

ON THE LARGE SCALE GEOMETRY OF BIG MAPPING CLASS GROUPS OF SURFACES WITH A UNIQUE MAXIMAL END

RITA JIMÉNEZ ROLLAND AND ISRAEL MORALES

ABSTRACT. Building on the work of K. Mann and K. Rafi [MR23], we analyze the large scale geometry of big mapping class groups of surfaces with a unique maximal end. We obtain a complete characterization of those that are globally CB, which does not require the *tameness* condition. We prove that, for surfaces with a unique maximal end, any locally CB big mapping class group is CB generated, and we give an explicit criterion for determining which big mapping class groups are CB generated. Finally, we answer [MR23, Problem 6.12] by giving an example of a non-tame surface whose mapping class group is CB generated but is not globally CB.

1. INTRODUCTION

Let Σ be an infinite-type surface and $\text{Map}(\Sigma)$ be the group of all isotopy classes of orientation-preserving self-homeomorphisms of Σ . This group is called the *(big) mapping class group* of Σ . If we equip the homeomorphism group with the compact-open topology then $\text{Map}(\Sigma)$ is a Polish group with respect to the quotient topology. For an overview about big mapping class groups the reader may consult [AV20].

Recently, C. Rosendal [Ros21] used the notion of *coarsely bounded* sets in order to extend the framework of geometric group theory to the broader context of topological groups.

Definition 1. Let G be a topological group. A subset of G is *CB (coarsely bounded)* if it has finite diameter for every compatible left-invariant metric on G . The group G is *locally CB* if it admits a CB neighborhood of the identity. If G is generated by a CB subset we say that G is *CB generated*.

With this language, Rosendal extends Gromov's fundamental theorem of geometric group theory showing that if A_1 and A_2 are two CB generating sets of a Polish group G then the respective word metrics on G are quasi-isometric [Ros21, Proposition 2.72]. In other words, a CB generated Polish group has a well-defined quasi-isometric type. Moreover, it was proved that being locally CB is a necessary condition for a Polish group to be CB generated [Ros21, Theorem 1.2].

Based on Rosendal's framework, K. Mann and K. Rafi started in [MR23] the study of the large scale geometry of big mapping class groups. They obtained a classification of locally CB big mapping class groups [MR23, Theorem 1.4] and, under the assumption of *tameness*, the classification of CB generated big mapping class groups [MR23, Theorem 1.6].

Studying the mapping class group of an infinite-type surface Σ can be complicated since it involves understanding the homeomorphism group of the space of ends of Σ , which is denoted

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by $E(\Sigma)$. In order to address this, Mann and Rafi introduced a preorder on the space of ends of a surface and showed that the induced partial order always has maximal elements. We say that an open subset U of an infinite-type surface Σ is a *neighborhood* of an end $x \in E(\Sigma)$ if its end space contains x , that is, $x \in E(U)$. A surface Σ has a *unique maximal end* if there is only one end x of Σ such that any neighborhood of x has a copy of some neighborhood of any other end y (see Section 2 for a precise definition).

In this paper we focus our study on the large scale geometry of big mapping class groups of surfaces with a unique maximal end. These surfaces have mapping class groups that have exhibited a different behavior from the rest of big mapping class groups. For instance, $\text{Map}(\Sigma)$ has a dense conjugacy class if and only if every finite-type subsurface of Σ is displaceable and has a unique maximal end, see [HHHM⁺, LV22]. In addition, some examples of big mapping class groups that do not satisfy the *automatic continuity* property are in this class [MT23, Man23].

We start our study by giving an alternative characterization of locally CB big mapping class groups for surfaces with a unique maximal end.

Theorem 1.1. *Let Σ be an infinite-type surface with a unique maximal end x . Then $\text{Map}(\Sigma)$ is locally CB if and only if there is a connected finite-type subsurface K of Σ with the following properties:*

- (1) $\Sigma \setminus K = \Sigma_0 \sqcup \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$, where the closure of each Σ_i in Σ is of infinite-type with one boundary component, $g(\Sigma_i) \in \{0, \infty\}$ and $x \in E(\Sigma_0)$.
- (2) For any subsurface $U \subseteq \Sigma_0$ that is a neighborhood of x there is $f_U \in \text{Homeo}^+(\Sigma)$ such that either $f_U(\Sigma_0) \subseteq U$ or $f_U(\Sigma \setminus U) \subseteq U$.

Moreover, if $\text{Map}(\Sigma)$ is locally CB then $\mathcal{V}_K := \{f \in \text{Map}(\Sigma) : f|_K = \text{Id}_K\}$ is a CB neighborhood of the identity.

Remark 1. Notice that our condition (1) in Theorem 1.1 is the same as item (1) in [MR23, Theorem 1.4]. The differences between the statements lie in item (2). Briefly, for surfaces with a unique maximal end, item (2) in [MR23, Theorem 1.4] consists of the following: *i)* $E(\Sigma_0)$ is self-similar, *ii)* each complementary component of K different from Σ_0 is mapped inside Σ_0 by a homeomorphism and, *iii)* if K is not empty, for each neighborhood $U \subseteq \Sigma_0$ of the unique maximal end there is a homeomorphism f such that $f(\Sigma_0) \subset U$. In our result, we do not require conditions *i)* and *ii)*. Instead, we consider condition *iii)* and include the possibility that for some neighborhoods $U \subseteq \Sigma_0$ of the unique maximal end there exists a homeomorphism f_U such that $f_U(\Sigma \setminus U) \subseteq U$. This last consideration does not appear in the statement of item (2) of [MR23, Theorem 1.4]. We prove that this possibility can only appear if $\text{Map}(\Sigma)$ is globally CB and the subsurface K of Σ can be taken to be empty; see Lemma 3.3.

We use Theorem 1.1 to give an alternative proof of the self-similarity (see Definition 6) of the space of ends of an infinite-type surface with a unique maximal end whose mapping class group is locally CB; see also [MR23, Proposition 5.4].

Theorem 1.2. *Let Σ be an infinite-type surface with a unique maximal end and suppose $\text{Map}(\Sigma)$ is locally CB. Then, the space of ends $E(\Sigma)$ is self-similar.*

The following notions were introduced in [MR23] to have some control on the topology of the space of ends $E(\Sigma)$.

Definition 2. A neighborhood U of an end x is *stable* [MR23, Definition 4.14] if for every neighborhood $U' \subseteq U$ of x there is a homeomorphic copy of U inside U' .

Definition 3. We say that a surface Σ is *tame* if every end of Σ which is either of maximal type or any immediate predecessor of an end of maximal type has a stable neighborhood.

Remark 2. This notion was originally introduced in this form by Mann and Rafi, and it has been used in the literature; see for instance [HQR22, Section 5], [FGM21, Section 5.1], [SC22, Definition 2.15]. However, it differs from how it is stated in [MR23, Definition 6.11]. In Section 7 we show that these two definitions are equivalent, when the surface Σ has a unique maximal end and $\text{Map}(\Sigma)$ is locally CB but not globally CB.

In [MR23], Mann and Rafi give a characterization of globally CB big mapping class groups under the hypothesis of tameness [MR23, Theorem 1.7]. We prove that in the case of surfaces with a unique maximal end, the tameness hypothesis is not needed.

