A Brief Motivation to the Course "Introduction to Symbolic Dynamics and its Applications"

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Abstract

A brief motivation to the course "Symbolic Dynamics" on NMV, July 14-28.

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1 Discrete dynamical systems

A (discrete) dynamical system consists of a *space* X and a *transformation* $f\colon X \to X$. In general, X carries some sort of structure:

- X is a topological space.
- X is a measurable space (Borel), and comes with a probability measure.
- X is a metric space.
- X is a *compact* metric space.
- X is a differentiable manifold.

Also, in general f is required to preserve the structure:

- f is continuous.
- f is measurable, and it is measure preserving.
- f is expanding, contracting, an isometry, etc.
- f is differentiable.

Each case above gives rise to a whole area of study: topological dynamics, measurable dynamics, ergodic theory, dynamics of compact metric spaces, differentiable dynamics, etc. Also, we have the following two cases:

- f is invertible (a bijection).
- f is *not* invertible.

We will be mainly interested in *topological dynamical systems*[1] (TDS), that is, systems (X, f) where X is a topological space and $f\colon X \to X$ is continuous. Furthermore, in general, the topology will be metrizable and X will be a compact metric space. In addition, we will mainly focus on the case when f is invertible. Henceforth, unless otherwise stated, by a *dynamical system* we will mean a pair (X, f) where $X = (X, d)$ is a compact metrizable space ($d\colon X \times X \to \mathbb{R}^+$ is a metric that induces the topology), and $f\colon X \to X$ is a continuous function or a homeomorphism (automorphism).

2 Orbits and natural extensions

Let (X, f) be a dynamical system and let $x \in X$.

- *Positive semiorbit* (or *forward orbit*):

$$\mathcal{O}_f^+(x) = \{x, f(x), f^2(x), \dots\} \in X^{\mathbb{N}}.$$

- *Negative semiorbit* (or *backward orbit*): If f is invertible,

$$\mathcal{O}_f^-(x) = \{\dots, f^{-3}(x), f^{-2}(x), f^{-1}(x)\} \in X^{\mathbb{Z}_{<0}}.$$

[1]Later, if there is time and interest, we will consider *measurable dynamical systems*.

- *Orbit*: If f is invertible,

$$\mathcal{O}_f(x) = \mathcal{O}_f^+(x) \uplus \mathcal{O}_f^+(x) = \{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\} \in X^{\mathbb{Z}}.$$

We shall view (semi)orbits both as sets and as (either one-sided –left or right–, or double-sided) infinite sequences (the context should always be clear[2]).

A point $x \in X$ is *transitive* for f if $\overline{\mathcal{O}_f^+(x)} = X$.

Proposition 2.1 (See [1], pp. 32)**.** *If (X, f) is invertible, X has no isolated points, and there is a dense orbit, then there exists a dense forward orbit.*

A TDS (X, f) is *minimal* if it possesses no closed non-empty invariants subsets (i.e. $Y \subsetneq X$, $Y \neq \varnothing$, and $f^n(Y) \subseteq Y$ for all n)[3]. If (X, d) is compact, then (X, f) is minimal if and only if every point $x \in X$ is transitive.

The *natural extension* of an (invertible) TDS (X, f) is the (invertible) TDS $(\mathbf{X}, \mathbf{f})$ where

$$\mathbf{X} = \{\mathcal{O}_f(x) \mid x \in X\} \subseteq X^{\mathbb{Z}}$$

(the orbits here are thought as sequences), and $\mathbf{f} \colon \mathbf{X} \to \mathbf{X}$ is defined as follows: for every $x \in X$,

$$\mathbf{f}\big(\mathcal{O}_f(x)\big)_n = f\big(\mathcal{O}_f(x)_n\big).$$

Here $X^{\mathbb{Z}}$ inherits the product topology. Since X is compact metrizable and f is continuous (and invertible), $(\mathbf{X}, \mathbf{f})$ is a compact metrizable (invertible) TDS. Observe that $\mathbf{f}$ is the *left shift map* automorphism, that is $\mathbf{f} = \sigma$ where $\sigma \colon \mathbf{X} \to \mathbf{X}$ acts as follows:

$$\begin{array}{ccccccccccccccc}
\mathcal{O}_f(x) = & \cdots & , & f^{-2}(x) & , & f^{-1}(x) & , & x & , & f^1(x) & , & f^2(x) & , & \cdots \\
\downarrow \sigma & & & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & & & . \\
\mathbf{f}\big(\mathcal{O}_f(x)\big) = & \cdots & , & f^{-1}(x) & , & x & , & f^1(x) & , & f^2(x) & , & f^3(x) & , & \cdots
\end{array}$$

A similar definition applies to non-invertible TDS.

