Cycle–Pancyclism in Tournaments III

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Abstract

Let T be a hamiltonian tournament with n vertices and γ a hamiltonian cycle of T. For a cycle C_k of length k in T we denote $\mathcal{I}_\gamma(C_k) = |A(\gamma) \cap A(C_k)|$, the number of arcs that γ and C_k have in common. Let $f(n,k,T,\gamma) = \max\{\mathcal{I}_\gamma(C_k) | C_k \subset T\}$ and $f(n,k) = \min\{f(n,k,T,\gamma) | T$ is a hamiltonian tournament with n vertices, and γ a hamiltonian cycle of $T\}$. In a previous paper [3] we studied the case of $n \geq 2k-4$ and proved that

- $f(n,3) = 1$, $f(n,4) = 1$ and $f(n,5) = 2$ if $n \neq 2k-2$;
- $f(n,k) = k-1$ if and only if $n = 2k-2$;
- for $k > 5$, $f(n,k) = k-2$ if and only if $n \geq 2k-4$, $n \neq 2k-2$ and $n \equiv k \pmod{k-2}$;
- for $k > 5$, $f(n,k) = k-3$ if and only if $n \geq 2k-4$ and $n \not\equiv k \pmod{k-2}$.

In this paper we consider the case of $n \leq 2k-5$ and complete the description of $f(n,k)$ by proving that $f(n,k) = k-4$ if and only if $n \leq 2k-5$.

1 Introduction

The subject of pancyclism in tournaments has been studied by several authors (e.g. [1],[2]). Two types of pancyclism have been considered. A tournament T is *vertex-pancyclic* if given any vertex v there are cycles of every length containing v. Similarly, a tournament T is *arc-pancyclic* if given any arc e there are cycles of every length containing e. It is well known that a hamiltonian tournament is vertex-pancyclic, but not necessarily arc-pancyclic. In a previous paper [3] we introduced the concept of *cycle-pancyclism* to study questions such as the following. Given a cycle C, what is the maximum number of arcs which a cycle of length k contained in C has in common with C? Clearly, to study this kind of questions it is sufficient to consider a hamiltonian tournament where C is a hamiltonian cycle of T.

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Let T be a hamiltonian tournament with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set A. Assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian cycle of T. Let C_k denote a directed cycle of length k. For a cycle C_k we denote $\mathcal{I}_\gamma(C_k) = |A(\gamma) \cap A(C_k)|$, or simply $\mathcal{I}(C_k)$, when γ is understood. Let $f(n, k, T, \gamma) = \max\{\mathcal{I}_\gamma(C_k) | C_k \subset T\}$, or simply $f(n, k, T)$, when γ is understood. Define $f(n, k) = \min\{f(n, k, T, \gamma) | T$ is a hamiltonian tournament with n vertices, and γ a hamiltonian cycle of $T\}$. In a previous paper ([3]) we studied the case of $n \geq 2k-4$, and proved that

- $f(n, 3) = 1$, $f(n, 4) = 1$ and $f(n, 5) = 2$ if $n \neq 2k-2$;
- $f(n, k) = k-1$ if and only if $n = 2k-2$;
- for $k > 5$, $f(n, k) = k-2$ if and only if $n \geq 2k-4$, $n \neq 2k-2$ and $n \equiv k \pmod{k-2}$;
- for $k > 5$, $f(n, k) = k-3$ if and only if $n \geq 2k-4$ and $n \not\equiv k \pmod{k-2}$.

In this paper we use methods and results of [4] to complete the characterization of $f(n, k)$ by proving that $f(n, k) = k-4$ if and only if $n \leq 2k-5$.

2 Preliminaries

A *chord* of a cycle C is an arc not in C with both terminal vertices in C. The *length* of a chord $g = (u, v)$ of C, denoted $l(g)$, is equal to the length of $\langle u, C, v \rangle$, where $\langle u, C, v \rangle$ denotes the uv–directed path contained in C. We say that g is a c-chord if $l(g) = c$ and $g = (u, v)$ is a $-c$-chord if $l\langle v, C, u \rangle = c$. Observe that if g is a c-chord then it is also a $-(n-c)$-chord. Unless otherwise stated if the cycle is not specified, it will be assumed that the chord is of cycle γ.

In the rest of this paper T is a tournament of n vertices with a hamiltonian cycle $\gamma = (0, 1, \ldots, n-1, 0)$. Also we assume that $\frac{n+5}{2} \leq k \leq n-1$. Thus $k > 5$. The following theorem of [4] will be used.