Theorem 1.3. *Let Σ be an infinite-type surface with a unique maximal end and suppose that $\text{Map}(\Sigma)$ is locally CB. Then $\text{Map}(\Sigma)$ is globally CB if and only if the genus of Σ is zero or infinite.*

Thanks to Theorem 1.3 we have a better understanding of those surfaces with locally but not globally CB mapping class group. In Section 5 below we prove the following result; see also Proposition 7.3 for a more refined statement about the space of ends.

Corollary 1.4. *Let Σ be an infinite-type surface with a unique maximal end and suppose that $\text{Map}(\Sigma)$ is locally CB but not globally CB. Then Σ has finite nonzero genus and the space of ends of Σ is uncountable.*

Under the assumption of tameness, [MR23, Theorem 1.6] states that a locally CB big mapping class group is CB generated if and only if the space of ends of the surface is not of *limit type* and is of *finite rank*, see [MR23, Definitions 6.2 & 6.5]. For surfaces Σ with a unique maximal end, it can be shown that $\text{Map}(\Sigma)$ is always of finite rank and not of limit type. We see that the tameness hypothesis is actually not needed in order to be CB generated.

Theorem 1.5. *Let Σ be an infinite-type surface with a unique maximal end and suppose that $\text{Map}(\Sigma)$ is locally CB. Then $\text{Map}(\Sigma)$ is CB generated.*

Recall that a necessary condition for a group to be CB generated is that the group is locally CB. Our Theorem 1.5 shows that it is also a sufficient condition for big mapping class groups of surfaces with a unique maximal end. Recently, T. Hill [Hil23] observed the same phenomena for pure mapping class groups.

Additionally, it follows that our Theorem 1.1 and [MR23, Theorem 1.4] give explicit criteria for determining which big mapping class groups are CB generated.

Remark 3. It follows from Theorem 1.5 and the work of Horbez, Qing and Rafi [HQR22, Theorem 1 and Corollary 2] that for CB generated big mapping class groups of surfaces Σ with a unique maximal end, either

- (1) $\text{Map}(\Sigma)$ is globally CB and therefore the quasi-isometric type of $\text{Map}(\Sigma)$ is trivial or,

- (2) $\text{Map}(\Sigma)$ admits a continuous and non-elementary action by isometries on a hyperbolic space; in this situation, the space of non-trivial quasi-morphisms of $\text{Map}(\Sigma)$ has infinite dimension.

Answering a question of Mann and Rafi. Thanks to our Theorem 1.5 we get a positive answer to [MR23, Problem 6.12].

Theorem 1.6. *There exists a non-tame infinite-type surface whose mapping class group is CB generated but it is not globally CB.*

Remark 4. In [MR23, Example 6.13] Mann and Rafi constructed a non-tame surface with a unique maximal end whose mapping class group is globally CB. We note that in this example the set of immediate predecessors of the unique maximal end is countable. We can modify the Mann-Rafi example by adding finite nonzero genus to obtain a non-tame surface with a unique maximal end. However, by Proposition 7.3 below, this new surface is not locally CB because the unique maximal end has countably many immediate predecessors. Our example (Section 7) satisfies that the equivalence class of each immediate predecessor of the unique maximal end is uncountable, the unique maximal end has stable neighborhoods, and some of the immediate predecessors of the unique maximal end do not have stable neighborhoods.

Outline. In Section 2 we present the preliminaries, Section 3 is devoted to the proof of Theorem 1.1, Section 4 to the proof of Theorem 1.2 and Theorem 1.3, Section 5 to the proof of Corollary 1.4. Finally, in Section 6 we give the proof of Theorem 1.5, and in Section 7 we prove Theorem 1.6.

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2. PRELIMINARIES

Topological surfaces. All our surfaces are assumed to be connected, orientable and possibly with non-empty boundary. The boundary of a surface Σ is denoted by $\partial\Sigma$ and always supposed to be compact. A surface is of *finite type* if its fundamental group is finitely generated. Otherwise, we say that it is of *infinite type*. Unless otherwise specified, infinite-type surfaces will be assumed to have empty boundary. Finite-type surfaces are classified, up to homeomorphisms, by their genus, number of punctures and number of boundary components. An infinite-type surface Σ with empty boundary is classified, up to homeomorphisms, by their genus (which can be infinite) and a pair of nested topological spaces $E_\infty(\Sigma) \subseteq E(\Sigma)$. The space $E(\Sigma)$ is called the *space of ends* of Σ and it is homeomorphic to a clopen subset of the Cantor space. The space $E_\infty(\Sigma)$ is a closed subspace of the space of ends and it encodes all ends of Σ which are accumulated by genus. Moreover, $\hat{\Sigma} := \Sigma \cup E(\Sigma)$ is compact and it is called the *Freudenthal compactification* of Σ . We refer the

reader to the work of Richards [Ric63] and the book of Ahlfors and Sario [AS15] for a detailed discussion about classification of surfaces.

Any homeomorphism $f : \Sigma \rightarrow \Sigma$ has a unique homeomorphism extension $\hat{f} : \hat{\Sigma} \rightarrow \hat{\Sigma}$. In particular, the restriction of \hat{f} to $E(\Sigma)$ induces a homeomorphism of the nested pair $(E(\Sigma), E_\infty(\Sigma))$ to itself. From [Ric63] we have that if the nested pair of spaces $(A, B) \subseteq (E(\Sigma), E_\infty(\Sigma))$ is homeomorphic to the nested pair $(A', B') \subseteq (E(\Sigma), E_\infty(\Sigma))$ then there is a homeomorphism $f : \Sigma \rightarrow \Sigma$ such that its extension \hat{f} sends the nested pair (A, B) into (A', B') . We assume that any homeomorphism between subsets of $E(\Sigma)$ is induced by a homeomorphism of the surface Σ . Abusing of notation, we will usually refer simply by f to the homeomorphism of the surface Σ or to its extension to $\hat{\Sigma}$.

A *simple closed curve* in Σ is an embedding of the circle into Σ . A simple closed curve is *essential* if it is not homotopic to a point, a puncture, or a boundary component. All curves we consider in this paper will be essential, so we refer to them simply as *curves*. We say that a curve α in Σ is *separating* if $\Sigma \setminus \alpha$ is disconnected.

By a subsurface of Σ we mean a subspace $S \subseteq \Sigma$ that is a surface itself (possibly with non-empty boundary). If we do not specify it in the paper, we assume that all boundary curves of a subsurface are separating curves in Σ . Furthermore, any subsurface of finite type is assumed to have non-empty boundary.

Big mapping class groups. The *mapping class group* of a surface (of finite or infinite type) Σ , denoted by $\text{Map}(\Sigma)$, is the group of all isotopy classes of orientation-preserving self-homeomorphisms of Σ ; if $\partial\Sigma \neq \emptyset$, then we require that all homeomorphisms and isotopies fix $\partial\Sigma$ pointwise. If we equip the homeomorphism group of Σ with the compact-open topology then $\text{Map}(\Sigma)$ is a Polish group with respect to the quotient topology. In recent literature, the mapping class groups of infinite-type surfaces are often called *big mapping class groups*. For the rest of the paper, Σ denotes an infinite-type surface. Moreover, any homeomorphism f of Σ to itself is assumed to be orientation-preserving.

