Independent sets and non-augmentable paths in generalizations of tournaments

Hortensia Galeana-Sánchez and Ricardo Gómez

Instituto de Matemáticas de la Universidad Nacional Autónoma de México. Área de la Investigación Científica. Circuito Exterior, Ciudad Universitaria. México D.F. 04510 MÉXICO

Abstract

We study different classes of digraphs, which are generalizations of tournaments, to have the property of possessing a maximal independent set intersecting every non-augmentable path (in particular, every longest path). The classes are the arc-local tournament, quasi-transitive, locally in-semicomplete (out-semicomplete), and semicomplete k-partite digraphs. We present results on strongly internally and finally non-augmentable paths as well as a result that relates the degree of vertices and the length of longest paths. A short survey is included in the introduction.

Key words: Independent set; non-augmentable path; longest path; generalization of tournament

1 Introduction

The conjecture of Laborde, Payan and Xuong can be stated as follows: *In every digraph, there exists a maximal independent set that intersects every longest path* (see [21]). The conjecture is true for every digraph having a kernel, that is, an independent and absorbing set of vertices, e.g., every transitive digraph (many other classes of digraphs have kernels, for instance, see [11], [15]). In [21], Laborde, Payan and Xuong showed that in every symmetric digraph, there exists an independent set intersecting every longest path and with the property that each of its vertices is the origin of a longest path (they conjectured that this holds for all digraphs). In [9], Bang-Jensen, Huang and Prisner proved that every strongly connected (i.e. strong) locally in-semicomplete

Email addresses: hgaleana@math.unam.mx, rgomez@math.unam.mx (Hortensia Galeana-Sánchez and Ricardo Gómez).

digraph has a hamiltonian cycle (hence every longest path intersects every independent set). They showed that a locally in-semicomplete digraph has a hamiltonian path if and only if it contains a vertex that can be reached by all other vertices by a directed path, a result that constituted a sufficient condition for any independent set to intersect every longest path for this class of digraphs. In [16], Galeana-Sánchez and Rincón-Mejía proved several sufficient conditions for a digraph to have an independent set intersecting every longest path. Later, in [14], Galeana-Sánchez investigates sufficient conditions for a digraph to have the property that each of its induced subdigraphs has a maximal independent set intersecting all its non-augmentable paths. Moreover, Galeana-Sánchez finds necessary and sufficient conditions for this property to hold in case that the digraph is asymetrical, and also finds necessary and sufficient conditions for any orientation of a graph to have this property. More recently, in [20], F. Havet proved that if a digraph has stability number at most two, then there exists a stable set that intersects every longest path. Here, the stability number is the cardinality of a largest stable set, i.e., the cardinality of a largest independent set.

The conjecture of Laborde, Payan and Xuong is a particular instance of what is called the *Path Partition Conjecture* which states the following: *For every digraph D and any choice of positive integers λ_1 and λ_2 with $\lambda(D) = \lambda_1 + \lambda_2$, where $\lambda(D)$ is the number of vertices of a longest path in D, there exists a partition of D into two digraphs D_1 and D_2 such that $\lambda(D_i) \leq \lambda_i$ for $i = 1, 2$.* In [7] this conjecture is proved for several classes of digraphs which are generalizations of tournaments, namely quasi-transitive, extended semicomplete and locally in-semicomplete digraphs (for the last two classes, the authors show that equality holds in the statement of the conjecture). Both conjectures deal with longest paths. We, however, consider non-augmentable paths for which longest paths are a particular case. Hence the results in [7] and the results we present in this paper for the coinciding classes of digraphs differ except for the fact that both have the Laborde, Payan and Xuong conjecture holding as a particular case.

In this paper, we exhibit classes of digraphs having the property of possessing maximal independent sets intersecting every longest path. In particular, we show that the Laborde, Payan and Xuong conjecture is true for arc-local tournament digraphs, line digraphs, quasi-transitive digraphs, path-mergeable digraphs, in-semicomplete (out-semicomplete) digraphs, and semicomplete k-partite digraphs, all of them being generalizations of tournaments except for line digraphs (see [6]). We prove that there always exists a maximal independent set intersecting every non-augmentable path in a semicomplete digraph (proposition 25). For arc-local tournament digraphs (section §2), we show that there exists a maximal independent set that intersects every non-augmentable path (theorem 15). Actually, we show that in an arc-local tournament digraph, every maximal independent set intersects every non-augmentable path of even

length, and exhibit arc local tournament digraphs with maximal independent sets and non-augmentable paths of arbitrary odd length which do not intersect (proposition 17). We show that line digraphs satisfy a hypothesis quite similar to the one defining arc local tournament digraphs (hypothesis 19), and prove that every maximal independent set in a digraph satisfying this hypothesis intersects every non-augmentable path (theorem 20). For quasi-transitive digraphs (section §3), using a structural theorem of Bang-Jensen and Huang taken from [8] (theorem 27 in this paper), we show that there exists a maximal independent set that intersects every non-augmentable path (theorem 30), in particular, every longest path (see [7]). Moreover, we show that if the quasi-transitive digraph is strong, then this maximal independent set has a natural decomposition according to Bang-Jensen and Huang's structural theorem. Next (section §4), we define (definition 32) a path to be strongly internally and finally non-augmentable (it is easy to see that a longest path in a path-mergeable digraph is strongly internally and finally non-augmentable). Finally, we show that in any strong digraph, every maximal independent set intersects every strongly internally and finally non-augmentable path (theorem 34). Finally, we show that in any digraph, there exists a maximal independent set intersecting every strongly internally and finally non-augmentable path (theorem 35). We state without proofs (section §5) the following elementary results for locally in-semicomplete (out-semicomplete) digraphs and semicomplete k-partite digraphs: both classes have the property of possessing a maximal independent set that intersects every non-augmentable path (theorems 39, 40 and 41). The last section contains a theorem that relates the degree of vertices and the length of longest paths.

All these generalizations of tournaments have been taken from a survey by Bang-Jensen and Gutin (see [6]). We refer the reader to it for a detailed exposition of results concerning them and restrict ourselves to briefly present the following summary. Arc-local tournament digraphs were introduced by Bang-Jensen in [1] as an extension of the idea of a generalization of semicomplete digraphs called locally semicomplete digraphs. Some properties of arc local tournament digraphs have been studied by Bang-Jensen in [1] and [4], and by Bang-Jensen and Gutin in [6]. Galeana-Sánchez characterized all kernel-perfect and critical kernel-imperfect arc local tournament digraphs in [13], both classes introduced by Berge and Duchet in [10]. Quasi-transitive digraphs were introduced by Ghouilà-Houri (see [17]). They are related to comparability digraphs in the sense that a graph can be oriented as a quasi-transitive digraph if and only if it is a comparability digraph. In [8], Bang-Jensen and Huang extensively study quasi-transitive digraphs. Path-mergeable digraphs were introduced by Bang-Jensen in [2]. They can be recognized in polynomial time and the merging of two internally disjoint paths can be done in a particular nice way in the sense that it is always possible to respect the order of one of the paths. Locally in-semicomplete (out-semicomplete) digraphs were introduced by Bang-Jensen in [3], and in [2] he proved that locally in-semicomplete

(out-semicomplete) digraphs are path mergeable (in particular, every tournament is path-mergeable). Semicomplete k-partite digraphs have been recently studied. In [13], Gutin presents a survey on this kind of digraphs. See [5] for a unified and comprehensive survey on digraphs.

2 Arc local tournament digraphs

In this paper, a digraph D will consist of a vertex set $V(D)$ and an arc set $A(D) \subset V(D) \times V(D)$. All digraphs will be simple, that is, there will be no loops nor multiple arcs between any pair of distinct vertices. For $u, v \in V(D)$, we will write $\overrightarrow{uv}$ or $\overleftarrow{vu}$ if $(u, v) \in A(D)$, and also, we will write $\overline{uv}$ if $\overrightarrow{uv}$ or $\overrightarrow{vu}$. Given $K \subset V(D)$, let $D(K)$ be the subdigraph induced by K and let $D \setminus K$ be the digraph that results from D by removing the vertices in K.

