Ergod. Th. & Dynam. Sys. (2003), **23**, 1485–1504
DOI: 10.1017/S0143385702001700

Positive K-theory for finitary isomorphisms of Markov chains

RICARDO GÓMEZ

Instituto de Matemáticas, Area de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, México DF 04510, México
(e-mail: rgomez@math.unam.mx)

(*Received* 10 *September* 2001 *and accepted in revised form* 6 *February* 2003)

Abstract. We use matrices over formal power series to represent irreducible and positive recurrent Markov chains and identify a natural class of good finitary isomorphisms (magic word isomorphisms) as those arising from elementary matrix operations. This extends the positive K-theory framework which has been used for other classification problems in symbolic dynamics.

1. *Introduction*

One of the significant recent developments in symbolic dynamics is the 'positive K-theory' approach to classification and isomorphism [**5**], inaugurated in [**13, 14**] following earlier applications of polynomial matrices [**2, 3, 20, 31**]. In the most important example, very roughly, a shift of finite type is represented by a matrix A whose entries are polynomials with $\mathbb{Z}_+$ coefficients; and multiplication of $I - A$ by an elementary matrix satisfying a positivity condition induces an isomorphism of shifts of finite type. This gives both a new framework for classification [**8**] and a powerful method for constructing isomorphisms [**13, 14**]. This approach has been generalized to the study of flow equivalence and group extensions of shifts of finite type and continuous isomorphism of finite state Markov chains [**4, 6–8**]. All of these cases involve continuous maps on compact spaces.

In this paper, we extend the approach to obtain an algebraic framework for good finitary isomorphisms of positive recurrent irreducible Markov chains. A finitary isomorphism is a measure-preserving isomorphism which is a homeomorphism after removal of a null set; in §2, we review the place of finitary isomorphism in the classification theory of Markov chains. Our 'good' finitary isomorphisms will be the magic word isomorphisms. We will show that the class of magic word isomorphisms is exactly the class of isomorphisms induced by elementary matrix operations in our positive K-theory setup.

Magic word isomorphisms are a very natural class. The few general techniques to date for producing isomorphisms with finite expected coding time have produced magic word isomorphisms [**1, 19–21, 23, 31, 32**]; magic word isomorphisms are particularly natural for exponentially recurrent chains (§7); and it is expected that finite state chains which are isomorphic via a finitary isomorphism with finite expected coding time will be isomorphic by magic word isomorphisms, so in this case our framework can be viewed as an algebraic framework for (good) isomorphisms with finite expected coding time. The setup we present is distinguished from previous 'positive K-theory' work in the following ways: our matrix entries are formal power series rather than polynomials (corresponding to the natural non-compactness of the finitary situation); and we use 'intrinsically irreducible' matrices to 'mod out' submatrices of $I - A$ (corresponding to dropping a null set).

Let us be more explicit about our classification. We will describe a class $\mathcal{M}$ of $\mathbb{N} \times \mathbb{N}$ finitely supported matrices A representing our Markov chains. Such a matrix A will be intrinsically irreducible (there will be a unique 'maximal' irreducible component, defined by having the maximum Perron value). An entry $A(i, j)$ will be zero or the sum of countably many terms of the form $n[p]t^d$. In such a term, d is a non-negative integer, p is a positive real number (corresponding to a transition probability), n is a positive integer and $n[p]$ is an element of the integral group ring of the multiplicative real numbers. (The use of this ring allows the matrix to track paths with weights. It was Parry and Tuncel [**27**] who made this advance using the ring $\mathbb{Z}$[exp], which is isomorphic to the integral group ring we use.) In the special case that every d is positive, the corresponding Markov chain will be isomorphic to the chain represented by a graph in which a summand $[p]t^d$ of $A(i, j)$ corresponds to a path from i to j of d successive and otherwise isolated edges, with p the product of the transition probabilities along the path. In particular, we may think of d as a transition time. However, as in the continuous case [**8**], in order to represent all our isomorphisms algebraically, it will be necessary for us to allow the possibility $d = 0$. To clarify the corresponding finitary isomorphisms, we will define the Markov chain associated to A as the *path chain* $\sum_A$.

We say a matrix is a basic elementary matrix if it equals the identity matrix except perhaps in a single off-diagonal entry. If A and B are two matrices in $\mathcal{M}$ and E is a basic elementary matrix such that $E(I - A) = I - B$ or $(I - A)E = I - B$, then we will produce a finitary isomorphism from $\sum_A$ to $\sum_B$, which we call an elementary isomorphism. We show such isomorphisms are magic word isomorphisms (Proposition 6.1) and in our main result (Theorem 6.5) we show that all magic word isomorphisms are compositions of elementary isomorphisms.

We can indicate now how the algebraic structure represented in this paper parallels the structure of K_1 in algebraic K-theory. (In the better-developed positive K-theory over $\mathbb{Z}$, there is a more than formal connection with algebraic K-theory [**34**].) It follows from the definition of the class $\mathcal{M}$ that $I - \mathcal{M}$ is a subset of $GL(\infty, \mathbf{R})$, where $\mathbf{R}$ is a certain ring of a formal power series and $GL(\infty, \mathbf{R})$ consists of the images under the map

$$M \hookrightarrow \begin{pmatrix} M & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \in GL(\infty, \mathbf{R})$$

of all $n \times n$ and invertible matrices $M \in GL(n, \mathbf{R})$, for all $n \geq 1$. The Whitehead Lemma (see [**28**]) states that the abelianization of $GL(\infty, \mathbf{R})$ is

$$GL(\infty, \mathbf{R})/E(\mathbf{R}) = \mathrm{K}_1(\mathbf{R}),$$

where $E(\mathbf{R})$ denotes the group generated by the basic elementary matrices. So for all $M, N \in GL(\infty, \mathbf{R})$, M and N are the same in $\mathrm{K}_1(\mathbf{R})$ if and only if they are connected by a finite sequence of basic elementary matrix multiplications (and it is not too difficult to check directly that two of our matrices $I - A$ and $I - B$ are the same in $\mathrm{K}_1(\mathbf{R})$ if and only if they have the same determinant). The equivalence of $I - A$ and $I - B$ for the positive K-theory differs in two respects: in the sequence of basic elementary matrix multiplications, each multiplication step must satisfy the positivity condition that it relates two elements of $I - \mathcal{M}$; and in each step, we only require equality to hold in entries corresponding to the unique maximal components. The latter difference is consistent with the fact that $\det(I - A)$ is not an invariant of the magic word isomorphism. (For finite state chains, the residue of this determinant invariant is the beta function of S. Tuncel.)

The paper is organized as follows. In §2, we give definitions and recall background. In §3, we explain how to represent Markov chains with matrices over power series, and we define the path chains which we choose as the systems corresponding to the matrices. In §4, we explain how multiplication by a basic elementary matrix induces an elementary finitary isomorphism. In §5, we show how finitary isomorphisms associated with state splitting can be given as compositions of elementary isomorphisms. We apply these results and further construction in §6 to prove our main result. In §7, we make a few remarks about exponentially recurrent chains.

2. *Markov chains and their classification*

Let $P = P(i, j)_{i,j \in \mathcal{S}}$ be a stochastic matrix having its rows and columns indexed by a countable set $\mathcal{S}$ called the *state space*. A *state* $i \in \mathcal{S}$ is *recurrent* if

$$\sum_{n=1}^{\infty} P^n(i, i) = \infty;$$

otherwise it is *transient*. For every pair of states $i, j \in \mathcal{S}$, let $f_0(i, j) = 0$, $f_1(i, j) = P(i, j)$ and $f_{n+1}(i, j) = \sum_{k \neq i} f_n(i, k)P(k, j)$ for all $n \geq 1$. A state $i \in \mathcal{S}$ is *positive recurrent* if its *mean recurrence time* is finite, that is if

$$\sum_{n=1}^{\infty} n f_n(i, i) < \infty;$$

otherwise it is *null recurrent*. The matrix P is *positive recurrent* if every state $i \in \mathcal{S}$ is positive recurrent. The matrix P is *irreducible* if, for every $i, j \in \mathcal{S}$, there exists $n = n(i, j) \geq 1$ such that $P^n(i, j) > 0$. If P is irreducible, then P is positive recurrent if and only if there exists a positive recurrent state $i \in \mathcal{S}$ (see [**11**] and [**16**]). The (*vertex*) *shift space* defined by P is the topological space

$$\sum_{P} = \{x = \dots x_{-1} \centerdot x_0 x_1 \dots \in \mathcal{S}^{\mathbb{Z}} \mid P(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{Z}\}$$

with the relative topology obtained from the product topology on $\mathcal{S}^{\mathbb{Z}}$ and it represents all doubly infinite paths in the graph $G(P) = (\mathcal{S}, \mathcal{E})$ with vertex set $\mathcal{S}$ and edge set $\mathcal{E}$ formed by putting an edge from $i \in \mathcal{S}$ to $j \in \mathcal{S}$ labelled by $P(i, j)$ if and only if $P(i, j) > 0$. Let $\mathcal{W}_P = \{i_0 \dots i_n \mid n \geq 0 \text{ and } P(i_0, i_1) \dots P(i_{n-1}, i_n) > 0\}$ be the set of all finite paths in $G(P)$ and let the *length* of a finite path $\gamma = i_0 \dots i_n \in \mathcal{W}_P$ be $\ell(\gamma) = n + 1$. If $\tau \in \mathbb{Z}$, then let the *cylinder set* determined by γ and τ be $[\gamma, \tau] = \{x \in \sum_P \mid x_\tau \dots x_{\tau+n} = \gamma\}$. The collection of all cylinder sets forms a basis for the topology on $\sum_P$. If P is irreducible and positive recurrent, then there exists a unique *stationary distribution*, i.e. a strictly positive and stochastic vector $\pi = \pi(i)_{i \in \mathcal{S}}$ satisfying $\pi P = \pi$ (in fact, $\pi(i)^{-1}$ is the mean recurrence time of $i \in \mathcal{S}$). In this case, let μ_P be the Borel probability measure defined on cylinder sets by the rule

$$\mu_P([\gamma, \tau]) = \pi(i_0) P(i_0, i_1) \dots P(i_{n-1}, i_n).$$

Then μ_P is ergodic with respect to the *left-shift* automorphism $\sigma_P \colon \sum_P \to \sum_P$ defined by setting $(\sigma_P x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$. The triple $(\sum_P, \sigma_P, \mu_P)$ is the *irreducible and positive recurrent Markov chain with transition matrix* P.

Let P and Q be the transition matrices of two irreducible and positive recurrent Markov chains. A measure-preserving transformation $\varphi\colon \sum_P \to \sum_Q$ is an *isomorphism* if it is invertible and shift-commuting, i.e. $\varphi \circ \sigma_P = \sigma_Q \circ \varphi$. A measure-preserving and shift-commuting transformation $\varphi\colon \sum_P \to \sum_Q$ is a *finitary map* (or simply *finitary*) if it is continuous after restriction to the complement of a null set. For a finitary map $\varphi\colon \sum_P \to \sum_Q$ and almost every $x, y \in \sum_P$, there exists a minimal positive integer $n = n(x)$ such that $\varphi(y)_0 = \varphi(x)_0$ if $y_{-n} \dots y_n = x_{-n} \dots x_n$. In this case, φ has *finite expected coding time* (FECT) if and only if

$$\int n(x)\, d\mu_P < \infty.$$

If $\varphi\colon \sum_P \to \sum_Q$ is a measure-preserving and shift-commuting transformation, then a path $\omega \in \mathcal{W}_P$ is a *magic word* for φ if for all $\gamma \in \mathcal{W}_P$ such that $\omega\gamma\omega \in \mathcal{W}_P$, there exists $\varphi(\gamma) \in \mathcal{W}_Q$ such that $\ell(\varphi(\gamma)) = \ell(\gamma)$ and $\varphi(x) \in [\varphi(\gamma), 0]$ for almost all $x \in [\omega\gamma\omega, -\ell(\omega)]$. Because σ_P is ergodic and $\mu_P([\omega, 0]) > 0$, the set of points in $\sum_P$ that eventually stop visiting $\omega \in \mathcal{W}_P$ is a set of measure zero and, therefore, φ is a finitary map. A finitary isomorphism $\varphi\colon \sum_P \to \sum_Q$ is an *FECT isomorphism* (respectively a *magic word isomorphism*) if both φ and φ^{-1} have FECT (respectively a magic word). A magic word isomorphism between exponentially recurrent irreducible Markov chains (in particular, finite state irreducible chains) is an FECT isomorphism (see §7).

