# ON THE DIMENSION OF HARER'S SPINE FOR THE DECORATED TEICHMÜLLER SPACE

## NESTOR COLIN, RITA JIMÉNEZ ROLLAND, PORFIRIO L. LEÓN ÁLVAREZ, AND LUIS JORGE SÁNCHEZ SALDAÑA

ABSTRACT. In [Har86] Harer explicitly constructed a spine for the decorated Teichmüller space of orientable surfaces with at least one puncture and negative Euler characteristic. In this paper we point out some instances where his computation of the dimension of this spine is off by 1 and give the correct dimension.

#### 1. INTRODUCTION

Let F be an orientable closed connected surface of genus g and consider  $\{p_1, \ldots, p_s\}$  a collection of distinguished points in F. The mapping class group  $\operatorname{Mod}_g^s$  is the group of isotopy classes of all orientation-preserving diffeomorphisms  $f: F_g^s \to F_g^s$ , where  $F_g^s := F - \{p_1, \ldots, p_s\}$ . The group  $\operatorname{Mod}_g^s$  permutes the set  $\{p_1, \ldots, p_s\}$  and the kernel of such action is the pure mapping class group  $\operatorname{PMod}_g^s$ . These groups are related by the following short exact sequence

$$1 \to \operatorname{PMod}_q^s \to \operatorname{Mod}_q^s \to \mathfrak{S}_s \to 1,$$

where  $\mathfrak{S}_s$  denotes the symmetric group on *s* letters. The groups  $\operatorname{Mod}_g^s$  and  $\operatorname{PMod}_g^s$  act on the *Teichmüller space*  $\mathscr{T}_g^s$  of conformal equivalence classes of marked Riemann surfaces of genus *g* with *s* distinguished points.

In the influential paper [Har86], J. L. Harer established several cohomology properties of  $PMod_a^s$ , including the computation of its virtual cohomological dimension:

$$\operatorname{vcd}(\operatorname{PMod}_{g}^{s}) = \begin{cases} s-3 & \text{if } g = 0 \text{ and } s \ge 3\\ 1 & \text{if } g = 1 \text{ and } s = 0\\ 4g-5 & \text{if } g \ge 2 \text{ and } s = 0\\ 4g-4+s & \text{if } g \ge 1 \text{ and } s \ge 1. \end{cases}$$

In particular, for  $s \geq 1$ , 2g + s > 2, and  $\Delta = \{p_1, \ldots, p_m\}$  with  $1 \leq m \leq s$ , Harer explicitly described a cell complex  $\mathcal{Y}(=\mathcal{Y}_g^{s,m})$  inside the *decorated Teichmüller space*  $\mathscr{T}_g^s(\Delta)$  onto which  $\mathscr{T}_g^s(\Delta)$  may be  $\operatorname{PMod}_g^s$ -equivariantly retracted. In [Har86, Theorem 2.1] Harer states that the dimension of  $\mathcal{Y}$  is 4g - 5 + s + m. However, while revising the details of his construction we observed that this computation is off by 1 when the parameter m < s; see for instance Example 3.4 below. In this paper we compute the correct dimension of  $\mathcal{Y}$ .

In what follows we refer to  $\mathcal{Y}$  as Harer's spine of the decorated Teichmüller space.

**Theorem 1.1** (Dimension of Harer's spine). Let  $s \ge 1$ , 2g + s > 2, and  $1 \le m \le s$ . The dimension of the spine  $\mathcal{Y}(=\mathcal{Y}_q^{s,m})$  is given by

$$\dim(\mathcal{Y}) = \begin{cases} 4g - 4 + s + m & \text{if } m < s \\ 4g - 5 + s + m & \text{if } m = s. \end{cases}$$

In the literature, Harer's spine  $\mathcal{Y}$  (when m = 1) has been cited to justify the existence of a model for  $\underline{E} \operatorname{Mod}_a^s$ , the classifying space for proper actions of  $\operatorname{Mod}_a^s$ , of minimal dimension.

2020 Mathematics Subject Classification. 57K20, 55R35, 20J05, 57M07, 57M50, 57M60.

Key words and phrases. Mapping class groups, Teichmüller space, spines, classifying spaces for proper actions.

