A Conjecture on Cycle–Pancyclism in Tournaments

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Abstract

Let T be a hamiltonian tournament with n vertices and γ a hamiltonian cycle of T. In previous works we introduced and studied the concept of cycle–pancyclism to capture the following question: What is the maximum intersection with γ of a cycle of length k? More precisely, for a cycle C_k of length k in T we denote $\mathcal{I}_\gamma(C_k) = |A(\gamma) \cap A(C_k)|$, the number of arcs that γ and C_k have in common. Let $f(k,T,\gamma) = \max\{\mathcal{I}_\gamma(C_k)|C_k \subset T\}$ and $f(n,k) = \min\{f(k,T,\gamma)|T$ is a hamiltonian tournament with n vertices, and γ a hamiltonian cycle of $T\}$. In previous papers we gave a characterization of $f(n,k)$. In particular, the characterization implies that $f(n,k) \geq k-4$.

The purpose of this paper is to give some support to the following original conjecture: for any vertex v there exists a cycle of length k containing v with $f(n,k)$ arcs in common with γ.

Key Words. Tournaments, pancyclism, cycle–pancyclism.

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1 Introduction

Recall that a *tournament* is a digraph in which each pair of vertices is connected by exactly one arc, that is, a complete asymmetric digraph. Quoting from the classical textbook by Behzad, Chartrand and Lesniak-Foster [3] (p. 353), among the various classes of digraphs, the tournaments are probably the most studied and most applicable. The book by Moon [9] treats these digraphs in great detail. The book by Robinson and Foulds [11], and the book [3] itself dedicate one chapter to tournaments.

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The subject of pancyclism in tournaments is a classical subject in the study of tournaments; it has been treated in textbooks (e.g. [3]) and in many papers (e.g. [1, 2, 4, 10, 12]). Two types of pancyclism have been considered. A tournament T is *vertex-pancyclic* if given any vertex v there are cycles of every length containing v. Similarly, a tournament T is *arc-pancyclic* if given any arc e there are cycles of every length containing e. It is well known, and perhaps surprising, that if a tournament has a cycle going through all of its vertices (i.e. it has a *hamiltonian cycle* or the tournament is *hamiltonian*) then it is vertex-pancyclic. This result was first proved by Moon [8], and a proof by C. Thomassen can be found in [3] p. 358. It is easy to see that a vertex-pancyclic tournament is not necessarily arc-pancyclic.

In a previous paper, [5], we introduced the concept of *cycle-pancyclism* to try to understand in more detail the structure of a pancyclic tournament; to explore how are the cycles of the various lengths positioned with respect to each other. We considered questions such as the following. Given a cycle C of a tournament T with n vertices, what is the maximum number of arcs which a cycle of length k contained in C has in common with C? In [5, 6, 7] we discovered that, for every k, there is always a cycle of length k, with its vertices contained in C, and all of its arcs contained in C except for at most 4: "almost" completely contained in C. This result implies that for any given hamiltonian cycle γ_n of T, there is a cycle γ_{n-1} of length $n-1$ contained in γ_n with at most 4 edges not in γ_n. By considering the subtournament of T with $n-1$ vertices induced by γ_{n-1}, we can repeat this argument and obtain cycles $\gamma_{n-2}, \gamma_{n-3}, \ldots$, such that each γ_i is "almost" completely contained in γ_{i+1}.

In this paper we suggest –and present some evidence– that a similar result may hold, even if we add the requirement that the cycle "almost" completely contained in C passes through a specified vertex. Informally, assume that a hamiltonian cycle γ of a tournament T, and a vertex 0 are given, and we ask what is the maximum number of arcs that γ and a cycle of length k going through 0 have in common. This kind of result would considerably strengthen the vertex-pancyclism classical result.

We proceed with a formal description of the problem. Let T be a hamiltonian tournament with vertex set V and arc set A. Assume without loss of generality that $V = \{0, 1, \ldots, n-1\}$ and $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian cycle of T. Let C_k denote a directed cycle of length k. For a cycle C_k we denote $\mathcal{I}_\gamma(C_k) = |A(\gamma) \cap A(C_k)|$, or simply $\mathcal{I}(C_k)$ when γ is known, the number of arcs that γ and C_k have in common. Let $f(k, T, \gamma) = \max\{\mathcal{I}_\gamma(C_k) | C_k \subset T\}$ and $f(n, k) = \min\{f(k, T, \gamma) | T$ is a hamiltonian tournament with n vertices, and γ a hamiltonian cycle of $T\}$. In [5, 6, 7] we gave a characterization of $f(n, k)$:

- $f(n, 3) = 1$, $f(n, 4) = 1$ and $f(n, 5) = 2$ if $n \neq 2k - 2$;
- $f(n, k) = k - 1$ if and only if $n = 2k - 2$.