We finish this section with a brief sketch of the classification theory of finite state mixing Markov chains. Ornstein and Friedman (see [**23**] and [**10**]) have shown that entropy is a complete invariant for the measurable isomorphism of these chains. Keane and Smorodinsky [**12**] have shown that this isomorphism could be constructed to be finitary. This result both improves the niceness of the measurable isomorphism and indicates that a finitary isomorphism can be terrifically complicated. FECT isomorphisms were suggested as a class of more effective codes in 1979 by Parry [**25**], who showed that, in general, it is not possible to find an FECT isomorphism between two finite state mixing Markov chains with the same entropy [**25**]. The known invariants of FECT isomorphisms are the multiplicative group Δ of ratios of weights of cycles of the same length [**17**], the multiplicative group Γ generated by the weights of cycles and the distinguished generator $c\Delta$ of the cyclic group Γ/Δ [**26**], and the β function, introduced in [**32**] as an invariant of finite equivalence and proved to be an invariant of an FECT isomorphism in [**30**]. It has been conjectured that the quadruple $(\Delta, \Gamma, c\Delta, \beta)$ constitutes a complete invariant of an FECT isomorphism between finite state irreducible Markov chains.

3. *Path chains and matrices over power series*

In this section, we explain how certain matrices over formal power series represent all positive recurrent Markov chains as *path chains*. Path chain representations elaborate the representations of shifts of finite type as *path shifts* [**5, 8**], which are constructed from polynomial matrices with entries in $\mathbb{Z}_+[t]$. The path chain representation is needed for conveniently modelling isomorphisms to be produced from elementary matrix operations.

Let $\mathbb{R}_+^*$ be the multiplicative group of positive real numbers. Let $\mathbb{Z}[[\mathbb{R}_+^*]]$ be the set of formal sums $\sum_{p\in\mathbb{R}_+^*} n_p[p]$, where $n_p \in \mathbb{Z}$ and $n_p = 0$ for all but countably many $p \in \mathbb{R}_+^*$. Let $\mathbb{Z}_+[[\mathbb{R}_+^*]]$ be the subset of $\mathbb{Z}[[\mathbb{R}_+^*]]$ obtained by requiring $n_p \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of positive integers. Let $\mathbf{R} = \mathbb{Z}[[\mathbb{R}_+^*]][[t]]$ be the set of formal power series with coefficients in $\mathbb{Z}[[\mathbb{R}_+^*]]$ and let $\mathbf{R}_+$ be the subset of $\mathbf{R}$ obtained by requiring the coefficients to be in $\mathbb{Z}_+[[\mathbb{R}_+^*]]$.

Let A be an infinite square matrix over $\mathbf{R}_+$ and let $\mathbb{N}$ index the rows and columns of A. Write $A = A(i,j)_{i,j\in\mathbb{N}}$, where

$$A(i,j) = A(i,j)(t) = \sum_{d=0}^{\infty} a_d(i,j)t^d$$

for some $a_d(i,j) \in \mathbb{Z}_+[[\mathbb{R}_+^*]]$, say

$$a_d(i,j) = \sum_{p\in\mathbb{R}_+^*} n_{p,d}(i,j)[p],$$

with $n_{p,d}(i,j) = 0$ for all but countably many $p \in \mathbb{R}_+^*$. The matrix A induces a directed and labelled graph $\mathcal{G}_A = (\mathcal{V}_A, \mathcal{E}_A)$ as follows. First, let $\mathbb{N}_A \subset \mathcal{V}_A$ be such that $i \in \mathbb{N}_A$ if and only if there is at least one non-zero entry in the row or the column of A corresponding to i. For every $i, j \in \mathbb{N}_A$, $p \in \mathbb{R}_+^*$ and $d \geq 1$, put $n_{p,d}(i,j)$ paths of length d from i to j and call them *routes* (the *intermediate vertices* of distinct routes of length two or more are disjoint), and also put $n_{p,0}(i,j)$ edges from i to j and call them *zero-length routes*. Every route r is labelled by its *weight* $wt_A(r) = p$. The vertex set $\mathcal{V}_A$ is such that $\mathcal{V}_A - \mathbb{N}_A$ is the set of intermediate vertices. The set of routes, including the zero-length routes, is denoted by $\mathcal{R}_A$ and the edge set $\mathcal{E}_A$ consists of all the edges that form the routes in $\mathcal{R}_A$. For every edge $e \in \mathcal{E}_A$, if e is not a zero-length route, then let $\ell(e) = 1$, otherwise let $\ell(e) = 0$. For every $n \geq 1$, let $\mathcal{W}_A(n) = \{\gamma = e_1 \dots e_n \mid e_1, \dots, e_n \in \mathcal{E}_A\}$. Let $\mathcal{W}_A = \bigcup_{n\geq 1} \mathcal{W}_A(n)$ and if $\gamma = e_1 \dots e_n \in \mathcal{W}_A$, then let $i(\gamma)$ and $\tau(\gamma)$ be the initial and terminal vertices of e_1 and e_n respectively, and also let the *length* of γ be $\ell(\gamma) = \ell(e_1) + \cdots + \ell(e_n)$. If $i, j \in \mathcal{V}_A$, then let $\mathcal{W}_A(i,j) = \{\gamma \in \mathcal{W}_A \mid i(\gamma) = i \text{ and } \tau(\gamma) = j\}$, let $\mathcal{W}_A(i,\cdot) = \bigcup_{k\in\mathcal{V}_A} \mathcal{W}_A(i,k)$ and let $\mathcal{W}_A(\cdot,j) = \bigcup_{k\in\mathcal{V}_A} \mathcal{W}_A(k,j)$. If $\mathcal{X}, \mathcal{Y} \subset \mathcal{W}_A$, then $\mathcal{X}\mathcal{Y}$ denotes the set of paths formed by the consecutive concatenation of an element in $\mathcal{X}$ followed by an element in $\mathcal{Y}$ and $\mathcal{X}^*$ denotes the collection of paths which are finite concatenations of elements in $\mathcal{X}$, including the *empty path* consisting of no edges. Let $\mathcal{R}_A^*(0) = \varnothing$ and for every $n \geq 1$, let $\mathcal{R}_A^*(n) \subset \mathcal{R}_A^*$ be the set of paths formed by the consecutive concatenation of n routes (for example, $\mathcal{R}_A^*(1) = \mathcal{R}_A$). If $i, j \in \mathbb{N}_A$, then let $\mathcal{R}_A(i, j) = \mathcal{R}_A \cap \mathcal{W}_A(i, j)$, $\mathcal{R}_A(i, \cdot) = \mathcal{R}_A \cap \mathcal{W}_A(i, \cdot)$, $\mathcal{R}_A(\cdot, j) = \mathcal{R}_A \cap \mathcal{W}_A(\cdot, j)$ and $\mathcal{R}_A^*(i, j) = \mathcal{R}_A^* \cap (\mathcal{W}_A(i, j) \cup \{i, j\})$, and if $\gamma \in \mathcal{W}_A(i, j)$, then $wt_A(\gamma)$ denotes the *weight of* γ defined as the product of the weights of the routes that form γ.

Let $\underline{A}$ be the matrix that results from A after replacing each coefficient $a_d(i, j)$ by its image under the map defined by the rule $n[p] \longmapsto np$ for all $n \in \mathbb{Z}_+$ and $p \in \mathbb{R}_+^*$. If $t \in \mathbb{R}_+$ is a non-negative real number, then let $\underline{A}(t)$ be the matrix over $\mathbb{R}_+ \cup \{\infty\}$ that is obtained when evaluating $\underline{A}$ at t. We will always assume that the *no* $\mathbb{Z}_+$*-cycles condition* of Boyle and Wagoner is satisfied (see [**8**] or [**5**]):

$$\sum_{n=1}^{\infty} \operatorname{tr}(\underline{A}^n(0)) = \sum_{n=1}^{\infty} \sum_{i \in \mathbb{N}} \underline{A}^n(0)(i, i) = 0$$

(i.e. there exist no zero-length cycles).

To describe the set of all doubly infinite paths in $\mathcal{G}_A$, let a *symbol* be a pair $(r, \tau) \in \mathcal{R}_A \times \mathbb{Z}$. If (r_1, τ_1) and (r_2, τ_2) are two symbols such that $\tau(r_1) = i(r_2)$ and $\tau_2 = \tau_1 + \ell(r_1)$, then say that (r_2, τ_2) *follows* (r_1, τ_1) and write $(r_1, \tau_1) \to (r_2, \tau_2)$. A doubly infinite sequence of symbols $x = \dots x_{-1} x_0 x_1 \dots = \dots (r_{-1}, \tau_{-1})(r_0, \tau_0)(r_1, \tau_1) \dots$ such that $(r_n, \tau_n) \to (r_{n+1}, \tau_{n+1})$ for all $n \in \mathbb{Z}$ represents a doubly infinite path in $\mathcal{G}_A$, with (r_n, τ_n) indicating for all $n \in \mathbb{Z}$ that at time τ_n the route r_n begins. However, more than one of these sequences may represent the same doubly infinite path, so consider the equivalence relation that makes two of these sequences x and x' equivalent if and only if for some $r \in \mathbb{Z}$, we have $x'_n = x_{n+r}$ for all $n \in \mathbb{Z}$. Let $[x]$ denote the equivalence class of x and define the *path space* $\sum_A$ as the set of all the equivalence classes. Give to $\sum_A$ the quotient topology obtained from the relative topology of the product topology on the set

$$\prod_{\tau \in \mathbb{Z}} (\mathcal{E}_A \times \mathbb{Z}).$$

Let $\gamma = r_1 \dots r_n \in \mathcal{R}_A^*(n)$, with $n \geq 1$ and $r_1, \dots, r_n \in \mathcal{R}_A$, and let $\tau \in \mathbb{Z}$. The *cylinder set* determined by γ and τ is the set $[\gamma, \tau] \subset \sum_A$, where $[x] \in [\gamma, \tau]$ if and only if there exists $m \in \mathbb{Z}$ such that $x_{m+1} \dots x_{m+n} = (r_1, \tau) \dots (r_n, \tau + \ell(r_1) + \dots + \ell(r_{n-1}))$. The collection of cylinder sets forms a basis for the topology on $\sum_A$. If $\gamma \in \mathcal{W}_A(i, j)$ with $i, j \in \mathcal{V}_A$, then there exists a unique path in $\gamma \in \mathcal{R}_A^*$ containing γ and having minimal length, say $\widehat{\gamma} = u\gamma v \in \mathcal{R}_A^*$ for certain $u, v \in \mathcal{W}_A$ and, in this case, let $[\gamma, \tau] = [\widehat{\gamma}, \tau - \ell(u)]$. Also, if $i \in \mathcal{V}_A$, then let $[i, \tau] = \{[x] \in \sum_A \mid [x] \in [e, \tau]$ for some $e \in \mathcal{E}_A(i, \cdot)\}$. The *left-shift map* $\sigma_A \colon \sum_A \to \sum_A$ is defined for every $[x] = [\dots (r_{-1}, \tau_{-1})(r_0, \tau_0)(r_1, \tau_1) \dots] \in \sum_A$ by $\sigma_A([x]) = [\dots (r_{-1}, \tau_{-1} + 1)(r_0, \tau_0 + 1)(r_1, \tau_1 + 1) \dots]$.

