Cycle–Pancyclism in Tournaments I

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Abstract

Let T be a hamiltonian tournament with n vertices and γ a hamiltonian cycle of T. In this paper we start the study of the following question: What is the maximum intersection with γ of a cycle of length k? This number is denoted $f(n,k)$. We prove that for k in the range, $3 \leq k \leq \frac{n+4}{2}$, $f(n,k) \geq k-3$, and that the result is best possible; in fact, a characterization of the values of n, k, for which $f(n,k) = k-3$ is presented.

In a forthcoming paper we study $f(n,k)$ for the case of cycles of length $k > \frac{n+4}{2}$.

1 Introduction

The subject of pancyclism in tournaments has been studied by several authors (e.g. [1],[2]). Two types of pancyclism have been considered. A tournament T is *vertex-pancyclic* if given any vertex v there are cycles of every length containing v. Similarly, a tournament T is *arc-pancyclic* if given any arc e there are cycles of every length containing e. It is well known that a hamiltonian tournament is vertex-pancyclic, but not necessarily arc-pancyclic. In this paper we introduce the concept of *cycle-pancyclism* to study questions such as the following. Given a cycle C, what is the maximum number of arcs which a cycle of length k contained in T has in common with C? Clearly, to study this kind of question it is sufficient to consider a hamiltonian tournament where C is a hamiltonian cycle of T.

Let T be a tournament with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set A. Assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian cycle of T. Let C_k denote a directed cycle of length k. For a cycle C_k we denote $\mathcal{I}_\gamma(C_k) = |A(\gamma) \cap A(C_k)|$, or simply $\mathcal{I}(C_k)$ when γ is understood. Let $f(n,k,T) = \max\{\mathcal{I}_\gamma(C_k) | C_k \subset T\}$ and $f(n,k) = \min\{f(n,k,T) | T$ is a hamiltonian tournament with n vertices$\}$. This paper is the first part of a study of $f(n,k)$;

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it is devoted to k in the range $3 \leq k \leq \frac{n+4}{2}$. It is proved that $f(n,k) = k-3$ if and only if $n \geq 2k-4$, and $n \not\equiv k \pmod{k-2}$. Also, $f(n,k) \geq k-2$ if and only if $n \equiv k \pmod{k-2}$. In a forthcoming paper we study $f(n,k)$ for $k > \frac{n+4}{2}$.

The rest of this paper is organized as follows. In Section 2 some notation and basic results needed in the rest of the paper are introduced. The proof of the main result, i.e., that $f(n,k) \geq k-3$ for $n \geq 2k-4$ appears in Section 3, 4, 5 and 6. Sections 3 through 5 contain special cases (for particular values of n and k). The general case is left to Section 6, where it is proved that $f(n,k) \geq k-3$. In Section 7 it is proved that $f(n,k) \leq k-3$, when $n \not\equiv k \pmod{k-2}$, and that the results are best possible; namely, for $n < 2k-4$, $f(n,k) < k-3$. Thus, a characterization is presented of the values of n, k for which $f(n,k) = k-3$ and for which $f(n,k) = k-2$.

2 Preliminaries

A *chord* of a cycle C is an arc not in C with both terminal vertices in C. The *length* of a chord $f = (u,v)$ of C, denoted $l(f)$, is equal to the length of $\langle u, C, v\rangle$, where $\langle u, C, v\rangle$ denotes the uv–directed path contained in C. We say that f is a c-chord if $l(f) = c$ and $f = (u,v)$ is a $-c$-chord if $l\langle v, C, u\rangle = c$. Observe that if f is a c-chord then it is also a $-(n-c)$-chord.

In what follows all notation is taken modulo n.

For any a, $2 \leq a \leq n-2$, denote by t_a the largest integer such that $a + t_a(k-2) < n-1$. The important case of t_{k-1} is denoted by t in the rest of the paper. Let r be defined as follows: $r = n - [k-1+t(k-2)]$.

Notice the following facts.

- If $a \leq b$, then $t_a \geq t_b$.
- $t \geq 0$.
- $2 \leq r \leq k-1$.

Lemma 2.1 *If the a-chord with initial vertex 0 (recall that 0 is an arbitrary vertex of T) is in A, then at least one of the two following properties holds.*

(i) $f(n,k,T) \geq k-2$.