Given a subsurface $S \subseteq \Sigma$ we denote by \mathcal{V}_S the subgroup of $\text{Map}(\Sigma)$ defined by all the homeomorphisms $f : \Sigma \rightarrow \Sigma$ such that $f|_S = \text{Id}_S$ up to isotopy. By the Alexander's method [FM12], if S is a finite-type subsurface of Σ then \mathcal{V}_S is an open subgroup. Moreover, the collection of open subgroups \mathcal{V}_S where S runs over all finite-type subsurfaces of Σ forms a base of neighborhoods of the identity. Therefore, $\text{Map}(\Sigma)$ is first countable and, in particular, $\text{Map}(\Sigma)$ is a non-Archimedean¹ group. Throughout the article, we often use the following fact: for any neighborhood V of the identity in $\text{Map}(\Sigma)$ there is a finite-type subsurface $S \subseteq \Sigma$ such that $\mathcal{V}_S \subseteq V$.

Large scale geometry of Polish groups. We use the following characterization of coarsely bounded sets (CB sets).

Theorem 2.1 (Proposition 2.7, [Ros21]). *Let G be a Polish group and A be a subset of G . The following are equivalent*

- (1) A is CB.
- (2) For every neighborhood V of the identity in G , there is a finite subset $F \subseteq G$ and some $k \geq 1$ such that $A \subseteq (FV)^k$.

¹A Polish group is *non-Archimedean* if the identity has a basis of open subgroups, see [BK96].

Partial order on the space of ends. We recall the partial order on the space of ends $E(\Sigma)$ of a surface Σ introduced by Mann and Rafi in [MR23].

Definition 4. Let $x, y \in E(\Sigma)$. We define the binary relation on $E(\Sigma)$ where $y \leq x$ if for any neighborhood U_x of x in $E(\Sigma)$ there is a neighborhood U_y of y and a homeomorphism f of the surface Σ such that $f(U_y) \subseteq U_x$.

We obtain an equivalence relation on $E(\Sigma)$ declaring that two ends $x, y \in E(\Sigma)$ are of the *same type* if $y \leq x$ and $x \leq y$. Equivalently, x and y are of the same type if and only if there exists a homeomorphism h of Σ such that $h(x) = y$; see [MR24, Theorem 1.2]. Define $y < x$ if $y \leq x$ but x and y are not of the same type. The relation $<$ defines a partial order on the set of equivalence classes of ends.

Proposition 2.2 (Proposition 4.7, [MR23]). *The partial order $<$ has maximal elements. Moreover, the equivalence class of a maximal element is either finite or a Cantor set.*

We denote by $E(x)$ the equivalence class of $x \in E(\Sigma)$ and by $\mathcal{M}(\Sigma)$ the set of all maximal ends for $<$.

Definition 5 (Unique maximal end). If $|\mathcal{M}(\Sigma)| = 1$ we say that Σ has a *unique maximal end*.

Mann and Rafi also introduced the notion of self-similar space of ends that we recall now along with some of their results that will be needed for our proofs below.

Definition 6. We say that the space of ends $(E(\Sigma), E_\infty(\Sigma))$ of an infinite-type surface Σ is *self-similar* if for any decomposition of $E(\Sigma)$ into pairwise disjoint clopen sets

$$E(\Sigma) = E_1 \sqcup E_2 \sqcup \dots \sqcup E_n$$

there exists a clopen set D in some E_i such that $(D, D \cap E_\infty(\Sigma))$ is homeomorphic to $(E(\Sigma), E_\infty(\Sigma))$.

Definition 7. Let Σ be an infinite-type surface. A finite-type subsurface K of Σ , possibly disconnected, is *nondisplaceable* if for each homeomorphism f of Σ we have that $f(K) \cap K \neq \emptyset$.

Theorem 2.3 (Theorem 1.9, [MR23]). *Let Σ be an infinite-type surface. If Σ contains a nondisplaceable finite-type subsurface then $\text{Map}(\Sigma)$ is not globally CB.*

Proposition 2.4 (Proposition 3.1, [MR23]). *Let Σ be an infinite-type surface of infinite or zero genus. If $E(\Sigma)$ is self-similar then $\text{Map}(\Sigma)$ is globally CB.*

Lemma 2.5 (Lemma 4.12, [MR23]). *Suppose Σ is an infinite-type surface with a unique maximal end and such that it has no nondisplaceable finite-type subsurfaces. Then $E(\Sigma)$ is self-similar.*

Theorem 2.6. *Let Σ be an infinite-type surface with zero or infinite genus and with one maximal end. Then $\text{Map}(\Sigma)$ is globally CB if and only if $E(\Sigma)$ is self-similar.*

Proof. Suppose that $\text{Map}(\Sigma)$ is globally CB. By Theorem 2.3, Σ does not have displaceable finite-type subsurfaces, then by Lemma 2.5, $E(\Sigma)$ is self-similar. The sufficiency part is given by Proposition 2.4. \square

Definition 8. Suppose Σ has a unique maximal end x and α is a separating curve in Σ .

- The *interior* of α is defined as the only connected component of $\Sigma \setminus \alpha$ that is a neighborhood of the unique maximal end x . We denote it by $\text{Int}(\alpha)$.
- The complement of $\text{Int}(\alpha) \cup \alpha$ in Σ is called the *exterior* of α and it is denoted by $\text{Ext}(\alpha)$.

Observe that for each homeomorphism f of Σ

$$\text{Int}(f(\alpha)) = f(\text{Int}(\alpha)) \quad \text{and} \quad \text{Ext}(f(\alpha)) = f(\text{Ext}(\alpha)).$$

If Σ has a unique maximal end x , then any neighborhood of x contains a subsurface with one boundary component whose interior is a neighborhood of x . This fact is often used throughout the work. Recall that we are assuming that subsurfaces have separating boundary curves.

3. PROOF OF THEOREM 1.1

First we prove the necessity condition of Theorem 1.1 and after three preparatory lemmas we give the proof of the sufficiency part.

For the necessity part we use the following lemma that is a consequence of [MR23, Lemma 5.2].

Lemma 3.1. *Let Σ be an infinite-type surface and K be a finite-type subsurface of Σ . If \mathcal{V}_K is CB then every finite-type subsurface S (possibly disconnected) contained in $\Sigma \setminus K$ is $\text{Map}(\Sigma)$ -displaceable.*

Proof of the necessity part of Theorem 1.1. We assume that $\text{Map}(\Sigma)$ is locally CB. Let V be a CB neighborhood of the identity in $\text{Map}(\Sigma)$. Take a connected finite-type subsurface K of Σ such that $\mathcal{V}_K \subseteq V$. We have that \mathcal{V}_K is CB. By enlarging K (and therefore, shrinking \mathcal{V}_K) if it were necessary, we can assume that K satisfies item (1), that is, the closure of each complementary component of K in Σ is of infinite-type with one boundary component either with zero or infinite genus. Without loss of generality, we can assume that the unique maximal end x is an end of Σ_0 .