3 Periodic points and zeta functions

Let (X, f) be a dynamical system. A point $x \in X$ is *periodic* if the following holds:

$$\exists\, n \geq 1 \text{ such that } f^n(x) = x.$$

If the above holds, then we say that x is a *periodic point of period* n. Observe that if x is periodic of period n, then it is also periodic of period nk for every $k \geq 1$. Thus it makes sense to define the *minimal period* of x as the smallest $n \geq 1$ such that $f^n(x) = x$.

A point $x \in X$ is *eventually periodic* if the following holds:

$$\exists k \geq 0 \text{ and } n \geq 1 \text{ such that } f^{n+k}(x) = f^k(x).$$

If f is invertible, then eventual periodicity $\Longrightarrow$ periodicity.

---

[2]For example, if we write something like $\mathcal{O}_f(x)_k$, then we will be refering to the kth entry of the *sequence* determined by the orbit of x, i.e. $\mathcal{O}_f(x)_k = f^k(x)$.

[3]Otherwise $(Y, f|_Y)$ is a TDS, a *subsystem* of (X, f). Thus a minimal system has no proper subsystems.

Proposition 3.1. *Let (X, f) be an invertible dynamical system.*

- *A point $x \in X$ is periodic if and only if $|\mathcal{O}_f(x)| < \infty$.*
- *A point $x \in X$ is periodic of minimal period $n \geq 1$ if and only if $|\mathcal{O}_f(x)| = n$.*
- *If $x \in X$ is periodic of period $m \geq 1$ and minimal period $n \geq 1$, then $n|m$.*

Proof. **Exercise**. □

For every positive integer $n \geq 1$, let the set of periodic points of period n and the set of periodic points of minimal period n be

- $\mathcal{P}_n(X, f) = \{x \in X \mid f^n(x) = x\}$
- $\mathcal{Q}_n(X, f) = \{x \in \mathcal{P}_n(X, f) \mid f^k(x) \neq x \text{ for all } k = 1, \ldots, n-1\}$

respectively (observe that $\mathcal{Q}_n(X, f) \subseteq \mathcal{P}_n(X, f)$).

Notation. We will also write $\mathcal{P}_n(f)$, $\mathcal{P}_n(X)$, and sometimes simply $\mathcal{P}_n$, and similarly for $\mathcal{Q}_n$, whenever there is no room for confusion (similar abuse of notation will occur in other instances).

Also, let $\mathcal{P}(X, f) = \bigcup_{n\geq 1} \mathcal{P}_n(X, f)$ be the set of all periodic points. If f is invertible, then $x \in X$ is periodic if and only if $f(x)$ is periodic, and in this case

$$(X \setminus \mathcal{P}(X), f|_{X\setminus\mathcal{P}(X)})$$

is a (generally non-compact) dynamical system called the *free part* of (X, f). Let

- $p_n(X, f) = |\mathcal{P}_n(X)|$
- $q_n(X, f) = |\mathcal{Q}_n(X)|$

Finite assumption. p_n is finite for every $n \geq 1$.

Clearly,

$$p_n = \sum_{k|n} q_k. \tag{1}$$

On the other hand, Möbius inversion yields the converse (see Appendix):

$$q_n = \sum_{k|n} \mu\left(\frac{n}{k}\right) p_k. \tag{2}$$

The counting sequence $\{p_n\}$ is coded in the *Artin-Mazur* (or *exp-log*, or *dynamic*) zeta function defined by

$$\zeta_{X,f}(z) = \exp\left(\sum_{n\geq 1} \frac{p_n}{n} z^n\right). \tag{3}$$

We say that a dynamical system (X, f) has a *well defined zeta function* if, in addition to the condition $p_n < \infty$ for every $n \geq 1$, we also have $R^{-1} = \overline{\lim}(p_n)^{1/n} < \infty$. In this case, $\zeta(z)$ is an analytic function in the disc centered at the origin and radius R.