For $n \geq 2k - 4$ and $k > 5$,

- $f(n, k) = k - 2$ if and only if $n \neq 2k - 2$ and $n \equiv k \pmod{k-2}$;

- $f(n,k) = k-3$ if and only if $n \not\equiv k \pmod{k-2}$.

For $n \leq 2k-5$,

- $f(n,k) = k-4$.

That is, we showed that there is always a cycle C_k almost completely contained in γ; except for at most 4 arcs. The purpose of this paper is to conjecture that the same results hold if we in addition require that the cycles pass through a fixed vertex; that is, that for any vertex v there exists a cycle of length k containing v with $f(n,k)$ arcs in common with γ. As evidence for the conjecture, we present various particular cases in which this equality holds.

More precisely, for a vertex v of a hamiltonian tournament T with n, let

$$\tilde{f}(k,T,\gamma,v) = \max\{\mathcal{I}_\gamma(C_k) | C_k \subset T\},$$

for short be denoted sometimes $\tilde{f}(n,k,T)$, and to stress that T has n vertices. Let $\tilde{f}(n,k) = \min\{\tilde{f}(k,T,\gamma,v) | T,\ v \in T$, and γ a hamiltonian cycle of $T\}$. Clearly, $\tilde{f}(n,k) \leq f(n,k)$. We conjecture that $\tilde{f}(n,k) = f(n,k)$.

We know that the conjecture is true in the following particular cases. When

- $k = 3,4,5,6$;
- $n = 2k-2, 2k-3, 2k-4$;
- $r = k-1, k-2$, where $n-k+1 \equiv r \pmod{k-2}$.

The proofs are identical to the ones in [5], except for the proof of case $r = k-2$, which is similar, and the case $k = 6$ which is new. For completeness we include all the proofs here.

2 Preliminaries

In the rest of this paper we consider an arbitrary tournament T with n vertices, with some fixed vertex 0, and a hamiltonian cycle $\gamma = (0,1,\ldots,n-1,0)$.

A *chord* of a cycle C is an arc not in C with both terminal vertices in C. The *length* of a chord $f = (u,v)$ of C, denoted $l(f)$, is equal to the length of $\langle u,C,v\rangle$, where $\langle u,C,v\rangle$ denotes the uv–directed path contained in C. We say that f is a c-chord if $l(f) = c$ and $f = (u,v)$ is a $-c$-chord if $l\langle v,C,u\rangle = c$. Observe that if f is a c-chord then it is also a $-(n-c)$-chord.

In what follows every integer is taken modulo n.

For any a, $2 \leq a \leq n-2$, denote by t_a the largest integer such that $a + t_a(k-2) < n-1$. The important case of t_{k-1} is denoted by t in the rest of the paper. Let r be defined as follows: $r = n - [k-1+t(k-2)]$.

Notice the following facts.

- If $a \leq b$, then $t_a \geq t_b$.
- $t \geq 0$.
- $2 \leq r \leq k-1$.

Lemma 2.1 *If the a-chord with initial vertex* 0 *is in* A*, then at least one of the two following properties holds.*

(i) $\tilde{f}(n,k,T) \geq k-2$.

(ii) *For every* $0 \leq i \leq t_a$*, the* $a+i(k-2)$*-chord with initial vertex* 0 *is in* A*.*

Proof: Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1,2,\ldots,t_a\} \mid (a+i(k-2),0) \in A\},$$

then

$$C_k = (0, a+(j-1)(k-2)) \cup \langle a+(j-1)(k-2), \boldsymbol{\gamma}, a+j(k-2)\rangle \cup (a+j(k-2),0)$$

is a cycle such that $\mathcal{I}(C_k) = k-2$ with $0 \in C_k$, and hence (i) in the lemma is true. ∎

3 The Cases $k = 3, 4, 5$

Theorem 3.1 $\tilde{f}(n,3) \geq 1$.