Now we prove item (2). Let U be the interior of a connected subsurface of Σ_0 with one boundary component that is a neighborhood of x in Σ . If U is isotopic Σ_0 it is enough to find f_U isotopic to Id_Σ such that $f_U(U) = \Sigma_0$. Now, suppose that $U \subseteq \Sigma_0$ is not isotopic to Σ_0 . Then there is a pair of pants $P \subset \Sigma_0$ such that $\partial U \sqcup \partial \Sigma_0 \subseteq \partial P$ and $\Sigma \setminus P = (\Sigma \setminus \overline{\Sigma_0}) \sqcup U \sqcup W$. As \mathcal{V}_K is CB, by Lemma 3.1 there is a homeomorphism f such that $f(P) \cap P = \emptyset$.

We claim that, up to replacing f by its inverse, we can assume that $f(P) \subset U$. Indeed, observe that either $f(P) \subset U$ or $f(P) \subset \Sigma \setminus U$. If $f(P) \subset \Sigma \setminus U$ then $P \subset \text{Int}(f(\partial U))$. Given that $U = \text{Int}(\partial U)$ and U is a neighborhood of the unique maximal end then $\text{Int}(f(\partial U)) = f(\text{Int}(\partial U)) = f(U)$; then $f^{-1}(P) \subset U$.

Assume $f(P) \subseteq U$ and we set $f_U := f$. Again, as U is a neighborhood of x and ∂U is a separating curve in Σ , there are two possibilities for $f_U(\partial U)$: either $\text{Int}(f_U(\partial U)) \subseteq U$ or $\text{Ext}(f_U(\partial U)) \subseteq U$. Since $f_U(\Sigma_0)$ is a neighborhood of the unique maximal end x and $\text{Int}(\partial U) \subseteq \text{Int}(\partial \Sigma_0)$, if $\text{Int}(f_U(\partial U)) \subseteq U$ then necessarily $\text{Int}(f_U(\partial \Sigma_0)) \subseteq U$. In this case we obtain that $f(\Sigma_0) \subseteq U$ because $\Sigma_0 = \text{Int}(\partial \Sigma_0)$, see Figure 1 a). Finally, since $\text{Ext}(\partial U) = \Sigma \setminus (U \cup \partial U)$ and $\text{Ext}(f_U(\partial U)) = f_U(\text{Ext}(\partial U))$, if $\text{Ext}(f_U(\partial U)) \subseteq U$ then $f_U(\Sigma \setminus U) \subseteq U$, see Figure 1 b). In conclusion, either $f_U(\Sigma_0) \subseteq U$ or $f_U(\Sigma \setminus U) \subseteq U$. \square

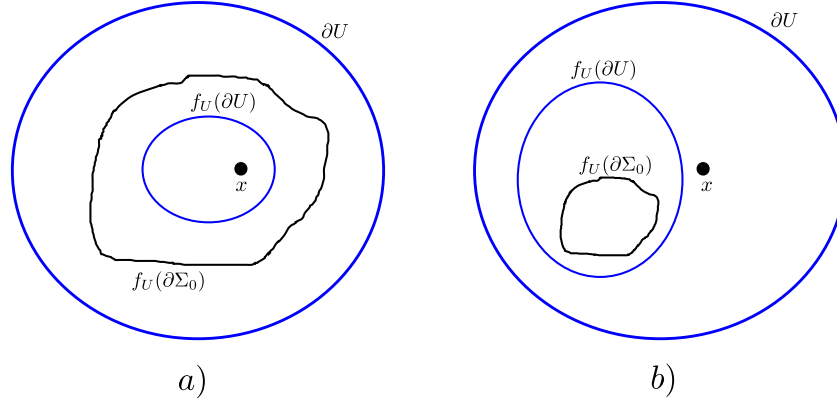


FIGURE 1. a) If $\text{Int}(f_U(\partial U)) \subseteq U$ then $f_U(\Sigma_0) \subseteq U$. b) If $\text{Ext}(f_U(\partial U)) \subseteq U$ then $f_U(\Sigma \setminus U) \subseteq U$.

For the proof of the sufficiency part of Theorem 1.1 we use three lemmas. Assume that Σ has a unique maximal end x and let K be a connected finite-type subsurface of Σ with complementary subsurfaces $\Sigma_0, \Sigma_1, \dots, \Sigma_n$, i.e.,

$$\Sigma \setminus K = \Sigma_0 \sqcup \Sigma_1 \sqcup \dots \sqcup \Sigma_n.$$

Additionally, suppose that the closure in Σ of each Σ_i is an infinite-type surface of zero or infinite genus with one boundary component and that Σ_0 is a neighborhood of the unique maximal end x .

In what follows, the *support* of a homeomorphism $f : \Sigma \rightarrow \Sigma$, denoted by $\text{supp}(f)$, is defined as the closure in Σ of the set $\{s \in \Sigma \mid f(s) \neq s\}$.

Lemma 3.2. *Suppose $U \subseteq \Sigma_0$ is a neighborhood of x such that for each subsurface $\tilde{U} \subseteq U$ that is a neighborhood of x there is a homeomorphism $f_{\tilde{U}}$ such that $f_{\tilde{U}}(\Sigma_0) \subseteq \tilde{U}$. Then for each $1 \leq i \leq n$ there exists a homeomorphism f_i such that $f_i(\Sigma_i) \subseteq U$.*

Remark 5. The hypothesis of Lemma 3.2 implies that U is a stable neighborhood of x , see Definition 2. We point out that Lemma 3.2 can be obtained using [MR23, Lemma 4.18]. Here we provide a self-contained proof.

Proof. Let $1 \leq i \leq n$ be fixed. Observe that $E(U)$ contains a homeomorphic copy of $E(\Sigma_i)$. Indeed, given that U is a neighborhood of the unique maximal end x and $E(\Sigma_i)$ is a compact subset of $E(\Sigma)$ then there is a finite collection $\{N_j\}_{j=1}^m$ of disjoint clopen subsets that covers $E(\Sigma_i)$ and such that each N_j is mapped inside $E(U)$ by a homeomorphism h_j . Now, we use the hypothesis of the lemma to make the collection $\{h_j(N_j)\}_{j=1}^m$ disjoint inside of $E(U)$.

Now, since Σ_i has zero or infinite genus, we can find a homeomorphic copy Σ'_i of Σ_i contained in U . In order to obtain the desired homeomorphism f_i , let $P \subseteq \Sigma$ be a pair of pants such that ∂P contains the boundary curves of Σ_i and Σ'_i , see Figure 2. Then f_i is the homeomorphism supported on $P \cup \Sigma_i \cup \Sigma'_i$ and sends Σ_i onto Σ'_i .

□

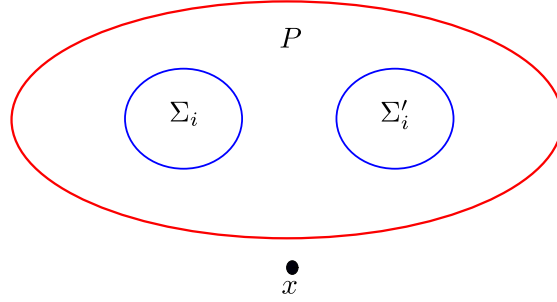


FIGURE 2. Σ_i and Σ'_i are homeomorphic subsurfaces in Σ . Then there is a homeomorphism f with support in $P \cup \Sigma_i \cup \Sigma'_i$ that sends Σ_i onto Σ'_i .

