Cycle–Pancyclism in Tournaments II

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Abstract

Let T be a Hamiltonian tournament with n vertices and γ a Hamiltonian cycle of T. In this paper we develop a general method to find cycles of length k, $\frac{n+4}{2} < k < n$, intersecting γ in a large number of arcs. In particular we can show that if there does not exist a cycle C_k intersecting γ in at least $k-3$ arcs then for any arc e of γ there exists a cycle C_k containing e and intersecting γ in at least $k - \frac{2(n-3)}{n-k+3} - 2$ arcs. In a previous paper [3] the case of cycles of length k, $k \leq \frac{n+4}{2}$ was studied.

1 Introduction

The subject of pancyclism in tournaments has been studied by several authors (e.g. [1],[2]). Two types of pancyclism have been considered. A tournament T is *vertex-pancyclic* if given any vertex v there are cycles of every length containing v. Similarly, a tournament T is *arc-pancyclic* if given any arc e there are cycles of every length containing e. It is well known that a Hamiltonian tournament is vertex-pancyclic, but not necessarily arc-pancyclic. In a previous paper [3] we introduced the concept of *cycle-pancyclism* to study questions such as the following. Given a cycle C, what is the maximum number of arcs which a cycle of length k, with vertex set contained in C, has in common with C? Clearly, to study this kind of questions it is sufficient to consider a Hamiltonian tournament where C is a Hamiltonian cycle of T.

Let T be a Hamiltonian tournament with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set A. Assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a Hamiltonian cycle of T. Let C_k denote a directed cycle of length k. In [3] we proved that for $k = 4, 5$ and for every k, such that $n > 2k-5$, there exists a cycle C_k intersecting γ in at least $k-3$ arcs. For $k = 3$ it was proved that there exists a cycle C_3 intersecting γ in at least one arc.

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In this paper we assume that $k+1 \leq n \leq 2k-5$, $k > 5$, and develop methods for finding a cycle C_k intersecting γ in a large number of arcs. In particular we can show that if there does not exist a cycle C_k intersecting γ in at least $k-3$ arcs then for any arc e of γ there exists a cycle C_k containing e and intersecting γ in at least $k - \frac{2(n-3)}{n-k+3} - 2$. The methods developed in this paper are the basis for our subsequent work in which we study the maximum intersection of a cycle C_k with γ.

2 Preliminaries

A *chord* of a cycle C is an arc not in C with both terminal vertices in C. The *length* of a chord $f = (u,v)$ of C, denoted $l(f)$, is equal to the length of $\langle u, C, v\rangle$, where $\langle u, C, v\rangle$ denotes the uv–directed path contained in C. We say that f is a c-chord if $l(f) = c$ and $f = (u,v)$ is a $-c$-chord if $l\langle v, C, u\rangle = c$. Observe that if f is a c-chord then it is also a $-(n-c)$-chord. Unless otherwise stated if the cycle is not specified, it will be assumed that the chord is of cycle γ.

In what follows T is a tournament of n vertices with a Hamiltonian cycle γ. For a cycle C_k of length k with vertex set contained in γ we denote $\mathcal{I}_\gamma(C_k) = |A(\gamma) \cap A(C_k)|$, or simply $\mathcal{I}(C_k)$ when γ is understood. Let $f(n,k,T) = \max\{\mathcal{I}(C_k) | C_k \subset T\}$.

Lemma 2.1 *At least one of the following properties holds.*

(i) $f(n,k,T) \geq k-3$.

(ii) *All the following chords are in A.*

- **(a)** *Every $(k-1)$-chord.*
- **(b)** *Every $(k-2)$-chord.*

Proof: Suppose that (i) is not true.

(a) If $(k-1,0)$ is a $-(k-1)$-chord then $C_k = (0,1,\ldots,k-1,0)$ satisfies $\mathcal{I}(C_k) = k-1 > k-3$, and hence $f(n,k,T) \geq k-3$.