Proof: Let $i = \min\{j \in V | (j,0) \in A\}$. Observe that i is well defined since $(n-1,0) \in A$. Clearly $i \neq 1$, so $i-1 > 0$ and then $(0, i-1, i, 0)$ is a cycle C_3 with $\mathcal{I}(C_3) \geq 1$. ∎

Theorem 3.2 $\tilde{f}(n,4) \geq 1$.

Proof: We proceed by contradiction. Taking $a = 3$ and $x_0 = 0$ in Lemma 2.1 we get that for each i, $0 \leq i \leq t_a$, the $(3+2i)$-chord $(0, 3+2i)$ is in A. Recall that t_a is the greatest integer such that $3 + 2t_a < n-1$.

When n is even, it holds that $t_a = (n-4)/2 - 1$, $(0, 3+2t_a) \in A$. That is, $(0, n-3) \in A$ and $C_4 = (0, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_4) = 3$. When n is odd, it holds that $t_a = \lfloor \frac{n-4}{2} \rfloor$ and $(0, 3+2t_a) \in A$, namely $(0, n-2) \in A$.

Now, we may assume that $(n-3, 0) \in A$, because otherwise the cycle $C_4 = (0, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(C_4) = 3$. If $(n-1, n-3) \in A$ then $C_4 = (n-1, n-3, 0, n-2, n-1)$ is a cycle with $\mathcal{I}(C_4) = 1$. Else, $(n-3, n-1) \in A$ and $C_4 = (n-3, n-1, 0, n-4, n-3)$ is a cycle with $\mathcal{I}(C_4) = 1$. ∎

Theorem 3.3 $\tilde{f}(n,5) \geq 2$.

Proof: We consider the three cases $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$.

Case $n \equiv 2 \pmod 3$. Taking $a = 4$ in Lemma 2.1, we get that $(0, n-4) \in A$ and $C_5 = (0, n-4, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 4$.

Case $n \equiv 1 \pmod 3$. Taking $a = 4$ in Lemma 2.1, we get that $4 + 3t_4 = n - 3$. Hence $(0, n-3) \in A$ and $(0, n-6) \in A$. Observe that $(n-4, 0) \in A$. Otherwise $(0, n-4) \in A$ and $C_5 = (0, n-4, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 4$.

Now, if $(n-2, n-5) \in A$ then $C_5 = (n-2, n-5, n-4, 0, n-3, n-2)$ is a cycle with $\mathcal{I}(C_5) = 2$. Else $(n-5, n-2) \in A$ and $C_5 = (0, n-6, n-5, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 3$.

Case $n \equiv 0 \pmod 3$. If $(0,3) \in A$ then taking $a = 3$ in Lemma 2.1, we obtain that $(0, n-6) \in A$ and $(0, n-3) \in A$. The proof proceeds exactly as in the proof for the case $n \equiv 1 \pmod 3$. Hence, let us assume that $(3, 0) \in A$.

Observe that $(5, 0) \in A$, because otherwise $(0, 5) \in A$ and taking $a = 5$ in Lemma 2.1, we get that $(0, n-4) \in A$ and $C_5 = (0, n-4, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 4$.

Therefore we have that $(5, 0) \in A$ and $(3, 0) \in A$. Considering the cycle $(0, 1, 2, 3, 4, 5, 0)$ it is easy to check that $(5, 3) \in A$ and $(1, 5) \in A$ (or else the proof follows). Analyzing the direction of the arc joining 2 and 5 we see that in any case there is a cycle C_5 with $\mathcal{I}(C_5) = 2$: If $(5, 2) \in A$ then the cycle is $C_5 = (3, 0, 1, 5, 2, 3)$, else, if $(2, 5) \in A$ then the cycle is $C_5 = (3, 0, 1, 2, 5, 3)$. ■

4 The case of $n = 2k - 4$

In this section it is proved that if $n = 2k - 4$ then $\tilde{f}(n, k) \geq k - 3$.