(ii) *For every $0 \leq i \leq t_a$, the $a + i(k-2)$-chord with initial vertex 0 is in A.*

Proof: Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1,2,\ldots,t_a\} \,|\, (a+i(k-2),0) \in A\},$$

then

$$C_k = (0, a + (j-1)(k-2)) \cup \langle a + (j-1)(k-2), \gamma, a + j(k-2)\rangle \cup (a + j(k-2), 0)$$

is a cycle such that $\mathcal{I}(C_k) = k-2$ and hence (i) in the lemma is true. ∎

The following is a consequence of Lemma 2.1.

Corollary 2.2 *At least one of the two following properties holds.*

(i) $f(n, k, T) \geq k-2$.

(ii) *For every* $0 \leq i \leq t$, *every* $((k-1)+i(k-2))$*-chord is in* A.

Proof: Clearly, for any vertex 0, $(0, k-1) \in A$ since otherwise $(k-1, 0) \in A$ and $C_k = (0, 1, \ldots, k-1, 0)$ is a cycle with $\mathcal{I}(C_k) = k-1$ and thus (i) holds.

Now applying Lemma 2.1 with $a = k-1$ we have that (i) or (ii) hold. ∎

3 The Cases $k = 3, 4, 5$

Theorem 3.1 $f(n, 3) \geq 1$.

Proof: Let $i = \min\{j \in V | (j, 0) \in A\}$. Observe that i is well defined since $(n-1, 0) \in A$. Clearly $i \neq 1$, so $i - 1 > 0$ and then $(0, i-1, i, 0)$ is a cycle C_3 with $\mathcal{I}(C_3) \geq 1$. ∎

Theorem 3.2 $f(n, 4) \geq 1$.

Proof: We proceed by contradiction. Taking $a = 3$ and $x_0 = 0$ in Lemma 2.1 we get that for each i, $0 \leq i \leq t_a$, the $(3+2i)$-chord $(0, 3+2i)$ is in A. Recall that t_a is the greatest integer such that $3 + 2t_a < n-1$.

When n is even, it holds that $t_a = (n-4)/2 - 1$, $(0, 3+2t_a) \in A$. That is, $(0, n-3) \in A$ and $C_4 = (0, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_4) = 3$. When n is odd, it holds that $t_a = \lfloor \frac{n-4}{2} \rfloor$ and $(0, 3+2t_a) \in A$, namely $(0, n-2) \in A$.

Now, we may assume that $(n-3, 0) \in A$, because otherwise the cycle $C_4 = (0, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(C_4) = 3$. If $(n-1, n-3) \in A$ then $C_4 = (n-1, n-3, 0, n-2, n-1)$ is a cycle with $\mathcal{I}(C_4) = 1$. Else, $(n-3, n-1) \in A$ and $C_4 = (n-3, n-1, 0, n-4, n-3)$ is a cycle with $\mathcal{I}(C_4) = 1$. ∎

Theorem 3.3 $f(n, 5) \geq 2$.

Proof: We consider the three cases $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$.

Case $n \equiv 2 \pmod 3$. Taking $a = 4$ in Lemma 2.1, we get that $(0, n-4) \in A$ and $C_5 = (0, n-4, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 4$.

Case $n \equiv 1 \pmod 3$. Taking $a = 4$ in Lemma 2.1, we get that $4 + 3t_4 = n - 3$. Hence $(0, n-3) \in A$ and $(0, n-6) \in A$. Observe that $(n-4, 0) \in A$. Otherwise $(0, n-4) \in A$ and $C_5 = (0, n-4, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 4$.

Now, if $(n-2, n-5) \in A$ then $C_5 = (n-2, n-5, n-4, 0, n-3, n-2)$ is a cycle with $\mathcal{I}(C_5) = 2$. Else $(n-5, n-2) \in A$ and $C_5 = (0, n-6, n-5, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 3$.

Case $n \equiv 0 \pmod 3$. If $(0, 3) \in A$ then taking $a = 3$ in Lemma 2.1, we obtain that $(0, n-6) \in A$ and $(0, n-3) \in A$. The proof proceeds exactly as in the proof for the case $n \equiv 1 \pmod 3$. Hence, let us assume that $(3, 0) \in A$.

Observe that $(5, 0) \in A$, because otherwise $(0, 5) \in A$ and taking $a = 5$ in Lemma 2.1, we get that $(0, n-4) \in A$ and $C_5 = (0, n-4, n-3, n-2, n-1, 0)$ is a cycle with $\mathcal{I}(C_5) = 4$.