Definition 1 *An* independent set *in a digraph D is a subset of vertices $\mathcal{I} \subset V(D)$ with no $x, y \in \mathcal{I}$ such that $\overline{xy}$, and is* maximal *if there exists no $z \in V(D) - \mathcal{I}$ such that $\mathcal{I} \cup \{z\}$ is an independent set.*

Definition 2 *A* path *in a digraph D is a finite sequence of distinct vertices $\gamma = (x_0, \ldots, x_n)$ such that $\overrightarrow{x_{i-1}x_i}$ for every $1 \leq i \leq n$, and its* length *is n (zero-length path consists of a single vertex). We let $V(\gamma) = \{x_0, \ldots, x_n\}$.*

Definition 3 *A path $\gamma = (x_0, \ldots, x_n)$ in a digraph D is* non-augmentable *if there exists no path $(y_0, \ldots, y_k)$ with $y_0, \ldots, y_k \in V(D) - V(\gamma)$ and such that $\overrightarrow{y_k x_0}$, or $\overrightarrow{x_n y_0}$, or $\overrightarrow{x_{i-1} y_0}$ and $\overrightarrow{y_k x_i}$ for some $1 \leq i \leq n$. More generally, γ is* non-augmentable *if there exist no path $(z_0, \ldots, z_m)$ in D with $m > n$, a function $\sigma \colon \{0, \cdots, n\} \to \{0, \cdots, m\}$ and $0 \leq r \leq n$ satisfying:*

(1) For every $i \in \{0, \ldots, n\}$, $x_i = z_{\sigma(i)}$ (hence σ is injective).
(2) If $i \neq r$, then $\sigma(i) < \sigma(i+1)$.
(3) If $r < n$, then $\sigma(n) < \sigma(0)$.

Otherwise, γ is augmentable.

Remark 4 *The first definition of non-augmentability is a particular case of the second one, with $m = n + k + 1$, $r = n$ and $\sigma(i) = i + k + 1$ for every $0 \leq i \leq n$ if $\overrightarrow{y_k x_0}$, $\sigma(i) = i$ for every $0 \leq i \leq n$ if $\overrightarrow{x_n y_0}$, and $\sigma(j) = j$ for $j < i$ and $\sigma(j) = j + k + 1$ for $j \geq i$ if $\overrightarrow{x_{i-1} y_0}$ and $\overrightarrow{y_k x_i}$ for some $1 \leq i \leq n$, so that if γ is augmentable, then $(y_0, \ldots, y_k, x_0, \ldots, x_n)$, or $(x_0, \ldots, x_n, y_0, \ldots, y_k)$, or $(x_0, \ldots, x_{i-1}, y_0, \ldots, y_k, x_i, \ldots, x_n)$ are paths in D.*

Example 5 *Consider the digraph in figure 1. The path $\gamma = (x_0, x_1, x_2, x_3)$ is augmentable by $(z_0, z_1, z_2, z_3, z_4) = (x_3, x_0, x_1, x_2, x_5)$, with $\sigma(0) = 1$, $\sigma(1) = 2$,*

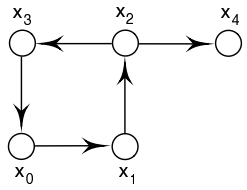

Fig. 1. The path (x_0, x_1, x_2, x_3) is augmentable by $(x_3, x_0, x_1, x_2, x_4)$.

$\sigma(2) = 3$, $\sigma(3) = 0$ *and* $r = 2$ *(according to the first definition of non-augmentability,* γ *would be non-augmentable).*

Definition 6 *A path* γ *in a digraph* D *is a* longest path *if there exists no path in* D *of bigger length.*

Clearly, a longest path is non-augmentable and the converse is not true.

Definition 7 *A path* γ *in a digraph* D *and a subset of vertices* $\mathcal{I} \subset V(D)$ intersect *if* $V(\gamma) \cap \mathcal{I} \neq \varnothing$*, otherwise they* do not intersect.

In this section, if D is a digraph and $u, v, x, y \in V(D)$ are such that $\overline{uv}$, $\overrightarrow{xu}$ and $\overleftarrow{vy}$, then we will write $\overrightarrow{x}\,\overline{uv}\,\overleftarrow{y}$, and similarly, if $\overline{uv}$, $\overleftarrow{xu}$ and $\overrightarrow{vy}$, then we will write $\overleftarrow{x}\,\overline{uv}\,\overrightarrow{y}$.

Definition 8 *A digraph* D *is an* arc local tournament *if whenever* $u, v, x, y \in V(D)$ *are such that* $\overrightarrow{x}\,\overline{uv}\,\overleftarrow{y}$ *or* $\overleftarrow{x}\,\overline{uv}\,\overrightarrow{y}$*, then* $\overline{xy}$.

Proposition 9 *Let* D *be an arc local tournament digraph. Let* $\mathcal{I}$ *be a maximal independent set and let* $\gamma = (x_0, \ldots, x_n)$ *be a non-augmentable path in* D *such that* $V(\gamma) \cap \mathcal{I} = \varnothing$*. If there exists* $z \in \mathcal{I}$ *such that* $\overrightarrow{zx_i}$ *for some* $i = 0, \ldots, n$*, then* $i_0 = \min\{i \mid \overrightarrow{zx_i}\} = 1$.

PROOF. If $i_0 \geq 2$, then $\overline{zx_{i_0-2}}$ because $\overrightarrow{z}\,\overline{x_{i_0}x_{i_0-1}}\,\overleftarrow{x_{i_0-2}}$. Since i_0 is minimal, $\overleftarrow{zx_{i_0-2}}$. There exists $y \in \mathcal{I}$ such that $\overline{yx_{i_0-1}}$ because $\mathcal{I}$ is a maximal independent set. Suppose that $y = z$. Then $\overleftarrow{zx_{i_0-1}}$ becuase i_0 is minimal, and therefore $(x_0, \ldots, x_{i_0-1}, z, x_{i_0}, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Suppose that $y \neq z$. If $\overrightarrow{yx_{i_0-1}}$, then $\overline{yz}$ because $\overrightarrow{y}\,\overline{x_{i_0-1}x_{i_0}}\,\overleftarrow{z}$, contradicting that $\mathcal{I}$ is an independent set, and if $\overleftarrow{yx_{i_0-1}}$, then $\overline{yz}$ because $\overleftarrow{z}\,\overline{x_{i_0-2}x_{i_0-1}}\,\overrightarrow{y}$, contradicting again that $\mathcal{I}$ is an independent set. Now, if $i_0 = 0$, then $\overrightarrow{zx_0}$ and therefore $(z, x_0, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Hence $i_0 = 1$. $\square$

Definition 10 *Let* D *be a digraph. For every vertex* $v \in V(D)$*, let the* in-degree *of* v *be the number of incoming edges to* v *and let the* out-degree *of* v *be the number of outgoing edges from* v*. Denote them by* $in(v)$ *and* $out(v)$ *respectively. We let* $\mathcal{O}(D) = \{v \in V(D) \mid out(v) = 0\}$.

Lemma 11 *Let D be an arc local tournament digraph. Let $\mathcal{I}$ be a maximal independent set and $\gamma = (x_0, \ldots, x_n)$ be a non-augmentable path in D such that $V(\gamma) \cap \mathcal{I} = \varnothing$. Then $z \in \mathcal{O}(D)$ for every $z \in \mathcal{I}$ such that $\overleftarrow{zx_0}$.*

PROOF. Suppose that $out(z) > 0$. If $\overrightarrow{zx_i}$ for some $i = 0, \ldots, n$, then, by proposition 9, $\overrightarrow{zx_1}$ and hence $(x_0, z, x_1, \ldots x_n)$ is a path in D contradicting that γ is non-augmentable. Let $w \in V(D) - V(\gamma)$ be such that $\overrightarrow{zw}$. Then $\overline{wx_1}$ because $\overleftarrow{w}\overline{zx_0}\overrightarrow{x_1}$. If $\overrightarrow{wx_1}$, then $(x_0, z, w, x_1, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Thus $\overleftarrow{wx_1}$. We claim that for every $0 \leq i \leq n$, $\overleftarrow{zx_i}$ if i is even and $\overleftarrow{wx_i}$ if i is odd. The claim is true for $i = 0, 1$. Suppose that for some m with $1 \leq m \leq n$, the claim is true for all $i \leq m$. If m is odd, then $\overline{zx_{m+1}}$ because $\overleftarrow{z}\overline{x_{m-1}x_m}\overrightarrow{x_{m+1}}$. If $\overrightarrow{zx_{m+1}}$, then, by proposition 9, $\overrightarrow{zx_1}$ and hence $(x_0, z, x_1, \ldots x_n)$ is a path in D contradicting that γ is non-augmentable. Thus $\overleftarrow{zx_{m+1}}$. If m is even, then $\overline{wx_{m+1}}$ because $\overleftarrow{w}\overline{zx_m}\overrightarrow{x_{m+1}}$. If $\overrightarrow{wx_{m+1}}$, then $(x_0, \ldots, x_m, z, w, x_{m+1}, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Thus $\overleftarrow{wx_{m+1}}$. Therefore the claim is proved. If n is even, then $(x_0, \ldots, x_n, z)$ is a path in D contradicting that γ is non-augmentable. If n is odd, then $(x_0, \ldots, x_n, w)$ is a path in D contradicting that γ is non-augmentable. The contradiction comes from the assumption $out(z) > 0$. Henceforth the lemma is proved. $\square$

