Symbolic Dynamics, Ricardo Gómez, Nesin Mathematics Village

Exercises

1. Let $\mathcal{A}$ be an alphabet of size m. How many points $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ are fixed points? How many points are periodic of periodo $n \geq 2$? How many periodic points have minimal period 12?

2. Let $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{A}^*$.

   (a) Show that $\mathsf{X}_{\mathcal{F}_1} \cap \mathsf{X}_{\mathcal{F}_2} = \mathsf{X}_{\mathcal{F}_1 \cup \mathcal{F}_2}$. Use this fact to show that the intersection of two shift spaces over the same alphabet is a shift space. Extend this result to arbitrary intersections of shidt spaces over the same alphabet.

   (b) Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $X_{\mathcal{F}_1} \supseteq X_{\mathcal{F}_2}$. What is the relationship between $X_{\mathcal{F}_1 \cup X_{\mathcal{F}_2}}$ and $X_{\mathcal{F}_1 \cap \mathcal{F}_2}$?

3. Let $\mathcal{L}_1 \subset \mathcal{A}^*$ y $\mathcal{L}_2 \subset \mathcal{B}^*$ be the languages of two shift spaces over the alphabets $\mathcal{A}$ y $\mathcal{B}$. Hence we already know that the following two properties are satisfied:

   - **Extension:** $\forall\, w \in \mathcal{L}$, $\exists\, u, v \in \mathcal{L} \setminus \{\epsilon\}$ such that $uwv \in \mathcal{L}$ (ϵ is the empty word, i.e. u and v must have length ≥ 1).
   - **Factor:** $\forall\; w = a_1 \dots a_n \in \mathcal{L}$, $a_k \dots a_{k+r} \in \mathcal{L}\; \forall\; k \geq 1$ and $r \geq 0$ such that $k + r \leq n$.

   Show that $\mathcal{L}_1 \cup \mathcal{L}_2$ satisfies both the extension and factor properties. Then show that the union of two shift spaces is a shift space. Explain why this result can not be extended to arbitrary unions of shift spaces, even when the alphabet is the same for all.

4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be as in the previous exercise. Determine whether or not $\mathcal{L}_1 \cap \mathcal{L}_2$ is the language of a shift space, i.e. determine if the intersections of this two languages satisfies both the extension and factor properties.

5. Let $\mathcal{A} = \{0, 1\}$ be a binary alphabet. Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be the set of points $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that there exists exactly on entry with a 1 and the rest of the entries are 0.

   (a) Show that X is σ-invariant.

   (b) Show that X is *not* a shift space.

   (c) Show that $X \cup \{0^\infty\}$ is a shift space, where $0^\infty \in \mathcal{A}^{\mathbb{Z}}$ is the point with all of its coordinates equal to 0.

   (d) Find a subset of forbidden words $\mathcal{F} \subset \mathcal{A}^*$ such that $X \cup \{0^\infty\} = \mathsf{X}_{\mathcal{F}}$.

   (e) Determine if $X \cup \{0^\infty\}$ is a shift of finite type.

   (f) Find the periodic points of $X \cup \{0^\infty\}$.

6. Let $B_n = |\mathcal{B}_n(X_{\{11\}}|$ be the amount of n-blocks of the golden mean shift. Identify the counting sequence $(B_n)_{n\geq 0}$ and then find the generating function $f(z) = \sum\limits_{n\geq 0} B_n z^n$.

7. For the full $\{-1,+1\}$-shift and $k \geq 1$, find the number of k-blocks having the property that the sum of the symbols is 0.

8. Let $x \in X$ be a periodic point of minimal period k in a dynamical system (X, T). Show that if $n \geq 1$ is such that $T^n(x)$, then $k|n$, that is, k divides n.

9. Let $S \subseteq \{0, 1, 2, 3, \ldots\}$ be a *finite* subset of the nonnegative integers. Let $X(S) \subseteq \{0,1\}^{\mathbb{Z}}$ be the set of points in $\{0,1\}^{\mathbb{Z}}$ such that 1 occurs infinitely often both in the past and future, and between any two conecutive occurrences of 1, the amount of 0s belongs to S. If S is infinite, then we define $X(S)$ as before but we also add the point $0^\infty = \ldots 000 \ldots$ (why?).