Therefore we have that $(5, 0) \in A$ and $(3, 0) \in A$. Considering the cycle $(0, 1, 2, 3, 4, 5, 0)$ it is easy to check that $(5, 3) \in A$ and $(1, 5) \in A$ (or else the proof follows). Analyzing the direction of the arc joining 2 and 5 we see that in any case there is a cycle C_5 with $\mathcal{I}(C_5) = 2$: If $(5, 2) \in A$ then the cycle is $C_5 = (3, 0, 1, 5, 2, 3)$, else, if $(2, 5) \in A$ then the cycle is $C_5 = (3, 0, 1, 2, 5, 3)$. ∎

4 The case of $n = 2k - 4$

In this section it is proved that if $n = 2k - 4$ then $f(n, k) \geq k - 3$.

Theorem 4.1 *If $n = 2k - 4$ then $f(n, k) \geq k - 3$.*

Proof: Let x and y be two vertices of T such that $l\langle x, \gamma, y\rangle = l\langle y, \gamma, x\rangle = k - 2$. Without loss of generality we can assume that $x = 0$, $y = k - 2$ and $(0, k-2) \in A$. Hence $(k-1, 2)$ is a $(k-1)$-chord, $l\langle 2, \gamma, k-1\rangle = k - 3$, $(1, k)$ is a $(k-1)$-chord and $l\langle 2, \gamma, k+1\rangle = k - 1$.

- $(k, 2) \in A$. Otherwise $(2, k) \in A$ and then $C_k = (k-2, k-1, 2, k) \cup \langle k, \gamma, 0\rangle \cup (0, k-2)$ is a cycle with $\mathcal{I}(C_k) = k - 3$.
- $(1, k-1) \in A$. Otherwise $(k-1, 1) \in A$ and then $C_k = (k-1, 1, k) \cup \langle k, \gamma, 0\rangle \cup (0, k-2, k-1)$ is a cycle with $\mathcal{I}(C_k) = k - 3$.

Therefore, since $(k, 2) \in A$ and $(1, k-1) \in A$ then $C_k = (1, k-1, k, 2, k+1) \cup \langle k+1, \gamma, 1\rangle$ is a cycle with $\mathcal{I}(C_k) = k - 3$. ∎

5 The case of $r = k-1$ and $r = k-2$

In this section it is proved that if $r = k-1$ or $r = k-2$ then $f(n,k) \geq k-3$.

Theorem 5.1 *If $r = k-1$ or $r = k-2$ then $f(n,k) \geq k-3$.*

Proof: Assume $r = k-1$. By Corollary 2.2 (taking $i = 0$) either $f(n,k,T) \geq k-2$ or $(0, k-1) \in A$. In the latter case we have that $\langle k-1+t(k-2), \gamma, 0\rangle \cup (0, k-1+t(k-2))$ is a cycle of length k intersecting γ in $k-1$ arcs. Thus, in both cases, $f(n,k,T) \geq k-2$.

Now, assume $r = k-2$ and $f(n,k,T) < k-3$.

We consider the vertices $x = k-1+t(k-2)$, $y = k-1+(t-1)(k-2)$. Observe that when $t = 0$ we obtain $y = 1$.

(i) $(0,x) \in A$. It follows from Corollary 2.2.

(ii) $(y, x+1) \in A$. If $(y, x+1) \notin A$ then $(x+1, y) \in A$ and $(x+1, y) \cup \langle y, \gamma, x+1\rangle$ is a cycle of length k which intersects γ in $k-1$ arcs.

(iii) $(x,y) \in A$. If $(x,y) \notin A$ then $(y,x) \in A$ and $(y,x) \cup \langle x, \gamma, 0\rangle \cup (0,y)$ (Corollary 2.2 implies $(0,y) \in A$) is a cycle of length k intersecting γ in at least $k-2$ arcs.

It follows from (i), (ii) and (iii) that $(x,y) \cup (y, x+1) \cup \langle x+1, \gamma, 0\rangle \cup (0,x)$ is a cycle of length k which intersects γ in at least $k-3$ arcs. A contradiction. ∎

Corollary 5.2 *If $t = 0$ then $f(n,k) \geq k-3$.*

Proof: If $t = 0$ then $n = k-1+r$, where $k-3 \leq r \leq k-1$ since $n \geq 2k-4$. When $r = k-1$ or $r = k-2$, Theorem 5.1 implies that $f(n,k) \geq k-3$. If $r = k-3$ then $n = 2k-4$ and Theorem 4.1 implies that $f(n,k) \geq k-3$. ∎

6 The General Case

In this section we assume that $r \leq k-3$, since the case $r > k-3$ has been considered in Theorem 5.1, and that $t \geq 1$, since the case of $t = 0$ has been considered in Corollary 5.2. The next lemma follows directly from Lemma 2.1.