Corollary 12 *Let D be an arc local tournament digraph. If $\mathcal{O}(D) = \varnothing$, then every maximal independent set intersects every non-augmentable path in D.*

Proposition 13 *Let D be an arc local tournament digraph and let $D_0 = D \setminus \mathcal{O}(D)$. Let $\mathcal{I}_0$ be a maximal independent set in D_0 and let $\gamma = (x_0, \ldots, x_n)$ be a non-augmentable path in D such that $\{x_0, \ldots, x_r\} \cap \mathcal{I}_0 = \varnothing$ for some $r < n$. If $z \in \mathcal{I}_0$, then there exists no $i \leq r$ such that $\overrightarrow{zx_i}$.*

PROOF. First, observe that $x_i \in V(D_0)$ for every $0 \leq i < n$. Suppose that there exists $i \leq r$ such that $\overrightarrow{zx_i}$ and let $i_0 = \min\{i \mid \overrightarrow{zx_i}\}$. Suppose that $i_0 > 1$. Then $\overline{zx_{i_0-2}}$ because $\overrightarrow{x_{i_0-2}}\overline{x_{i_0-1}x_{i_0}}\overleftarrow{z}$. In fact, $\overleftarrow{zx_{i_0-2}}$ because i_0 is minimal. Since $\mathcal{I}_0$ is a maximal independent set in D_0, there exists $y \in \mathcal{I}_0$ such that $\overline{yx_{i_0-1}}$. If $y = z$, then $\overleftarrow{zx_{i_0-1}}$ because i_0 is minimal, but then $(x_0, \ldots, x_{i_0-1}, z, x_{i_0}, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Suppose that $y \neq z$. If $\overrightarrow{yx_{i_0-1}}$, then $\overline{yz}$ because $\overrightarrow{y}\overline{x_{i_0-1}x_{i_0}}\overleftarrow{z}$, contradicting that $\mathcal{I}_0$ is an independent set. Now, if $\overleftarrow{yx_{i_0-1}}$, then $\overline{yz}$ because $\overleftarrow{z}\overline{x_{i_0-2}x_{i_0-1}}\overrightarrow{y}$, contradicting that $\mathcal{I}_0$ is an independent set. If $i_0 = 0$, then $(z, x_0, \ldots, x_n)$ is a path in D contradicting that γ_0 is non-augmentable. Suppose that $i_0 = 1$. There exists $y \in \mathcal{I}_0$ such that $\overline{yx_0}$ because $\mathcal{I}_0$ is a maximal independent set in D_0. Suppose that $y = z$. In this case, $\overleftarrow{zx_0}$ because i_0 is minimal, but then $(x_0, z, x_1, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Suppose that $y \neq z$. If $\overrightarrow{yx_0}$, then $(y, x_0, \ldots, x_n)$ is a path

in D contradicting that γ is non-augmentable. Suppose that $\overleftarrow{yx_0}$. Then $\overline{yx_2}$ because $\overleftarrow{y}\,\overline{x_0x_1x_2}$ (observe that $n \geq 2$ because $n > r \geq i_0 = 1$). If $\overrightarrow{yx_2}$, then $\overline{yz}$ because $\overrightarrow{z}\,\overline{x_1x_2}\,\overleftarrow{y}$, contradicting that $\mathcal{I}_0$ is an independent set. Suppose that $\overleftarrow{yx_2}$. Since $y \in V(D_0)$, $out(y) > 0$ when we consider y as a vertex of the digraph D. If there exists a vertex $w \in V(D) - V(\gamma)$ such that $\overrightarrow{yw}$, then $\overline{wx_1}$ because $\overleftarrow{w}\,\overline{yx_0}\,\overrightarrow{x_1}$, but then $\overline{yz}$ because $\overrightarrow{y}\,\overline{wx_1}\,\overleftarrow{z}$, contradicting that $\mathcal{I}_0$ is an independent set. Hence there exists s such that $0 \leq s \leq n$ and $\overrightarrow{yx_s}$. If $s = 0$, then $(y, x_0, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. If $s = 1$, then $(x_0, y, x_1, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. If $s > 1$, then $\overline{x_1x_s}$ becuase $\overleftarrow{x_1}\,\overline{x_0y}\,\overrightarrow{x_s}$, but then $\overline{yz}$ because $\overrightarrow{z}\,\overline{x_1x_s}\,\overleftarrow{y}$, contradicting that $\mathcal{I}_0$ is an independent set. $\square$

Lemma 14 *Let D be an arc local tournament digraph and let $D_0 = D \backslash \mathcal{O}(D)$. If $\mathcal{I}_0$ is a maximal independent set in D_0 that intersects every non-augmentable path in D_0, then $\mathcal{I}_0$ intersects every non-augmentable path in D.*

PROOF. Let γ be a non-augmentable path in D. If $V(\gamma) \cap \mathcal{O}(D) = \varnothing$, then γ is a non-augmentable path in D_0 and hence $V(\gamma) \cap \mathcal{I}_0 \neq \varnothing$. Henceforth we assume that γ ends in a vertex in $\mathcal{O}(D)$. If γ becomes a non-augmentable path in D_0 after removal of $\mathcal{O}(D)$, then $V(\gamma) \cap \mathcal{I}_0 \neq \varnothing$. Suppose that γ does not become a non-augmentable path in D_0. Then there exists a non-augmentable path $\gamma_0 = (x_0, \ldots, x_n)$ in D_0 and $k \geq 0$ with $k < n$ such that γ becomes the path $(x_0, \ldots, x_k)$ after removal of $\mathcal{O}(D)$ (so the length of γ is $k + 1$). Since $V(\gamma_0) \cap \mathcal{I}_0 \neq \varnothing$, there exists $i \geq 0$, with $i \leq n$, such that $x_i \in \mathcal{I}_0$. Let $i_0 = \min\{i \mid\ x_i \in \mathcal{I}_0\}$. If $i_0 \leq k$ (in particular if $i_0 = 0$), then $V(\gamma) \cap \mathcal{I}_0 \neq \varnothing$. Henceforth we suppose that $i_0 > k$.

Let $r = i_0 - 1 < n$. Then $x_i \notin \mathcal{I}_0$ for all $i = 0, \ldots, r$ and hence, by proposition 13, there exist no $i \leq r$ such that $\overrightarrow{zx_i}$ for any $z \in \mathcal{I}_0$. Let $x \in \mathcal{O}(D)$ be such that $\gamma = (x_0, \ldots, x_k, x)$. Suppose that $k \geq 1$. There exists $z \in \mathcal{I}_0$ such that $\overline{zx_{k-1}}$ because $\mathcal{I}_0$ is a maximal independent set in D_0. By proposition 13, $\overleftarrow{zx_{k-1}}$ since $k \leq r$. Then $\overline{zx}$ because $\overleftarrow{z}\,\overline{x_{k-1}x_k}\,\overrightarrow{x}$. Since $x \in \mathcal{O}(D)$, $\overrightarrow{zx}$. There exists $y \in \mathcal{I}_0$ such that $\overline{yx_k}$ because $\mathcal{I}_0$ is a maximal independent set in D_0. Suppose that $y = z$. By proposition 13, $\overleftarrow{zx_k}$ since $k \leq r$, but then $(x_0, \ldots, x_k, z, x)$ is a path in D contradicting that γ is non-augmentable. Suppose that $y \neq z$. By proposition 13, $\overleftarrow{yx_k}$ since $k < r$, and hence $\overline{yz}$ because $\overleftarrow{z}\,\overline{x_{k-1}x_k}\,\overrightarrow{y}$, contradicting that $\mathcal{I}_0$ is an independent set. Suppose that $k = 0$ (hence $\gamma = (x_0, x)$). If $n \geq 2$, then $\overline{xx_2}$ because $\overleftarrow{x}\,\overline{x_0x_1}\,\overrightarrow{x_2}$. Since $x \in \mathcal{O}(D)$, $\overleftarrow{xx_2}$, but then (x_0, x_1, x_2, x) is a path in D contradicting that γ is non-augmentable. Suppose that $n = 1$. Then $i_0 = 1$ and hence $x_1 \in \mathcal{I}_0$. Now, $out(x_1) > 0$ when we consider x_1 as a vertex of the digraph D. If $\overrightarrow{x_1x_0}$, then (x_1, x_0, x) is a path in D contradicting that γ is non-augmentable (here we are using the general definition of non-augmentability in definition 3). If $\overrightarrow{x_1x}$, then (x_0, x_1, x) is a path in D contradicting that γ is non-augmentable. Suppose that there exists $w \in V(D) - V(\gamma)$ such that $\overrightarrow{x_1w}$.