Exercise: We have seen that $\{p_n\}$ determines $\{q_n\}$ and viceversa (see equations (1) and (2)). Equation (3) expresses the zeta function in terms of the counting sequence of periodic points, namely $\{p_n\}$. For the counting sequences of *minimal* periodic points, i.e. $\{q_n\}$, show that the zeta functions admits the following product formula:

$$\zeta(z) = \prod_{n \geq 1} (1 - z^{q_n/n})^{-1}$$

(observe that q_n/n is the number of orbits of size n).

4 Examples

- Interval maps. Let $f\colon [0,1] \to [0,1]$ be a continuous function. Then $([0,1], f)$ is a TDS.

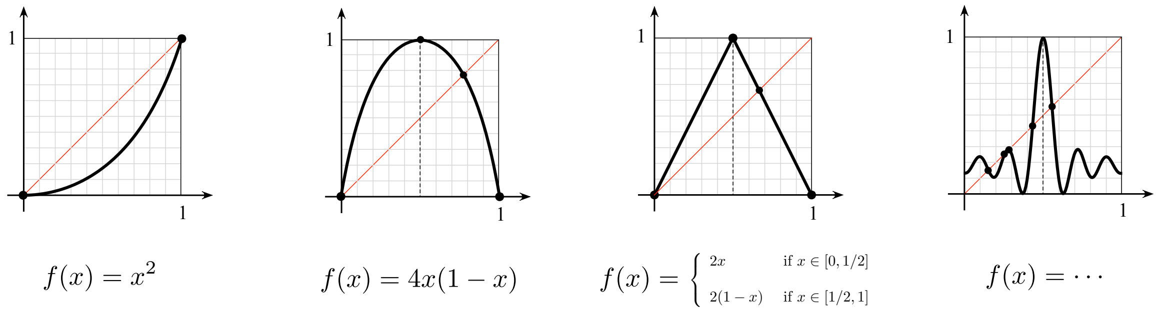

Figure 1: Interval maps.

Exercise. Study the periodic points of the maps above.

- Circle rotations. Let

$$\mathbb{S}^1 = \{z \in \mathbb{C} \ : \ |z| = 1\} = \{e^{2\pi i x} \ : \ x \in [0,1]\} = [0,1]/\sim$$

be the unitary circle (here $\sim$ means that 0 and 1 are identified). For $\alpha \in \mathbb{R}$, let $f = R_\alpha\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation of $\mathbb{S}^1$ by angle $2\pi\alpha$, i.e.

$$R_\alpha(x) = x + \alpha \mod 1.$$

Then $(\mathbb{S}^1, R_\alpha)$ is a TDS.

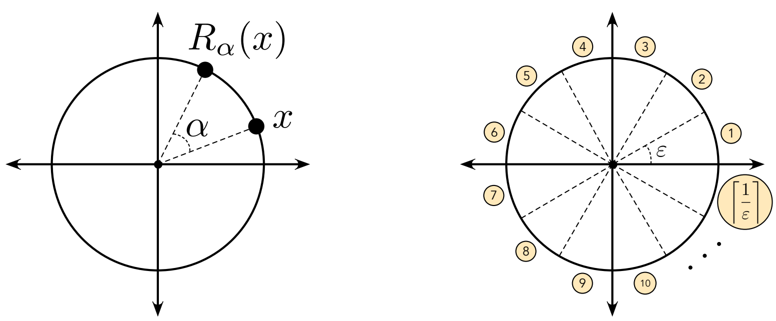

Figure 2: Circle rotations.

If $\alpha = p/q$ is rational, then $R_\alpha^q = \mathrm{Id}$, so every point is periodic (and the zeta function is not well defined, in fact the finite assumption is not even satisfied! Why?). Otherwise, if α is irrational, then every positive semiorbit is dense in $\mathbb{S}^1$. Indeed, the pigeon-hole principle implies that, for every $\varepsilon > 0$, there exist $m, n < 1/\varepsilon$ such that $m < n$ and $d(R_\alpha^m, R_\alpha^n) < \varepsilon$. Then R^{n-m} is a rotation by an angle less than ε, so every positive semiorbit is ε-dense in $\mathbb{S}^1$.

Thus, when α is irrational, $(\mathbb{S}^1, R_\alpha)$ is minimal (in particular, R_α has no periodic points).

- Expanding endomorphisms of the circle. Again let $\mathbb{S}^1 = [0,1]/\sim$ be the circle. For every $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$, let $f = E_m \colon \mathbb{S}^1 \to \mathbb{S}^1$ be

$$f(x) = mx \mod 1.$$

Then $(\mathbb{S}^1, E_m)$ is a TDS.