Theorem 4.1 *If $n = 2k - 4$ then $\tilde{f}(n, k) \geq k - 3$.*

Proof: Let x and y be two vertices of T such that $l\langle x, \gamma, y\rangle = l\langle y, \gamma, x\rangle = k - 2$. Without loss of generality we can assume that $x = 0$, $y = k - 2$ and $(0, k-2) \in A$. Hence $(k-1, 2)$ is a $(k-1)$-chord, $l\langle 2, \gamma, k-1\rangle = k - 3$, $(1, k)$ is a $(k-1)$-chord and $l\langle 2, \gamma, k+1\rangle = k - 1$.

- $(k, 2) \in A$. Otherwise $(2, k) \in A$ and then $C_k = (k-2, k-1, 2, k) \cup \langle k, \gamma, 0\rangle \cup (0, k-2)$ is a cycle with $\mathcal{I}(C_k) = k - 3$.
- $(1, k-1) \in A$. Otherwise $(k-1, 1) \in A$ and then $C_k = (k-1, 1, k) \cup \langle k, \gamma, 0\rangle \cup (0, k-2, k-1)$ is a cycle with $\mathcal{I}(C_k) = k - 3$.

Therefore, since $(k, 2) \in A$ and $(1, k-1) \in A$ then $C_k = (1, k-1, k, 2, k+1) \cup \langle k+1, \gamma, 1\rangle$ is a cycle with $\mathcal{I}(C_k) = k - 3$. Notice that $0 \in \langle k+1, \gamma, 1\rangle$. ■

5 The case of $r = k - 1$ and $r = k - 2$

In this section it is proved that if $r = k - 1$ or $r = k - 2$ then $\tilde{f}(n, k) \geq k - 3$.

Theorem 5.1 *If $r = k - 1$ or $r = k - 2$ then $\tilde{f}(n, k) \geq k - 3$.*

Proof: Assume $r = k - 1$. By Lemma 2.1 (taking $i = 0$) either $\tilde{f}(n, k, T) \geq k - 2$ or $(0, k - 1) \in A$. In the latter case we have that $\langle k - 1 + t(k - 2), \gamma, 0\rangle \cup (0, k - 1 + t(k - 2))$ is a cycle of length k intersecting γ in $k - 1$ arcs. Thus, in both cases, $\tilde{f}(n, k, T) \geq k - 2$.

Now, assume $r = k - 2$ and $\tilde{f}(n, k, T) < k - 3$.

We consider the vertices $x = k - 1 + t(k - 2)$, $y = k - 1 + (t - 1)(k - 2)$. Observe that when $t = 0$ we obtain $y = 1$.

(i) $(0, x) \in A$. It follows from Lemma 2.1.

(ii) $(x - 1, 0) \in A$. It follows directly from the case $r = k - 1$.

(iii) $(x, y) \in A$. If $(x, y) \notin A$ then $(y, x) \in A$ and $(y, x) \cup \langle x, \gamma, 0\rangle \cup (0, y)$ (Lemma 2.1 implies $(0, y) \in A$) is a cycle of length k intersecting γ in at least $k - 2$ arcs.

It follows from (i), (ii) and (iii) that $(0, x, y) \cup \langle y, \gamma, x - 1\rangle \cup (x - 1, 0)$ is a cycle of length k which intersects γ in at least $k - 3$ arcs. A contradiction. ■

The case of $n = 2k - 3$ follows from this theorem because in this case $r = k - 2$.

The case of $n = 2k - 2$ is trivial.

6 The Case $k = 6$

Theorem 6.1 *$\tilde{f}(7, 6) = 2$.*

Proof: By Theorem 7.5 of [5], $f(7, 6) < 3$, and therefore $\tilde{f}(7, 6) < 3$. We proceed to prove that $\tilde{f}(7, 6) \geq 2$.

We consider $\gamma = (0, 1, 2, 3, 4, 5, 6)$, and construct a cycle C_6 going through 0 with at least 2 arcs in common with γ. Clearly, we can assume that the arcs $(2, 0)$, $(4, 2)$, $(6, 4)$ and $(0, 5)$ are in A because otherwise there exists a cycle C_6 passing through 0 with $\mathcal{I}(C_6) = 5$.