Then $\overline{xw}$ because $\overleftarrow{x}\,\overrightarrow{x_0x_1}\,\overrightarrow{w}$, but since $x \in \mathcal{O}(D)$, $\overleftarrow{xw}$. Then (x_0, x_1, w, x) is a path in D contradicting that γ is non-augmentable.

It follows that supposing $i_0 > k$ leads a contradiction. Hence $i_0 \leq k$ and therefore $V(\gamma) \cap \mathcal{I}_0 \neq \varnothing$. $\square$

Theorem 15 *If D is an arc local tournament digraph, then there exists a maximal independent set that intersects every non-augmentable path.*

PROOF. We proceed by induction on the number of vertices. Clearly, the theorem is true if $|V(D)| = 1$. Let $m > 1$ and suppose that the theorem is true for every arc local tournament digraph with $k < m$ vertices. Suppose that $|V(D)| = m$. Let $D_0 = D \setminus \mathcal{O}(D)$. By lemma 11, if $\mathcal{O}(D) = \varnothing$, then every maximal independent set intersects every non-augmentable path in D. Suppose that $\mathcal{O}(D) \neq \varnothing$. Then $|V(D_0)| < m$ and the induction hypothesis implies that there exists a maximal independent set $\mathcal{I}_0 \subset V(D_0)$ in the digraph D_0 that intersects every non-augmentable path in D_0. By lemma 14, if γ is a non-augmentable path in D, then $V(\gamma) \cap \mathcal{I}_0 \neq \varnothing$. Hence any maximal independent set $\mathcal{I} \subset V(D)$ containing $\mathcal{I}_0$ intersects every non-augmentable path in D. $\square$

Remark 16 *Clearly, $\mathcal{O}(D)$ is an independent set. If every non-augmentable path in D ends in a vertex in $\mathcal{O}(D)$, then any maximal independent set $\mathcal{I} \subset V(D)$ containing $\mathcal{O}(D)$ intersects every non-augmentable path in D. On the other hand, if no non-augmentable path in D ends in a vertex in $\mathcal{O}(D)$, then the set of non-augmentable paths in D_0 corresponds to the set of non-augmentable paths in D, and hence any maximal independent set $\mathcal{I} \subset V(D)$ containing a maximal independent set $\mathcal{I}_0$ intersecting every non-augmentable path in D_0 intersects every non-augmentable path in D.*

Proposition 17 *The following statements are true.*

(1) In any arc local tournament digraph, every maximal independent set intersects every non-augmentable path of even length.
(2) For every odd number n, there exists an arc local tournament digraph in which there exist a maximal independent set and a non-augmentable path of length n which do not intersect.

PROOF. Let D be an arc local tournament digraph. Let $\gamma = (x_0, \ldots, x_n)$ be a non-augmentable path in D and let $\mathcal{I}$ be a maximal independent set. Suppose that $V(\gamma) \cap \mathcal{I} = \varnothing$. There exists $y \in \mathcal{I}$ such that $\overline{yx_0}$ because $\mathcal{I}$ is a maximal independent set. If $\overrightarrow{yx_0}$, then $(y, x_0, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable, so $\overleftarrow{yx_0}$. There exists $z \in \mathcal{I}$ such that $\overline{zx_n}$ because $\mathcal{I}$ is a maximal independent set. If $\overleftarrow{zx_n}$, then $(x_0, \ldots, x_n, z)$ is a path in D

contradicting that γ is non-augmentable, so $\overrightarrow{zx_n}$. Suppose that $n = 1$. If $y = z$, then (x_0, y, x_1) is a path in D contradicting that γ is non-augmentable. If $y \neq z$, then we obtain an arc local tournament digraph in which there is a maximal independent set $\mathcal{I} = \{y, z\}$ and a path $\gamma = (x_0, x_1)$ of length 1 which do not intersect, so (2) is true for $n = 1$ (see figure 2).

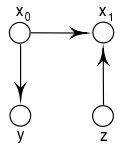

Fig. 2. A digraph satisfying 2 in proposition 17 for $n = 1$.

Suppose that $n \geq 2$. Then $\overline{yx_2}$ because $\overleftarrow{y}\,\overline{x_0x_1}\overrightarrow{x_2}$. In fact, $\overline{yx_k}$ as long as $\overleftarrow{yx_{k-2}}$ with $k \geq 2$ and $k \leq n$ since $\overleftarrow{y}\,\overline{x_{k-2}x_{k-1}}\overrightarrow{x_k}$. Suppose that $\overleftarrow{yx_{k-2}}$ but $\overrightarrow{yx_k}$. There exists $w \in \mathcal{I}$ such that $\overline{wx_{k-1}}$ because $\mathcal{I}$ is an maximal independent set. Suppose that $w = y$. If $\overrightarrow{yx_{k-1}}$, then $(x_0, \ldots, x_{k-2}, y, x_{k-1}, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable, otherwise if $\overleftarrow{yx_{k-1}}$, then $(x_0, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Suppose that $w \neq y$. If $\overrightarrow{wx_{k-1}}$, then $\overline{yw}$ because $\overrightarrow{w}\,\overline{x_{k-1}x_k}\overleftarrow{y}$, contradicting that $\mathcal{I}$ is an independent set, and if $\overleftarrow{wx_{k-1}}$, then $\overline{yw}$ because $\overleftarrow{y}\,\overline{x_{k-2}x_{k-1}}\overrightarrow{w}$, contradicting that $\mathcal{I}$ is an independent set. For every $k \leq n$ even, $\overleftarrow{yx_k}$ because $\overleftarrow{yx_0}$. If n is even, then $\overleftarrow{yx_n}$ and hence $(x_0, \ldots, x_n, y)$ is a path in D contradicting that γ is non-augmentable. So (1) follows. Figures 3 and 4 describe the arc local tournament digraphs D_3 and D_5 which satisfy (2) for $n = 3$ and $n = 5$. In general, in figure 5 we describe the digraph D_n for n odd, with $\mathcal{I} = \{y, z\}$ as the maximal independent set and $\gamma = (x_0, \ldots, x_n)$ as the non-augmentable path. Such digraph is defined as follows. For n odd, we have $V(D_n) = \{y, z\} \cup \{x_0, \ldots, x_n\}$ with arc set $A(D_n)$ defined by the following rules: (1) $\overrightarrow{x_iy}$ for every i even, (2) $\overleftarrow{x_iz}$ for every i odd and (3) $\overrightarrow{x_ix_{i+k}}$ for every odd number $k < n$ and every $i \in \{0, \ldots, n-k\}$. $\square$

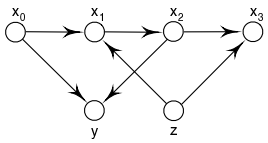

Fig. 3. A digraph satisfying 2 in proposition 17 for $n = 3$.

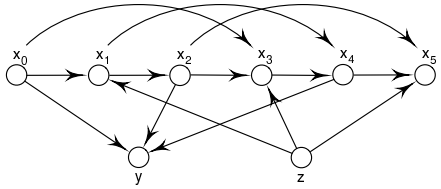

Fig. 4. A digraph satisfying 2 in proposition 17 for $n = 5$.

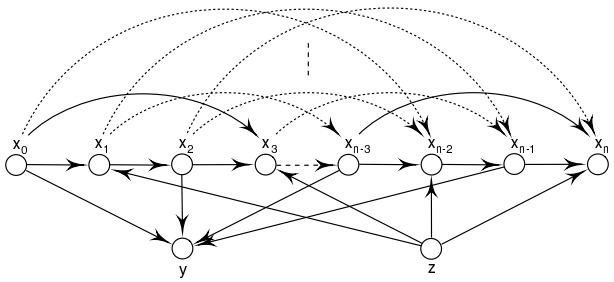

Fig. 5. A digraph satisfying 2 in proposition 17 for n odd.

Definition 18 *Let D be a digraph. Let the* line digraph *of D be the digraph $\mathcal{L}(D)$ with vertex set $V(\mathcal{L}(D)) = A(D)$ and arc set $A(\mathcal{L}(D))$ defined by the following rule. If $x, y, z \in V(D)$ are such that $\overrightarrow{xy}$ and $\overrightarrow{yz}$, then $\overrightarrow{(x,y)(y,z)}$.*

Line digraphs are similar to arc local tournament digraphs in the sense that line digraphs satisfy the following hypothesis.