(b) Suppose that there exists a $-(k-2)$-chord, $f = (y,x)$. We can assume w.l.o.g. that $x = 1$, $y = k-1$. It follows from (a) that $(0,k-1) \in A$. Also (a) implies that $(n-(k-1),0) \in A$ (notation modulo n). Note that that the hypothesis $n \leq 2k-5$ implies that $1 < n-(k-1) < k-1$. Let $z \in V$ be the maximum in $\langle 2, \gamma, k-2\rangle$ such that $(z,0) \in A$. Since $(0,k-1) \in A$ then $(z,0) \in A$ and $(0,z+1) \in A$. For $C_k = (z,0,z+1) \cup \langle z+1, \gamma, k-1\rangle \cup (k-1,1) \cup \langle 1, \gamma, z\rangle$ it holds $\mathcal{I}(C_k) = k-3$. ■

3 Lower Bounds

Lemma 3.1 *Let $P = (0, 1, \ldots, l)$, $l \geq 2$, be a directed path contained in γ, $z \in V - V(P)$, and $\{(0,z), (1,z), (z,l), (z, l-1), \ldots, (z, l-a+1)\} \subseteq A$ with $1 \leq a \leq l-1$. Then there exists i, $0 \leq i \leq l-(a+1)$, such that $\{(i,z), (z, i+a+1)\} \subseteq A$.*

Proof: First we will prove that there exists $j > 1$, such that $j \equiv b \pmod{a+1}$, $b \in \{0,1\}$, and $(z,j) \in A$. Since $l, l-1, \ldots, l-(a-1)$ are consecutive numbers, it follows that there exists $j \in \{l, l-1, \ldots, l-(a-1)\}$ with $j \equiv b \pmod{a+1}$ and $b \in \{0,1\}$; the hypothesis of the lemma implies $(z,j) \in A$, and $j > 1$ because $\{(0,z),(1,z)\} \subseteq A$. Now, let $j_0 = \min\{j | j \equiv b \pmod{a+1}, j > 1, \text{and } (z,j) \in A\}$. It follows that $(z, j_0 - (a+1)) \notin A$. Hence $(z, j_0) \in A$ and $(j_0 - (a+1), z) \in A$. Clearly, taking $i = j_0 - (a+1)$ we have $\{(i,z), (z, i+a+1)\} \subseteq A$. ■

Lemma 3.2 *If all the $k-2, k-1, \ldots, p$-chords, $k-1 \leq p < n-2$, are in T then at least one of the two following properties hold.*

(i) $f(n,k,T) \geq k-3$.

(ii) *Every $(p+1)$-chord is in T.*

Proof: We show that if (ii) is false then (i) holds. Let (s_1, s_2) be a $-(p+1)$-chord and z a vertex in $\langle s_1, \gamma, s_2 \rangle$. Assume w.l.o.g. that $s_2 = 0$. Let $x \equiv z + n - p \pmod{n}$. Observe that

$$\{(x,z), (x+1,z), \ldots, (x+p-(k-2), z)\} \subseteq A \tag{1}$$

since these are the $p, p-1, \ldots, k-2$-chords of γ ending in z. Similarly

$$\{(z, z+p), (z, z+p-1), \ldots, (z, z+k-2)\} \subseteq A. \tag{2}$$

Observe that the start points of the arcs in the set (1) are consecutive in γ and less than the end points of the arcs in the set (2), which are also consecutive in γ. This is because the largest start point of an arc in (1) is $z + n - (k-2)$ and the least end point of an arc in (2) is $z + (k-2)$, and $z + (k-2) > z + n - (k-2)$. See Fig. 1.

Now, consider the directed path $\langle x, \gamma, z+p \rangle$. Note that the cardinality of (1) is at least 2 and the cardinality of (2) is $p-k+3$. Thus letting $a = p-k+3$ it follows from Lemma 3.1 that there exist j, $x \leq j < z + (k-2)$ such that $\{(j,z), (z, j+a+1)\} \subseteq A$. It follows that $C = (s_1, s_2) \cup \langle s_2, \gamma, j \rangle \cup (j,z) \cup (z, j+a+1) \cup \langle j+a+1, \gamma, s_1 \rangle$ is a cycle. In order to see that $l(C) = k$ note that $l\langle s_1, \gamma, s_2 \rangle = n - (p+1)$, and thus $l\langle s_2, \gamma, s_1 \rangle = p+1$. Clearly, $l\langle j, \gamma, j+a+1 \rangle = a+1$. Therefore $l(C) = p+1-(a+1)+3 = k$ and C is a cycle with $\mathcal{I}(C) = k-3$. ■

It follows directly from Lemma 3.2 the following.