Consider two cases: $(0, 3) \in A$ or $(3, 0) \in A$. For the case $(0, 3) \in A$, we first prove that $(2, 6) \in A$. Otherwise, $(6, 2) \in A$ and $C_6 = (0, 3, 4, 5, 6, 2, 0)$ goes through 0 and has $\mathcal{I}(C_6) = 3$. Thus $(2, 6) \in A$, and we show that also $(2, 5)$ must also be in A. If $(5, 2) \in A$ then $C_6 = (0, 3, 4, 5, 2, 6, 0)$ goes through 0 and has $\mathcal{I}(C_6) = 3$. Since $(0, 3) \in A$ and $(2, 5) \in A$ we have $C_6 = (0, 3, 4, 2, 5, 6, 0)$ that goes through 0 and has $\mathcal{I}(C_6) = 3$.

The case where $(3, 0) \in A$ we have $C_6 = (0, 5, 6, 4, 2, 3, 0)$ that goes through 0 and has $\mathcal{I}(C_6) = 2$. ■

Theorem 6.2 *$\tilde{f}(n,6) \geq 3$ if $n \geq 8$.*

Proof: We consider the four cases $n \equiv i \pmod 4$, $i = 0,1,2,3$.

Case $n \equiv 3 \pmod 4$.

First notice that $(n-1,4) \in A$, since otherwise $C_6 = (0,1,2,3,4,n-1,0)$ goes through 0 and has $\mathcal{I}(C_6) = 5$. Also, $(6,0) \in A$, because otherwise, if $(0,6) \in A$ by Lemma 2.1, $(0,n-5) \in A$ and $C_6 = (0,n-5,n-4,n-3,n-2,n-1,0)$ goes through 0 and has $\mathcal{I}(C_6) = 5$. Again by Lemma 2.1, $(0,n-2) \in A$. We conclude the proof if this case with $C_6 = (0,n-2,n-1,4,5,6,0)$ that goes through 0 and has $\mathcal{I}(C_6) = 3$.

Case $n \equiv 2 \pmod 4$. Taking $a = 5$ in Lemma 2.1, we get that $(0,n-5) \in A$ and $C_6 = (0,n-5,n-4,n-3,n-2,n-1,0)$ is a cycle with $\mathcal{I}(C_6) = 5$.

Case $n \equiv 1 \pmod 4$. Taking $a = 5$ in Lemma 2.1, we get that $5 + 4t_5 = n-4$. Hence $(0,n-4) \in A$ and $(0,n-8) \in A$. Observe that $(n-5,0) \in A$. Otherwise $(0,n-5) \in A$ and $C_6 = (0,n-5,n-4,n-3,n-2,n-1,0)$ is a cycle with $\mathcal{I}(C_6) = 5$.

Now, if $(n-2,n-6) \in A$ then $C_6 = (n-2,n-6,n-5,0,n-4,n-3,n-2)$ is a cycle with $\mathcal{I}(C_6) = 3$. Else $(n-6,n-2) \in A$ and $C_6 = (0,n-8,n-7,n-6,n-2,n-1,0)$ is a cycle with $\mathcal{I}(C_6) = 4$. Notice that this cycle is well defined, since $n \geq 9$. This is so because $n \equiv 1 \pmod 4$ and $n \geq 8$.

Case $n \equiv 0 \pmod 4$. If $(0,4) \in A$ then taking $a = 4$ in Lemma 2.1, we obtain that $(0,n-4) \in A$. The proof proceeds exactly as in the proof for the case $n \equiv 1 \pmod 4$. Hence, let us assume that $(4,0) \in A$.

Observe that $(6,0) \in A$, because otherwise $(0,6) \in A$ and taking $a = 6$ in Lemma 2.1, we get that $(0,n-2) \in A$, and the proof proceeds exactly as in the proof for the case $n \equiv 3 \pmod 4$. It follows that $(5,3) \in A$, because if $(3,5) \in A$ then $C_6 = (0,1,2,3,5,6,0)$ is a cycle C_6 with $\mathcal{I}(C_6) = 4$.

Now, $(5,2) \in A$, because if $(2,5) \in A$ then $C_6 = (0,1,2,5,3,4,0)$ is a cycle C_6 with $\mathcal{I}(C_6) = 3$. Therefore, $(5,1) \in A$, because if $(1,5) \in A$ then $C_6 = (0,1,5,2,3,4,0)$ is a cycle C_6 with $\mathcal{I}(C_6) = 3$.

Finally, using the chords $(0,5), (5,1), (4,0)$ we get $C_6 = (0,5,1,2,3,4,0)$ is a cycle C_6 with $\mathcal{I}(C_6) = 3$. ∎

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