Hypothesis 19 *For a digraph D, whenever $u, v, x, y \in V(D)$ are such that $\overrightarrow{xu}$, $\overleftarrow{uv}$ and $\overrightarrow{vy}$, then $\overline{xy}$.*

For $u, v, w, x, y \in V(D)$, if $\overrightarrow{(x,u)(u,v)}$, $\overleftarrow{(u,v)(w,u)}$ and $\overrightarrow{(w,u)(u,y)}$, then an arrangement described in figure 6 occurs and therefore $\overrightarrow{(x,u)(u,y)}$.

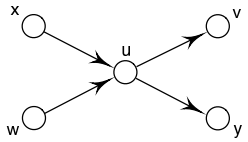

Fig. 6. Configuration described in hypothesis 19 for line digraphs.

Theorem 20 *In any digraph satisfying hypothesis 19, every maximal independent set intersects every non-augmentable path.*

PROOF. Let D be a digraph satisfying hypothesis 19. Suppose that $\mathcal{I}$ is a maximal independent set and $\gamma = (x_0, \ldots, x_n)$ is a non-augmentable path in D such that $V(\gamma) \cap \mathcal{I} = \varnothing$. There exists $z \in \mathcal{I}$ such that $\overline{zx_0}$ because $\mathcal{I}$ is a maximal independent set. If $\overrightarrow{zx_0}$, then $(z, x_0, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Suppose that $\overleftarrow{zx_0}$. Let $k_0 = \max\{k \mid \overleftarrow{zx_k}\}$. If $k_0 = n$, then $(x_0, \ldots, x_n, z)$ is a path in D contradicting that γ is non-augmentable. Suppose that $k_0 < n$. There exists $y \in \mathcal{I}$ such that $\overline{yx_{k_0+1}}$ because $\mathcal{I}$ is a maximal independent set. If $y = z$, then $\overrightarrow{zx_{k_0+1}}$ because k is maximal, and hence $(x_0, \ldots, x_{k_0}, z, x_{k_0+1}, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable. Suppose that $y \neq z$. If $\overrightarrow{yx_{k_0+1}}$, then $\overline{yz}$ because $\overleftarrow{x_{k_0+1}x_{k_0}}$ and $\overrightarrow{x_{k_0}z}$, contradicting that $\mathcal{I}$ is an independent set. Suppose that $\overleftarrow{yx_{k_0+1}}$ and repeat the argument as many times as necessary until we find an element $w \in \mathcal{I}$ such that $\overleftarrow{wx_{n-1}}$. Now, since $\mathcal{I}$ is a maximal independent set,

there exists $u \in \mathcal{I}$, such that $\overline{ux_n}$. If $\overleftarrow{ux_n}$, then $(x_0, \ldots, x_n, u)$ is a path in D contradicting that γ is non-augmentable. Suppose that $\overrightarrow{ux_n}$. If $u = w$, then $(x_0, \ldots, x_{n-1}, w, x_n)$ is a path in D contradicting that γ is non-augmentable. If $u \neq w$, then $\overline{uw}$ because $\overrightarrow{ux_n}$, $\overleftarrow{x_n x_{n-1}}$ and $\overrightarrow{x_{n-1}w}$, contradicting that $\mathcal{I}$ is an independent set. $\square$

Corollary 21 *Let D be a digraph. Then every maximal independent set $\mathcal{I} \subset V(\mathcal{L}(D))$ intersects every non-augmentable path in $\mathcal{L}(D)$.*

3 Quasi-transitive digraphs

Definition 22 *A digraph D is* transitive *if whenever $u, v, w \in V(D)$ are such that $\overrightarrow{uv}$ and $\overrightarrow{vw}$, then $\overrightarrow{uw}$. The digraph D is* quasi-transitive *if whenever $u, v, w \in V(D)$ are such that $\overrightarrow{uv}$ and $\overrightarrow{vw}$, then $\overline{uw}$.*

Definition 23 *Let D be a digraph. If for every $u, v \in V(D)$ there exists a path that starts in u and ends in v and a path that starts in v and ends in u, then D is* strong*, otherwise it is* non-strong*. The digraph D is* oriented *if it contains no cycles of length two, that is, if there exist no $u, v \in V(D)$ such that $\overrightarrow{uv}$ and $\overleftarrow{uv}$. The digraph D is* semicomplete *if $\overline{uv}$ for every $u, v \in V(D)$.*

Lemma 24 *If D is a digraph and γ is a non-augmentable path in D, then there exists no $z \in V(D) - V(\gamma)$ such that $\overline{zx}$ for every $x \in V(\gamma)$.*

PROOF. Suppose that $\gamma = (x_0, \ldots, x_n)$ and $z \in V(D) - V(\gamma)$ are such that $\overline{zx}$ for every $x \in V(\gamma)$. If $\overrightarrow{zx_0}$, then $(z, x_0, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable, so $\overleftarrow{zx_0}$. If $\overrightarrow{zx_1}$, then $(x_0, z, x_1, \ldots, x_n)$ is a path in D contradicting that γ is non-augmentable, so $\overleftarrow{zx_1}$. Continuing in this way, it follows that $\overleftarrow{zx_n}$, but then $(x_0, \ldots, x_n, z)$ is a path in D contradicting that γ is non-augmentable. $\square$

Proposition 25 *Let D be a semicomplete digraph. Then every maximal independent set consists of a single vertex and intersects every non-augmentable path.*

PROOF. Clearly, if $\mathcal{I} \subset V(D)$ is a maximal independent set, then $|\mathcal{I}| = 1$ because D is semicomplete, say $\mathcal{I} = \{z\}$ with $z \in V(D)$ arbitrary. Since D is semicomplete, $\overline{zx}$ for every $x \in V(D) - \{z\}$. By lemma 24, $\mathcal{I}$ intersects every non-augmentable path in D. $\square$

Definition 26 *Let D be a digraph and let $\{\alpha_u\}_{u\in V(D)}$ be a family of digraphs indexed by $u \in V(D)$. The* sum *of D and $\{\alpha_u\}_{u\in V(D)}$ is the digraph $\sigma(D, \alpha_u)$ with vertex set $\bigcup_{u\in V(D)}\{u\}\times V(\alpha_u)$, and for every $(u, x), (v, y) \in V(\sigma(D, \alpha_u))$, $\overrightarrow{(u, x)(v, y)}$ if $u = v$ and $\overrightarrow{xy}$, or if $u \neq v$ and $\overrightarrow{uv}$.*

Theorem 27 (Bang-Jensen and Huang [8]) *Let Q be a quasi-transitive digraph. There exist a digraph D and a family of digraphs $\{\alpha_u\}_{u\in V(D)}$ such that $Q = \sigma(D, \alpha_u)$ and satisfying the following.*

(1) If Q is non-strong, then D is transitive oriented and α_u is strong quasi-transitive for all $u \in V(D)$.
(2) If Q is strong, then D is strong semicomplete and α_u is non-strong quasi-transitive for all $u \in V(D)$.

For a quasi-transitive digraph Q, we will always write $Q = \sigma(D, \alpha_u)$ where D and $\{\alpha_u\}_{u\in V(D)}$ are as in theorem 27.

Proposition 28 *Let $H = \sigma(D, \alpha_u)$. If D is transitive oriented and if for every $u \in V(D)$, there exists a maximal independent set $\mathcal{I}_u \subset V(\alpha_u)$ that intersects every non-augmentable path in α_u, then there exists a maximal independent set that intersects every non-augmentable path in H.*

PROOF. The digraph D has a kernel because it is transitive. Therefore, there exists a maximal independent set $\mathcal{I} \subset V(D)$ that intersects every non-augmentable path in D. Clearly, $\mathcal{J} = \bigcup_{u\in\mathcal{I}}\{u\} \times \mathcal{I}_u$ is a maximal independent set. Let $\gamma = ((u_0, x_0^{(0)}), \ldots, (u_0, x_{n(0)}^{(0)}), \ldots, (u_n, x_0^{(n)}), \ldots, (u_n, x_{n(n)}^{(n)}))$ be a non-augmentable path in $\sigma(D, \alpha_u)$. Then $\overrightarrow{u_{i-1}u_i}$ for every $1 \leq i \leq n$, and hence $\overrightarrow{u_iu_j}$ for all $i < j$ because D is transitive. It follows that $u_i \neq u_j$ for all $i \neq j$ since D is oriented. Therefore $\gamma_D = (u_0, \ldots, u_n)$ is a non-augmentable path in D and $\gamma_i = (x_0^{(i)}, \ldots, x_{n(i)}^{(i)})$ is a non-augmentable path in α_{u_i}. Hence there exist $u = u_i \in V(\gamma_D) \cap \mathcal{I}$ and $x \in V(\gamma_i) \cap \mathcal{I}_{u_i}$ and thus $(u, x) \in V(\gamma) \cap \mathcal{J} \neq \varnothing$. □