Lemma 3.3. *Let W be a subsurface of Σ that is a neighborhood of x (may be equal to Σ) and suppose that for each subsurface $U \subseteq W$ that is a neighborhood of x there is a homeomorphism f_U such that $f_U(\Sigma \setminus U) \subseteq U$. Then $\text{Map}(\Sigma)$ is globally CB.*

Proof. We prove that any finite-type subsurface of Σ is displaceable; in particular, Σ has zero or infinite genus. Let S be a finite-type subsurface of Σ . Then we can construct a connected subsurface $U_S \subseteq W$ that defines a neighborhood of x and such that $U_S \cap S = \emptyset$. Applying the hypothesis of the lemma, there is a homeomorphism f such that $f(\Sigma \setminus U_S) \subseteq U_S$. As $S \subseteq \Sigma \setminus U_S$ we obtain that $f(S) \cap S = \emptyset$.

Now, by Lemma 2.5, $E(\Sigma)$ is self-similar and, by Theorem 2.6, $\text{Map}(\Sigma)$ is globally CB. \square

Lemma 3.4. *Let T be a finite-type subsurface of Σ containing K and let U_T be the only connected component of $\Sigma \setminus T$ which is a neighborhood of x . Suppose that there is a homeomorphism f_0 with $f_0(\Sigma_0) \subseteq U_T$, and assume that for each $1 \leq i \leq n$ there is a homeomorphism f_i with $f_i(\Sigma_i) \subseteq \Sigma_0$. Then $\mathcal{V}_K \subseteq (F\mathcal{V}_T)^{4n+2}$ where $F = \{f_i^{\pm 1}\}_{i=0}^n$.*

Proof. Let $g \in \mathcal{V}_K$. Then $g = g_0 g_1 \cdots g_n$ where each g_i has its support in Σ_i . Observe that $f_0^{-1}(T) \subseteq \Sigma \setminus f_0^{-1}(U_T) \subseteq \Sigma \setminus \Sigma_0$. Since g_0 has its support on Σ_0 then $f_0 g_0 f_0^{-1} \in \mathcal{V}_T$. Therefore $g_0 \in (F\mathcal{V}_T)^2$.

Now, let $1 \leq i \leq n$. We notice that $f_i g_i f_i^{-1}$ has its support on Σ_0 . This is because g_i has its support on Σ_i and $\text{supp}(f_i g_i f_i^{-1}) = f_i(\text{supp}(g_i))$. As in the previous paragraph we have that $f_i g_i f_i^{-1} \in (F\mathcal{V}_T)^2$ and therefore $g_i \in (F\mathcal{V}_T)^4$. Putting all together we obtain the desired result. \square

Proof of the sufficient part of Theorem 1.1. Let K be a finite-type subsurface of Σ satisfying the requirements of the statement of Theorem 1.1. We prove that \mathcal{V}_K is CB. Let V be an arbitrary neighborhood of the identity in $\text{Map}(\Sigma)$. By Theorem 2.1, we need to show that there exists a finite set $F \subset \text{Map}(\Sigma)$ and $m \geq 1$ such that $\mathcal{V}_K \subseteq (FV)^m$.

Take T a finite-type subsurface of Σ where each boundary component is a separating curve, with $K \subseteq T$ and such that $\mathcal{V}_T \subseteq V \cap \mathcal{V}_K$. Define U_T as the unique connected component of $\Sigma \setminus T$ that is a neighborhood of the unique maximal end x .

There are two cases for U_T :

- 1) There is a connected subsurface $U \subseteq U_T$ that is a neighborhood of x for which there exists a homeomorphism f_U such that $f_U(\Sigma_0) \subseteq U$.
- 2) If item 1) does not hold then for every $U \subseteq U_T$ a connected subsurface with one boundary component and $x \in E(U)$ there is a homeomorphisms f_U with $f_U(\Sigma \setminus U) \subseteq U$.

Suppose we are in item 1). Applying again the hypothesis to U we have two possibilities:

- i) there is a connected subsurface $\tilde{U} \subseteq U$ that is a neighborhood of x for which there exists a homeomorphism $f_{\tilde{U}}$ such that $f_{\tilde{U}}(\Sigma \setminus \tilde{U}) \subseteq \tilde{U}$.
- ii) Item i) does not hold. Then for every connected subsurface $\tilde{U} \subseteq U$ that is a neighborhood of x there is a homeomorphism $f_{\tilde{U}}$ such that $f_{\tilde{U}}(\Sigma_0) \subseteq \tilde{U}$.

If item i) holds, then letting $f_0 := f_U$ and $f_i := f_{\tilde{U}}$ for each $1 \leq i \leq n$ in Lemma 3.4 we have that $\mathcal{V}_K \subseteq (F\mathcal{V}_T)^{4n+2} \subseteq (FV)^{4n+2}$ where $F := \{f_i^{\pm 1}\}_{i=0}^n$.

Now, suppose item ii) holds. We set $f_0 := f_U$ and we use Lemma 3.2 to obtain for each $1 \leq i \leq n$ a homeomorphism f_i such that $f_i(\Sigma_i) \subseteq U \subseteq \Sigma_0$. Applying Lemma 3.4 we conclude that $\mathcal{V}_K \subseteq (F\mathcal{V}_T)^{4n+2} \subseteq (FV)^{4n+2}$ where $F := \{f_i^{\pm 1}\}_{i=0}^n$.

Finally, if item 2) holds then by Lemma 3.3 $\text{Map}(\Sigma)$ is globally CB and, in particular, it is locally CB. \square

4. PROOF OF THEOREMS 1.2 AND 1.3

We use the following result.

Lemma 4.1 (Lemma 4.10, [MR23]). *$E(\Sigma)$ is self-similar if and only if for any decomposition $E(\Sigma) = A_1 \sqcup A_2$ into clopen subsets there is some A_i that contains a homeomorphic copy of $E(\Sigma)$.*

Proof of Theorem 1.2. If Σ has no nondisplaceable subsurfaces of finite type then the result is given by Lemma 2.5.

Suppose Σ has a nondisplaceable finite-type subsurface $S \subseteq \Sigma$. We use Lemma 4.1 to prove that $E(\Sigma)$ is self-similar. Let $E(\Sigma) = A_1 \sqcup A_2$ be a decomposition of $E(\Sigma)$ into clopen subsets. Let K be a finite-type subsurface as in Theorem 1.1. Since S is of finite-type then there is $U \subseteq \Sigma_0$ a neighborhood of the unique maximal end x such that $U \cap S = \emptyset$. Given that S is a nondisplaceable subsurface of Σ , the subset U satisfies that for any $\tilde{U} \subseteq U$ that is a neighborhood of x there is a homeomorphism $f_{\tilde{U}}$ such that $f_{\tilde{U}}(\Sigma_0) \subseteq \tilde{U}$. Hence, U is a stable neighborhood of x . Without lost of generality we can suppose that A_1 contains x . If necessary, we can take U small enough such that $E(U) \subseteq A_1$.