Lemma 6.1 *If the $k-1+\alpha$-chord, $\alpha \leq r$, with initial vertex 0 is in A, then at least one of the two following properties holds.*

(i) $f(n,k,T) \geq k-2$.

(ii) *For every $0 \le i \le t-1$, the $k-1+\alpha+i(k-2)$-chord with initial vertex 0 is in A.*

Lemma 6.2 *At least one of the two following properties holds.*

(i) $f(n,k,T) \ge k-3$.

(ii) *All the following chords are in A.*

- **(a)** *Every $(k-1)$-chord.*
- **(b)** *Every $(-r)$-chord.*
- **(c)** *Every $(k-2)$-chord.*
- **(d)** *Every $-(r+1)$-chord.*

Proof: The proof of (a) follows directly from Corollary 2.2.

The proof of (b) follows from Corollary 2.2, observing that $n-r = k-1+t(k-2)$.

To prove (c) assume that there is a $-(k-2)$-chord, say $f=(y,x)$. Consider the vertex $x-1$. It follows from (a) that $(x-1,y)$ is in A, and it follows from (b) that $(x-1+r, x-1)$ is in A. Therefore, there exists a vertex z in $\langle x-1+r, \gamma, y-1\rangle$ such that $(z, x-1)$ and $(x-1, z+1)$ are in A. Then $C_k = (y,x) \cup \langle x,\gamma,z\rangle \cup (z,x-1) \cup (x-1,z+1) \cup \langle z+1,\gamma,y\rangle$ is a cycle with $\mathcal{I}(C_k) = k-3$, and (i) holds.

Finally, to prove (d) let (y,x) be a $(r+1)$-chord. It follows from (c) and Lemma 2.1 that every $t(k-2)$-chord is in A. In particular, $(x+k-2, x+(t+1)(k-2))$ is in A. Observe that $y = x+(t+1)(k-2)$ since $n = (k-1)+t(k-2)+r$. It follows that $C_k = (y,x) \cup \langle x,\gamma,x+k-2\rangle \cup (x+k-2,y)$ is a cycle with $\mathcal{I}(C_k) = k-2$. Hence (i) holds. ■

Lemma 6.3 *Let $-1 \le i \le r$. If all the $-r$-chords, $-(r+1)$-chords, $(k-2+i)$-chords and $(k-1+i)$-chords are in T then at least one of the following properties holds.*

(i) $f(n,k,T) \ge k-3$.

(ii) *All the $-(2r-i+1)$-chords, $-(2r-i+2)$-chords and $-(2r-i+3)$-chords are in T.*

Proof: Assume that the hypothesis of the lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the $[(k-2)+i]$-chords and all the $[(k-1)+i]$-chords are in T, it follows from Lemma 6.1 (taking $\alpha = i-1$) that every $[k-2+i+(t-1)(k-2)]$-chord is in T, and that (taking $\alpha = i$) every $[k-1+i+(t-1)(k-2)]$-chord is in T. Thus the following arcs are in T: $(r,0)$, $(r+1,0)$, $(0, k-1+(t-1)(k-2)+i)$, $(0, k-1+(t-1)(k-2)+i-1)$.

Let $x_1 = r$, $x_2 = r+1$, $x_3 = k-1+(t-1)(k-2)+i-1$, $x_4 = x_3+1$, $x_5 = x_4+k-2$, $x_6 = x_5+1$, $x_7 = x_5-1$ and $x_8 = x_7-1$. Therefore $(0,x_4)$ and $(0,x_3)$ are in A.

Observe that:

- It follows from $x_5 = k - 1 + t(k-2) + i$, and $n = k - 1 + t(k-2) + r$ that $l\langle x_5, \gamma, 0\rangle = n - x_5 = r - i$.
- $l\langle x_6, \gamma, 0\rangle = r - i - 1$.
- $l\langle x_6, \gamma, x_1\rangle = 2r - i - 1$.
- $l\langle x_7, \gamma, x_1\rangle = 2r - i + 1$.
- $l\langle x_7, \gamma, x_2\rangle = 2r - i + 2$.
- $l\langle x_8, \gamma, x_2\rangle = 2r - i + 3$.
- $l\langle x_4, \gamma, x_7\rangle = k - 3$.
- $l\langle x_3, \gamma, x_8\rangle = k - 3$.