Definition 29 *Let D be a digraph. A path $\gamma = (x_0, \ldots, x_n)$ in D is* internally non-augmentable with respect to $B \subset V(D) - V(\gamma)$ *if there exists no path $(y_0, \ldots, y_r)$ in D with $r \geq 0$ and $y_j \in B$ for every $0 \leq j \leq r$ and such that $\overrightarrow{x_{i-1}y_0}$ and $\overrightarrow{y_kx_i}$ for some $1 \leq i \leq n$.*

Theorem 30 *Let $Q = \sigma(D, \alpha_u)$ be a quasi-transitive digraph. There exists a maximal independent set $\mathcal{J} \subset V(Q)$ that intersects every non-augmentable path in Q. Moreover, if Q is strong and $\mathcal{I}_u \subset V(\alpha_u)$ is a maximal independent set intersecting every non-augmentable path in α_u, then $\mathcal{J} = \{u\} \times \mathcal{I}_u$ is a maximal independent set intersecting every non-augmentable path in Q.*

PROOF. We proceed by induction on the number of vertices. Clearly, the result is true if $|V(Q)| = 1, 2$. Suppose that the result is true for every quasi-transitive digraph with at most $m-1$ vertices. Suppose that $|V(Q)| = m$.

If Q is non-strong, then, by (1) in theorem 27, D is transitive oriented and α_u is strong quasi-transitive with $|V(\alpha_u)| < m$ for every $u \in V(D)$. By the induction hypothesis, there exists a maximal independent set $\mathcal{I}_u \subset V(\alpha_u)$ that intersects every non-augmentable path in α_u for every $u \in V(D)$. Hence, in this case, the result follows from proposition 28.

If Q is strong, then, by (2) in theorem 27, D is strong semicomplete and α_u is non-strong quasi-transitive with $|V(\alpha_u)| < m$ for every $u \in V(D)$. By proposition 25, a maximal independent set is of the form $\mathcal{J} = \{u\} \times \mathcal{I}_u$ for some $u \in V(D)$ and some maximal independent set $\mathcal{I}_u \subset V(\alpha_u)$. By the induction hypothesis, we can suppose that $\mathcal{I}_u$ intersects every non-augmentable path in α_u. Let γ be a non-augmentable path in Q. If $V(\gamma) \cap (\{v\} \times V(\alpha_v)) = \varnothing$ for some $v \in V(D) - \{u\}$, then $\overline{(v,y)(w,z)}$ for every $(w,z) \in V(\gamma)$ and $y \in V(\alpha_v)$, contradicting, by lemma 24, that γ is non-augmentable. Then $\{u \in V(D) \mid (u,x) \in V(\gamma) \text{ for some } x \in V(\alpha_u)\} = V(D)$. Moreover, let $v \in V(D) - \{u\}$ and $y \in V(\alpha_v)$. Clearly, $(v,y) \in V(\gamma)$ if $|V(\alpha_v)| = 1$. Suppose that $|V(\alpha_v)| > 1$. Let $Q' = Q \setminus \{(v,y)\}$, so that $|V(Q')| < |V(Q)|$. If $(v,y) \notin V(\gamma)$, then γ and $\mathcal{J}$ remain the same in Q', and therefore, by the induction hypothesis, $V(\gamma) \cap \mathcal{J} \neq \varnothing$. So $\{y \in V(\alpha_v) \mid (v,y) \in V(\gamma)\} = V(\alpha_v)$ when $V(\gamma) \cap \mathcal{J} = \varnothing$. We will use the following lemma.

Lemma 31 *Let $x, y \in V(\alpha_u) - \mathcal{I}_u$. If there exists a path ρ of length at least two, starting at $(u,x) \in V(Q)$, ending at $(u,y) \in V(Q)$, with $\{z \in V(\alpha_u) \mid (u,z) \in V(\rho) \text{ for some } z \in V(\alpha_u)\} = \{x,y\}$, and internally non-augmentable with respect to $\{u\} \times \mathcal{I}_u$, then α_u is semicomplete.*

PROOF. [Proof of lemma 31] We proceed by induction on the length of ρ. First, let $\rho = ((u,x),(v_1,z_1),(u,y))$ be a path of length two in Q, with $v_1 \in V(D) - \{u\}$ and $z_1 \in V(\alpha_{v_1})$. For every $x', y' \in \alpha_u$, the definition of the sum implies that if $\overrightarrow{x'z_1}$ and $\overrightarrow{z_1y'}$, then $\overrightarrow{x'y'}$ because Q is quasi-transitive. Suppose that the result is true for every path of length $k \geq 2$ satisfying the hypothesis of the lemma. Let $\mathbf{x} = (u,x)$, $\mathbf{y} = (u,y)$ and $\mathbf{z}_i = (v_i, z_i)$ with $v_i \in V(D) - \{u\}$ and $z_i \in V(\alpha_{v_i})$ for every $1 \leq i \leq k$ so that $\rho = (\mathbf{x}, \mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{y})$ is a path of length $k+1$ satisfying the hypothesis of the lemma. Since $\overrightarrow{\mathbf{z}_{k-1}\mathbf{z}_k}$ and $\overrightarrow{\mathbf{z}_k\mathbf{y}}$, $\overline{\mathbf{z}_{k-1}\mathbf{y}}$ because Q is quasi-transitive. If $\overrightarrow{\mathbf{z}_{k-1}\mathbf{y}}$, then $\rho' = (\mathbf{x}, \mathbf{z}_1, \ldots, \mathbf{z}_{k-1}, \mathbf{y})$ is a path that starts at $\mathbf{x} = (u,x)$, ends at $\mathbf{y} = (u,y)$, with $\{z \in V(\alpha_u) \mid (u,z) \in V(\rho') \text{ for some } z \in V(\alpha_u)\} = \{x,y\}$. Since $\mathcal{I}_u$ is independent, a path in D with its vertices in $\{u\} \times \mathcal{I}_u$ is necessarily a zero-length path. Suppose that $\mathbf{y}_0 \in \{u\} \times \mathcal{I}_u$ is such that $\overrightarrow{\mathbf{z}_{i-1}\mathbf{y}_0}$ and $\overrightarrow{\mathbf{y}_0\mathbf{z_i}}$ for some $1 < i < k$, or $\overrightarrow{\mathbf{x}\mathbf{y}_0}$ and $\overrightarrow{\mathbf{y}_0\mathbf{z_1}}$, or $\overrightarrow{\mathbf{z}_{k-1}\mathbf{y}_0}$ and $\overrightarrow{\mathbf{y}_0\mathbf{y}}$. Clearly, having the first or the

second of these cases holding contradicts that ρ is internally non-augmentable with respect to $\{u\} \times \mathcal{I}_u$. Suppose that $\overrightarrow{\mathbf{z}_{k-1}\mathbf{y}_0}$ and $\overrightarrow{\mathbf{y}_0\mathbf{y}}$. Since $\overrightarrow{\mathbf{z}_k\mathbf{y}}$ and $v_k \neq u$, the definition of the sum implies that $\overrightarrow{\mathbf{z}_k\mathbf{y}_0}$, contradicting, together with $\overrightarrow{\mathbf{y}_0\mathbf{y}}$, that ρ is internally non-augmentable with respect to $\{u\} \times \mathcal{I}_u$. Therefore ρ' is a path of length $k \geq 2$ internally non-augmentable with respect to $\{u\} \times \mathcal{I}_u$ so that the induction hypothesis implies that α_u is semicomplete. Suppose that $\overleftarrow{\mathbf{z}_{k-1}\mathbf{y}}$. Suppose that $\overrightarrow{\mathbf{z}_j\mathbf{y}}$ for some $j < k-1$ and let $j_0 = \max\{j \mid \overrightarrow{\mathbf{z}_j\mathbf{y}}\}$ so that $\overleftarrow{\mathbf{z}_{j_0+1}\mathbf{y}}$. The definition of the sum implies that for every $w \in \mathcal{I}_u$, $\overrightarrow{\mathbf{z}_{j_0}\mathbf{w}}$ and $\overleftarrow{\mathbf{z}_{j_0+1}\mathbf{w}}$, where $\mathbf{w} = (u, w)$, contradicting that ρ is internally non-augmentable with respect to $\{u\} \times \mathcal{I}_u$. Therefore $\overleftarrow{\mathbf{z}_j\mathbf{y}}$ for every $j < k$. In particular, $\overleftarrow{\mathbf{z}_2\mathbf{y}}$ and hence the definition of the sum implies that $\overleftarrow{\mathbf{z}_2\mathbf{x}}$. Then $\rho'' = (\mathbf{x}, \mathbf{z}_2, \ldots, \mathbf{z}_k, \mathbf{y})$ is a path of length k that starts at $\mathbf{x} = (u, x)$, ends at $\mathbf{y} = (u, y)$, with $\{z \in V(\alpha_u) \mid (u, z) \in V(\rho') \text{ for some } z \in V(\alpha_u)\} = \{x, y\}$ and internally non-augmentable with respect to $\{u\} \times \mathcal{I}_u$, as shown by an argument similar to the one above, so that the induction hypothesis implies that α_u is semicomplete. $\square$