Our next task is to define a Borel probability measure on the path space $\sum_A$. We first suppose that A is *irreducible*, which means that for every $i, j \in \mathbb{N}_A$, there exists $n = n(i, j)$ such that $A^n(i, j) \neq 0$. For every $i, j \in \mathbb{N}_A$ and $n \geq 0$, let

$$g_n(i, j) = \frac{1}{n!}\left(\frac{d^n}{dt^n} \sum_{m=1}^{\infty} \underline{A}^m(i, j)(0)\right).$$

If $i \in \mathbb{N}_A$, then $\lambda_A = \lim_{n\to\infty} \sqrt[n]{g_n(i,i)}$ exists, is independent of $i \in \mathbb{N}_A$ and is called the *Perron value of* A (see [**16**]). We assume that $\lambda_A < \infty$. The matrix A is *recurrent* if there exists $i \in \mathbb{N}_A$ such that

$$\sum_{n=1}^{\infty} \underline{A}^n(i,i)(\lambda_A^{-1}) = \infty.$$

For every $i, j \in \mathbb{N}_A$, let $f_0(i,j) = g_0(i,j)$, $f_1(i,j) = g_1(i,j)$ and $f_{n+1}(i,j) = \sum_{k\neq i} f_n(i,k) f_1(k,j)$. A recurrent matrix A is *positive recurrent* if there exists $i \in \mathbb{N}_A$ such that its *mean recurrence time* is finite, i.e. if

$$\sum_{n=1}^{\infty} n f_n(i,i)/\lambda_A^n < \infty;$$

otherwise it is *null recurrent*. If A is positive recurrent, then there exists a non-negative vector π such that $\pi(i) > 0$ for all $i \in \mathbb{N}_A$ and $A(\lambda_A^{-1})\pi = \pi$. Let D and D^{-1} be the diagonal matrices having $D(i,i) = [\pi(i)]$ and $D^{-1}(i,i) = [\pi(i^{-1})]$ for all $i \in \mathbb{N}$. Let

$$\tilde{A} = D^{-1} A D \left(\frac{t}{[\lambda_A]} \right).$$

The submatrix obtained from $\underline{\tilde{A}}(1)$ by removing the rows and columns corresponding to the elements not in $\mathbb{N}_{\tilde{A}}$ is stochastic, irreducible and positive recurrent and, hence, there exists a unique, non-negative and stochastic vector $\tilde{\pi}$ such that $\tilde{\pi}(i) > 0$ for all $i \in \mathbb{N}_{\tilde{A}}$ and $\tilde{\pi}\underline{\tilde{A}}(1) = \tilde{\pi}$. In fact, $\mathcal{G}_A$ and $\mathcal{G}_{\tilde{A}}$ are the same except for the weights assigned to the routes. For every $\gamma \in \mathcal{R}_A^*(n)$, where $n \geq 1$, define the *transition probabilities* (see [**20**])

$$p_A(\gamma) = wt_{\tilde{A}}(\gamma) = \frac{wt_A(\gamma)\pi(\tau(\gamma))}{\lambda_A \pi(i(\gamma))}$$

and for every $\tau \in \mathbb{Z}$, let

$$\mu_A([\gamma,\tau]) = \tilde{\pi}(i(\gamma)) p_A(\gamma).$$

Then μ_A determines a Borel probability measure on $\sum_A$ which is invariant with respect to σ_A. The triple $\left(\sum_A, \sigma_A, \mu_A\right)$ is the *path chain* defined by A.

Path chains are isomorphic to Markov chains as follows. If A is an irreducible and positive recurrent matrix over $\mathbf{R}_+$, then the corresponding path chain is conjugate to the Markov chain with transition matrix $A^{\#}$, the matrix over $\mathbb{R}_+$ associated to the graph $\mathcal{G}_{\tilde{A}}$ (see [**5**] for the way to construct a conjugacy that transforms matrices satisfying the no $\mathbb{Z}_+$-cycles condition into matrices with no zero-length paths). Conversely, for an irreducible and positive recurrent Markov chain with transition matrix P, let

$$\widehat{P} = \begin{pmatrix} tP & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\widehat{P}$ is an irreducible and positive recurrent matrix over $\mathbf{R}_+$ and the path chain it defines is conjugate to the Markov chain defined by P via the conjugacy defined for every $x = \dots x_{-1}x_0x_1 \dots \in \sum_P$ by the map

$$x \mapsto [x] = [\dots(e(x_{-1},x_0),-1)(e(x_0,x_1),0)(e(x_1,x_2),1)\dots] \in \sum_{\widehat{P}}, \tag{3.1}$$

where $e(x_n, x_{n+1})$ denotes the edge of $G(P)$ from x_n to x_{n+1} for all $n \in \mathbb{Z}$. Observe that $(\widehat{P})^\# = P$. Henceforth, the definitions concerning path chains and with a Markov chain counterpart will be consistent (see [**5**], [**11**] and [**16**]).

Let A be a matrix over $\mathbf{R}_+$. An irreducible component of A is a maximal irreducible submatrix of A. Say that A is *intrinsically irreducible* if there exists a unique irreducible component A_0 with maximal Perron value. In this case, define the *Perron value of* A as $\lambda_A = \lambda_{A_0}$. An intrinsically irreducible matrix A is *positive recurrent* if every irreducible component of A is positive recurrent. In this case, define an ergodic measure μ_A on $\sum_A$ by requiring

$$\mu_A|_{\sum_{A_0}} = \mu_{A_0}.$$

Any two intrinsically irreducible and positive recurrent matrices having the same distinguished irreducible component determine the same path space modulo a set of measure zero and will be considered the same.

Let $\mathcal{M}$ be the set of all the infinite matrices A over $\mathbf{R}_+$ which are intrinsically irreducible and positive recurrent and have finite support, that is $|\mathbb{N}_A| < \infty$ (we will often abuse notation and write elements in $\mathcal{M}$ as finite matrices). Then $\mathbb{N}_A$ is a finite subset of vertices containing at least one vertex of each cycle in $\mathcal{G}_A$. Subsets of vertices of this kind are called *cycle-passage domains* of $\mathcal{G}_A$ (see [**11**]).

For every pair of matrices $A, B \in \mathcal{M}$, we translate word-by-word the definitions of an isomorphism, a finitary map, a finitary map having FECT, an FECT isomorphism, a finitary map having a magic word and a magic word isomorphism. For example, if $\varphi\colon \sum_A \to \sum_B$ is a measure-preserving and shift-commuting transformation, then a path $\omega \in \mathcal{W}_A$ is a *magic word* for φ if for all $\gamma \in \mathcal{W}_A$ such that $\omega\gamma\omega \in \mathcal{W}_A$, there exists $\varphi(\gamma) \in \mathcal{W}_B$ such that $\ell(\varphi(\gamma)) = \ell(\gamma)$ and $\varphi([x]) \in [\varphi(\gamma), 0]$ for almost all $[x] \in [\omega\gamma\omega, -\ell(\omega)]$.

For every $i \in \mathcal{V}_A$, let $\mathcal{L}_A(i)$ be the set of cycles that start and end at i and do not visit i otherwise.

PROPOSITION 3.1. *Let P be the transition matrix of an irreducible and positive recurrent Markov chain. Then there exist a matrix $A \in \mathcal{M}$ with at most one non-zero entry and a continuous magic word isomorphism $\pi\colon \sum_A \to \sum_{\widehat{P}}$.*

Proof. Let $i \in \mathbb{N}_{\widehat{P}}$ and $f \in \mathcal{M}$ be the power series representing $\mathcal{L}_{\widehat{P}}(i)$. Let $A = f$ and choose a length- and weight-preserving bijection $\Phi\colon \mathcal{L}_{\widehat{P}}(i) \to \mathcal{R}_A$. If $[x] \in \sum_A$ visits i infinitely many times in the past and future, then abuse notation and write $[x] = [\ldots(\gamma_{-1}, \tau_{-1})(\gamma_0, \tau_0)(\gamma_1, \tau_1)\ldots]$, where $\gamma_n \in \mathcal{L}_{\widehat{P}}(i)$ and $\tau_n \in \mathbb{Z}$, with $\tau_{n+1} = \tau_n + \ell(\gamma_n)$. Let $\pi([x]) = [\ldots(\Phi(\gamma_{-1}), \tau_{-1})(\Phi(\gamma_0), \tau_0)(\Phi(\gamma_1), \tau_1)\ldots]$. Then π is a finitary isomorphism with the desired properties. □

The map defined by equation (3.1) is a functor that assigns isomorphisms between two irreducible and positive recurrent Markov chains with transition matrices P and Q to isomorphisms between the path chains defined by $\widehat{P}$ and $\widehat{Q}$, and respects magic word isomorphisms (see Proposition 6.2).

4. *Elementary isomorphisms*

We now come to the generators, the 'elementary isomorphisms'.

A *basic elementary matrix* E is an infinite square matrix over $\mathbf{R}$ such that every diagonal entry is equal to 1 and there is at most one non-zero off-diagonal entry. Elementary isomorphisms are constructed from a certain type of matrix multiplications involving basic elementary matrices. The following proposition guarantees that the class $\mathcal{M}$ is preserved under such operations.

PROPOSITION 4.1. *Let A and B be two matrices in $\mathbf{R}_+$. If $A \in \mathcal{M}$ and there exists a basic elementary matrix E such that either $E(I-A) = I-B$ or $(I-A)E = I-B$, then $B \in \mathcal{M}$.*

Proof. The matrix B satisfies the no $\mathbb{Z}_+$-cycles condition because

$$\sum_{n=1}^{\infty} \operatorname{tr}(\underline{B}^n)(0) = \sum_{n=1}^{\infty} \operatorname{tr}(\underline{A}^n)(0) = 0.$$

The matrix B is intrinsically irreducible and positive recurrent because $\det(I-A) = \det(I-B)$. □

For a formal power series with zero constant term $a = \sum_{n=1}^{\infty} a_n t^n$, let

$$a^* = \frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots .$$

For the rest of this section, E will denote a basic elementary matrix and if $E \neq I$, then $i_0, j_0 \in \mathbb{N}$ will denote the corresponding row and column of the non-zero off-diagonal entry of E.