   (a) Describe $X(S)$ when $S = \{1\}$.
   (b) Describe $X(S)$ when $S = \{4, 5, 6, 7\}$.
   (c) Show that the full shift is an S-gap shift (find S).
   (d) Show that the golden mean shift an S-gap shift (find S).
   (e) Show that if S is infinite, then $X(S)$ is no longer a shift space.
   (f) Show that $X(S) \cup \{0^\infty\}$ is always a shift space (even when S is infinite).
   (g) When S is infinite, we let the S-gap shift be $X(S) \cup \{0^\infty\}$. Show that the even shift is an S-gap shift (find S).
   (h) For which sets S does the S-gap shift have infinitely many periodic points?

10. Consider the binary alphabet $\{0, 1\}$. Describe the shift space X_{01}.

11. Let X be the full $\mathcal{A}$-shift.

   (a) Show that if X_1 and X_2 are shift spaces such that $X_1 \cup X_2 = X$, then $X_1 = X$ or $X_2 = X$ (or both).
   (b) Extend your argument to show that if X is the union of any collection $\{X_\alpha\}$ of shift spaces, then there is an α such that $X = X_\alpha$.
   (c) Explain why these statements no longer hold if we merely assume that X is a shift space (not full).

12. Let X be a shift space and let $N \geq 1$. Shoe that there is a collection $\mathcal{F}$ of forbidden blocks of length at least N such that $X = X_{\mathcal{F}}$.

13. Show that there are uncountably many shift spaces contained in the full 2-shift. (*Hint*: consider the S-gap shift.)

14. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the languages of two shift spaces (i.e. they both satisfy both the factor and extension properties). Show that $\mathcal{L}_1 \cup \mathcal{L}_2$ satisfies both the factor and extension properties. Use this to show that the union of two shift spaces is also a shift space. Now let $\mathcal{L}_1\mathcal{L}_2, \mathcal{L}_3 \ldots$ be a countable collection of languages of shift spaces over the same alphabet. Show that $\cup_{n\geq 1}\mathcal{L}_n$ is also a language of a shift space. Why can't you use this to show that the union of an infinite number of shift spaces over the same alphabet is also a shift space?

15. A shift space X is *irreducible* if for every two legal words $u, v \in \mathcal{B}(X)$, there exists a legal word $w \in \mathcal{B}(X)$ such that uwv is legal, that is, $uwv \in \mathcal{B}(X)$.

    (a) Is the intersection of two irreducible shift spaces always irreducible?

    (b) Let $\mathcal{A} = \{0, 1\}$. Is $X_{\{01\}}$ irreducible?

    (c) If X is irreducible, then show that for any pair $u, v \in \mathcal{B}(X)$ there exists a *non-empty* word $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.

16. Let $\phi: X \to Y$ and $\psi: Y \to Z$ be two sliding block codes.

    (a) Show that $\psi \circ \phi$ is a sliding block code.

    (b) Show that if both are factor codes, then the composition is also a factor code, and similarly for embeddings and conjugacies.

17. Let $X = \{0, 1\}^{\mathbb{Z}}$ and $\Phi: \{0, 1\} \to \{0, 1\}$ be the 1-block map given by $\Phi(0) = 1$ and $\Phi(1) = 0$. Show that $\phi = \Phi_\infty: X \to X$ is a conjugacy of the full two-shift to itself.

18. Let $X = \{0, 1\}^{\mathbb{Z}}$ be (again) the full 2-shift (over the alphabet $\{0, 1\}$). Define the block map $\Phi\{0, 1\}^4 \to \{0, 1\}$ by

    $$\Phi(abcd) = b + a(c + 1)d \pmod 2$$

    and put $\phi = \Phi_\infty^{[-1,2]}$.

    (a) Describe the action of ϕ on $x \in X$ in terms of the blocks 1001 and 1101 appearing in x.