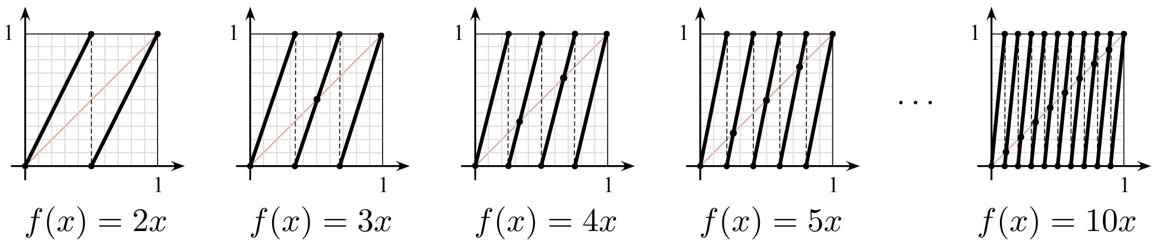

Figure 3: Expanding endomorphisms of the circle.

- Full shifts and shift spaces. Let $\mathcal{A}$ be an *alphabet*, i.e a finite discrete set, e.g.

$$\mathcal{A} = \{0, \dots, m-1\}.$$

Let

$$\mathcal{A}^{\mathbb{Z}} = \{(x_n)_{n\in\mathbb{Z}} \ : \ x_n \in \mathcal{A} \ \ \forall\ n \in \mathbb{Z}\}$$

be the set of bi-infinite sequences of symbols in $\mathcal{A}$. The set $\mathcal{A}^{\mathbb{Z}}$ is equipped with the product topology. A basis for such topology consists of the *cylinder sets*: Given $\omega = a_1 \dots a_n \in \mathcal{A}^n$ and $k \in \mathbb{Z}$,

$$C[\omega; k] = \{\mathbf{x} = (x_n)_{n\mathbb{Z}} \ : \ x_{[k,k+n-1]} = \omega\}.$$

Actually, the topology is metrizable, it is generated by the *Cantor metric*:

$$\forall \ \ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \quad \text{and} \quad \forall \ \ \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}},$$

let

$$d(\mathbf{x}, \mathbf{y}) = \frac{1}{2^{n(\mathbf{x},\mathbf{y})}}$$

where

$$n(\mathbf{x}, \mathbf{y}) = \begin{cases} \infty & \text{if } \mathbf{x} = \mathbf{y} \\ \min\{k \geq 0 \ : \ x_k \neq y_k \text{ or } x_{-k} \neq y_{-k}\} & \text{otherwise} \end{cases}.$$

Exercise. Prove that the Cantor metric generates the topology.

The dynamics is given by the left shift map $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, i.e. the automorphism defined for every

$$\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$$

by the rule

$$\sigma(\mathbf{x})_n = x_{n+1} \ \ \forall\ n \in \mathbb{Z}$$

(compare with the natural extension: What differences do you see?).

Exercise. Prove that $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a homeomorphism.

Then $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is the TDS known as the (two-sided, or invertible) *full shift over the alphabet* $\mathcal{A}$.

One-sided full shifts are defined similarly. In this case, $\sigma\colon \mathcal{A}^{\mathbb{N}} \to A^{\mathbb{N}}$ is no longer a homeomorphism, it is onto though, actually it is m-to-1.

- Subshifts. Let $\mathcal{F} \subset \mathcal{A}^* = \cup_{n\geq 1}\mathcal{A}^n$ be a set of finite words. Then define

  $$X = \mathsf{X}_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}}$$

  as follows:

  $$\mathsf{X}_{\mathcal{F}} = \{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \ : \ \forall\ k \in \mathbb{Z} \text{ and } \forall\ w \in \mathcal{F}, \ \ x_{[k,k+|\omega|-1]} \neq \omega\}.$$

  Then $\mathsf{X}_{\mathcal{F}}$ is a closed σ-invariant subset (prove it!) and

  $$(\mathsf{X}_{\mathcal{F}}, \sigma|_{\mathsf{X}_{\mathcal{F}}})$$

  is a TDS known as a *subshift*, or *shift space.*

  One sided shift spaces are defined similarly.