LEMMA 4.2. *Let A and B be two matrices over $\mathbf{R}_+$. Suppose that there is a basic elementary matrix E such that either $E(I-A) = I-B$ or $(I-A)E = I-B$. If $E = I$, then $A(i,j) = B(i,j)$ for all $i, j \in \mathbb{N}$. Otherwise,*

- *if $E(I-A) = I-B$, then $A(i,j) = B(i,j)$ for every $i \neq i_0$, and $j \neq j_0$, then $A(i_0,j) + A(i_0,j_0)A(j_0,j_0)^*A(j_0,j) = B(i_0,j) + B(i_0,j_0)B(j_0,j_0)^*B(j_0,j)$; and*
- *if $(I-A)E = I-B$, then $A(i,j) = B(i,j)$ for every $j \neq j_0$, and if $i \neq i_0$, then $A(i,j_0) + A(i,i_0)A(i_0,i_0)^*A(i_0,j_0) = B(i,j_0) + B(i,i_0)B(i_0,i_0)^*B(i_0,j_0)$.*

Proof. Suppose that $E(I-A) = I-B$. Then $A(i,j) = B(i,j)$ for all $i \neq i_0$, $B(i_0,j_0) = A(i_0,j_0) + E(i_0,j_0)A(j_0,j_0) - E(i_0,j_0)$ and, for $j \neq j_0$, $B(i_0,j) = A(i_0,j) + E(i_0,j_0)A(j_0,j)$. Then $B(i_0,j) + B(i_0,j_0)B(j_0,j_0)^*B(j_0,j) = A(i_0,j) + E(i_0,j_0)A(j_0,j) + B(i_0,j_0)A(j_0,j_0)^*A(j_0,j)$ and, thus, it is enough to prove that $A(i_0,j_0)A(j_0,j_0)^*A(j_0,j) = E(i_0,j_0)A(j_0,j) + B(i_0,j_0)A(j_0,j_0)^*A(j_0,j)$. This happens if and only if $A(i_0,j_0)A(j_0,j_0)^* = E(i_0,j_0) + (A(i_0,j_0) + E(i_0,j_0)A(j_0,j_0) - E(i_0,j_0))A(j_0,j_0)^*$, if and only if $E(i_0,j_0)(1 + A(j_0,j_0)A(j_0,j_0)^* - A(j_0,j_0)^*) = 0$, and this happens if and only if $(1 - A(j_0,j_0))A(j_0,j_0)^* = 1$, which holds by the definition of $A(j_0,j_0)^*$.

A similar argument applies if $(I-A)E = I-B$. □

If $A \in \mathcal{M}$ and $i, j, k \in \mathbb{N}_A$, then the power series $A(i,k) + A(i,j)A(j,j)^*A(j,k)$ represents $\mathcal{S}_A(i,j,k) = \mathcal{R}_A(i,k) \cup \mathcal{R}_A(i,j)\mathcal{R}_A^*(j,j)\mathcal{R}_A(j,k)$. Let $B \in \mathcal{M}$ and suppose that either $E(I - A) = I - B$ or $(I - A)E = I - B$. If $E = I$, let $\mathcal{P}_A = \mathcal{R}_A$ and $\mathcal{P}_B = \mathcal{R}_B$. Otherwise, either let

$$\mathcal{P}_A = \Big(\bigcup_{i,j\in\mathbb{N}_A, i\neq i_0} \mathcal{R}_A(i,j) \Big) \bigcup \Big(\bigcup_{j\in\mathbb{N}_A, j\neq j_0} \mathcal{S}_A(i_0, j_0, j) \Big)$$

and

$$\mathcal{P}_B = \Big(\bigcup_{i,j\in\mathbb{N}_B, i\neq i_0} \mathcal{R}_B(i,j) \Big) \bigcup \Big(\bigcup_{j\in\mathbb{N}_B, j\neq j_0} \mathcal{S}_B(i_0, j_0, j) \Big)$$

or let

$$\mathcal{P}_A = \Big(\bigcup_{i,j\in\mathbb{N}_A, j\neq j_0} \mathcal{R}_A(i,j) \Big) \bigcup \Big(\bigcup_{i\in\mathbb{N}_A, i\neq i_0} \mathcal{S}_A(i, i_0, j_0) \Big)$$

and

$$\mathcal{P}_B = \Big(\bigcup_{i,j\in\mathbb{N}_B, j\neq j_0} \mathcal{R}_B(i,j) \Big) \bigcup \Big(\bigcup_{i\in\mathbb{N}_B, i\neq i_0} \mathcal{S}_B(i, i_0, j_0) \Big).$$

Hence, by lemma 4.2, we can choose a bijection $\Phi\colon \mathcal{P}_A \to \mathcal{P}_B$ such that for every $\gamma \in \mathcal{P}_A$, $\ell(\gamma) = \ell(\Phi(\gamma))$ and $wt_A(\gamma) = wt_B(\Phi(\gamma))$ (that is, Φ is length- and weight-preserving), and such that

$$\Phi(\mathcal{R}_A(i,j)) = \mathcal{R}_B(i,j) \tag{4.1}$$

holds for all $i, j \in \mathbb{N}$; otherwise either (4.1) holds only if $i \neq i_0$ but still if $j \neq j_0$, so that

$$\Phi(\mathcal{S}_A(i_0, j_0, j)) = \mathcal{S}_B(i_0, j_0, j)$$

or (4.1) holds only if $j \neq j_0$ but still if $i \neq i_0$, so that

$$\Phi(\mathcal{S}_A(i, i_0, j_0)) = \mathcal{S}_B(i, i_0, j_0).$$

If $[x] \in \sum_A$ is formed by doubly infinite concatenations of paths in $\mathcal{P}_A$, then abuse notation and write

$$[x] = [\dots(\gamma_{-1}, \tau_{-1})(\gamma_0, \tau_0)(\gamma_1, \tau_1)\dots]$$

where $\gamma_n \in \mathcal{P}_A$ and $\tau_n \in \mathbb{Z}$, with $\tau_{n+1} = \tau_n + \ell(\gamma_n)$, and let

$$\varphi_E([x]) = [\dots(\Phi(\gamma_{-1}), \tau_{-1})(\Phi(\gamma_0), \tau_0)(\Phi(\gamma_1), \tau_1)\dots].$$

The set corresponding to all doubly infinite concatenations of paths in $\mathcal{P}_A$ is a full measure subset of the path space $\sum_A$ and a similar statement holds for B. Then $\varphi_E\colon \sum_A \to \sum_B$ is a finitary isomorphism of Markov chains.

Definition 4.3. Let $A, B \in \mathcal{M}$ and let $\varphi\colon \sum_A \to \sum_B$ be a finitary isomorphism. Then φ is an *elementary isomorphism* if there exists an elementary matrix E such that $\varphi = \varphi_E$ or $\varphi = \varphi_E^{-1}$.

Remark 4.4. An elementary isomorphism depends on the defining bijection Φ and is uniquely determined by the matrices E and A only up to the choice of this bijection.

5. *State splitting and loop systems*

In this section we do most of the technical work for the main theorem.

Let $A \in \mathcal{M}$, $i \in \mathbb{N}_A$ and $\mathcal{P} = (\mathcal{X}, \mathcal{Y})$ be a partition of $\mathcal{R}_A(i, \cdot)$. Then $\mathcal{P}$ induces subpartitions $\mathcal{P}_j = (\mathcal{X}_j, \mathcal{Y}_j)$ of $\mathcal{R}_A(i, j)$ for all $j \in \mathbb{N}_A$. Write $A(i, j) = x_j + y_j$, where x_j and y_j are the power series corresponding to $\mathcal{X}_j$ and $\mathcal{Y}_j$. Let $A^{\mathcal{P}} \in \mathcal{M}$ be the matrix obtained from A when *out-splitting* i into two vertices $i[\mathcal{X}], i[\mathcal{Y}] \in \mathbb{N}_{A^{\mathcal{P}}}$, setting $A^{\mathcal{P}}(i[\mathcal{X}], i[\mathcal{X}]) = A^{\mathcal{P}}(i[\mathcal{X}], i[\mathcal{Y}]) = x_i$, $A^{\mathcal{P}}(i[\mathcal{Y}], i[\mathcal{X}]) = A^{\mathcal{P}}(i[\mathcal{Y}], i[\mathcal{Y}]) = y_i$, and for all other $j \in \mathbb{N}$, $A^{\mathcal{P}}(i[\mathcal{X}], j) = x_j$, $A^{\mathcal{P}}(i[\mathcal{Y}], j) = y_j$ and $A^{\mathcal{P}}(j, i[\mathcal{X}]) = A^{\mathcal{P}}(j, i[\mathcal{Y}]) = A(j, i)$. Given a labelling on $\mathcal{E}_A$, the induced labelling on $\mathcal{E}_{A^{\mathcal{P}}}$ is such that the label of $\gamma \in \mathcal{W}_{A^{\mathcal{P}}}$ begins with a label corresponding to an element in $\mathcal{X}$ if and only if $i(\gamma) = i[\mathcal{X}]$ and similarly for $\mathcal{Y}$. The labelling map sends $\sum_{A^{\mathcal{P}}}$ into $\sum_A$ and is a finitary isomorphism called a *simple out-split map of i according to $\mathcal{P}$*.

PROPOSITION 5.1. *Let $A \in \mathcal{M}$, $i \in \mathbb{N}_A$ and $\mathcal{P} = (\mathcal{X}, \mathcal{Y})$ be a partition of $\mathcal{R}_A(i, \cdot)$. Let $\pi : \sum_{A^{\mathcal{P}}} \to \sum_A$ be a simple out-split map of i according to $\mathcal{P}$. Then π is a composition of elementary isomorphisms.*

Proof. Suppose, without losing generality, that $i = 1$ and that $A(h, k) = 0$ for all $h, k \notin \{1, \ldots, n\}$. For each $j = 1, \ldots, n$, write $A(1, j) = x_j + y_j$, where x_j and y_j are the power series corresponding $\mathcal{X}_j$ and $\mathcal{Y}_j$, respectively. Let E_0 be the elementary matrix with $E_0(1, 0) = -1$. For each $j = 1, \ldots, n$, let E_j be the elementary matrix with $E_j(0, j) = -x_j$. Let E_{n+1} be the elementary matrix with $E_{n+1}(1, 0) = 1$. Let $E_0(I - A) = I - A_1$. For each $j = 1, \ldots, n$, let $(I - A_j)E_j = I - A_{j+1}$. Finally, let $(I - A_{n+1})E_{n+1} = I - A_{n+2}$. Then $A_j \in \mathcal{M}$ for all $j = 1, \ldots, n + 2$. Moreover, $A_{n+2} = A^{\mathcal{P}}$ and, by choosing the right bijections defining the elementary isomorphisms induced by these elementary matrix multiplications, the inverse of the map that results when composing these elementary isomorphisms coincides with π, the simple out-split map of i according to $\mathcal{P}$. □

Similarly, let $A \in \mathcal{M}$, $j \in \mathbb{N}_A$ and $\mathcal{P} = (\mathcal{X}, \mathcal{Y})$ be a partition of $\mathcal{R}_A(\cdot, j)$. Then $\mathcal{P}$ induces subpartitions $\mathcal{P}_i = (\mathcal{X}_i, \mathcal{Y}_i)$ of $\mathcal{R}_A(i, j)$ for all $i \in \mathbb{N}$. Write $A(i, j) = x_i + y_i$, where x_i and y_i are the power series corresponding to $\mathcal{X}_i$ and $\mathcal{Y}_i$ respectively. Let $A_{\mathcal{P}} \in \mathcal{M}$ be the matrix obtained from A when *in-splitting* j into two vertices $[\mathcal{X}]j, [\mathcal{Y}]j \in \mathbb{N}$, setting $A_{\mathcal{P}}([\mathcal{X}]j, [\mathcal{X}]j) = A_{\mathcal{P}}([\mathcal{Y}]j, [\mathcal{X}]j) = x_j$, $A_{\mathcal{P}}([\mathcal{Y}]j, [\mathcal{Y}]j) = A_{\mathcal{P}}([\mathcal{X}]j, [\mathcal{Y}]j) = y_j$, and for all other $i \in \mathbb{N}$, $A_{\mathcal{P}}(i, [\mathcal{X}]j) = x_i$, $F(i, [\mathcal{Y}]j) = y_i$ and $A_{\mathcal{P}}([\mathcal{X}]j, i) = A_{\mathcal{P}}([\mathcal{Y}]j, i) = A(j, i)$. Given a labelling on $\mathcal{E}_A$, the induced labelling on $\mathcal{E}_{A_{\mathcal{P}}}$ is such that the label of $\gamma \in \mathcal{W}_{A_{\mathcal{P}}}$ ends with a label corresponding to an element in $\mathcal{Y}$ if and only if $\tau(\gamma) = [\mathcal{Y}]j$ and similarly for $\mathcal{X}$. The labelling map sends $\sum_{A_{\mathcal{P}}}$ into $\sum_A$ and is a finitary isomorphism called a *simple in-split map of j according to $\mathcal{P}$*.