By the property satisfied by U , there is a homeomorphic copy Σ'_0 of Σ_0 contained in U . On the other hand, by Lemma 3.2, for each $i = 1, \dots, n$, there is $\Sigma'_i \subseteq U$ homeomorphic to Σ_i . Finally, let P denote the set of ends of Σ contained into K . Remember that P consist of a finite number of punctures. Given that Σ has a unique maximal end there is a copy P' of P inside $E(U)$. Now, using again the property of U we can make that the collection $\{\Sigma'_i\}_{i=0}^n$ pairwise disjoint and disjoint from P' . This implies that $E(U)$ (and therefore A_1) contains a homeomorphic copy of $E(\Sigma)$. \square

Proof of Theorem 1.3. If Σ has zero or infinite genus, combine Theorems 1.2 and 2.6 to conclude that $\text{Map}(\Sigma)$ is globally CB. If Σ has finite non-zero genus then $\text{Map}(\Sigma)$ is not globally CB by Theorem 2.3. \square

5. PROOF OF COROLLARY 1.4

We first prove the following corollary of Theorem 1.1.

Corollary 5.1. *Let Σ be an infinite-type surface with a unique maximal end x and $0 < g(\Sigma) < \infty$. Then $\text{Map}(\Sigma)$ is locally CB if and only if there is a connected finite-type subsurface K of Σ with the following properties:*

- (1) $\Sigma \setminus K = \Sigma_0 \sqcup \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ where the closure of each Σ_i is a surface of infinite-type with one boundary component, $g(\Sigma_i) \in \{0, \infty\}$ and Σ_0 is a neighborhood of x and,
- (2) for any subsurface $U \subseteq \Sigma_0$ that is a neighborhood of x there is a homeomorphism f_U such that $f_U(\Sigma_0) \subseteq U$.

Proof. Suppose that $\text{Map}(\Sigma)$ is locally CB and K is a finite-type subsurface as in Theorem 1.1. If additionally $0 < g(\Sigma) < \infty$, then all the genus of Σ is contained in K and therefore for any subsurface $U \subseteq \Sigma_0$ whose interior is a neighborhood of the unique maximal end x does not exist a homeomorphism f satisfying that $f(\Sigma \setminus U) \subseteq U$. \square

Proof of Corollary 1.4. (By contradiction) Suppose that $E(\Sigma)$ is countable. We prove that $\text{Map}(\Sigma)$ is not locally CB by showing that item (2) in Corollary 5.1 does not occur for any finite-type subsurface K satisfying item (1). So, let K be a finite-type subsurface of Σ satisfying (1) of Corollary 5.1. As $E(\Sigma)$ is homeomorphic to $\omega^\alpha + 1$ with α a countable ordinal and Σ_0 is a neighborhood of the unique maximal end x . The ordinal α can be either a successor ordinal or a limit ordinal. In either case, $E(\Sigma \setminus \Sigma_0)$ has a finite number of maximal ends. Taking $U \subseteq \Sigma_0$ neighborhood of x such that $E(\Sigma \setminus U)$ contains more maximal ends than $E(\Sigma \setminus \Sigma_0)$ we have that for this U there does not exist a homeomorphism f_U such that $f_U(\Sigma_0) \subseteq U$. \square

6. PROOF OF THEOREM 1.5

A globally CB group is in particular CB generated. Suppose that $\text{Map}(\Sigma)$ is locally CB but not globally CB. By Theorem 1.3, the surface Σ has finite nonzero genus. Let $K \subseteq \Sigma$ be a finite type subsurface as in Theorem 1.1, that is,

$$\Sigma \setminus K = \Sigma_0 \sqcup \Sigma_1 \sqcup \cdots \sqcup \Sigma_n,$$

where each Σ_i is of infinite type and has genus zero, Σ_0 is a neighborhood of the unique maximal end of Σ and, \mathcal{V}_K is a CB neighborhood of the identity. Denote by P_K to the union of all Σ_i for $i = 1, \dots, n$. Observe that $E(\Sigma)$ is self-similar (by Theorem 1.2), K is compact with $g(K) = g(\Sigma)$ and it has $n + 1$ boundary components.

We use the following lemma that appears in [MR23, Observation 6.9]. By abuse of notation, given a finite-type subsurface S of Σ with compact boundary we think of an element of $\text{Map}(S)$ as an element of $\text{Map}(\Sigma)$ by extending it by the identity on the complement of S in Σ .

Lemma 6.1 (Observation 6.9, [MR23]). *Let Σ be an infinite-type surface possibly with nonempty boundary and $S \subseteq \Sigma$ be a finite-type subsurface. Then there is $D_S \subseteq \text{Map}(\Sigma)$ a finite set of Dehn twist such that for any finite-type subsurface $S' \subseteq \Sigma$*

$$\text{Map}(S') \subseteq \langle D_S \cup \mathcal{V}_S \rangle \subseteq \text{Map}(\Sigma).$$

Proof of Theorem 1.5. Applying Lemma 6.1 to the subsurface K , let D_K be a finite set of Dehn twists such that for every finite type subsurface S' of Σ , $\text{Map}(S')$ is contained in the group generated by $D_K \cup \mathcal{V}_K$. As $E(\Sigma)$ is self-similar, there is $g_K \in \text{Map}(\Sigma)$ such that $g_K(P_K) \subseteq \Sigma_0$. Let G be the subgroup of $\text{Map}(\Sigma)$ generated by the CB set $\{g_K\} \cup D_K \cup \mathcal{V}_K$. We show that $\text{Map}(\Sigma)$ coincides with G .

Let $f \in \text{Map}(\Sigma)$. First we prove that there exist $f', f'' \in G$ such that $f'f^{-1}f''|_{P_K} = \text{Id}_{P_K}$. Indeed, take $U \subseteq \Sigma_0$ a neighborhood of the unique maximal end x of Σ such that $f(U) \subseteq \Sigma_0$. As the space of ends of Σ is self-similar, $E(U)$ contains a homeomorphic copy of $E(\Sigma)$, and therefore there is $P'_K \subseteq U$ homeomorphic to P_K . In particular, $f(P'_K) \subseteq \Sigma_0$. Now, as Σ_0 has genus zero there is $h \in \text{Map}(\Sigma_0)$ such that $hf(P'_K) = g_K(P_K)$. So, $f^{-1}h^{-1}g_K(P_K) \subseteq \Sigma_0$. As $g_K(P_K)$ is also contained into Σ_0 , there is $h' \in \text{Map}(\Sigma_0)$ such that $h'f^{-1}h^{-1}g_K(P_K) = g_K(P_K)$ and $h'f^{-1}h^{-1}g_K|_{\partial P_K} = g_K|_{\partial P_K}$, in other words, $g_K^{-1}h'f^{-1}h^{-1}g_K(P_K) = P_K$ and $g_K^{-1}h'f^{-1}h^{-1}g_K|_{\partial P_K} = \text{Id}_{\partial P_K}$. Finally, we can find an element $w \in \text{Map}(P_K) \subseteq \mathcal{V}_K$ such that the restriction of $wg_K^{-1}h'f^{-1}h^{-1}g_K$ to P_K is equal to Id_{P_K} . Letting $f' := wg_K^{-1}h'$ and $f'' := h^{-1}g_K$ we obtain the desired result.