Theorem 2.1 ([4]) *At least one of the following conditions holds.*

(i) $f(n, k, T) \geq k-3$.

(ii) *For each p, $k-2 \leq p \leq n-2$, every p-chord of γ is in T.*

3 The Lower Bound

In this section we prove the following theorem.

Theorem 3.1 *For every k such that $\frac{n+5}{2} \leq k \leq n-1$, $f(n, k, T) \geq k-4$.*

Proof: It follows from Theorem 2.1 that we can assume that for each p, $k-2 \leq p \leq n-2$, every p-chord is in T; i.e., for each q, $2 \leq q \leq n-(k-2)$, every $(-q)$-chord is in T. This assumption will be maintained along the rest of the proof.

Let s be the least integer such that γ has an s-chord. Note that $s \geq n-k+3$, and thus $-s \leq k-3$. This is because for each p, $k-2 \leq p \leq n-2$, every p-chord is in T, and therefore for each p, $2 \leq n-p \leq n-k+2$, every $(n-p)$-chord is not in T. Let $g=(u,v)$ be an s-chord of γ. Denote by w the last vertex of $\langle v+1, \gamma, u-1\rangle$ such that there exists an arc (w,z) with $z \in \langle u+2, \gamma, v-1\rangle$. Notice that $l\langle u, \gamma, v\rangle \geq 3$, and thus $\langle u+2, \gamma, v-1\rangle$ has at least one vertex. Also, the vertex w is well defined because $(v+1, v-1)$ is a (-2)-chord and hence it is in A. Hence for every vertex $x \in \langle w+1, \gamma, u-1\rangle$, every (x', x) arc with $x' \in \langle u+1, \gamma, v-1\rangle$ is in A, by the definition of w. Also for any x, x' such that $2 \leq l\langle x, \gamma, x'\rangle < s$ and $\{x, x'\} \subseteq V(\langle u, \gamma, v-1\rangle)$ the arc (x', x) is in A because of the definition of f. Therefore we have the following lemma.

Lemma 3.2

(a) *For every $z_1 \in \langle z, \gamma, v-1\rangle$ and $u_1 \in \langle u, \gamma, z-1\rangle$ such that $l\langle u_1, \gamma, z_1\rangle \geq 2$, it holds $(z_1, u_1) \in A$.*

(b) *For every $u_2 \in \langle u+1, \gamma, v-1\rangle$ and $w_1 \in \langle w+1, \gamma, u\rangle$ such that $l\langle w_1, \gamma, u_2\rangle \geq 2$, it holds $(u_2, w_1) \in A$.*

As a direct consequence we have the following lemma.

Lemma 3.3

(a) *If $z_1 \in \langle z, \gamma, v-1\rangle$ and $w_1 \in \langle w+1, \gamma, u\rangle$ such that $l\langle w_1, \gamma, z_1\rangle \geq 2$, then*

$$C^1 = (z_1, w_1) \cup \langle w_1, \gamma, u\rangle \cup (u, v) \cup \langle v, \gamma, w\rangle \cup (w, z) \cup \langle z, \gamma, z_1\rangle$$

is a cycle of T of length p, with $\mathcal{I}(C^1) = p-3$.

(b) *If $z_1 \in \langle z, \gamma, v-1\rangle$, $w_1 \in \langle w+1, \gamma, u\rangle$, $u_1 \in \langle u+1, \gamma, z-1\rangle$ and $u_2 \in \langle u_1, \gamma, z-1\rangle$ such that $l\langle u_1, \gamma, z_1\rangle \geq 2$ and $l\langle w_1, \gamma, u_2\rangle \geq 2$, then*

$$C^2 = (z_1, u_1) \cup \langle u_1, \gamma, u_2\rangle \cup (u_2, w_1) \cup \langle w_1, \gamma, u\rangle \cup (u, v) \cup \langle v, \gamma, w\rangle \cup (w, z) \cup \langle z, \gamma, z_1\rangle$$

is a cycle of T of length q, with $\mathcal{I}(C^2) = q-4$.

Observe that C^2 is a cycle because $\langle u+1, \gamma, v-1\rangle$ is non-empty, and because $z \neq u_2$, $u \neq u_1$, $w \neq w_1$, and $v \neq z_1$. A similar observation holds for C^1.