PROPOSITION 5.2. *Let $A \in \mathcal{M}$, $j \in \mathbb{N}_A$ and $\mathcal{P} = (\mathcal{X}, \mathcal{Y})$ be a partition of $\mathcal{R}_A(\cdot, j)$. Let $\pi : \sum_{A_{\mathcal{P}}} \to \sum_A$ be a simple in-split map of j according to $\mathcal{P}$. Then π is a composition of elementary isomorphisms.*

Proof. The proof is similar to the proof of Proposition 5.1. □

We now give a version of the higher block representation of a Markov chain (see, for example, [**19**]), where the edges are replaced by routes. Let $A \in \mathcal{M}$ and suppose that $\mathcal{P}$ is a finite partition of $\mathcal{R}_A$ formed by subpartitions $(\mathcal{X}_1(i, j), \dots, \mathcal{X}_{n(i,j)}(i, j))$ of $\mathcal{R}_A(i, j)$, with $i, j \in \mathbb{N}_A$ and $n(i, j) \geq 1$. Write $A(i, j) = x_1(i, j) + \cdots + x_{n(i,j)}(i, j)$, where $x_1(i, j), \dots, x_{n(i,j)}(i, j)$ are the power series representing $\mathcal{X}_1(i, j), \dots, \mathcal{X}_{n(i,j)}(i, j)$, respectively. Let $A_{\mathcal{P}}^{[0,0]} = A$ and for every two non-negative integers $m, a \geq 0$ such that $m + a > 0$, let $A_{\mathcal{P}}^{[m,a]} \in \mathcal{M}$ have its rows and columns determined by lists of $m + a$ adjacent components of $\mathcal{P}$ and write

$$i = [\mathcal{X}_{k_{-m}}(i_{-m}, i_{-m+1}) \dots]i_0[\dots \mathcal{X}_{k_{a-1}}(i_{a-1}, i_a)] \in \mathbb{N}_{A_{\mathcal{P}}^{[m,a]}}$$

where $i_{-m}, \dots, i_a \in \mathbb{N}_A$ and $1 \leq k_s \leq n(i_s, i_{s+1})$ for all $-m \leq s < a$. If

$$j = [\mathcal{X}_{k_{-m+1}}(i_{-m+1}, i_{-m+2}) \dots]i_1[\dots \mathcal{X}_{k_a}(i_a, i_{a+1})] \in \mathbb{N}_{A_{\mathcal{P}}^{[m,a]}}$$

where $i_{a+1} \in \mathbb{N}_A$ and $1 \leq k_a \leq n(i_a, i_{a+1})$, then $A_{\mathcal{P}}^{[m,a]}(i, j) = x_{k_0}(i_0, i_1)$, otherwise $A_{\mathcal{P}}^{[m,a]}(i, j) = 0$. Given a labelling on $\mathcal{E}_A$, the induced labelling on $\mathcal{E}_{A_{\mathcal{P}}^{[m,a]}}$ is such that if $\gamma = \gamma_- \gamma_+ \in \mathcal{W}_{A_{\mathcal{P}}^{[m,a]}}$, where $\gamma_-, \gamma_+ \in \mathcal{W}_{A_{\mathcal{P}}^{[m,a]}}$, then the label of γ_- ends with a label corresponding to an element in $\mathcal{X}_{k_{-m}}(i_{-m}, i_{-m+1}) \dots \mathcal{X}_{k_{-1}}(i_{-1}, i_0)$ and γ_+ begins with a label corresponding to an element in $\mathcal{X}_{k_0}(i_0, i_1) \dots \mathcal{X}_{k_{a-1}}(i_{a-1}, i_a)$ if and only if $i(\gamma_+)$ (or $\tau(\gamma_-)$ if $a = 0$) is precisely $[\mathcal{X}_{k_{-m}}(i_{-m}, i_{-m+1}) \dots]i_0[\dots \mathcal{X}_{k_{a-1}}(i_{a-1}, i_a)]$. The labelling map sends $\sum_{A_{\mathcal{P}}^{[m,a]}}$ into $\sum_A$ and is a finitary isomorphism called an *$[m, a]$-split map according to $\mathcal{P}$*.

PROPOSITION 5.3. *Let $A \in \mathcal{M}$ and $\mathcal{P}$ be a finite partition of $\mathcal{R}_A$ formed by subpartitions $(\mathcal{X}_1(i, j), \dots, \mathcal{X}_{n(i,j)}(i, j))$ of $\mathcal{R}_A(i, j)$, with $i, j \in \mathbb{N}_A$ and $n(i, j) \geq 1$, and let $m, a \geq 0$ be non-negative integers. Let $\pi\colon \sum_{A_{\mathcal{P}}^{[m,a]}} \to \sum_A$ be an $[m, a]$-split map. Then π is a composition of simple in-split and out-split maps and, hence, of elementary isomorphisms.*

Proof. First, a total of a rounds of simple out-split maps are performed. The first round goes as follows. Choose $i_0, j_0 \in \mathbb{N}_A$ and $1 \leq s_0 \leq n(i_0, j_0)$. For every $i, j \in \mathbb{N}_A$ and $1 \leq s \leq n(i, j)$ such that $(i, j, s) \neq (i_0, j_0, s_0)$, apply a simple out-split map of i according to the partition of the set of outgoing edges of i formed by the set of elements corresponding to $\mathcal{X}_r(i, j)$ and its complement. Then vertex i out-splits into two new vertices, one denoted by $i[\mathcal{X}_r(i, j)]$ and with the property that the label of a path γ begins with a label that corresponds to an element in $\mathcal{X}_k(i, j)$ if and only if $i(\gamma) = i[\mathcal{X}_k(i, j)]$, and the other denoted again by i so that we can apply the same kind of simple out-split map of i, except that when the last simple out-split map occurs, the other vertex has a similar property for $\mathcal{X}_k(i, j)$ and, thus, is denoted by $i[\mathcal{X}_r(i, j)]$. When the first round of simple out-split maps is completed, the resulting matrix is $A_{\mathcal{P}}^{[0,1]}$. Each of the other $a - 1$ rounds of simple out-split maps starts with the matrix $A_{\mathcal{P}}^{[0,n]}$, with $1 \leq n < a$, and is carried out similarly, with the partition of the corresponding set of routes being such that for every pair of vertices $i_0[\mathcal{X}_{k_0}(i_0, i_1) \dots \mathcal{X}_{k_{n-1}}(i_{n-1}, i_n)], i_1[\mathcal{X}_{k_1}(i_1, i_2) \dots \mathcal{X}_{k_n}(i_n, i_{n+1})] \in \mathbb{N}_{A_{\mathcal{P}}^{[0,n]}}$, with $i_0, \dots, i_{n+1} \in \mathbb{N}_A$ and $1 \leq k_s \leq n(i_s, i_{s+1})$ for all $0 \leq s \leq n$, the corresponding subpartition simply consists of one component denoted by $\mathcal{X}_{k_0}(i_0, i_1) \dots \mathcal{X}_{k_n}(i_n, i_{n+1})$. When the a rounds of simple out-split maps are completed, the resulting matrix is $A_{\mathcal{P}}^{[0,a]}$.

Next, a total of m rounds of simple in-split maps are performed. The first round starts with the matrix $A_{\mathcal{P}}^{[0,a]}$ and goes as follows. For every pair of vertices $i_{-1}[\mathcal{X}_{k_{-1}}(i_{-1},i_0)\ldots\mathcal{X}_{k_{a-2}}(i_{a-2},i_{a-1})]$ and $i_0[\mathcal{X}_{k_0}(i_0,i_1)\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ in $\mathbb{N}_{A_{\mathcal{P}}^{[0,a]}}$, with $i_{-1},\ldots,i_a\in\mathbb{N}_A$ and $1\le k_s\le n(i_s,i_{s+1})$ for all $-1\le s<a$, apply a simple in-split map of $i_0[\mathcal{X}_{k_0}(i_0,i_1)\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ according to the partition of the set of incoming edges of $i_0[\mathcal{X}_{k_0}(i_0,i_1)\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ formed by the set of elements corresponding to $\mathcal{X}_{k_{-1}}(i_{-1},i_0)$ and its complement. Then vertex $i_0[\mathcal{X}_{k_0}(i_0,i_1)\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ in-splits into two new vertices, one denoted by $[\mathcal{X}_{k_{-1}}(i_{-1},i_0)]i_0[\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ and with the property that if $\gamma=\gamma_-\gamma_+$ is a path in the corresponding graph formed by the consecutive concatenation of the paths γ_- and γ_+, then the label of γ_- ends with a label corresponding to an element in $\mathcal{X}_{k_{-1}}(i_{-1},i_0)$ and the label of γ_+ begins with a label corresponding to an element in $\mathcal{X}_{k_0}(i_0,i_1)\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)$ if and only if $\tau(\gamma_-)=[\mathcal{X}_{k_{-1}}(i_{-1},i_0)]i_0[\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$, and the other denoted again by $i_0[\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ so that we can apply the same kind of simple in-split map of $i_0[\mathcal{X}_{k_0}(i_0,i_1)\ldots X_{k_{a-1}}(i_{a-1},i_a)]$. When the first round of simple in-split maps is completed, the resulting matrix is $A_{\mathcal{P}}^{[1,a]}$. Each of the other $m-1$ rounds of simple in-split maps starts with the matrix $A_{\mathcal{P}}^{[n,a]}$, with $1\le n<m$, and is carried out similarly, with the partition of the corresponding set of routes being such that for every pair of vertices $[\mathcal{X}_{k_{-n-1}}(i_{-n-1},i_{-n})\ldots]i_{-1}[\ldots\mathcal{X}_{k_{a-2}}(i_{a-2},i_{a-1})]$ and $[\mathcal{X}_{k_{-n}}(i_{-n},i_{-n+1})\ldots]i_0[\ldots\mathcal{X}_{k_{a-1}}(i_{a-1},i_a)]$ in $\mathbb{N}_{A_{\mathcal{P}}^{[n,a]}}$, with $i_{-n-1},\ldots,i_a\in\mathbb{N}_A$ and $1\le k_s\le n(i_s,i_{s+1})$ for all $-n-1\le s<a$, the corresponding subpartition simply consists of one component denoted by $\mathcal{X}_{k_{-n-1}}(i_{-n-1},i_{-n})\ldots\mathcal{X}_{k_{-1}}(i_{-1},i_0)$. When the m rounds of simple in-split maps are completed, the resulting matrix is $A_{\mathcal{P}}^{[m,a]}$.