Theorem 3.3 *At least one of the following conditions holds.*

(i) $f(n, k, T) \geq k - 3$.

(ii) *For each* p*,* $k - 2 \leq p \leq n - 2$*, every* p*-chord of* γ *is in* T*.*

Let l and r be integers such that $n = l(n - k + 3) + r$ and $0 \leq r < n - k + 3$. The following theorem shows that if $f(n, k, T) < k - 3$ then there exist cycles with a large intersection with γ.

Theorem 3.4

(a) *If* $r = 0$ *or if* $r = 1$ *and* $k < n - 1$ *then* $f(n, k, T) \geq k - 2l$*.*

(b) *If* $r = 1$ *and* $k = n - 1$ *then* $f(n, k, T) \geq k - 2l - 1$*.*

(c) *If* $r = 2$ *then* $f(n, k, T) \geq k - 2l - 1$*.*

(d) *If* $r > 2$ *then* $f(n, k, T) \geq k - 2l - 2$*.*

Proof: Notice that for $l = 1$, $0 \leq r \leq 2$ it holds that $3 \leq k \leq 5$. For these cases the theorem follows from [3]. When $l = 1$ we can assume that $r > 2$. Therefore we can assume that the bounds of the theorem ($k - 2l$, $k - 2l - 1$, $k - 2l - 2$) are at most $k - 4$. Therefore if there exists a cycle C_k with $\mathcal{I}(C_k) \geq k - 3$ the theorem follows. Assume that this is not the case, namely, $f(n, k, T) < k - 3$. By Theorem 3.3, for each p, $k - 2 \leq p \leq n - 2$, every p-chord of γ is in T.

We construct a cycle C_k intersecting γ in the required number of arcs. For this goal we specify the following vertices of T, through which C_k passes. Let

$$\begin{aligned} x_1 &= k - 3 \\ x_2 &= x_1 - (n - k + 3) \\ &\vdots \\ x_l &= x_{l-1} - (n - k + 3). \end{aligned}$$

Observe that $l\langle x_{i+1}, \gamma, x_i \rangle = n - k + 3$, and hence by the definition of l, $x_l \geq 0$. It follows from Theorem 3.3 that for every i, $1 \leq i \leq l - 1$, the $(k - 2)$-chord $(x_i, x_{i+1} + 1)$ is in T. And also, if $z \in V(\gamma)$ such that $2 \leq l\langle x_i, \gamma, z \rangle \leq n - k + 2$, then $(z, x_i) \in T$. For any fixed election of $y_i \in \gamma$, $1 \leq i \leq l - 1$, such that $2 \leq l\langle x_i, \gamma, y_i \rangle \leq n - k + 2$, consider the following path $T_{k'}$ of length k' from 0 to x_{l-1} (Fig. 2)

$$T_{k'} = (0, x_1 + 1) \cup \langle x_1 + 1, \gamma, y_1 \rangle \cup (y_1, x_1, x_2 + 1) \cup \langle x_2 + 1, \gamma, y_2 \rangle \cup (y_2, x_2, x_3 + 1) \cup \cdots \cup (y_{l-1}, x_{l-1}).$$

We now describe how to complete $T_{k'}$ into a cycle C for the different possible values of r. Observe that $l(C)$ will vary depending on the election of the y_i's, i.e. depending on $l\langle x_i, \gamma, y_i \rangle$.

In each case we prove that C can be constructed in a way such that $l(C) = k$ by a suitable election of y_i, $1 \leq i \leq l-1$, and that $\mathcal{I}(C)$ takes the required values. We shall use the fact that $l(T_k') \geq 3(l-1)$ and that the number of arcs of T_k' not in γ is $2(l-1)$. Recall that we are assuming that $k > 5$.