Symbolic dynamics is the area of topological dynamics that studies shift spaces. The main focus of the course will be to study shift spaces!

5 Classification of dynamical systems

Let (X, f) and (Y, g) be two TDSs. A (*topological*) *homomorphism* is a continuous function $\phi\colon Y \to X$ such that $f\circ\phi = \phi\circ g$, that is, the following diagram is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow[\ f\ ]{} & X \end{array}$$

- If ϕ is injective, then it is an (*topological*) *embedding.* In this case we can think of (Y, g) as a subsystem of (X, f).

- If ϕ is surjective, then it is a (*topological*) *semiconjugacy*, in which case we say that (X, f) is a *factor* of (Y, g). In this case we can think of (Y, g) as a supersystem of (X, f), meaning that we can "squeeze" (Y, g) and as a result get (X, f), so to speak.
- If ϕ is bijective, then it is a (*topological*) *conjugacy*. In this case we can think of (X, f) and (Y, g) as being "essentially" the same.

Classification problems in dynamics. Classify TDSs with respect to embeddings, semiconjugacies and conjugacies. More precisely, given two TDSs (X, f) and (Y, g), find necessary and sufficient conditions for

- the existence of an embedding $\phi\colon Y \to X$;
- the existence of a semiconjugacy $\phi\colon Y \to X$; and
- the existence of a conjugacy $\phi\colon Y \to X$.

6 Invariants

We will associate special mathematical objects to dynamical systems, widely called *invariants* because they are meant to help addressing classification problems in dynamics. For example, consider the number of periodic points of period n, or moreover the zeta function itself:

Exercise.

- If $\phi\colon X \to Y$ is an embedding, then $p_n(X) \le p_n(Y)$ for every $n \ge 1$.
- If $\phi\colon X \to Y$ is a factor, then $p_n(X) \ge p_n(Y)$ for every $n \ge 1$.
- If $\phi\colon X \to Y$ is a conjugacy, then $\zeta_X(z) = \zeta_Y(z)$, or equivalently, $p_n(X) = p_n(Y)$ for every $n \ge 1$.

One central and fundamental invariant is the *topological entropy*. In words, the topological entropy is defined as the logarithm of the exponential growth rate of the number of essentially different orbit segments of length n (thus it is a non-negative number or $+\infty$). It is a topological invariant that measures the complexity of the orbit structure of a dynamical system. For symbolic systems, the topological entropy is easily defined and it can be computed in several general cases (as we shall see). For general TDSs, the definition of topological entropy is more intricate, and computing the entropy can be quite difficult.

Given a TDS (X, f), we will denote by $h(X)$ its topological entropy (we have not defined it yet). The following will hold:

- If $\phi\colon X \to Y$ is an embedding, then $h(X) \le h(Y)$.
- If $\phi\colon X \to Y$ is a factor, then $h(X) \ge h(Y)$.
- If $\phi\colon X \to Y$ is a conjugacy, then $h(X) = h(Y)$.

These facts exhibit the importance of the topological entropy, particularly with respect to classification problems in dynamics. Let us jump into it.

7 Topological entropy

Let (X, f) be a TDS with (X, d) a compact metric space. For each $n \geq 1$, let

$$d_n(x, y) = \max_{0 \leq k < n} d\big(f^k(x), f^k(y)\big) \quad \forall\, x, y \in X.$$

Exercise. Prove the following:

- (X, d_n) is a metric on X.
- $d_1 = d$ and $d_n \geq d_{n-1}$ for all $n > 1$.
- All these metrics are equivalent.

Let $\varepsilon > 0$.

- SPANNING SETS. A subset $A \subseteq X$ is (n, ε)-spanning if

  $$\forall\, x \in X, \quad \exists\, y \in A, \quad d_n(x, y) < \varepsilon.$$

  Compactness $\Longrightarrow$ there are *finite* (n, ε)-spanning sets. Then we define

  $$\operatorname{span}(n, \varepsilon, f) = \min\{|A| \;:\; A \subseteq X \text{ is } (n, \varepsilon)\text{-spanning}\}$$

- SEPARATED SETS. A subset $A \subseteq X$ is (n, ε)-separated if

  $$\forall\, x, y \in A, \quad d_n(x, y) \geq \varepsilon.$$

  Compactness $\Longrightarrow$ any (n, ε) separated set is finite. Then we define

  $$\begin{aligned} \operatorname{sep}(n, \varepsilon, f) &= \sup\{|A| \;:\; A \subseteq X \text{ is } (n, \varepsilon)\text{-separated}\} \\ &= \max\{|A| \;:\; A \subseteq X \text{ is } (n, \varepsilon)\text{-separated}\} \end{aligned}$$

  (the second equality also follows from compactness).