We first prove that every $-(2r-i+1)$-chord is in T. Suppose that there exists a $(2r-i+1)$-chord. We can assume w.l.o.g. that (x_7, x_1) is such a chord. Hence $C_k = (x_7, x_1, 0, x_4) \cup \langle x_4, \gamma, x_7\rangle$ is a cycle with $\mathcal{I}(C_k) = k - 3$.

Now we prove that every $-(2r-i+2)$-chord is in T. Assume the contrary and let (x_7, x_2) be a $(2r-i+2)$-chord. Then $C_k = (x_7, x_2, 0, x_4) \cup \langle x_4, \gamma, x_7\rangle$ is a cycle with $\mathcal{I}(C_k) = k - 3$.

Finally we show that every $-(2r-i+3)$-chord is in T. Assuming the opposite let (x_8, x_2) be a $(2r-i+3)$-chord. Then $C_k = (x_8, x_2, 0, x_3) \cup \langle x_3, \gamma, x_8\rangle$ is a cycle with $\mathcal{I}(C_k) = k - 3$. ■

Lemma 6.4 *At least one of the following properties holds.*

(i) $f(n, k, T) \geq k - 2$.

(ii) *For any vertex x, there exist at most $k-3$ consecutive vertices in γ which are in–neighbors of x.*

Proof: Assume that (i) does not hold. Assume without loss of generality that $x = 0$. The vertices $k - 1 + i(k-2)$, for $0 \leq i \leq t$, are not in–neighbors of 0. This follows from Lemma 6.2 part (a), and Lemma 2.1. Thus, there are at most $k-3$ consecutive vertices in $\langle k-1, \gamma, 0\rangle$ which are in–neighbors of 0. Since $(0, 1) \in A$, also in $\langle 0, \gamma, k-1\rangle$ there are at most $k-3$ consecutive in–neighbors of 0. ■

Observe that in the Lemma 6.4 the general assumption of this section that $n \geq 2k - 4$ is not needed. The following corollary is a direct consequence of this lemma.

Corollary 6.5 *Let T be a tournament with n vertices and γ a hamiltonian cycle of T. For each vertex x of T such that the number of consecutive in–neighbors of x in γ is at least $k-2$, $3 \leq k \leq n$, there exists a cycle C_k containing the vertex x, with $\mathcal{I}(C_k) \geq k - 2$.*

Lemma 6.6 *If every k-chord and every $(-r)$-chord is in A then at least one of the two following properties holds.*

(i) $f(n,k,T) \geq k-3$.

(ii) *For every α, $0 < \alpha r < k$, every $-(\alpha+1)r$-chord is in A.*

Proof: Assume that (ii) does not hold; we show that (i) holds. Let α be the least integer for which an $(\alpha+1)r$-chord is in A, and let (x_2, x_1) be an $(\alpha+1)r$-chord.

Let $x_0 \in V$ such that $l\langle x_2, \gamma, x_0\rangle = r$. It follows that $(x_1, x_0) \in A$ because it is an $-\alpha r$-chord. Let $x_3 \in V$ such that $l\langle x_0, \gamma, x_3\rangle = k + (t-1)(k-2)$. Observe that $x_3 \in \langle x_1, \gamma, x_0\rangle$ because $\alpha r < k$ and $t \geq 1$.

Lemma 6.1 and the fact that every k-chord is in A imply that either $f(n,k,T) \geq k-2$ or every $k+(t-1)(k-2)$-chord is in A. In the latter case $(x_0, x_3) \in A$ and $l\langle x_3, \gamma, x_2\rangle = k-3$. We conclude that $C_k = \langle x_3, \gamma, x_2\rangle \cup (x_2, x_1, x_0, x_3)$ is a cycle with $\mathcal{I}(C_k) = k-3$, and hence $f(n,k,T) \geq k-3$. ■

Lemma 6.7 *At least one of the following properties holds.*

(i) $f(n,k,T) \geq k-3$.

(ii) *For $-1 \leq i \leq r$, every $-(2r+1-i)$-chord and every $(k-1+i)$-chord is in A.*

Proof: Suppose that $f(n,k,T) < k-3$. We shall prove that property (ii) holds by induction on i. We start with $i=-1$ and $i=0$, namely, we prove that the following chords are in A:

(a) Every $(k-2)$-chord.

(b) Every $(k-1)$-chord.

(c) Every $-(2r+2)$-chord.

(d) Every $-(2r+1)$-chord.