Let $\gamma = ((u_0, x_0^{(0)}), \ldots, (u_0, x_{n(0)}^{(0)}), \ldots, (u_n, x_0^{(n)}), \ldots, (u_n, x_{n(n)}^{(n)}))$. Suppose that $u_i = u_j = u$ for some $0 \leq i < j+1 \leq n$ and $u_k \neq u$ for every $i < k < j$. The path obtained from γ that starts in $\mathbf{x} = (u_i, x_{n(i)}^{(i)}) \in V(\alpha_u)$ and ends in $\mathbf{y} = (u_j, x_0^{(j)}) \in V(\alpha_u)$ satisfies the hypothesis of lemma 31, implying that α_u is semicomplete. Therefore, by proposition 25, $\mathcal{I}_u = \{z\}$ for some $z \in V(\alpha_u)$. Since $\mathcal{I}_u$ is a maximal independent set in α_u and D is semicomplete, $\overleftarrow{(u, z)(u_i, x_j^{(i)})}$ for every $0 \leq i \leq n$ and $0 \leq j \leq n(i)$, contradicting, by lemma 24, that γ is non-augmentable. It follows that there exists a unique $i \in \{0, \ldots, n\}$ such that $u_i = u$, and hence $(x_0^{(i)}, \ldots, x_{n(i)}^{(i)})$ is a non-augmentable path in α_u because otherwise γ would be augmentable. Therefore there exists $0 \leq j \leq n(i)$ such that $x_j^{(i)} \in \mathcal{I}_0$ and hence $(u_i, x_j^{(i)}) \in \mathcal{J}$, i.e., $\gamma \cap \mathcal{J} \neq \varnothing$ $\square$

4 Strongly internally and finally non-augmentable pahts

Definition 32 *A path $\gamma = (x_0, \ldots, x_n)$ in a digraph D is* strongly internally non-augmentable *if for every $0 \leq i < j \leq n$, there exists no path ρ of length at least two, starting at x_i, ending at x_j and with $V(\gamma) \cap V(\rho) = \{x_i, x_j\}$. We say that γ is* finally non-augmentable *if there exists no $y \in V(D) - V(\gamma)$ such that $\overrightarrow{x_n y}$.*

Lemma 33 *Let D be a strong digraph and let $\gamma = (x_0, \ldots, x_n)$ be a strongly internally and finally non-augmentable path in D. If $z \in V(D)$ is such that $\overrightarrow{z x_n}$, then $z \in V(\gamma)$.*

PROOF. Suppose that $z \notin V(\gamma)$. There exists a path $\rho = (y_0 = x_0, \ldots, y_m = z)$ starting at x_0 and ending at z because D is strong. Let $i_0 = \max\{i \mid x_i \in V(\rho)\}$ so that $y_{j_0} = x_{i_0}$ for some $j_0 < m$. If $i_0 < n$, then $\lambda = (y_{j_0}, \ldots, y_m, x_n)$ is a path in D of length at least two, starting at x_{i_0}, ending at x_n and with $V(\gamma) \cap V(\lambda) = \{x_{i_0}, x_n\}$, contradicting that γ is strongly internally non-augmentable. If $i_0 = n$, then $\overrightarrow{x_n y_{j_0+1}}$, contradicting that γ is finally non-augmentable because $y_{j_0+1} \notin V(\gamma)$. $\square$

Theorem 34 *Let D be a strong digraph. Then every maximal independent set intersects every strongly internally and finally non-augmentable path.*

PROOF. Let $\mathcal{I}$ be a maximal independent set. Suppose that $\gamma = (x_0, \ldots, x_n)$ is a strongly internally and finally non-augmentable path such that $V(\gamma) \cap \mathcal{I} = \varnothing$. There exists $z \in \mathcal{I}$ such that $\overline{zx_n}$ because $\mathcal{I}$ is a maximal independent set. Since γ is finally non-augmentable, $\overrightarrow{zx_n}$, and hence, by lemma 33, $z \in V(\gamma)$, contradicting that $V(\gamma) \cap \mathcal{I} = \varnothing$. $\square$

Theorem 35 *If D is a digraph, then there exists a maximal independent set that intersects every strongly internally and finally non-augmentable path.*

PROOF. We proceed by induction on the number of vertices. Clearly, the theorem is true if $|V(D)| = 1, 2$. Let $m > 2$ and suppose that the theorem is true for every digraph with $k < m$ vertices. Suppose that $|V(D)| = m$. If D is strong, the result follows from theorem 34. Suppose that D is not strong. Consider the acyclic condensation digraph D^* that has a vertex for every maximal strong component of D, and for two vertices $u, v \in V(D^*)$, an arc from u to v if there exists an arc from a vertex in the corresponding component of u to a vertex in the corresponding component of v. Since D^* is acyclic, there exists $u_0 \in V(D^*)$ with $in(u_0) = 0$. Let $D' = D \setminus V(C_0)$, where C_0 is the component corresponding to u_0. Since D is not strong, D' is not the empty digraph and hence $|V(D')| < m$. Therefore there exists an independent set $\mathcal{I}' \subset V(D')$ that intersects every strongly internally and finally non-augmentable path in D'. Let $\mathcal{I} \subset V(D)$ be a maximal independent set in D containing $\mathcal{I}'$, say $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}_0$ for some $\mathcal{I}_0 \subset V(C_0)$. Let $\gamma = (x_0, \ldots, x_n)$ be a strongly internally and finally non-augmentable path in D. If $V(\gamma) \cap V(C_0) = \varnothing$, then γ is a strongly internally and finally non-augmentable path in D', and therefore $V(\gamma) \cap \mathcal{I} \neq \varnothing$. Suppose that $V(\gamma) \cap \mathcal{I} = \varnothing$. Then $V(\gamma) \cap V(C_0) \neq \varnothing$ and hence $x_0 \in V(C_0)$ because $in(u_0) = 0$. Actually, $V(\gamma) \subset V(C_0)$ because otherwise, if $i_0 = \min\{i \mid x_i \notin V(C_0)\}$, then $\gamma' = (x_{i_0}, \ldots, x_n)$ is a strongly internally and finally non-augmentable path in D', and therefore $V(\gamma') \cap \mathcal{I}' \neq \varnothing$, contradicting that $V(\gamma) \cap \mathcal{I} = \varnothing$. There exists $z \in \mathcal{I}$ such that $\overline{x_n z}$ because $\mathcal{I}$ is a maximal independent set. Since γ is finally non-augmentable, $\overrightarrow{zx_n}$ and therefore $z \in V(C_0)$, that is, $z \in \mathcal{I}_0$. We have a

strong digraph C_0, a strongly internally and finally non-augmentable path γ in C_0 and $z \in V(C_0)$ such that $\overrightarrow{zx_n}$. By lemma 33, $z \in V(\gamma)$, contradicting that $V(\gamma) \cap \mathcal{I} = \varnothing$ because $z \in \mathcal{I}_0 \subset \mathcal{I}$. $\square$

5 Locally semicomplete and semicomplete k-partite digraphs

Definition 36 *Let D be a digraph. For every $u \in V(D)$, the* in-neighborhood *and* out-neighborhood *of u are the sets $\Gamma^-(u) = \{x \in V(D) \mid \overrightarrow{xu}\}$ and $\Gamma^+(u) = \{y \in V(D) \mid \overrightarrow{uy}\}$ respectively. Also, let $\delta^+(u) = |\Gamma^+(u)|$ and $\delta^-(v) = |\Gamma^-(v)|$.*

Definition 37 *A digraph D is* locally in-semicomplete *if for every $u \in V(D)$, the digraph induced by the in-neighborhood of u is semicomplete. A* locally out-semicomplete *digraph is defined similarly.*

Definition 38 *A digraph D is* semicomplete k-partite *if there exist disjoint independent sets $V_1, \ldots, V_k \subset V(D)$ with $V_1 \cup \ldots \cup V_k = V(D)$ and such that for every $i \neq j$, if $u \in V_i$ and $v \in V_j$, then $\overline{uv}$.*

The following results are elementary and we include them for completeness. The proofs are left to the reader as exercises.