- COVERING SETS. Let

  $$\operatorname{cov}(n, \varepsilon, f) = \min\{|\Lambda| \;:\; \mathcal{P} = \{P_k\}_{k \in \Lambda} \text{ is a covering of } X \text{ and } \operatorname{diam}_{d_n}(P_k) < \varepsilon\}.$$

  Compactness $\Longrightarrow \operatorname{cov}(n, \varepsilon, f) < \infty$.

These three quantities count the number of orbit segments of length n that are distinguishable at scale ε.

Lemma 7.1.

$$\operatorname{cov}(n, 2\varepsilon, f) \leq \operatorname{span}(n, \varepsilon, f) \leq \operatorname{sep}(n, \varepsilon, f) \leq \operatorname{cov}(n, \varepsilon, f).$$

Proof. First inequality: Let A be (n, ε)-spanning such that $|A| = \operatorname{span}(n, \varepsilon, f)$. Then the *open* d_n-balls of radius ε centered at the points of A cover X. Compactness $\Longrightarrow$ there exists $\varepsilon_1 < \varepsilon$ such that the *closed* balls of radius ε centered at the points of A also cover X. Their diameter is $2\varepsilon_1 < 2\varepsilon$, so $\operatorname{cov}(n, 2\varepsilon, f) \leq \operatorname{span}(n, \varepsilon, f)$.

The rest of the inequalities are left as **Exercises**. □

Let
$$h_\varepsilon(f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log\big(\mathrm{cov}(n,\varepsilon,f)\big).$$
$\mathrm{cov}(n,\varepsilon,f)\nearrow$ as $\varepsilon\searrow$ $\implies$ also $h_\varepsilon(f)\nearrow$ as $\varepsilon\searrow$ $\implies$ the following limit, called *topological entropy*, exists:
$$\boxed{h_{\mathrm{top}} = h(f) = \lim_{\varepsilon\to 0^+} h_\varepsilon(f)}$$
By Lemma 7.1, we have
$$\begin{aligned} h(f) &= \lim_{\varepsilon\to 0^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \log\big(\mathrm{span}(n,\varepsilon,f)\big) \\ &= \lim_{\varepsilon\to 0^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \log\big(\mathrm{sep}(n,\varepsilon,f)\big). \end{aligned}$$

Lemma 7.2. *The following limit exists and it is finite:*
$$\lim_{n\to\infty} \frac{1}{n} \log\big(\mathrm{cov}(n,\varepsilon,f)\big) = h_\varepsilon(f).$$
Moreover,
$$h_\varepsilon(f) = \inf_{n\geq 1} \frac{1}{n} \log\big(\mathrm{cov}(n,\varepsilon,f)\big).$$
Proof. Let $U,V\subseteq X$ be such that
$$\mathrm{diam}_{d_m}(U) < \varepsilon \qquad \& \qquad \mathrm{diam}_{d_n}(V) < \varepsilon$$
Then[4]
$$\mathrm{diam}_{d_{m+n}}\big(U\cap f^{-m}(V)\big) < \varepsilon.$$
Hence[5]
$$\mathrm{cov}(m+n,\varepsilon,f) \leq \mathrm{cov}(m,\varepsilon,f)\cdot \mathrm{cov}(n,\varepsilon,f)$$
and so the sequence $a_n = \log\big(\mathrm{cov}(n,\varepsilon,f)\big)$ is subadditive and thus the result follows from Fekete's subadditive lemma (see Appendix). □

Proposition 7.3. *$h(f)$ does not depend on the choice of a particular metric generating the topology of (X,d).*

Proof. Let d and d' be two (topologically) equivalent metrics. Given $\varepsilon>0$, let
$$\delta(\varepsilon) = \sup\{d'(x,y) \;:\; d(x,y)\leq\varepsilon\}.$$
Compactness $\implies$ $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$.
If $\mathrm{diam}_{d_n}(U)<\varepsilon$, then $\mathrm{diam}_{d'_n}(U)\leq\delta(\varepsilon)$.
Then $\mathrm{cov}'\big(n,\delta(\varepsilon),f\big)\leq \mathrm{cov}(n,\varepsilon,f)$.
Thus
$$\lim_{\delta\to 0^+}\lim_{n\to\infty}\frac{1}{n}\log\big(\mathrm{cov}'(n,\delta,f)\big) \leq \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\frac{1}{n}\log\big(\mathrm{cov}(n,\varepsilon,f)\big).$$
Interchanging d and d' gives the desired result. □