We proceed to prove the existence of a cycle of length k intersecting γ in at least $k-4$ arcs. We split the problem into several cases according to the position of z in $\langle u+1, \gamma, v-1\rangle$ and according to $l\langle u+1, \gamma, v-1\rangle$. With the exception of the harder case of $k=n-1$, $l\langle u, \gamma, v\rangle = 4$

and $z = v - 1$ (second part of Case 2.b.2), we are able to use constructions equal or similar to C^1 or C^2 (Case 1,2,2.a,2.b.1, first part of 2.b.2).

Consider the path $\alpha = (u, v) \cup \langle v, \gamma, w\rangle \cup (w, z)$, and let $r = k - l(\alpha)$. We now extend α to a cycle C_k with $\mathcal{I}(C_k) \geq k - 4$. Observe that since $l\langle v, \gamma, u\rangle \leq k - 3$ (by the definition of s) it follows that $l\langle v, \gamma, u - 1\rangle \leq k - 4$ and $l(\alpha) \leq k - 2$; hence $r \geq 2$.

Case 1: When $l\langle w, \gamma, u\rangle - 1 + l\langle z, \gamma, v\rangle - 1 \geq r - 1 > 0$.

Let r_1 and r_2 be such that $r_1 + r_2 = r - 1$, $0 \leq r_1 \leq l\langle w, \gamma, u\rangle - 1$, $0 \leq r_2 \leq l\langle z, \gamma, v\rangle - 1$, $w_1 = u - r_1$ and $z_1 = z + r_2$. The proof that $(z_1, w_1) \in A$ is as follows. We have that $l\langle w_1, \gamma, z_1\rangle = z + r_2 - (u - r_1) = z - u + r - 1$, since $r_1 + r_2 = r - 1$; the definition of z implies that $z - u \geq 2$, and therefore $z - u + r - 1 > r \geq 2$. It follows from Lemma 3.3 that $C_p = C^1$ is a cycle of length $l(\alpha) + r_1 + r_2 + 1 = k$ with $\mathcal{I}(C^1) = p - 3 = k - 3$.

Case 2: When $l\langle w, \gamma, u\rangle - 1 + l\langle z, \gamma, v\rangle - 1 < r - 1$.

Observe that $l\langle w, \gamma, u\rangle - 1 + l\langle z, \gamma, v\rangle - 1 + l\langle u, \gamma, z\rangle - 1 = |V(\gamma) - V(\alpha)| = n - l(\alpha) - 1 \geq k - l(\alpha) = r$. Thus $l\langle u, \gamma, z\rangle - 1 \geq r - (l\langle w, \gamma, u\rangle - 1 + l\langle z, \gamma, v\rangle - 1)$. From the hypothesis of Case 2, $l\langle u, \gamma, z\rangle \geq 3$.

Let $r_3 = r - (l\langle w, \gamma, u\rangle - 1 + l\langle z, \gamma, v\rangle - 1) - 2$. It follows from the hypothesis of Case 2 that $0 \leq r_3 \leq l\langle u, \gamma, z\rangle - 1$.

Denote:

$$\begin{aligned} w_1 &= w + 1, \\ u_1 &= u + 1, \\ u_2 &= u + r_3 + 1 = u_1 + r_3, \\ z_1 &= v - 1. \end{aligned}$$

We have that $l\langle u_1, \gamma, z_1\rangle = v - 1 - u - 1 = v - u - 2 = l\langle u, \gamma, v\rangle - 2 \geq 2$. The last equality follows because $l\langle u, \gamma, v\rangle = s$, $s \geq n - k + 3$ and $k \leq n - 1$, and hence $l\langle u, \gamma, v\rangle \geq 4$. Hence $(z_1, u_1) \in A$.

Now, $l\langle w_1, \gamma, u_2\rangle = u + r_3 + 1 - w - 1 = u - w + r_3$. Since by definition $u - w \geq 1$ and $r_3 \geq 0$; it is sufficient to consider two possibilities: $u - w + r_3 \geq 2$ and $u - w + r_3 = 1$.

Case 2.a: When $l\langle w_1, \gamma, u_2\rangle = u - w + r_3 \geq 2$.

In this case, $(u_2, w_1) \in A$. By Lemma 3.3 $C_q = C^2$ is a cycle with $\mathcal{I}(C^2) = q - 4$. Furthermore $q = k$:

$$l(C^2) = l(\alpha) + v - z - 1 + 1 + u_2 - u_1 + 1 + u - w - 1,$$

since $u_2 = u_1 + r_3$;

$$l(C^2) = l(\alpha) + v - z + r_3 + u - w.$$

The definition of r_3 implies

$$l(C^2) = l(\alpha) + v - z + u - w + r - (u - w - 1 + v - z - 1) - 2$$

$$
\begin{aligned}
&= l(\alpha) + v - z + u - w + r - u + w + 1 - v + z + 1 - 2 \\
&= l(\alpha) + r = k,
\end{aligned}
$$

and $q = k$.