- When $r = 0$ ($x_l = 0$) and when $r = 1$ ($x_l = 1$) and $k \leq n-2$, (see Figures 3 and 4)

$$C = T_{k'} \cup (x_{l-1}, x_l + 1) \cup \langle x_l + 1, \gamma, y_l \rangle \cup (y_l, 0).$$

The reason for not including the case of $k = n-1$ when $r = 1$ is the following. In the description of C the arc $(y_l, 0)$ is assumed to be in T. Theorem 3.3 guarantees the existence of this arc when $y_l \leq x_{l-1} - 2$; when $y_l = x_{l-1} - 1$ the arc $(y_l, 0)$ is a $(k-3)$-chord. This implies that the vertices $x_{l-1} - 1$ and 1 are not in C and thus k is at most $n-2$.

We proceed to prove that when $r = 0$ and when $r = 1$ ($x_l = 1$) and $k \leq n-2$, $\mathcal{I}(C) = l(C) - 2l$ and C can be constructed to have $l(C) = k$.

The description of C implies that exactly $2l$ arcs of C are not in γ. Therefore $\mathcal{I}(C) = l(C) - 2l$.

Next we show that C can be constructed to have $l(C) = k$. Notice that $C \cap \gamma$ is the union of the paths $\langle x_i + 1, \gamma, y_i \rangle$, for $1 \leq i \leq l$, and hence $l \leq \sum_{i=1}^{l} l\langle x_i + 1, \gamma, y_i \rangle = l(C) - 2l$. Hence, $3l \leq l(C)$. It follows that we can construct cycles C with any $l(C)$, $3l \leq l(C)$, and, by the definition of C, with $l(C) \leq n-1$ ($r = 0$) or $l(C) \leq n-2$ ($r = 1$). Since we want $l(C) = k$ it remains to show that it holds that $3l \leq k$. The proof is as follows. First observe that

$$\frac{3n}{n-k+3} \leq k,$$

because it is equivalent to

$$\begin{aligned} 3n &\leq k(n-k+3) = kn - k^2 + 3k \\ k^2 - 3k - nk + 3n &\leq 0 \\ k(k-3) &\leq n(k-3), \end{aligned}$$

and because $5 < k < n$. Now, if $r = 0$ then $n = l(n-k+3)$. Hence $3l = \frac{3n}{n-k+3}$ and then $3l \leq k$. If $r = 1$ then $n = l(n-k+3) + 1$. Hence $3l = \frac{3(n-1)}{n-k+3} < \frac{3n}{n-k+3} \leq k$.

- When $r = 1$ ($x_l = 1$) and $k = n-1$.

Notice that in this case $y_l = 4$ because $l\langle x_l, \gamma, x_{l-1} \rangle = 4$ and $x_{l-1} = 5$, we can define

$$C = T_{k'} \cup (5, 3, 1, 2, 0),$$

where $y_i = x_{i-1} - 1$ for $1 \leq i \leq l-1$. Clearly $l(C) = n-1$ because the only vertex not in C is 4. The description of C implies that exactly $2l+1$ arcs of C are not in γ. Therefore $\mathcal{I}(C) = k - 2l - 1$.

• When $r = 2$ ($x_l = 2$, Fig. 5)

$$C = T_{k'} \cup (x_{l-1}, x_l + 1) \cup \langle x_l + 1, \gamma, y_l \rangle \cup (y_l, x_l, 0).$$

There are exactly $2l + 1$ arcs of C not in γ. Hence $\mathcal{I}(C) = l(C) - 2l - 1$.

In this case $l(C) \geq 3l + 1$. Since $n = l(n - k + 3) + 2$, to prove that C can be constructed to have $l(C) = k$ it is sufficient to prove that $\frac{3(n-2)}{n-k+3} + 1 \leq k$. The proof is as follows.

$$\begin{aligned} \frac{3(n-2)}{n-k+3} &\leq k - 1 \\ 3(n-2) &\leq (k-1)(n-k+3) \\ k^2 - 4k - 3 &\leq n(k-4) \\ k(k-4) &\leq n(k-4), \end{aligned}$$

which holds because $5 < k < n$.