In fact we also prove that:

(e) Every $-(2r+3)$-chord is in A.

The proof of (a) and (b) follows directly from Lemma 6.2.

Let 0 be any vertex of T. By Lemma 6.2 (b) and (d) $(r,0)$ and $(r+1,0)$ are in A.

It follows from Lemma 6.2 (part (a) and part (c)), and Lemma 2.1 that the following two chords, whose end-points are consecutive in γ, are in A: $(0, k-1+(t-1)(k-2))$ and $(0, t(k-2))$.

Since 0 is an arbitrary vertex of T, we can prove that (c), (d) and (e) hold:

- Part (c): every $-(2r+2)$-chord is in A. If $(n-r-1, r+1) \in A$ then $C_k = (n-r-1, r+1) \cup (r+1, 0) \cup (0, k-1+(t-1)(k-2)) \cup \langle k-1+(t-1)(k-2), \gamma, n-r-1 \rangle$ is a cycle with $\mathcal{I}(C_k) = k-3$, a contradiction.

- Part (d): every $-(2r+1)$-chord is in A. If $(n-r-1, r) \in A$ then $C_k = (n-r-1, r) \cup (r, 0) \cup (0, k-1+(t-1)(k-2)) \cup \langle k-1+(t-1)(k-2), \gamma, n-r-1 \rangle$ is a cycle with $\mathcal{I}(C_k) = k-3$, a contradiction.

- Part (e): every $-(2r+3)$-chord is in A. If $(n-r-2, r+1) \in A$ then $C_k = (n-r-2, r+1) \cup (r+1, 0) \cup (0, t(k-2)) \cup \langle t(k-2), \gamma, n-r-2 \rangle$ is a cycle with $\mathcal{I}(C_k) = k-3$, a contradiction.

Assume that the lemma holds for each i', $i' \leq i$ and let us prove it for $i+1$; namely, we prove:

(α) Every $(k+i)$-chord is in A,

(β) Every $-(2r-i)$-chord is in A.

Proof of (α)

It follows from the inductive hypothesis that for each j, $0 \leq j \leq i$, every $(k-1)+j$-chord and every $(k-2)+j$-chord is in A. Hence, by Lemmas 6.2 and 6.3, every $-(2r-j+1)$-chord, $-(2r-j+2)$-chord and every $-(2r-j+3)$-chord is in A. That is, for each j, $0 \leq j \leq i+2$, every $-(2r-j+1)$-chord is in A. These are $(i+3)$-chords with initial vertices consecutive in γ.

Assume for contradiction that $(x_3, 0)$ is a $-(k+i)$-chord. Let $x_0 = n-(2r-i-1)$. Hence letting $x_2 = 2$, we have that (x_2, x_0), is a $-(2r-(i-1))$-chord.

Let us show that $x_0 \in \langle x_3+1, \gamma, n-1 \rangle$:

$$l\langle x_0, \gamma, 0 \rangle = 2r-i-1,$$

$$\begin{aligned} l\langle x_3, \gamma, x_0 \rangle &= n-(k+i+2r-i-1) \\ &= k-1+t(k-2)+r-(k+i+2r-i-1) \\ &\geq (k-1)+(k-2)+r-k-i-2r+i+1 = k-2-r. \end{aligned}$$

Since we are assuming $r \leq k-3$ then $l\langle x_3, \gamma, x_0 \rangle \geq 1$. Hence $l\langle x_0, \gamma, 0 \rangle \geq 1$, because $r \geq 1$.

Now, there exists an $x \in \Gamma^+(x_0)$ such that x is in $\langle x_2, \gamma, x_3 - 1 \rangle$. This is a direct consequence of Lemma 6.4 and the fact that the number of vertices in $\langle x_2, \gamma, x_3-1 \rangle$ is at least $k-2$. Let x_4 be the smallest (the nearest to 0 in γ) such vertex.

Let $x_1 = 0$. We will prove that $x_4 - i - 3 \in \langle x_1, \gamma, x_4 - 3 \rangle$. Since for each j, $0 \leq j \leq i+2$, every $-(2r-j+1)$-chord is in A, it follows that $\{(0, x_0), (1, x_0), (2, x_0), \ldots, (i+2, x_0)\} \subseteq A$. Hence, the election of x_4 implies $x_4 \geq i+3$ and then $x_4 - i - 3 \geq 0 = x_1$.