This use of Harer's spine is not correct when  $s \ge 2$ , and part of our motivation to write this paper is to clarify it. The misquotation comes from two directions:

- a) The complex  $\mathcal{Y}(=\mathcal{Y}_{g}^{s,1})$  is not a model for  $\underline{E} \operatorname{Mod}_{g}^{s}$  when  $s \geq 2$ . For 2g + s > 2, the Teichmüller space  $\mathscr{T}_{g}^{s}$  is known to be a model for  $\underline{E} \operatorname{Mod}_{g}^{s}$ ; see for instance [JW10, Proposition 2.3] and [Lüc05, Section 4.10]. When m = 1, Harer's spine  $\mathcal{Y}$  is a PMod\_{g}^{s}-equivariant deformation retraction of  $\mathscr{T}_{g}^{s}(\Delta) = \mathscr{T}_{g}^{s}$  and hence it is a model for  $\underline{E} \operatorname{PMod}_{g}^{s}$ . On the other hand, from the definition of Harer's spine, for m < s the complex  $\mathcal{Y}$  only admits a PMod\_{g}^{s}-action; see Section 3 for details. Therefore, despite  $\mathscr{T}_{g}^{s}$  admitting a Mod\_{g}^{s}-action, the spine  $\mathcal{Y}$  cannot be a Mod\_{g}^{s}-equivariant retraction of  $\mathscr{T}_{g}^{s}$ , and it is not a model for  $\underline{E} \operatorname{Mod}_{g}^{s}$  when  $s \geq 2$ . This misquotation appears in [JW10, Introduction], [AMP14, Proof of Theorem 1.1], [Ji14, Introduction], [FGM20, Subsection 3.2], and [JRLASS24, Proof of Theorem 1.5].
- b) The dimension of  $\mathcal{Y}(=\mathcal{Y}_g^{s,1})$  is not always equal to  $\operatorname{vcd}(\operatorname{PMod}_g^s)$ . For the case g = 0,  $s \geq 3$  and m = 1, and the case  $g \geq 1$  and s = m = 1, we have that  $\dim \mathcal{Y} = \operatorname{vcd}(\operatorname{PMod}_g^s)$ . However, for  $g \geq 1$  and  $s \geq 2$ , the computation in Theorem 1.1 shows that

$$\dim \mathcal{Y} = 4g - 4 + s + 1 = \operatorname{vcd}(\operatorname{PMod}_a^s) + 1$$

and hence  $\mathcal{Y}$  is not a model of minimal dimension in this case. This misquotation appears in [BV06, Section 1], [Mis10, Section 2.2] [AMP14, Introduction], [Ji14, Introduction], [HOP14, Section 10], [FGM20, Subsection 3.2], and [JRLASS24, Proof of Theorem 1.5]. We indicate in Corollary 3.5 how the dimension of Harer's spine when s = m = 1, and the Birman exact sequence can be used to show the existence of a model for  $\underline{E} \operatorname{PMod}_q^s$  of minimal dimension when  $s \geq 2$ .

To the best of our knowledge it is still unknown whether there exists a model for  $\underline{E} \operatorname{Mod}_g^s$  of dimension vcd(Mod<sup>s</sup><sub>g</sub>) for  $s \geq 2$ . Although [AMP14, Corollary 1.3] claims the existence of such models, their proof [AMP14, Remark 4.5] relies on an inductive argument using the Birman short exact sequence and it only applies for pure mapping class groups. We want to point out that [AMP14, Corollary 1.3] is cited for instance in [AJPTN18, Theorem 4.1, Proof of Theorem 1.5], [JPTN16, Introduction, Theorem 5.1], and [NP18, Proof of Proposition 5.3].

When s = m, it follows from Harer's construction that the retraction of  $\mathscr{T}_g^s(\Delta)$  onto Harer's spine  $\mathcal{Y}(=\mathcal{Y}_g^{s,s})$  is actually  $\operatorname{Mod}_g^s$ -equivariant. Such spine  $\mathcal{Y}$  is of dimension 4g - 5 + 2s and it can be shown that it gives a model for  $\underline{E} \operatorname{Mod}_g^s$ . Since  $\mathscr{T}_g^s(\Delta)$  is topologized as the product of  $\mathscr{T}_g^s$  with an open (s-1)-simplex, and  $\operatorname{Mod}_g^s$  acts diagonally, there is a natural  $\operatorname{Mod}_g^s$ -projection  $\pi : \mathscr{T}_g^s(\Delta) \to \mathscr{T}_g^s$ . Hence, the image  $\pi(\mathcal{Y})$  is a  $\operatorname{Mod}_g^s$ -equivariant deformation retract of  $\mathscr{T}_g^s$ and it is a model for  $\underline{E} \operatorname{Mod}_g^s$  of dimension at most 4g - 5 + 2s.