Theorem 39 *Let D be a locally in-semicomplete digraph. Then every maximal independent set intersects every non-augmentable path in D.*

Theorem 40 *Let D be a locally out-semicomplete digraph. Then every maximal independent set intersects every non-augmentable path in D.*

Theorem 41 *Let D be a semicomplete k-partite digraph. Then every maximal independent set intersects every non-augmentable path in D.*

6 Degrees of vertices and the length of longest paths

We finish with a result relating the length of longest paths and the degrees on their initial and terminal vertices. First let us prove the following lemma.

Lemma 42 *Let D be a digraph. Let γ be a longest path in D and $I \subset V(D)$ a maximal independent set. If there exists a Hamiltonian cycle in $D(V(\gamma))$, then $V(\gamma) \cap I \neq \varnothing$.*

PROOF. Let $\gamma = (x_0, \ldots, x_n)$. Suppose that $V(\gamma) \cap I = \varnothing$. Since I is a maximal independent set, there exists $z \in \mathcal{I}$ such that $\overline{zx_0}$. If $\overrightarrow{zx_0}$, then $(z, x_0, \ldots, x_n)$ is a path in D, contradicting that γ is a longest path. Suppose that $\overrightarrow{x_0 z}$. Since there exists a Hamiltonian cycle in $D(V(\gamma))$, there exists a longest path path $\gamma' = (y_0, \ldots, y_n)$ with $y_0 = x_0$. Therefore $(y_0, \ldots, y_n, z)$ is a path in D, contradicting that γ' is a longest path. $\square$

Definition 43 *Let D be a digraph. We define $L^+(D) = \{x \in V(D) \mid$ there exists a longest path in D starting at $x\}$ and $L^-(D) = \{y \in V(D) \mid$ there exists a longest path in D ending at $y\}$.*

Let $\overleftrightarrow{K}_3$ be the complete digraph on three vertices, with $V\left(\overleftrightarrow{K}_3\right) = \{x_1, x_2, x_3\}$ and $\overrightarrow{x_i x_j}$ for every $i \neq j$.

Theorem 44 *Let D be a digraph and let $n \geq 1$ be the length of a longest path in D. Suppose that for every $u \in L^+(D)$ and $v \in L^-(D)$ we have*

$$\delta^-(u) \geq \frac{2}{3}(n+1) \text{ and } \delta^+(v) \geq \frac{2}{3}(n+1).$$

Then every maximal independent set intersects every longest path.

PROOF. Let $\gamma = (x_0, x_1, \ldots, x_n)$ be a longest path in D and let $\mathcal{I} \subset V(D)$ be a maximal independent set. We will show that supposing $V(\gamma) \cap \mathcal{I} = \varnothing$ implies the existence of a Hamiltonian cycle in $D(V(\gamma))$. This will constitute a contradiction in virtue of lemma 42.

Claim 1. $n \geq 2$.

Proof. Since $\frac{2}{3}(n+1) \geq \frac{4}{3} > 1$, $\delta^+(u) \geq 2$ for all $u \in V(D)$. If $u \in V(D)$, then there exists two other vertices $v, w \in V(D)$ with $\overrightarrow{uv}$ and $\overrightarrow{vw}$ so that (u, v, w) is a path in D, whence $n \geq 2$.

Claim 2. If D is not the disjoint union of copies of $\overleftrightarrow{K}_3$, then $n \geq 3$.

Proof. Let D_0 be a connected component of D, that is, a maximal induced subdigraph of D with a connected underlying graph (by the underlying graph we mean the graph that results by removing the arrows from the arcs). Let (u, v, w) be a path in D_0 of length two. If $z \in \Gamma^+(w)$ is such that $z \neq u$ and $z \neq v$, then (u, v, w, z) is a path in D_0 of length three. Suppose that $\Gamma^+(w) = \{u, v\}$. Then (v, w, u) is a path in D_0 of length two. If $z \in \Gamma^+(v) \setminus \{w\}$ is such that $z \neq u$, then (w, u, v, z) is a path in D_0 of length three. Suppose that $\Gamma^+(v) = \{u, w\}$. Then (v, w, u) is a path in D_0 of length two. If $z \in \Gamma^+(u) \setminus \{v\}$

is such that $z \neq w$, then (v, w, u, z) is a path in D_0 of length three. Suppose that $\Gamma^+(u) = \{v, w\}$. Then, clearly, D_0 is isomorphic to $\overleftrightarrow{K}_3$.

The theorem is true if the connected components of D are all isomorphic to $\overleftrightarrow{K}_3$. Henceforth we assume that D possesses connected components non-isomorphic to $\overleftrightarrow{K}_3$ so that $n \geq 3$.

Claim 3. There exists $0 \leq j < i \leq n$ such that $\overrightarrow{x_n x_j}$ and $\overrightarrow{x_i x_0}$.

Proof. By hypothesis, $\Gamma^-(x_0)$ and $\Gamma^+(x_n)$ are non-empty. Moreover, both sets are contained in $V(\gamma)$ because γ is a longest path. Let

$$i = \max\{k \mid \overrightarrow{x_k x_0}\} \tag{1}$$

and

$$j = \min\{k \mid \overrightarrow{x_n x_k}\}. \tag{2}$$

Then $i \geq \frac{2}{3}(n+1) > \frac{1}{3}(n-2) = n - \frac{2}{3}(n+1) \geq j$.

If $i = n$ or $j = 0$, then there exists a Hamiltonian cycle in $V(\gamma)$. Henceforth we assume that $0 < j < i < n$.

Claim 4. Let $j > 0$ be defined by (2). Then there exists $k \in \{j+1, \ldots, n\}$ such that $\overrightarrow{x_{j-1} x_k}$ and $\overrightarrow{x_{k-1} x_0}$.

Proof. Let $i < n$ be defined by (1). Then $\delta^+(x_{j-1}) \geq \frac{2}{3}(n+1)$ because $(x_{i+1}, \ldots, x_n, x_j, x_{j+1}, \ldots, x_i, x_0, x_1, \ldots, x_{j-1})$ is a longest path in D ending at x_{j-1}. Therefore $\Gamma^+(x_{j-1}) \subset V(\gamma)$ because otherwise there would exist a path of length $n+1$. Let $(\Gamma^-(x_0))^{+1}$ be the set of vertices in $V(\gamma)$ which belong to the outer-neighborhood of a vertex in the inner-neighborhood of x_0, that is,

$$(\Gamma^-(x_0))^{+1} = \left(\bigcup_{z \in \Gamma^-(x_0)} \Gamma^+(z) \right) \cap V(\gamma).$$

Since $\Gamma^-(x_0) \subset V(\gamma)$ and $j > 0$, $|(\Gamma^-(x_0))^{+1}| = |\Gamma^-(x_0)|$. Letting $\gamma' = (x_j, x_{j+1}, \ldots, x_n)$ we get

$$\left| \left(\Gamma^+(x_{j-1}) \cap V(\gamma') \right) \cup (\Gamma^-(x_0))^{+1} \right| \leq n+1$$

since $(\Gamma^+(x_{j-1}) \cap V(\gamma')) \cup (\Gamma^-(x_0))^{+1} \subset V(\gamma)$ and $|V(\gamma)| \leq n+1$. We will show that

$$(\Gamma^+(x_{j-1}) \cap V(\gamma')) \cap (\Gamma^-(x_0))^{+1} \neq \varnothing.$$

Suppose that $(\Gamma^+(x_{j-1}) \cap V(\gamma')) \cap (\Gamma^-(x_0))^{+1} = \varnothing$. Then

$$\begin{aligned}|(\Gamma^+(x_{j-1}) \cap V(\gamma')) \cup (\Gamma^-(x_0))^{+1}| &= |\Gamma^+(x_{j-1}) \cap V(\gamma')| + |(\Gamma^-(x_0))^{+1}| \\ &= |\Gamma^+(x_{j-1}) \cap V(\gamma')| + |\Gamma^-(x_0)| \\ &> \tfrac{2}{3}(n+1) - \tfrac{1}{3}(n+1) + \tfrac{2}{3}(n+1) \\ &= n+1,\end{aligned}$$

a contradiction. Therefore there exists $k > j$ such that

$$x_k \in \left(\Gamma^+(x_{j-1}) \cap V(\gamma')\right) \cap (\Gamma^-(x_0))^{+1}$$

and the claim is proved.

Then $(x_k, \ldots, x_n, x_j, \ldots, x_{k-1}, x_0, \ldots, x_{j-1}, x_k)$, with k as in the last claim, is a Hamiltonian cycle in $V(\gamma)$. $\square$

Acknowledgements

We thank the anonymous referees for carefully reeding the original manuscript and for their suggestions to improve the paper.

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