Finally, since $l\langle x_4, \gamma, x_3\rangle + l\langle x_1, \gamma, x_4 - i - 3\rangle = k - 3$ then $C_k = (x_4 - i - 3, x_0, x_4) \cup \langle x_4, \gamma, x_3\rangle \cup (x_3, x_1) \cup \langle x_1, \gamma, x_4 - i - 3\rangle$ is a cycle with $\mathcal{I}(C_k) = k - 3$.

Proof of (β)

Part (β) follows from Lemma 6.3 (taking $i + 1$ instead of i) and the following facts.

- Every $(k + i)$-chord is in A. Follows from part (α).
- Every $(k - 1 + i)$-chord is in A. Follows from the inductive hypothesis.
- Every $(-r)$-chord and every $-(r + 1)$-chord is in A. Follows from Lemma 6.2.

■

Theorem 6.8 *If $n \geq 2k - 4$ then $f(n, k) \geq k - 3$.*

Proof: The case of $n = 2k - 4$ is considered in Section 4. Assume that $n > 2k - 4$ and assume for contradiction that $f(n, k, T) < k - 3$.

It follows from Lemma 6.7 that for each i, $-1 \leq i \leq r$, every $(k + i - 1)$-chord is in A. In particular

$$\{(0, k - 2), (0, k - 1), (0, k), \ldots, (0, k + r - 1)\} \in A. \tag{1}$$

It follows from Lemma 6.2 that every $(-r)$-chord is in A, and by Lemma 6.7 that every k-chord is in A. Therefore, by Lemma 6.6 for every α, $0 < \alpha r < k$, every $-(\alpha + 1)r$-chord is in A. Let $\alpha_0 = \max\{\alpha \in \mathcal{N} | \alpha r < k\}$. Clearly $\alpha_0 r < k$, and by Lemma 6.6 every $-(\alpha_0 + 1)r$-chord is in A. In particular $((\alpha_0 + 1)r, 0) \in A$ and $k \leq (\alpha_0 + 1)r < k + r$. Thus $y = (\alpha_0 + 1)r \in \{k - 2, k - 1, k, k + 1, \ldots, k + r - 1\}$. Therefore $(y, 0) \in A$. On the other hand, (1) implies that $(0, y) \in A$. A contradiction. ■

7 Upper Bounds

Two upper bounds are presented in this section. The first upper bound shows that for n, k, such that $n \geq 2k - 4$ the lower bounds on $f(n, k)$ presented previously are tight. In fact, a characterization of tournaments with $f(n, k) \geq k - 2$ is presented.

It has been shown that for $n \geq 2k - 4$, $f(n, k) \geq k - 3$. The second upper bound shows that for $n < 2k - 4$, $f(n, k) < k - 3$.

We start the proof of the first upper bound with the following simple lemma.

Lemma 7.1 *Let C_k be a cycle with $\mathcal{I}(C_k) = k-2$. If $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ are the arcs of C_k not in γ then $y_2 = y_1 + n - (k - 2 + x_1)$. Namely, f_2 is a $-(k-2+x_1)$-chord of γ.*

Theorem 7.2 *For $n \geq 5$, $k \geq 5$, such that $n \not\equiv k \pmod{k-2}$, $f(n,k) \leq k-3$.*

Proof: We prove the theorem by presenting a hamiltonian tournament T_n with no cycles C_k having $\mathcal{I}(C_k) = k-2$. We define T_n as follows.

$$
\begin{aligned}
A(T_n) \quad = \quad & \{(i, i+k-1+s(k-2)) | i \in \{0,1,\dots,n-1\}, s \in \{0,1,\dots,t\}\} \cup \\
& \{(i+j, i) | j \in \{\{2,3,\dots,\lfloor \frac{n-1}{2} \rfloor\} - \\
& \{k-1+s(k-2) | s \in \{0,1,\dots,t\}, i \in \{0,1,\dots,n-1\}\}\}\} \cup \\
& \{(i,i+1) | i \in \{0,1,\dots,n-1\}\}.
\end{aligned}
$$

(t was defined in Section 2) If n is even it remains to define the orientation of the $n/2$-chords. These are defined as follows. For $i \in \{0,1,\dots,n/2-1\}$, the arcs

$$(i+n/2, i)$$

are in A.

Assume for contradiction that C_k is a cycle of T_n with $\mathcal{I}(C_k) = k-2$, and let $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ the only arcs of C_k not in γ. Without loss of generality we can assume that $l(f_1) < n/2$. The definition of T_n implies that $x_1 = k-1+s(k-2)$, $s \in \{0,1,\dots,t\}$.