Let $g := f'f^{-1}f''$ and put $S' := K \cup g(K)$. Again, by Lemma 6.1, $\text{Map}(S')$ is contained in the group generated by $D_K \cup \mathcal{V}_K$ and then it is contained in the group G . Now, observe that $\partial \Sigma_0$ and $g(\partial \Sigma_0)$ are essential separating curves of the same topological type in S' . Then there is $g' \in \text{Map}(S') \subseteq G$ such that $g'g$ is the identity on K , that is, $g'g \in \mathcal{V}_K$. Therefore, $f \in G$. \square

7. PROOF OF THEOREM 1.6

In the first part of this section we recall the original definition [MR23, Definition 6.11] of a tame surface in the case when the surface has a unique maximal end. We show that this definition is equivalent to Definition 3 given in the Introduction of this paper. In the second part of the section we prove Theorem 1.6 by constructing a non-tame infinite-type surface with CB-generated but not globally CB mapping class group.

The following lemma is inspired by the first point of [MR23, Lemma 6.10].

Lemma 7.1. *Let Σ be an infinite-type surface with a unique maximal end x . Suppose that $\text{Map}(\Sigma)$ is locally CB and that for any $N \subseteq E(\Sigma_0)$ clopen neighborhood of x there is $f_N \in \text{Homeo}^+(\Sigma)$ such that $f_N(E(\Sigma) \setminus N) \subseteq E(\Sigma) \setminus E(\Sigma_0)$ (compare with Theorem 1.1, item 2)). Then there exists a clopen neighborhood $N(x) \subseteq E(\Sigma_0)$ of x such that the clopen subset $W_x := E(\Sigma_0) \setminus N(x)$ satisfies*

$$(1) \quad E(z) \cap W_x \neq \emptyset \text{ if and only if } E(z) \cap (E(\Sigma_0) \setminus \{x\}) \neq \emptyset,$$

in other words, W_x has representatives of each end in $E(\Sigma_0) \setminus \{x\}$.

Proof. The proof is done by contradiction. Suppose that for each clopen neighborhood $N \subseteq E(\Sigma_0)$ of x there is an end $z \in E(\Sigma_0) \setminus \{x\}$ such that $E(z) \cap (E(\Sigma_0) \setminus N) = \emptyset$. Then there exists a decreasing sequence of nested closed neighborhoods of x , $N_1 \supseteq N_2 \supseteq \dots$, with $\bigcap_{i \in \mathbb{N}} N_i = \{x\}$,

$N_i \subseteq E(\Sigma_0)$ and such that for each $i \in \mathbb{N}$ there is $z_i \in E(\Sigma_0) \setminus \{x\}$ with $E(z_i) \cap (E(\Sigma_0) \setminus N_i) = \emptyset$ and $z_i \in N_i \setminus N_{i+1}$. By hypothesis, for each $i \in \mathbb{N}$ there is a homeomorphism f_i such that $f_i(E(\Sigma) \setminus N_{i+1}) \subseteq E(\Sigma) \setminus E(\Sigma_0)$. Consequently, for every $i \in \mathbb{N}$ there is $z'_i = f_i(z_i)$ in $E(z_i)$ contained in $E(\Sigma) \setminus E(\Sigma_0)$. Let $y' \in E(\Sigma) \setminus E(\Sigma_0)$ be an accumulation point of the sequence $\{z'_i\}$. Since x is the unique maximal end of Σ there exists an end y equivalent to y' contained in N_1 . Let us take V a neighborhood of y such that V is disjoint from N_m for some $m \in \mathbb{N}$. Recall that y' is equivalent to y and the latter is an accumulation point of the sequence $\{z'_i\}$. Then, for some $j \geq m$, there exists $w \in E(z_j)$ contained in V and hence $w \in E(z_j) \cap (E(\Sigma_0) \setminus N_j)$, which is a contradiction. \square

Remark 6. In particular, the hypothesis of Lemma 7.1 are satisfied if $\text{Map}(\Sigma)$ is locally CB but not globally CB; see Theorem 1.3 and Corollary 5.1. Notice that there are cases when $\text{Map}(\Sigma)$ is globally CB and the argument in Lemma 7.1 cannot be applied to obtain a clopen W_x neighborhood of x . For example, take Σ to be the infinite-type surface of genus zero with space of ends homeomorphic to the ordinal number $\omega^\omega + 1$, then $\text{Map}(\Sigma)$ is globally CB. However, it is not possible to find a neighborhood of the unique maximal end of Σ such that its complement has representatives of all non maximal ends.

With the notation introduced in Lemma 7.1, we are able to state [MR23, Definition 6.11] for the case of surfaces with a unique maximal end. We refer the reader to Definition 2 for the definition of stable neighborhood.

Definition 9 (Tame surface). Let Σ be a surface with a unique maximal end x and suppose that $\text{Map}(\Sigma)$ is locally CB and not globally CB. We say that Σ is *tame* if the unique maximal end x has a stable neighborhood and every maximal end of W_x has a stable neighborhood.

Suppose Σ has a unique maximal end and $\text{Map}(\Sigma)$ is locally CB but not globally CB. The equivalence of Definition 9 and Definition 3 in this case follows from the Proposition 7.2 below. In the proof of this proposition we repeatedly use the following observation, which is a consequence of the fact that W_x is closed in $E(\Sigma)$.

Remark 7. If $y, z \in W_x$ then $y \leq z$ in W_x if and only if $y \leq z$ in $E(\Sigma)$.

Proposition 7.2. *Let Σ be an infinite-type surface with a unique maximal end x and suppose that $\text{Map}(\Sigma)$ is locally CB but not globally CB. Then y is an immediate predecessor of x if and only if there is a representative of y which is maximal in W_x .*

Proof. Suppose that y is an immediate predecessor of x . By Lemma 7.1 the clopen W_x has representatives of each end in $E(\Sigma_0) \setminus \{x\}$. Since x is the only accumulation point of $E(y)$, we can assume, up to equivalence, that $y \in W_x$. We show that y is maximal in W_x . Let $z \in W_x$ such that $y \leq z$ in W_x , then by Remark 7 we have that $y \leq z$ in $E(\Sigma)$. Given that y is an immediate predecessor of x we conclude that y is equivalent to z in $E(\Sigma)$. Again, by Remark 7, we have that y is equivalent to z in W_x and hence y is a maximal end in W_x .

Reciprocally, suppose that $y \in W_x$ is a maximal end of W_x . We show that y is an immediate predecessor of x . Let $z \in E(\Sigma) \setminus \{x\}$ such that $y \leq z$ in $E(\Sigma)$. Since $E(y)$ accumulates in $\{x\}$ (because x is the unique maximal end of Σ) and W_x has representatives of each end in $E(\Sigma_0) \setminus \{x\}$, without loss of generality we can assume that z is contained in W_x . By Remark 7, it follows that

$y \leq z$ in W_x . The maximality of y in W_x implies that z is equivalent to y in W_x . Applying again Remark 7, we have that z is equivalent to y in $E(\Sigma)$ and therefore y is an immediate predecessor of x . \square

The following result refines the conclusion of Corollary 1.4 about the space of ends. It gave us some insight for constructing the example for the proof of Theorem 1.6 below.