The composition of the simple in-split and out-split maps described earlier, with the right choice of bijections that define the elementary isomorphisms that compose each of them, results in π, the $[m,a]$-split map according to $\mathcal{P}$. □

Let $A\in\mathcal{M}$, $i\in\mathbb{N}_A$ and suppose that $\mu_A([i,0])>0$. Let f be the power series representing $\mathcal{L}_A(i)$ and choose a weight- and length-preserving bijection between $\mathcal{L}_A(i)$ and $\mathcal{R}_f$. Given a labelling on $\mathcal{E}_A$, the induced labelling map sends $\sum_f$ into the full-measure subset of $\sum_A$ consisting of all doubly infinite concatenations of elements in $\mathcal{L}_A(i)$, and then it is a finitary isomorphism called a *loop system map of* i.

PROPOSITION 5.4. *Let $A\in\mathcal{M}$, $i\in\mathbb{N}_A$ and suppose that $\mu_A([i,0])>0$. Let f be the power series representing $\mathcal{L}_A(i)$ and let $\pi\colon\sum_f\to\sum_A$ be a loop system map of i. Then π is a composition of elementary isomorphisms.*

Proof. Suppose, without losing generality, that $i=1$ and that $A(h,k)=0$ for all $h,k\notin\{1,\ldots,n\}$, and also that $n\ge 2$ (otherwise the result is trivial). Let $A_0=A$ and for every $r=0,\ldots,n-2$ and $s=1,\ldots,n-r-1$, let $m=rn-r(r+1)/2+s$ and $I-A_m=(I-A_{m-1})E_m$, where E_m is the basic elementary matrix defined by $E_m(n-r,n-s)=A_{m-1}(n-r,n-r)^*A_{m-1}(n-r,n-s)$. Then $A_m\in\mathcal{M}$ and for every $j=1,\ldots,n$, the power series $A_{n(n-1)/2}(j,j)$ corresponds to the first return loops to j that do not visit $1,\ldots,j-1$. Hence, $A_{n(n-1)/2}(1,1)=f$ is the irreducible component of maximal entropy and π is the inverse of the map that results when composing the right elementary isomorphisms induced by the elementary matrix multiplications described earlier. □

Let $A \in \mathcal{M}$, $\gamma = \gamma_- \gamma_+ \in \mathcal{R}_A^*(m+a)$, with $m+a > 0$, $\gamma_- \in \mathcal{R}_A^*(m)$ and $\gamma_+ \in \mathcal{R}_A^*(a)$, and suppose that $\mu([\gamma, 0]) > 0$. Let $\mathcal{L}_A(\gamma_-\gamma_+)$ be the set of cycles that begin at γ_+, end at γ_- and visit γ only once. (All cycles that start with γ_+ and end with γ_- visit γ at least once.) Let $f \in \mathcal{M}$ be the power series representing $\mathcal{L}_A(\gamma_-\gamma_+)$ and choose a weight- and length-preserving bijection between $\mathcal{L}_A(\gamma_-\gamma_+)$ and $\mathcal{R}_f$. Given a labelling on $\mathcal{E}_A$, the induced labelling on $\mathcal{R}_f$ is such that the label of every route in $\mathcal{R}_f$ begins with the label of γ_+ and ends with the label of γ_- and the label of γ occurs only when concatenating two routes in $\mathcal{R}_f$. The labelling map sends $\sum_f$ into the full-measure subset of $\sum_A$ consisting of all doubly infinite concatenations of elements in $\mathcal{L}_A(\gamma_-\gamma_+)$ and then it is a finitary isomorphism called a *loop system map of* $\gamma = \gamma_-\gamma_+$.

PROPOSITION 5.5. *Let $A \in \mathcal{M}$, $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A^*(m+a)$, with $m+a > 0$, $\gamma_- \in \mathcal{R}_A^*(m)$ and $\gamma_+ \in \mathcal{R}_A^*(a)$, and suppose that $\mu([\gamma, 0]) > 0$. Let $f \in \mathcal{M}$ be the power series representing $\mathcal{L}_A(\gamma_-\gamma_+)$ and $\pi \colon \sum_f \to \sum_A$ be a loop system map of $\gamma = \gamma_-\gamma_+$. Then π is a composition of elementary isomorphisms.*

Proof. Let $\mathcal{P}$ be the partition of $\mathcal{R}_A$ formed by the subpartitions of $\mathcal{R}_A(i, j)$, with $i, j \in \mathbb{N}_A$, induced by the partition of $\mathcal{R}_A$ formed by the singletons $\mathcal{X}_s = \{r_s\}$, with $-m \le s < a$, and the complement of their union. Let $i_0 = i(\gamma_+)$ if $a > 0$ or $i_0 = \tau(\gamma_-)$ if $a = 0$. There exist an $[m, a]$-split map, $\pi_1 \colon \sum_{A_{\mathcal{P}}^{[m,a]}} \to \sum_A$, and a loop system map of $[\mathcal{X}_{-m} \ldots]i_0[\ldots \mathcal{X}_{a-1}]$, $\pi_2 \colon \sum_f \to \sum_{A_{\mathcal{P}}^{[m,a]}}$, such that $\pi = \pi_2 \circ \pi_1$. Then the result follows from Propositions 5.3 and 5.4. □

Let $A \in \mathcal{M}$, $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A^*(m+a)$, with $\gamma_- \in \mathcal{R}_A^*(m)$ and $\gamma_+ \in \mathcal{R}_A^*(a)$, and suppose that $\mu([\gamma, 0]) > 0$ (if $m = a = 0$, then suppose that $\gamma = i_0 \in \mathbb{N}_A$). Let $\mathcal{X} \neq \varnothing$ be a proper subset of $\mathcal{L}_A(\gamma_-\gamma_+)$ and let $\mathcal{Y}$ be its complement. Let x and y be the power series representing $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then $\mathcal{X}\mathcal{Y}^*$ is represented by the power series xy^*. Choose a weight- and length-preserving bijection between $\mathcal{X}\mathcal{Y}^*$ and $\mathcal{R}_{xy^*}$. Then, given a labelling on $\mathcal{E}_A$, we get an induced labelling on $\mathcal{R}_{xy^*}$. If $\mu([\gamma, 0]) > 0$, then the labelling map sends $\sum_{xy^*}$ to the full-measure subset of $\sum_A$ consisting of all doubly infinite paths that never stop visiting elements of $\mathcal{X}$ and then it constitutes a finitary isomorphism called a *left-loop system map of* $\mathcal{X}$.

PROPOSITION 5.6. *Let $A \in \mathcal{M}$ and $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A^*(m+a)$, with $\gamma_- \in \mathcal{R}_A^*(m)$ and $\gamma_+ \in \mathcal{R}_A^*(a)$, and suppose that $\mu([\gamma, 0]) > 0$ (if $m = a = 0$, then suppose that $\gamma = i_0 \in \mathbb{N}_A$). Let $\mathcal{X} \neq \varnothing$ be a proper subset of $\mathcal{L}_A(\gamma_-\gamma_+)$ and let $\mathcal{Y}$ be its complement. Let x and y be the power series representing $\mathcal{X}$ and $\mathcal{Y}$, respectively, and let $\pi \colon \sum_{xy^*} \to \sum_A$ be a left-loop system map of $\mathcal{X}$. Then π is a composition of elementary isomorphisms.*

Proof. There exist a loop system map of $\gamma_-\gamma_+$, $\pi_1 \colon \sum_f \to \sum_A$, a simple out-split map of $i(\gamma_+)$ (or $\tau(\gamma_-)$ if $a = 0$ or $\gamma = i_0 \in \mathbb{N}_f$ if $m = a = 0$) according to the partition $\mathcal{P} = (\mathcal{X}, \mathcal{Y})$, $\pi_2 \colon \sum_{f_{\mathcal{P}}} \to \sum_f$ and $\pi_3 \colon \sum_{xy^*} \to \sum_{f^{\mathcal{P}}}$ a loop system map of $i_0[\mathcal{X}]$, such that $\pi = \pi_3 \circ \pi_2 \circ \pi_1$. Then the result follows from Propositions 5.5, 5.1 and 5.4. □

Similarly, using the previous notation above, the set $\mathcal{X}^*\mathcal{Y}$ is represented by the power series x^*y. Given a labelling on $\mathcal{E}_A$, we get an induced label on $\mathcal{R}_{x^*y}$. The labelling map sends $\sum_{x^*y}$ to the full-measure subset of $\sum_A$ consisting of all doubly infinite paths that never stop visiting elements of $\mathcal{Y}$ and then it constitutes a finitary isomorphism called a *right-loop system map of* $\mathcal{Y}$.

PROPOSITION 5.7. *Let $A \in \mathcal{M}$ and $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A^*(m+a)$, with $\gamma_- \in \mathcal{R}_A^*(m)$ and $\gamma_+ \in \mathcal{R}_A^*(a)$, and suppose that $\mu([\gamma, 0]) > 0$ (if $m = a = 0$, then suppose that $\gamma = i_0 \in \mathbb{N}_A$). Let $\mathcal{X} \neq \varnothing$ be a proper subset of $\mathcal{L}_A(\gamma_-\gamma_+)$ and let $\mathcal{Y}$ be its complement. Let x and y be the power series representing $\mathcal{X}$ and $\mathcal{Y}$, respectively, and let $\pi\colon \sum_{x^*y} \to \sum_A$ be a right-loop system map of $\mathcal{Y}$. Then π is a composition of elementary isomorphisms.*

Proof. The proof is similar to the proof of Proposition 5.6. $\square$

We finish this section by showing that if $A \in \mathcal{M}$, then, up to a conjugacy, any finite number of intermediate vertices can become elements of $\mathbb{N}_A$. More explicitly, let $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A$, with $\gamma_-, \gamma_+ \in \mathcal{W}_A$, $\ell(\gamma_-) = m > 0$ and $\ell(\gamma_+) = a > 0$. Let $A_0 \in \mathcal{M}$ be the matrix obtained from A by subtracting $[wt_A(\gamma)]t^{m+a}$ from $A(i(\gamma), \tau(\gamma))$ and by adding a new vertex $i_0 \in \mathbb{N}_{A_0}$, setting $A_0(i(\gamma), i_0) = [wt_A(\gamma)]t^m$ and $A_0(i_0, \tau(\gamma)) = [1]t^a$. The graphs defined by A and A_0 are identical but $\mathbb{N}_{A_0} = \mathbb{N}_A \cup \{i_0\}$, where $i_0 = \tau(\gamma_-)$. Given a labelling on $\mathcal{E}_A$, there is an induced labelling on $\mathcal{E}_{A_0}$, and the labelling map $\pi\colon \sum_A \to \sum_{A_0}$ is a topological conjugacy of Markov chains called the *promotion map of* $i_0 \in \mathcal{V}_A$.

PROPOSITION 5.8. *Let $A \in \mathcal{M}$ and $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A$, with $\gamma_-, \gamma_+ \in \mathcal{W}_A$, $\ell(\gamma_-) = m > 0$ and $\ell(\gamma_+) = a > 0$. Let $\pi\colon \sum_A \to \sum_{A_0}$ be a promotion map. Then π is a composition of elementary isomorphisms. Moreover, π is a homeomorphism.*

Proof. Let $i_0 = \tau(\gamma_-)$ and let E_1 and E_2 be the basic elementary matrices having $E_1(i_0, i(\gamma)) = [1]t^a$ and $E_2(i(\gamma), i_0) = wt_A(\gamma)t^m$. Then $I - A = E_2((I - A_0)E_1)$ and the composition of the elementary isomorphisms induced by these matrix multiplications, choosing the correct bijections that define them, results in π. Since the non-zero off-diagonal entries of E_1 and E_2 are both polynomials, π is a homeomorphism. $\square$

6. *Magic words are elementary*

In this section we prove our main result: magic word isomorphisms are compositions of elementary isomorphisms.