[4] Indeed, let $x,y\in U\cap f^{-m}(V)$. Since $x,y\in U$, $d_m(x,y)<\varepsilon$, but also $f^m(x),f^m(y)\in V$, and thus $d_n\big(f^m(x),f^m(y)\big)<\varepsilon$, and so the claim follows.

[5] Indeed, if $\mathcal{U}=\{U_k\}_{k\in\Lambda_{\mathcal{U}}}$ and $\mathcal{V}=\{V_j\}_{j\in\Lambda_{\mathcal{V}}}$ are two coverings of X, then $\{U\cap f^{-m}(V) \;:\; U\in\mathcal{U} \;\&\; V\in\mathcal{V}\}$ is also a covering of X.

Corollary 7.4. *Topological entropy is an invariant of topological conjugacy.*

Proof. Let $\phi\colon X \to Y$ be a topological conjugacy between the TDSs (X, f) and (Y, g). Let d be a metric in Y generating its topology. Then $d'(x_1, x_2) = d\big(\phi(x_1), \phi(x_2)\big)$ is a metric in X generating its topology. Since ϕ is an isometry between (X, d') and (Y, d) and the entropy is independent of the metric, $h(f) = h(g)$. □

8 Symbolic representations

The success of symbolic dynamics mainly relies on the fact that shift spaces serve as models of more general dynamical systems. Let us sketch an example (precise formal definitions will come later).

Consider the expanding automorphism $E_{10}\colon \mathbb{S}^1 \to \mathbb{S}^1$ for example. Let $\mathcal{A} = \{0, 1, \ldots, 9\}$ and consider the full $\mathcal{A}$-shift

$$X = \{0, 1, \ldots, 9\}^{\mathbb{N}}.$$

Subdivide $\mathbb{S}^1$ in ten equal subintervals $P_j = [j/10, (j + 1)/10)$ with $j = 0, \ldots, 9$.

For each $y \in \mathbb{S}^1$ and $n \geq 0$, define $x_n \in \mathcal{A}$ by the rule $E_{10}(y) \in P_j$. This results in a point $\mathbf{x} = x_0x_1 \ldots \in X$. The image of $\mathbb{S}^1$ on X under this correspondence is *not* a shift space: the sequence of points in $\mathbb{S}^1$ given by $y^{(1)} = .19$, $y^{(2)} = .199$, $y^{(3)}.1999$, … converges in $\mathbb{S}^1$ to $y = .2$, but their images in X, namely $\mathbf{x}^{(1)} = 19000\ldots$, $\mathbf{x}^{(2)} = 19900\ldots$, $\mathbf{x}^{(3)} = 19990\ldots$, … converges to $\mathbf{x} = 19999\ldots$, and the later is not the image of any point in $\mathbb{S}^1$ (the image of the point $y = .2$ is $20000\ldots$). So instead we will construct a semiconjugacy from X to $\mathbb{S}^1$, namely we will map both $19999\ldots$ and $20000\ldots$ to .2, and we will do something similar with all the 10-adic numbers.

The 10-adic numbers have Lebesgue measure zero, so they are kind of negligible. Thus the map from (X, σ) onto the TDS $(\mathbb{S}^1, E_{10})$ that results is 1-1 almost everywhere. Thus, we can claim that

$(\mathbb{S}^1, E_{10})$ *is essentially the (one-sided) full shift on 10 symbols.*

The above example is the preamble to a formal construction that we will see, known as *symbolic representations*.

9 Appendix

9.1 Möbius inversion

Here we prove equation (2).