• When $r > 2$ ($x_l > 2$)

$$C = T_{k'} \cup (x_{l-1}, x_l + 1) \cup \langle x_l + 1, \gamma, y_l \rangle \cup (y_l, x_l, 1) \cup \langle 1, \gamma, y_{l+1} \rangle \cup (y_{l+1}, 0).$$

There are exactly $2l + 2$ arcs of C not in γ. Hence $\mathcal{I}(C) = l(C) - 2l - 2$.

In this case $l(C) \geq 3l + 3$. Since $n = l(n - k + 3) + r$, to prove that C can be constructed to have $l(C) = k$ it is sufficient to prove that $3(\frac{n-r}{n-k+3} + 1) \leq k$. Clearly it suffices to prove the inequality for $r = 3$. The proof is as follows.

$$\begin{aligned} 3(\frac{n-3}{n-k+3} + 1) &\leq k \\ \frac{3(n-3+n-k+3)}{n-k+3} &\leq k \\ \frac{3(2n-k)}{n-k+3} &\leq k \\ 3(2n-k) &\leq k(n-k+3) \\ k(k-6) &\leq n(k-6), \end{aligned}$$

which holds since $5 < k < n$. ■

Notice that Theorem 3.4 guarantees the existence of a cycle C_k with $\mathcal{I}(C_k) \geq k - 2l - 2 \geq k - \frac{2n}{n-k+3} - 2$, for any $(n+5)/2 \leq k \leq n - 1$. Therefore, even in the extreme case of $k = n - 1$, $f(n, k, T) \geq n/2 - 3$.

Given an arc a of γ, in each of the constructions of the cycle C in the previous theorem, it is possible to construct C in such a way that it contains a. This is done by letting $a = (x_1 + 1, x_1 + 2)$. Hence we have the following

Corollary 3.5 *If $f(n, k, T) < k - 3$ then for any arc $a \in \gamma$ there exists a cycle C_k, $a \in C_k$, with $\mathcal{I}(C_k) \geq k - \frac{2n}{n-k+3} - 2$.*