We now divide the remaining case of $u - w + r_3 = 1$ into two subcases. The first subcase is easily proved using Lemma 3.2 as follows.

Case 2.b.1: When $u - w + r_3 = 1$ and $z \leq z_1 - 1$.

If $u - w + r_3 = 1$ then $w = u - 1$, $r_3 = 0$,$u = w_1$ and $u_2 = u_1 = u + 1$. Also, by the definition of r_3, $r = l\langle w, \gamma, u\rangle + l\langle z, \gamma, v\rangle$.

Notice that $l\langle u_2, \gamma, z\rangle \geq 2$ because if $l\langle u_2, \gamma, z\rangle = 1$ then $l\langle u, \gamma, z\rangle = 2$, and

$$
\begin{aligned}
k &= r + l(\alpha) \\
&= l\langle w, \gamma, u\rangle + l\langle z, \gamma, v\rangle + l(\alpha) \\
&= l\langle w, \gamma, u\rangle + l\langle z, \gamma, v\rangle + \langle v, \gamma, w\rangle + 2 \\
&= l\langle w, \gamma, u\rangle + l\langle z, \gamma, v\rangle + \langle v, \gamma, w\rangle + l\langle u, \gamma, z\rangle \\
&= l(\gamma) = n,
\end{aligned}
$$

i.e. $k = n$, a contradiction.

Therefore, since $z \leq z_1 - 1$ (hypothesis of Case 2.b.1) we have that $(z_1, u_2 + 1) \in A$. Also $(u_2 + 1, u) \in A$, and hence

$$C_q = \alpha \cup \langle z, \gamma, z_1\rangle \cup (z_1, u_2 + 1, u)$$

is a cycle of length q with $\mathcal{I}(C_q) = q - 4$. To prove that $q = k$ observe that $r = u - w + v - z$, $w = u - 1$ and $v = z_1 + 1$ imply that $r = z_1 - z + 2$. Thus $l(C_q) = l(\alpha) + z_1 - z + 2 = l(\alpha) + r = k$.

We conclude the proof of the Theorem with the next subcase.

Case 2.b.2: When $u - w + r_3 = 1$ and $z = z_1 = v - 1$.

From $r_3 = 0$, $w = u - 1$, $z = v - 1$ and $r = l\langle w, \gamma, u\rangle + l\langle z, \gamma, v\rangle$ it follows that $r = 2$.

In this case we can assume that $u_2 + 2 = z$ because otherwise $u_2 + 2 < z$ (recall that $l\langle u, \gamma, v\rangle \geq 4$) implies that $(z, u_2 + 1) \in A$, and since $(u_2 + 1, u) \in A$, we would have the cycle

$$C_q = \alpha \cup (z, u_2 + 1, u);$$

a cycle of length q with $\mathcal{I}(C_q) = q - 4$. Moreover, $l(C_q) = l(\alpha) + 2 = l(\alpha) + r = k$, and the theorem follows.

Summarizing, we are assuming that $w = u - 1$, $r_3 = 0$, $r = 2$, $z_1 = z = v - 1$, $u_1 = u + 1$, $u_2 = u_1$, $z = u_2 + 2 = u_1 + 2 = u + 3$. Thus $v = u + 4$, i.e. $l\langle u, \gamma, v\rangle = 4$. Also $n - k + 3 \leq s = 4$. Then $n - 1 \leq k$, and therefore

$$k = n - 1.$$

Lemma 3.4 *Under the previous assumptions, at least one of the following conditions holds.*

(i) $f(n, k, T) \geq k - 4$.

(ii) *Every* 4*-chord is in* T.

Proof: If (ii) is false then there exists a 4-chord (x, y) such that $(y-1, x-1)$ is a (-4)-chord in A. This is because there is at least one 4-chord, e.g. (u, v).

Redefining $x = u$ and $y = v$, the definition of w implies $w < u - 1$ and proceeding like in Case 2.a, we obtain a cycle C_k of length k with $\mathcal{I}(C_k) = k - 4$. ■

Lemma 3.5 *Let* $s \geq 5$ *and* $s - 3 \leq n - 2$. *Under the current assumptions at least one of the following conditions holds.*

(i) $f(n, k, T) \geq k - 4$.