It follows from Lemma 7.1 that $y_2 = y_1 + n - (k-1+(s+1)(k-2))$. If $s < t$ then $s+1 \leq t$ and f_2 is a $-(k-1+(s+1)(k-2))$-chord, contradicting the definition of T_n.

Assume now that $s = t$. Hence $x_1 = k-1+t(k-2)$, and $n = (k-1)+t(k-2)+r$ implies $l\langle x_1, \gamma, 0\rangle = r$. On the other hand, we have that $C_k - \{(0,x_1),(y_1,y_2)\} \in \langle x_1, \gamma, 0\rangle$. Thus $l\langle x_1, \gamma, 0\rangle \geq k-1$, and $r \geq k-1$. The definition of r implies $r \leq k-1$. Therefore $r = k-1$ and then $n \equiv k \pmod{k-2}$, a contradiction. ∎

It is easy to verify that if $n \equiv k \pmod{k-2}$, then $f(n,k) \geq k-2$. Hence as a consequence of the previous theorem we get the following characterization of $f(n,k) \geq k-2$.

Corollary 7.3 *$f(n,k) \geq k-2$ if and only if $n \equiv k \pmod{k-2}$.*

The next theorem follows from Theorem 6.8 and Theorem 7.2.

Theorem 7.4 *For each $n \geq 2k-4$, such that $n \not\equiv k \pmod{k-2}$ it holds that $f(n,k) = k-3$.*

We now present the proof of the second upper bound. The aim is to show that the range of k that we have been considering $(2k-4 \leq n)$ is as large as possible, with $f(n,k) \geq k-3$.

Theorem 7.5 *For $n \geq 5$, $k \geq 5$, such that $n \leq 2k-5$ it holds that $f(n,k) < k-3$.*

Proof: We prove the theorem by presenting a tournament T_n with no cycles C_k having $\mathcal{I}(C_k) \geq k-3$. We define T_n as follows. If n is odd then

$$\begin{aligned} A(T_n) &= \{(i,i+1) | i \in \{0,1,\ldots,n-1\}\} \cup \\ &\quad \{(i,i+j) | j \in \{\frac{n+1}{2}, \frac{n+1}{2}+1, \ldots, n-2\}\}. \end{aligned}$$

If n is even then

$$\begin{aligned} A(T_n) &= \{(i,i+1) | i \in \{0,1,\ldots,n-1\}\} \cup \\ &\quad \{(i,i+j) | j \in \{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n-2\}\} \cup \\ &\quad \{(i,i+\frac{n}{2}) | i \in \{0,1,\ldots,\frac{n}{2}-1\}\}. \end{aligned}$$

Consider a cycle C_k of length k. Observe that $\mathcal{I}(C_k) < k-2$. We prove that $\mathcal{I}(C_k) < k-3$, by showing that for any cycle C with $\mathcal{I}(C) = k-3$, it holds that $l(C) \leq k-1$.

Let $f_1 = (x_1, x_2)$, $f_2 = (x_3, x_4)$, and $f_3 = (x_5, x_6)$ be the three arcs of C not in γ. Hence, without loss of generality,

$$C = (x_1, x_2) \cup \langle x_2, \gamma, x_3 \rangle \cup (x_3, x_4) \cup \langle x_4, \gamma, x_5 \rangle \cup (x_5, x_6) \cup \langle x_6, \gamma, x_1 \rangle.$$

By the definition of T_n it follows that $l(f_i) \geq n/2$, for each $i \in \{1,2,3\}$. Moreover, there exists $j \in \{1,2,3\}$, such that $l(f_j) > n/2$. On the other hand,

$$\begin{aligned} l(C) &= l\langle x_2, \gamma, x_1 \rangle + l\langle x_6, \gamma, x_5 \rangle - l\langle x_3, \gamma, x_4 \rangle + 3 \\ &= n - l(f_1) + n - l(f_3) - l(f_2) + 3. \end{aligned}$$

Now we proceed with the proof for n even. The case of n odd is analogous. Since $l(f_j) > n/2$ and $l(f_i) \geq n/2$ it follows that

$$l(C) \leq n/2 + n/2 - (n/2+1) + 3 = \frac{n+4}{2}.$$

Therefore $l(C) \leq k-1$, since $n \leq 2k-5$. ■

Finally, the complete characterization of $f(n,k) = k-3$ is presented.

Theorem 7.6 (Main Result) $f(n,k) = k-3$ *if and only if* $n \geq 2k-4$, *and* $n \not\equiv k \pmod{k-2}$.

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