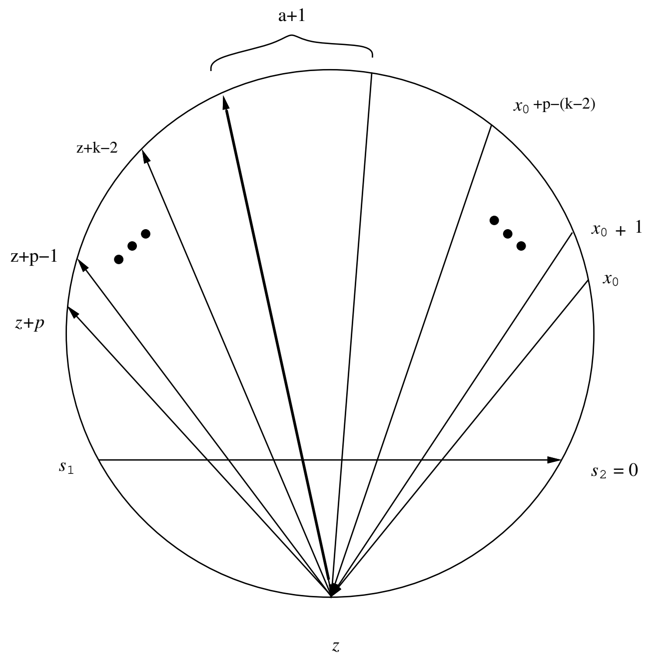

Figure 1: *Illustrating Lemma 3.2*

References

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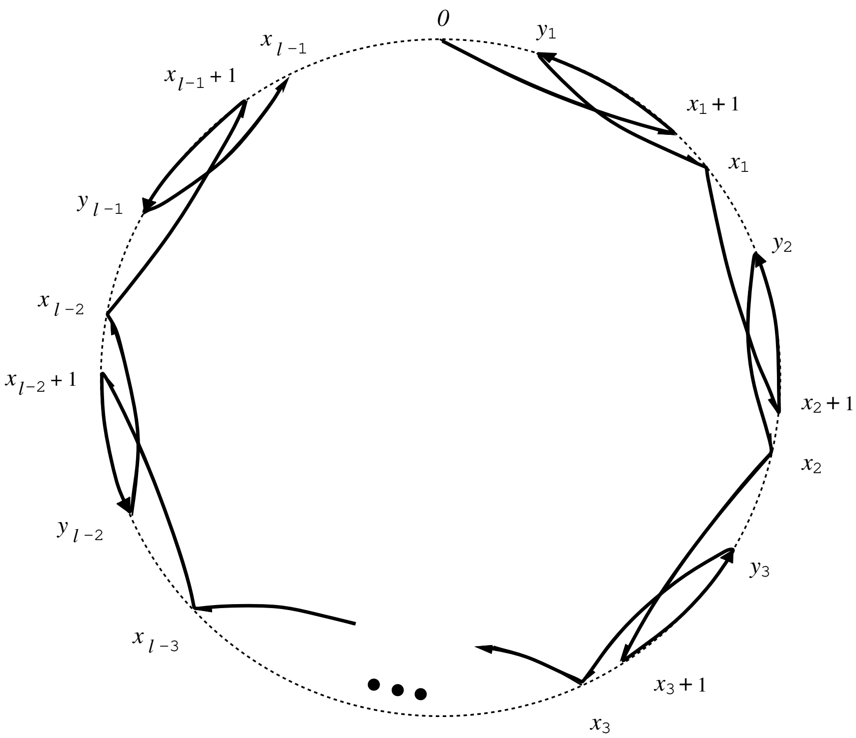

Figure 2: *The path* $T_{k'}$

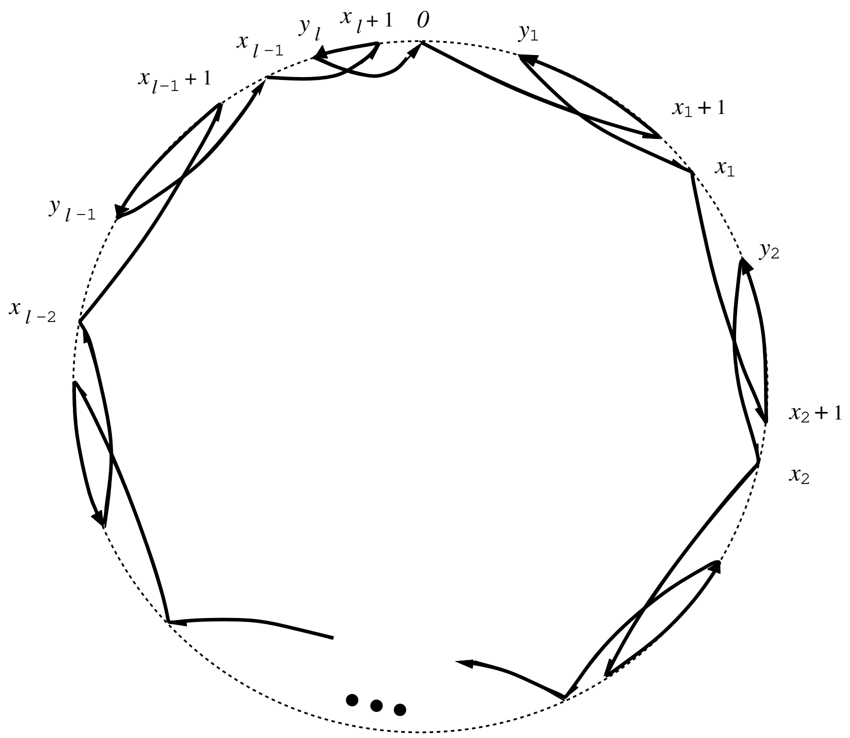

Figure 3: When $r = 0$ ($x_l = 0$)