(ii) *If every* s*-chord is in* T *then every* $(s-3)$*-chord is in* T.

Proof: Assume that (ii) is false and let (x, y) be a $-(s-3)$-chord. Since every s-chord is in T it follows that $(y-2, x+1)$ is in T. Also, by Lemma 3.4, $(y-3, y+1)$ is in T. Since we are assuming that every (-2)-chord is in T then $(y, y-2)$ is in T. It follows that

$$C_q = (y-2, x+1) \cup \langle x+1, \gamma, y-3 \rangle \cup (y-3, y+1) \cup \langle y+1, \gamma, x \rangle \cup (x, y, y-2)$$

is a cycle of length $n-1$, $n-1 = q = k$ (the only vertex of γ not in C_k is $y-1$) with $\mathcal{I}(C_k) = k - 4$. ■

Notice that for the Lemma 3.5 to be valid it is sufficient that every 4-chord and every (-2)-chord is in T.

Lemma 3.6 *Under the current assumptions at least one of the following conditions holds.*

(i) $f(n, k, T) \geq k - 4$.

(ii) *For every* $t \geq 0$ *s.t.* $n - (2+3t) \leq n-2$ *every* $-(2+3t)$*-chord is in* T*;*
for every $t \geq 0$ *s.t.* $n - (3+3t) \leq n-2$ *every* $-(3+3t)$*-chord is in* T.

Proof: We assume that (i) is false and prove by induction on t that (ii) holds. Since $k = n-1$, (Theorem 2.1 implies that) every (-2)-chord and every (-3)-chord is in T. Hence the lemma is true for $t = 0$.

Assume the lemma holds for t in the required range; we prove the lemma for $t+1$. Since every $-(2+3t)$-chord is in T, taking $s = -(2+3t)$ in Lemma 3.5, we have that every $(s-3)$-chord is in T; namely, every $-(2+3(t+1))$-chord is in T. Now, taking $s = -(3+3t)$ in Lemma 3.5, we have that every $(s-3)$-chord is in T; namely, every $-(3+3(t+1))$-chord is in T. ■

We conclude the proof of Case 2.b.2, and of the theorem, as follows. Clearly at least one of the following two cases holds: $n-2 \not\equiv 1 \pmod 3$ or $n-3 \not\equiv 1 \pmod 3$. Assume without loss of generality that $n-2 \not\equiv 1 \pmod 3$; namely, for some $t \geq 0$, either $n-2 = 3+3t$ or $n-2 = 2+3t$. In any case it follows from Lemma 3.6 that $f(n,k,T) \geq k-4$ or every $-(n-2)$-chord is in T. If the later holds, i.e., every 2-chord is in T, then it follows from Theorem 2.1 (of [4]) that $f(n,k,T) \geq k-3$.

■

4 Conclusions

Recall the following results from [3].

Theorem 4.1 ([3]) *$f(n,k) \geq k-2$ if and only if $n \equiv k \pmod{k-2}$.*

Theorem 4.2 ([3]) *$f(n,k) = k-3$ if and only if $n \geq 2k-4$, and $n \not\equiv k \pmod{k-2}$.*

Theorem 4.3 ([3]) *For $n \geq 5$, $k \geq 5$, such that $n \leq 2k-5$ it holds that $f(n,k) < k-3$.*

It is easy to verify that Theorem 4.1 implies that $f(n,k) = k-2$ if and only if $n \equiv k \pmod{k-2}$ and $n \neq 2k-2$, and $f(n,k) = k-1$ if and only if $n = 2k-2$.

The following theorem is a direct consequence of Theorem 3.1 and of Theorem 4.3.

Theorem 4.4 *$f(n,k) = k-4$ if and only if $n \leq 2k-5$ and $k < n$.*

The next theorem summarizes the previous results, giving the complete description of the function $f(n,k)$.

Theorem 4.5
- *$f(n,3) = 1$, $f(n,4) = 1$ and $f(n,5) = 2$ if $n \neq 2k-2$;*
- *$f(n,k) = k-1$ if and only if $n = 2k-2$;*
- *for $k > 5$, $f(n,k) = k-2$ if and only if $n \geq 2k-4$, $n \neq 2k-2$ and $n \equiv k \pmod{k-2}$;*
- *for $k > 5$, $f(n,k) = k-3$ if and only if $n \geq 2k-4$ and $n \not\equiv k \pmod{k-2}$;*
- *for $k > 5$, $f(n,k) = k-4$ if and only if $n \leq 2k-5$.*

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