Proposition 7.3. *Let Σ be an infinite-type surface with a unique maximal end x and suppose $\text{Map}(\Sigma)$ is locally CB but not globally CB. Then the set of immediate predecessors of x is not empty. Moreover, the equivalence class of each immediate predecessor of x is uncountable.*

Proof. The first part is consequence of Lemma 7.1 and Proposition 7.2.

Now, as $\text{Map}(\Sigma)$ is not globally CB, by Theorem 1.3, Σ has finite nonzero genus. Let K be the finite-type subsurface of Σ satisfying items 1) and 2) of Corollary 5.1. The proof of the moreover part is done by contradiction. Let y be an immediate predecessor of x and suppose that $E(y)$ is countable. Since x is the unique accumulation point of $E(y)$ then $E(\Sigma \setminus \Sigma_0)$ contains a finite number of elements of $E(y)$. Taking a neighborhood $U \subseteq \Sigma_0$ of x such that $E(\Sigma \setminus U)$ contains more elements of $E(y)$ than $E(\Sigma \setminus \Sigma_0)$, we have that for such U there does not exist a homeomorphism f_U such that $f_U(\Sigma_0) \subseteq U$. This contradicts item 2) of Corollary 5.1. \square

Proof of Theorem 1.6. Our example is inspired by the constructions carried out at [MR24, Sections 2 & 3]. For each $n \in \mathbb{N}$, let D_n be the surface of genus zero with one boundary component and such that: 1) $E(D_n) = C_n \sqcup Q_n$ where C_n is homeomorphic to a Cantor set and Q_n is a countable set, 2) $E(D_n)$ has Cantor-Bendixson rank n and, 3) for each derived set Ω of $E(D_n)$ that has isolated points, the accumulation set of the isolated points of Ω contains C_n ; see Figure 3. From the construction it follows that the ends in C_n are not comparable with the ends in C_m if $n \neq m$.

Let T be the surface obtained from the Cantor tree surface by placing 2^n copies of D_n at each n th-level of the tree, one for each bifurcation. Let C denote the set of ends of T coming from the ends of the Cantor tree surface. Observe that each end x in C does not have stable neighborhoods. Indeed, take U a neighborhood of x and let n_U be the smallest natural number such that $C_{n_U} \cap E(U) \neq \emptyset$. By construction, we can find a neighborhood $V \subseteq U$ of x with $E(V)$ not containing points of C_{n_U} . Given that the ends in C_n are not comparable with the ends in C_m if $n \neq m$, V cannot contain homeomorphic copies of U .

Let g be a nonzero natural number and F_g be the surface of genus g and space of ends homeomorphic to $\omega + 1$. Finally, we define Σ to be the surface obtained from F_g by replacing a neighborhood of each of its isolated ends with a copy of the surface T ; see Figure 4.

Note that the space of ends of Σ is self-similar, Σ has a unique maximal end and therefore the unique maximal end of Σ has stable neighborhoods. Moreover, each maximal end of T is an immediate predecessor end of the unique maximal end of Σ and, more importantly, some of them have no stable neighborhoods as we explained before. Therefore the surface Σ is not tame. Since Σ has finite nonzero genus, then $\text{Map}(\Sigma)$ is not globally CB. Let K be the compact subsurface of Σ with two boundary components, of genus g and complementary components Σ_0 and Σ_1 , where Σ_1 is exactly the first copy T and Σ_0 contains the remaining copies of T in the construction of Σ . The reader can verify that K is the desired subsurface in Theorem 1.1 for which \mathcal{V}_K defines a

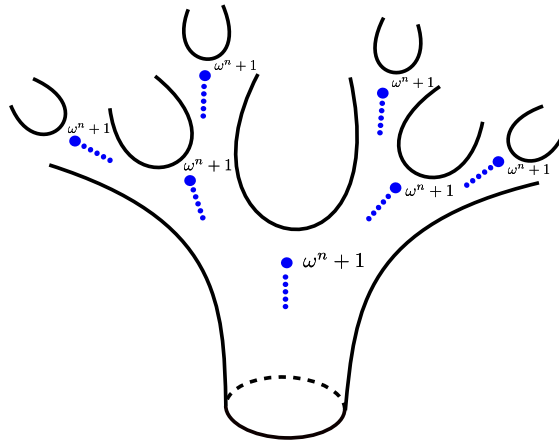


FIGURE 3. Surface D_{n+1} with one boundary component and whose space of ends has Cantor-Bendixson rank $n + 1$ with $(n + 1)$ th-derived set homeomorphic to a Cantor set C_{n+1} and, each point of C_{n+1} is accumulated by points homeomorphic to $\omega^n + 1$.

CB-neighborhood of the identity. So, $\text{Map}(\Sigma)$ is locally CB and therefore it is CB generated by Theorem 1.5. □

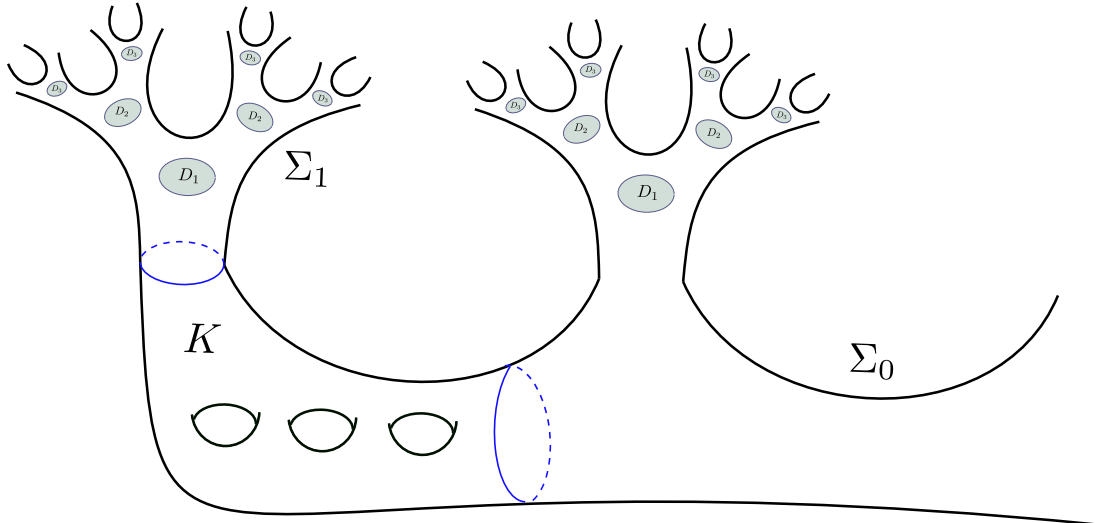


FIGURE 4. Non-tame surface with a unique maximal end and whose mapping class group is CB generated but not globally CB.

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INSTITUTO DE MATEMÁTICAS, UNAM, UNIDAD OAXACA, OAXACA DE JUÁREZ, OAXACA, C.P. 68000.
OAXACA, MÉXICO.

E-mail address: rita@im.unam.mx

E-mail address: imorales@im.unam.mx