Let $\mu\colon \mathbb{N}^* \to \{-1, 0, 1\}$ be the arithmetic function known as the *Möbius function*, defined by first letting $\mu(1) = 1$ and then, for every $n \geq 2$, if we consider the factorization of n as a product of powers of distinct primes, say $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then

$$\mu(n) = \begin{cases} 0 & \text{if } \alpha_k \geq 2 \text{ for some } k = 1, \ldots, r \\ (-1)^r & \text{otherwise (i.e. if } \alpha_1, \ldots, \alpha_r = 1) \end{cases}.$$

To prove equation (2), consider the Dirichlet function associated to the Möbius function μ, that is, let

$$M(s) = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$$

and also consider Riemann's zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Lemma 9.1 (Euler's formula for the Riemann zeta function)**.** *Let $\mathcal{P}$ denote the set of prime numbers. Then*

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}.$$

Proof. Start with

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \dots \tag{4}$$

and multiply by $\dfrac{1}{2^s}$ to get

$$\frac{1}{2^s} \cdot \zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \frac{1}{14^s} + \frac{1}{16^s} + \dots. \tag{5}$$

Subtracting equation (5) to equation (4) we get

$$\left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \dots. \tag{6}$$

Similarly, multiply the previous equation by $\dfrac{1}{3^s}$ and get

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \frac{1}{39^s} + \dots. \tag{7}$$

Subtracting equation (7) to equation (6) we get

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} + \frac{1}{19^s} + \frac{1}{23^s} + \dots.$$

Keep on going this process: multiplying by p^{-s} for every prime p and subtracting the corresponding previous equation and get the desired result. □

Lemma 9.2 (Inverses of zeta and Möbius functions)**.** $M(s) \cdot \zeta(s) = 1$.

Proof. Using Euler's formula we get

$$\begin{aligned}\frac{1}{\zeta(s)} &= \prod_{p \in \mathcal{P}} (1 - p^{-s}) \\ &= (1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s})(1 - 7^{-s})(1 - 11^{-s})(1 - 13^{-s}) \cdots \\ &= M(z).\end{aligned}$$

□

Lemma 9.3 (Convolution of Dirichlet series). *Let*

$$a(s) = \sum_{n\geq 1} \frac{a_n}{s^n} \qquad b(s) = \sum_{n\geq 1} \frac{b_n}{s^n} \qquad c(s) = \sum_{n\geq 1} \frac{c_n}{s^n}.$$

Then $c(s) = a(s) \cdot b(s)$ *if and only if*

$$c_n = \sum_{k|n} a_k b_{n/k} \quad \forall\, n \geq 1.$$

Proof. **Exercise**. □

Proof of equation (2). Let

$$a(s) = \sum_{n\geq 1} \frac{Q_n}{s^n} \qquad \text{and} \qquad c(s) = \sum_{n\geq 1} \frac{P_n}{s^n}.$$

Since equation (1) holds, the convolution of Dirichlet series implies $c(s) = a(s)\zeta(s)$, but then the inverses of the zeta and Möbius functions imply $a(s) = c(z)M(s)$, thus, again the convolution of Dirichlet series implies (2). □

9.2 Fekete's subadditive lemma

Lemma 9.4 (Fekete's subadditive lemma). *Let* $\{a_n\}_{n\geq 1}$ *be a subadditive sequence of real numbers, i.e.*

$$a_{n+m} \leq a_n + a_m \qquad \forall\, m, n \geq 1.$$

Then $\lim_{n\to\infty} a_n/n$ *exists and equals* $\inf_{n\geq 1} a_n/n$

Proof. Let $\alpha = \inf_{n\geq 1} a_n/n$. Then $a_n/n \geq \alpha \ \ \forall\, n \geq 1$. Let $\varepsilon > 0$. It suffices to show that $a_n/n < \alpha + \varepsilon$ for all $n \gg 1$.

By definition, α is the largest number less than or equal to all of the a_n/n. Thus

$$\exists\, k \geq 1, \quad \frac{a_k}{k} < \alpha + \frac{\varepsilon}{2}.$$

Then, for all $0 \leq j < k$ and $m \geq 1$ we have

$$\begin{aligned}\frac{a_{mk+j}}{mk+j} &\leq \frac{a_{mk}}{mk+j} + \frac{a_j}{mk+j} \leq \frac{a_{mk}}{mk} + \frac{a_j}{mk} \\ &\leq \frac{ma_k}{mk} + \frac{ja_1}{mk} \leq \frac{a_k}{k} + \frac{a_1}{m} < \alpha + \frac{\varepsilon}{2} + \frac{a_1}{m}.\end{aligned}$$

Hence, if $n = mk + j$ is large enough so that $a_1/m < \varepsilon/2$, then $a_n/n < \alpha + \varepsilon$. □

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