Let $A, B \in \mathcal{M}$ and suppose that $\varphi\colon \sum_A \to \sum_B$ is an isomorphism. Let $w = w_-w_+ \in \mathcal{W}_A$ and $v = v_-v_+ \in \mathcal{W}_B$, where $w_-, w_+ \in \mathcal{W}_A$ and $v_-, v_+ \in \mathcal{W}_B$, and suppose that if $[x] \in [w, -\ell(w_-)]$, then almost surely $\varphi([x]) \in [v, -\ell(v_-)]$. In this case, say that $w_- \centerdot w_+$ *codes* $v_- \centerdot v_+$ and write

$$\begin{array}{c} w_- \centerdot w_+ \\ \downarrow \\ v_- \centerdot v_+ \end{array}$$

Moreover, by writing

$$\begin{array}{c} w_1 \centerdot w_2 \centerdot \ldots \centerdot w_n \\ \downarrow \quad \downarrow \quad \downarrow \\ v_1 \centerdot v_2 \centerdot \ldots \centerdot v_n \end{array}$$

it is meant that $w_1 \ldots w_n \in \mathcal{W}_A$, with $w_1, \ldots, w_n \in \mathcal{W}_A$, $v_1 \ldots v_n \in \mathcal{W}_B$, with $v_1, \ldots, v_n \in \mathcal{W}_B$, $\ell(w_s) = \ell(v_s)$ for all $1 < s < n$ and that for all $1 \le s < n$,

$$\begin{array}{c} w_1 \ldots w_s \centerdot w_{s+1} \ldots w_n \\ \downarrow \\ v_1 \ldots v_s \centerdot v_{s+1} \ldots v_n \end{array}$$

So if $A, B \in \mathcal{M}$ and $\varphi\colon \sum_A \to \sum_B$ is an isomorphism with a magic word $w \in \mathcal{W}_A$, then for all $\gamma \in \mathcal{W}_A$ such that $w\gamma w \in \mathcal{W}_A$, there exists $\varphi(\gamma) \in \mathcal{W}_B$ such that $\ell(\varphi(\gamma)) = \ell(\gamma)$ and

$$\begin{array}{c} w \centerdot \ \gamma \ \centerdot w \\ \downarrow \qquad \downarrow \\ \centerdot\varphi(\gamma)\centerdot \end{array}$$

PROPOSITION 6.1. *Let $A, B \in \mathcal{M}$ and suppose that there is an elementary isomorphism $\varphi\colon \sum_A \to \sum_B$. Then φ is a magic word isomorphism.*

Proof. Let E be a basic elementary matrix such that $\varphi = \varphi_E$ (respectively $\varphi = \varphi_E^{-1}$). Let $i_0, j_0 \in \mathbb{N}$ correspond to the non-zero off-diagonal entry of E (if $E = I$, the result is trivial). If $E(I - A) = I - B$ (respectively $E(I - B) = I - A$) is the equation defining φ_E, then any element in $\mathcal{S}_A(i_0, j_0, j)$, with $j \neq j_0$, is a magic word for φ and any element in $\mathcal{S}_B(i_0, j_0, j)$ is a magic word for φ^{-1}. A similar argument applies when φ_E is defined by the equation $(I - A)E = I - B$ (respectively $(I - B)E = I - A$). □

PROPOSITION 6.2. *Let $A, B, C \in \mathcal{M}$ and suppose that $\varphi\colon \sum_A \to \sum_B$ and $\psi\colon \sum_B \to \sum_C$ are both magic word isomorphisms. Then $\psi \circ \varphi$ is a magic word isomorphism.*

Proof. Let $w \in \mathcal{W}_A$ and $v \in \mathcal{W}_B$ be magic words for φ and ψ respectively. We can find $\gamma = \gamma_- \gamma_+ \in \mathcal{W}_A$, with $\gamma_-, \gamma_+ \in \mathcal{W}_A$, such that $\gamma_- \centerdot \gamma_+$ codes $\centerdot v$, with $i(\gamma) = \tau(w)$, $\tau(\gamma) = i(w)$ and $\ell(\gamma_+) \ge \ell(v)$. Then $w\gamma w$ is a magic word for $\psi \circ \varphi$. A magic word for the inverse is found similarly. □

Let $A, B \in \mathcal{M}$. Let $\varphi\colon \sum_A \to \sum_B$ be an isomorphism and suppose that $w \in \mathcal{W}_A$ is a magic word for φ. Then any pair $w_-, w_+ \in \mathcal{W}_A$ containing w forms a *magic pair for φ*, that is for every $\gamma \in \mathcal{W}_A(n)$ such that $w_- \gamma w_+ \in \mathcal{W}_A$, there exists $\varphi(\gamma) \in \mathcal{W}_B(n)$ such that

$$\begin{array}{c} w_- \centerdot \ \gamma \ \centerdot w_+ \\ \downarrow \qquad \downarrow \\ \centerdot\varphi(\gamma)\centerdot \end{array}$$

Then for every $v = v_- v_+ \in \mathcal{R}_B^*(m + a)$, with $v_- \in \mathcal{R}_B^*(m)$ and $v_+ \in \mathcal{R}_B^*(a)$, we can always find $w = w_- w_+ \in \mathcal{W}_A$, with $w_-, w_+ \in \mathcal{W}_A$, such that $w_- \centerdot w_+$ codes $v_- \centerdot v_+$, with $(w_\pm, w_\pm)$ being magic pairs for φ. Moreover, up to recoding using Proposition 5.8,

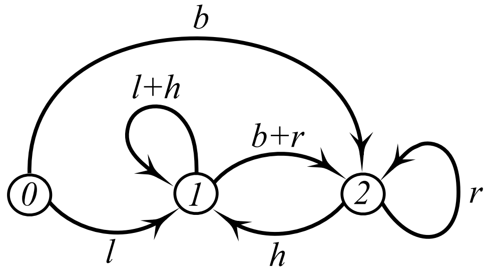

FIGURE 1. In this figure, the set of all paths from 0 to 2 corresponds to the loop system $\mathcal{L}_A(w_-w_+)$ in Lemma 6.3, and in this way, the relationship with the loop system $\mathcal{L}_B(v_-v_+)$ that produces the decomposition into elementary isomorphisms is exhibited.

we can assume that $w_-, w_+ \in \mathcal{R}_A^*$. Under these circumstances, for every $\gamma \in \mathcal{L}_A(w_-w_+)$, there is a unique image $\varphi(\gamma) \in (\mathcal{L}_B(v_-v_+))^*$ and, hence, for $\theta \in \mathcal{L}_B(v_-v_+)$, we consider the following two statements: (1) there exists $\gamma \in \mathcal{L}_A(w_-w_+)$ such that θ begins $\varphi(\gamma)$; and (2) there exists $\gamma \in \mathcal{L}_A(w_-w_+)$ such that θ ends $\varphi(\gamma)$. Let $\mathcal{P}_\varphi(w_-w_+, v_-v_+) = (\mathcal{B}, \mathcal{L}, \mathcal{R}, \mathcal{H})$ be the partition of $\mathcal{L}_A(v_-v_+)$ defined as follows:

$$\begin{aligned}
\mathcal{B} &= \{\theta \in \mathcal{L}_A(v_-v_+) \mid \text{both (1) and (2) hold}\};\\
\mathcal{L} &= \{\theta \in \mathcal{L}_A(v_-v_+) \mid \text{(1) holds and (2) does not hold}\};\\
\mathcal{R} &= \{\theta \in \mathcal{L}_A(v_-v_+) \mid \text{(2) holds and (1) does not hold}\};\\
\mathcal{H} &= \{\theta \in \mathcal{L}_A(v_-v_+) \mid \text{(1) does not hold and (2) does not hold}\}.
\end{aligned}$$

The sets $\mathcal{B}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are represented by power series $b, l, r, h \in \mathcal{M}$.

LEMMA 6.3. *Let $A, B \in \mathcal{M}$ and $\varphi\colon \sum_A \to \sum_B$ be a magic word isomorphism. Suppose that we can find $w = w_-w_+ \in \mathcal{R}_A^*$ and $v = v_-v_+ \in \mathcal{R}_B^*$, with $(w_\pm, w_\pm)$ and $(v_\pm, v_\pm)$ being magic pairs for φ and φ^{-1}, such that $w_- \centerdot w_+$ codes $v_- \centerdot v_+$ and if $\mathcal{P}_\varphi(w_-w_+, v_-v_+) = (\mathcal{B}, \mathcal{L}, \mathcal{R}, \mathcal{H})$, then for every $\theta_- \in \mathcal{B} \cup \mathcal{R}$ and $\theta_+ \in \mathcal{B} \cup \mathcal{L}$, we have that $\theta_- \centerdot \theta_+$ codes $w_- \centerdot w_+$. Then φ is a composition of elementary isomorphisms.*

Proof. If $\gamma \in \mathcal{L}_A(w_-w_+)$, then $\varphi(\gamma) \in (\mathcal{L}_B(v_-v_+))^*$ is a concatenation of routes that begins with an element of $\mathcal{B} \cup \mathcal{L}$ and ends with an element of $\mathcal{B} \cup \mathcal{R}$ and such that a concatenation in $(\mathcal{B} \cup \mathcal{R})(\mathcal{B} \cup \mathcal{L})$ never occurs in γ (otherwise γ would not be an element of $\mathcal{L}_A(w_-w_+)$). This subset of $(\mathcal{L}_B(v_-v_+))^*$ corresponds to the set of paths that start at 0 and end at 2 in Figure 1 (for clarity, the reader may assume that $b = 0$, justified by applying a left- or right-loop system map of $\mathcal{B}$).

Then $\mathcal{L}_A(w_-w_+)$ is represented by the power series

$$f = (b + l(l+h)^*(b+r))(r + h(l+h)^*(b+r))^*.$$

Given a labelling $\mathcal{R}_A$, we get an induced labelling on $\mathcal{L}_A(w_-w_+)$, which induces a labelling on $\mathcal{R}_f$. Hence, φ is the composition of a left-loop system map of $\mathcal{L} \cup \mathcal{H}$, $\pi_1\colon \sum_{(l+h)^*(b+r)} \to \sum_B$, followed by a left-loop system map of $\mathcal{R} \cup \mathcal{H}(\mathcal{L} \cup \mathcal{H})^*(\mathcal{B} \cup \mathcal{R})$, $\pi_2\colon \sum_f \to \sum_{(l+h)^*(b+r)}$, followed by $\pi_3\colon \sum_A \to \sum_f$, a loop system map of $[w_- \centerdot w_+]$ compatible with the labelling. $\square$

LEMMA 6.4. *Let $A, B \in \mathcal{M}$ and suppose that $\varphi\colon \sum_A \to \sum_B$ is a magic word isomorphism. Let $i \in \mathbb{N}_A$ be such that $\mu([i,0]) > 0$. Then, up to a conjugacy, there exist $W_-, W_+ \in (\mathcal{L}_A(i))^*$ and $V_-, V_+ \in (\mathcal{L}_B(\varphi(i)))^*$ for some $\varphi(i) \in \mathbb{N}_B$, such that*

(1) *$(W_\pm, W_\pm)$ and $(V_\pm, V_\pm)$ are magic pairs for φ and φ^{-1} respectively;*
(2) *$W_- \cdot W_+$ codes $V_- \cdot V_+$;*
(3) *$\ell(V_-V_+) \geq \ell(W_\pm)$; and*
(4) *the only self-overlap of V_-V_+ is trivial.*