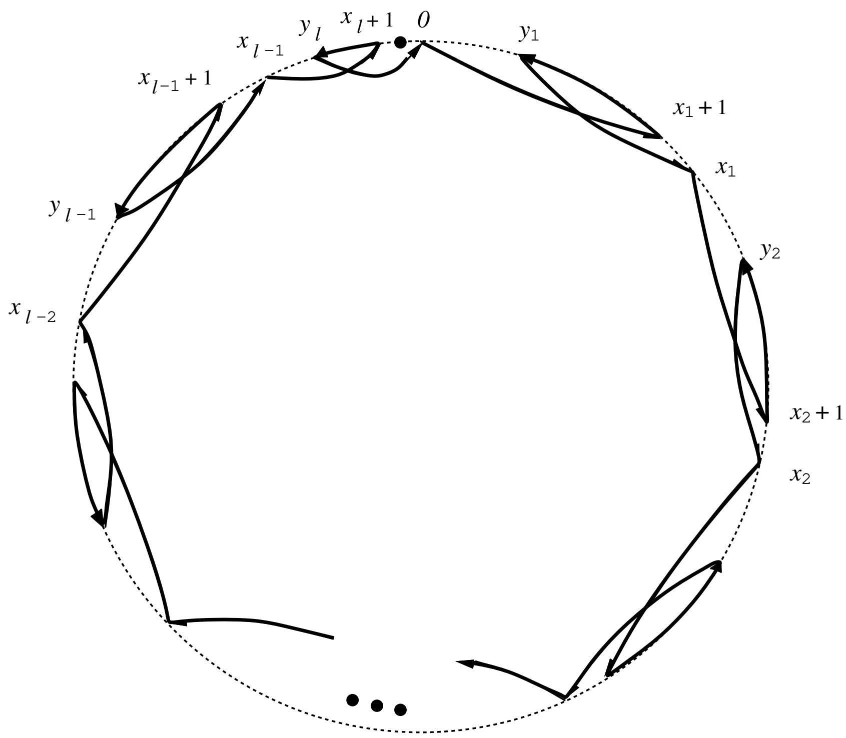

Figure 4: When $r = 1$ ($x_l = 1$) and $k \leq n - 2$

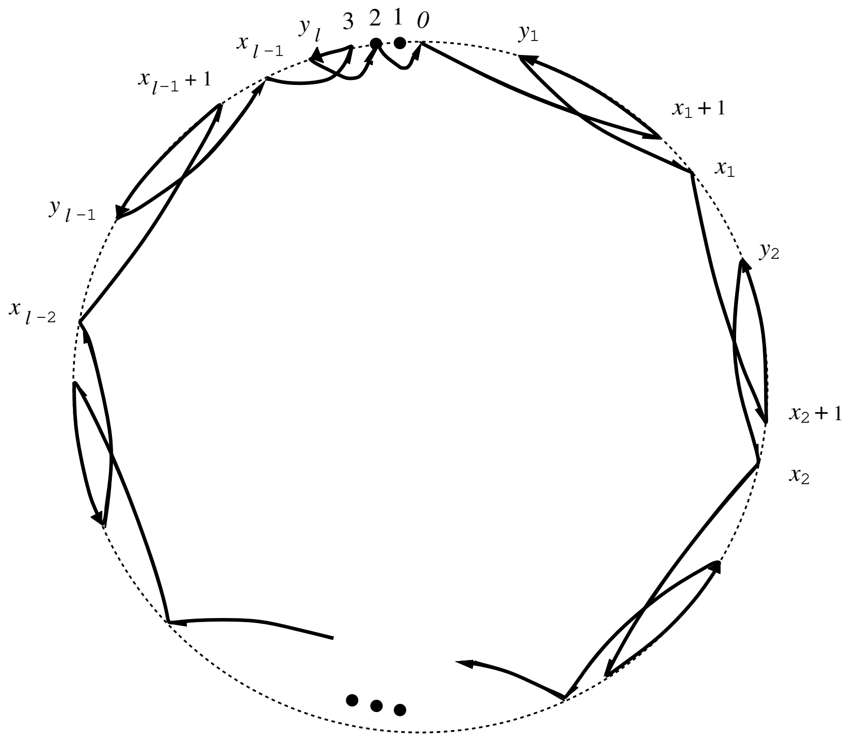

Figure 5: When $r = 2$