.

4.3. **Coherent States** & **Hyperbolic Metric**

Let us here calculate the quantum transition amplitude for two stochastic points of the quantum hyperbolic plane.

Proposition 4.1. *We have*

$$\langle z|w\rangle = \Big[\frac{\sqrt{1-|z|^2}\sqrt{1-|w|^2}}{1-z\bar{w}}\Big]^{1/c^2} \tag{4.4}$$

for every $z, w \in \mathbb{D}$. In particular, the classical hyperbolic distance between z and w is given by

$$\Big[\frac{1}{|\langle z|w\rangle|}\Big]^{c^2} = \cosh\big[d(z,w)/2\big]. \tag{4.5}$$

Proof. The expression (4.4) is a direct application of the definition of coherent states, and the calculated quantities via $\hbar$. The classical formula for the hyperbolic distance between z and w is

$$\tanh\big\{d(z,w)/2\big\} = \Big|\frac{z-w}{1-z\bar{w}}\Big| \qquad \frac{1}{\cosh\big\{d(z,w)/2\big\}} = \frac{\sqrt{1-|z|^2}\sqrt{1-|w|^2}}{|1-z\bar{w}|} \tag{4.6}$$

and from (4.4) the quantum distance formula follows. □

It is worth noticing how the amplitude of the transition approaches zero when $c \rightsquigarrow 0$. This is the classical limit. The complementary limiting case $c \rightsquigarrow 1$ gives the standard Toeplitz algebra presentation. It is the most fluctuating of all quantum hyperbolic models. It exhibits various singular purely quantum properties.

Coherent states provide a simple and elegant partition of unity.

Proposition 4.2. *We have*

$$\pi\nu = \int_{\mathbb{D}} |w\rangle\langle w| \tag{4.7}$$

where the integration is taken relative to the hyperbolic measure.

Proof. This is a direct calculation of matrix elements. Due to angular symmetry, it is clear that off-diagonal elements are zero. So we compute

$$\begin{gathered}\langle n|\Big\{\int_{\mathbb{D}}|w\rangle\langle w|\Big\}|n\rangle = \int_{\mathbb{D}}(1-|w|^2)^{1+1/\nu}|w|^{2n}\prod_{k=1}^{n}(1+\frac{1}{\nu k}) = \pi\int_0^1(1-t)^{1/\nu-1}t^n\,dt\times \\ \times\prod_{k=1}^{n}(1+\frac{1}{\nu k}) = \pi B(1/\nu,n+1)\prod_{k=1}^{n}(1+\frac{1}{\nu k}) = \\ = \pi\frac{\Gamma(1/\nu)\Gamma(n+1)}{\Gamma(1/\nu+1+n)}\prod_{k=1}^{n}(1+\frac{1}{\nu k}) = \pi n!\Big\{\prod_{k=0}^{n}\frac{1}{1/\nu+k}\Big\}\Big\{\prod_{k=1}^{n}(1+\frac{1}{\nu k})\Big\} = \pi\nu\end{gathered}$$

and thus (4.7) holds. □

It is interesting to see that in the limit of the standard Toeplitz extension $\nu \rightsquigarrow \infty$ the stochastic points fluctuate too much, and the integral diverges.

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Institute of Mathematics, Universidad Nacional Autonoma de México, Area de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, Mexico City, CP 04510, MEXICO

E-mail address: micho@matem.unam.mx, perla@matem.unam.mx