Proof. Let θ be a cycle in i which is a magic word for φ. Let $v \in (\mathcal{L}_B(\varphi(i)))^*$ be a magic word for φ^{-1}. Up to a conjugacy (see Proposition 5.8), there exist $w_-, w_+ \in (\mathcal{L}_A(i))^*$ such that $(w_\pm, w_\pm)$ are magic pairs for φ, with $\ell(w_\pm) \geq \ell(v)$, and such that

$$\begin{array}{c} w_- \cdot w_+ \\ \downarrow \\ \cdot v \end{array}$$

Let $u \in \mathcal{R}_B^*$ be such that

$$\begin{array}{ccc} w_- \cdot w_+ & w_- & \cdot w_+ \\ \downarrow & & \downarrow \\ \cdot v & u & \cdot v \end{array}$$

If $v = r_1 \ldots r_n \in \mathcal{R}_B^*(n)$, with $r_1, \ldots, r_n \in \mathcal{R}_B$, then let $\alpha \in (\mathcal{L}_B(\varphi(i)) - \{r_1\})^*$ be such that $\ell(\alpha) \geq \ell(vu)$. Let $\gamma = \gamma_-\gamma_+ \in \mathcal{R}_A^*$ be such that

$$\begin{array}{c} w_+\gamma_- \cdot \gamma_+w_- \\ \downarrow \\ \cdot \alpha v \end{array}$$

with $\ell(\gamma_+) \geq \ell(\alpha v)$. Let $\epsilon_1, \in \mathcal{L}_B(\varphi(i))$ be such that

$$\begin{array}{ccc} w_- \cdot w_+ & \gamma_- & \cdot \gamma_+w_- \\ \downarrow & & \downarrow \\ \cdot v & \epsilon_1 & \cdot \alpha v \end{array}$$

Let $\beta \in (\mathcal{L}_B(\varphi(i)) - \{r_1\})^*$ be such that $\ell(\beta) \geq \ell(v\epsilon_1)$. Let $\rho = \rho_-\rho_+ \in \mathcal{R}_A^*$ be such that

$$\begin{array}{c} w_-\rho_- \cdot \rho_+w_+ \\ \downarrow \\ v \cdot \beta \end{array}$$

Let $\epsilon_2 \in \mathcal{L}_B(\varphi(i))$ be such that

$$\begin{array}{ccc} w_+\gamma_- \cdot \gamma_+ & w_-\rho_- & \cdot \rho_+w_+ \\ \downarrow & & \downarrow \\ \cdot \alpha v & \epsilon_2 & v \cdot \beta \end{array}$$

Then for every positive integer $m \geq 1$, we have

$$\begin{array}{c} \overbrace{w_- \cdot w_+w_- \cdot w_+w_- \cdot \ \ldots\ \cdot w_+w_- \cdot w_+w_-}^{W_-} \cdot \overbrace{w_+\gamma_- \cdot \gamma_+w_-\rho_- \cdot \rho_+w_+}^{W_+} \\ \downarrow \quad \downarrow \quad \downarrow \cdots \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \underbrace{\cdot \underbrace{v\ \ u}_{1} \cdot \underbrace{v\ \ u}_{2} \cdot \ \ldots\ \cdot \underbrace{v\ \ u}_{m-1} \cdot \underbrace{v\ \ u}_{m}}_{V_-} \cdot \underbrace{v\ \ \epsilon_1 \cdot \alpha v\ \ \epsilon_2\ \ v \cdot \beta}_{V_+} \end{array}$$

From the construction, (1), (2) and (3) are satisfied for all $m \geq 1$. If m is large enough so that $m\ell(u) \geq \ell(\alpha v \epsilon_2)$, then (4) is also satisfied. □

THEOREM 6.5. *Let $A, B \in \mathcal{M}$ and suppose that $\varphi\colon \sum_A \to \sum_B$ is a magic word isomorphism. Then φ is a composition of elementary isomorphisms.*

Proof. Let $W_- W_+$ and $V_- V_+$ be as in Lemma 6.4. If $\theta_- \in \mathcal{B} \cup \mathcal{R}$ and $\theta_+ \in \mathcal{B} \cup \mathcal{L}$, then $\ell(\theta_- \theta_+) \geq (W_- W_+)$ because both θ_+ and θ_- begin and end with V_+ and V_-. Since $V_- \theta_- \centerdot \theta_+ V_+$ codes $W_- \centerdot W_+$, then $\theta_- \centerdot \theta_+$ codes $W_- \centerdot W_+$ because (V_+, V_-) is a magic pair. The result follows from Lemma 6.3. □

7. *Exponentially recurrent chains*

In this section we make some brief remarks about the exponentially recurrent irreducible Markov chains, which play a natural and prominent role in the theory of finitary isomorphisms of Markov chains [**9, 18, 29**].

Recall that an irreducible Markov chain with transition matrix P is *exponentially recurrent* if every open set $E \subset \sum_P$ is *exponentially recurrent*, i.e. if

$$\{\mu_P(E \cap (E - \sigma_P(E)) \cap \cdots \cap (E - \sigma_P^n(E)))\}_{n=1}^{\infty}$$

goes to zero exponentially fast for every open set $E \subset \sum_P$ (it is not hard to check that exponential recurrence is an invariant of finitary isomorphism). It is left as an exercise to verify that in the category of all exponentially recurrent Markov chains, magic word isomorphisms are FECT isomorphisms. One way to do this is to show that, in this category, magic word isomorphisms code exponentially fast (a finitary map $\varphi\colon \sum_P \to \sum_Q$ *codes exponentially fast* if $\{\mu_P(C_n)\}_{n=0}^{\infty}$ goes to zero exponentially fast, where C_n consists of all $x \in \sum_P$ such that it is not true that $\varphi(y) \in [\varphi(x)_0, 0]$ for almost all $y \in [x_{-n} \ldots x_0 \ldots x_n, -n]$) and then to show a finitary isomorphism coding exponentially fast has finite expected coding time. It is also left to the reader to check that, although, in general, compositions of FECT isomorphisms does not necessarily result in FECT isomorphisms, finitary isomorphisms coding exponentially fast are closed under composition.

Acknowledgement. I am very grateful to Mike Boyle from the University of Maryland for his guidance and help. I am also grateful for the hospitality of the Department of Mathematics, University of Washington and for the support of the IMATE and DGAPA of the University of México (UNAM). Finally, I kindly thank the anonymous referee for reading the paper and for making many observations and very helpful suggestions.

REFERENCES

[1] R. Adler and B. Marcus. Topological entropy and equivalence of dynamical systems. *Memoirs Amer. Math. Soc.* **219** (1979).

[2] L. Block, J. Guckenheimer, M. Misurewicz and L. Young. *Periodic Points and Topological Entropy of One-dimensional Maps* (*Springer Lecture Notes in Mathematics, 819*). Springer, New York, 1980, pp. 18–34.

[3] M. Boyle. Symbolic dynamics and matrices. *Combinatorial Graph Theory and Problems in Linear Algebra (IMA Volumes in Mathematics and its Applications, 50)*. Springer, New York, 1993.

[4] M. Boyle. Flow equivalence of shifts of finite type via positive factorizations. *Pacific J. Math.* **204** (2002), 273–317.

[5] M. Boyle. Positive K-theory and symbolic dynamics. *Dynam. Randomn.* (2002), 31–52.

[6] M. Boyle and D. Huang. Poset block equivalence of integral matrices. *Trans. Amer. Math. Soc.* to appear.

[7] M. Boyle and M. Sullivan. Equivariant flow equivalence for shifts of finite type. *Preprint*, 2003.

[8] M. Boyle and J. B. Wagoner. Positive algebraic K-theory and shift s of finite type. *Preprint*, 2003.

[9] U. R. Fiebig. A return time invariant for finitary isomorphisms. *Ergod. Th. & Dynam. Sys.* **4** (1984), 225–231.

[10] N. Friedman and D. S. Ornstein. On the isomorphism of weak Bernoulli transformations. *Adv. Math.* **5** (1970), 365–394.

[11] B. M. Gurevich and S. V. Savchenko. Thermodynamic formalism for countable symbolic Markov chains. *Russian Math. Surveys* **53**(2) (1998), 3–106.

[12] M. Keane and M. Smorodinsky. Finitary isomorphisms of irreducible Markov shifts. *Israel J. Math.* **34** (1979), 281–286.

[13] K. H. Kim, F. W. Roush and J.B. Wagoner. Characterization of inert actions on periodic points. Part I. *Forum Math.* **12**(6) (2000), 671–712.

[14] K. H. Kim, F. W. Roush and J.B. Wagoner. Characterization of inert actions on periodic points. Part II. *Forum Math.* **12**(5) (2000), 565–602.

[15] K. H. Kim and F. W. Roush. The Williams' conjecture is false for irreducible subshifts. *Ann. Math.* **149**(2) (1999), 545–558.

[16] B. P. Kitchens. *Symbolic Dynamics. One-sided, Two-sided and Countable State Markov Shifts*. Springer, New York, 1998.

[17] W. Krieger. On the finitary isomorphisms of Markov shifts that have finite expected coding time. *Z. Wahrscheinlichkeitstheorie* **65** (1983), 323–328.

[18] D. Lind. Ergodic group automorphisms are exponentially recurrent. *Israel J. Math.* **41**(4) (1982), 313–320.

[19] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, Cambridge, 1995.

[20] B. Marcus and S. Tuncel. The weight-per-symbol polytope and scaffolds of invariants associated with Markov chains. *Ergod. Th. & Dynam. Sys.* **11** (1991), 129–180.

[21] B. Marcus and S. Tuncel. Matrices of polynomials, positivity, and finite equivalence of Markov chains. *J. Amer. Math. Soc.* **6** (1993), 1131–147.

[22] B. Marcus and S. Tuncel. Resolving Markov chains onto Bernoulli shifts via positive polynomials. *Memoirs Amer. Math. Soc.* **150**(710) (2001), 99pp.

[23] D. S. Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Adv. Math.* **4** (1970), 337–352.

[24] W. Parry. A finitary classification of topological Markov chains and sofic systems. *Bull. London Math. Soc.* **9** (1977), 86–92.

[25] W. Parry. Finitary isomorphisms with finite expected coding-lengths. *Bull. London Math. Soc.* **11** (1979), 170–176.

[26] W. Parry and K. Schmidt. Natural coefficients and invariants for Markov-shifts. *Invent. Math.* **76** (1984), 15–32.

[27] W. Parry and S. Tuncel. On the classification of Markov chains by finite equivalence. *Ergod. Th. & Dynam. Sys.* **1** (1981), 303–335.

[28] J. Rosenberg. *Algebraic K-theory and its Applications*. Springer, New York, 1994.

[29] D. J. Rudolph. A mixing Markov chain with exponentially decaying return times is finitarily Bernoulli. *Ergod. Th. & Dynam. Sys.* **2**(1) (1982), 85–97.

[30] K. Schmidt. Invariants for finitary isomorphisms with finite expected code lengths. *Invent. Math.* **76** (1984), 33–40.

[31] C. Shannon. A mathematical theory of communication. *Bell Sys. Tech. J.* **27** (1948), 379–423; 623–656.

[32] S. Tuncel. Conditional pressure and coding. *Israel J. Math.* **39** (1981), 101–112.

[33] S. Tuncel. Faces of Markov chains and matrices of polynomials. *Contemp. Math.* **135** (1992), 391–422.

[34] J. B. Wagoner. Strong shift equivalence and K_2 of the dual numbers. *J. Reine Angew. Math.* **521** (2000), 119–160.