    (b) Show that $\phi^2(x) = x$ for all $x \in X$, and hence show that ϕ is a conjugacy of X to itself.

    (c) Use this method to find another conjugacies of the full 2-shift to itself.

    (d) Let $\phi: X \to Y$ be a sliding block code, and let $Z \subset Y$ be a shift space contained in Y. Show that $\varphi^{-1}(Z) = \{x \in X \ : \ \phi(x) \in Z\}$ is a shift space (contained in X).

19. Show that a forbidden list $\mathcal{F} \subseteq \mathcal{A}^*$ is such that $X_{\mathcal{F}} = \varnothing$ if and only if there exists N such that whenevcer u and v are blocks with $|u| = N$, then some subblock of uvu belongs to $\mathcal{F}$.

20. Let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ be shift spaces whose intersection is X. For each $N \geq 1$, use the Cantor diagonal argument to prove that there is a $K \geq 1$ such that $\mathcal{B}_N(X_k) = \mathcal{B}(X)$ for all $k \geq K$.

21. Decide which of the following is a shift of finite type (if so, exhibit a finite generating set of forbidden words):

    (a) The gap shift $X(S)$ when $S = 1, 2, 3$.

    (b) The gap shift $X(S)$ when S is the set of prime numbers (the so called *prime shift*).

    (c) The context free shift.

22. Let $\mathcal{A} = \{-1, +1\}$ and let $c \geq 0$. Let $X \subseteq \mathcal{A}$ be defined by the rule that for every word w occurring in any point of X, the algebraic sum s of the symbols in w satisfies $-c \leq s \leq c$. Show that X is a shift space. Is it a shift of finite type? Is it a sofic shift?

23. Let X and Y be shifts of finite type. Show that $X \cap Y$ is a shift of finite type. Must $X \cup Y$ be a shift of finite type?

24. Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{11, 00000\}$. Find a collection $\mathcal{F}_5$ of 5-blocks such that $X_{\mathcal{F}} = X_{\mathcal{F}_5}$.

25. If $\mathcal{F}$ is a collection of blocks for which $X_{\mathcal{F}} = \varnothing$, must there always be a *finite* subcollection $\mathcal{F}_0 \subseteq$ such that $X_{\mathcal{F}_0} = \varnothing$?

26. Let G be the graph of the full two shift (see Figure 1). Draw the graph of $G^{[3]}$.

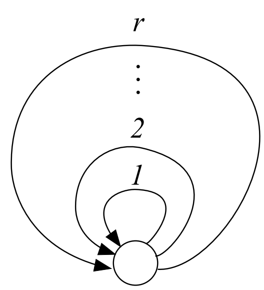

Figure 1: The graph of the full r-shift.

27. Let $\mathcal{A} = \{0, 1\}$, $\mathcal{F} = \{000, 111\}$ and $X = X_{\mathcal{F}}$.

    (a) Construct a graph G for which $X_G = X_{\mathcal{F}}^{[3]}$.

    (b) Use the adjacency matrix of G to compute the number of points in X with period 5.

    (c) Compute the entropy $h(X)$.

28. For each pair of adjacency matrices below, decide whether it is possible to obtain the graph of the second from the graph of the first by a finite sequence of splittings and amalgamations:

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(c) $\begin{pmatrix} 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

29. Let $\mathcal{G}$ and $\mathcal{H}$ be the labeled graphs with symbolic adjacency matrices

$$\begin{pmatrix} a & b \\ b & \varnothing \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & a \\ b & \varnothing \end{pmatrix}.$$

Show that $X_{\mathcal{G}}$ is not conjugate to $X_{\mathcal{H}}$.

30. For the labelled graphs $\mathcal{G}$ corresponding to each of the following symbolic adjacency matrices, compute the follower set graph of $X_{\mathcal{G}}$. Then find a right resolving presentation of $X_{\mathcal{G}}$.

$$\begin{pmatrix} \varnothing & a & a \\ \varnothing & \varnothing & b \\ a & b & \varnothing \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & a+b & a+c \\ a & \varnothing & \varnothing \\ a & \varnothing & \varnothing \end{pmatrix}.$$