Question 1.2. Is  $\pi(\mathcal{Y})$  a model for  $\underline{E} \operatorname{Mod}_q^s$  of minimal dimension when  $s \ge 1$  and 2g + s > 2?

In a forthcoming paper [CJRLASS] we use Harer's spine to construct a spine for the Teichmüller space of non-orientable punctured surfaces  $N_g^s$  (with  $s \ge 1$ ) with negative Euler characteristic. When s = 1 this spine gives us a model of minimal dimension for  $\underline{E} \operatorname{Mod}(N_q^1)$ .

Some comments on spines for Teichmüller space  $\mathscr{T}_g$ . A related question is whether Teichmüller space  $\mathscr{T}_g$  admits a Mod<sub>g</sub>-equivariant deformation retraction onto a cocompact spine whose dimension is equal to vcd(Mod<sub>g</sub>), see [BV06, Question 1]. Since Harer's construction needs at least one marked point in the surface, it cannot be used to address this question.

For  $g \geq 2$ , W. Thurston proposed a candidate for a spine for  $\mathscr{T}_g$  in a hard to find preprint [Thu85]; see also [Ji14, Remark 4.4]. M. Fortier Bourque proved in [FB24, Theorem 1.1] that in general this spine is not of minimal dimension. In a recent preprint [Irm24, Theorem 1] I. Irmer claims the existence of a Mod<sub>g</sub>-equivariant deformation retraction of the Thurston spine onto a CW-complex of dimension equal to vcd(Mod<sub>g</sub>).

Furthermore, in [AMP14, Theorem 1.1] J. Aramayona and C. Martínez-Pérez proved that there exists a cocompact model for  $\underline{E}Mod_g$  of dimension  $vcd(Mod_g)$ , for every  $g \ge 0$ . Their proof is not constructive and does not give a spine for  $\mathscr{T}_g$  when  $g \ge 2$ . For genus g = 2, their argument has a flaw: it relies on the fact that  $Mod_2$  is a central extension of  $Mod_0^6$  by  $\mathbb{Z}_2$ , and it assumes incorrectly that Harer's spine gives a co-compact model for  $\underline{E}Mod_0^6$ ; see item a) above. Instead, Ji's spine of minimal dimension for genus g = 2 can be cited.

**Organization of the paper.** In Section 2 we recall the definition of the arc complex  $\mathcal{A} = \mathcal{A}(\Delta)$  considered by Harer in [Har86, Section 1] and compute its dimension. This arc complex defines an ideal triangulation  $\mathcal{A}(\Delta) - \mathcal{A}_{\infty}(\Delta)$  of the decorated Teichmüller space  $\mathscr{T}_{g}^{s}(\Delta)$  which Harer used to construct his spine in [Har86, Section 2]. In Section 3 we review the definition of Harer's spine and compute its dimension proving Theorem 1.1.

Acknowledgments. We thank Javier Aramayona, Mladen Bestvina, Maxime Fortier Bourque, John Harer, Conchita Martínez-Pérez, and Andrew Putman for useful communication. The first author was funded by CONAHCyT through the program *Estancias Posdoctorales por México*. The third author's work was supported by UNAM *Posdoctoral Program (POSDOC)*. All authors are grateful for the financial support of DGAPA-UNAM grant PAPIIT IA106923.

#### 2. Arc systems and Harer's complex of arcs

We recall the definition of Harer's complex of arcs  $\mathcal{A} = \mathcal{A}(\Delta)$  from [Har86, Section1]. For the sake of completeness we include a full-detailed computation of its dimension in Theorem 2.2.

Let F be a closed, orientable surface of genus g and let  $\{p_1, \ldots, p_s\}$  be a collection of distinguished points in F. If  $\Delta = \{p_1, \ldots, p_m\}$ ,  $m \leq s$ , we write  $P = \{p_{m+1}, \ldots, p_s\}$  and  $F_0 = F - P$  so that the surface  $F_0$  has m distinguished points and s - m punctures. In what follows, the parameters r and n appearing in [Har86, Section 1] are assumed to be zero.

A properly imbedded path in  $F_0$  between two points of  $\Delta$  will be called a  $\Delta$ -arc. The isotopy class in  $F_0$  (rel  $\Delta$ )  $[\alpha_0, \ldots, \alpha_k]$  of a family of  $\Delta$ -arcs will be called a *rank-k arc system* if:

- (1)  $\alpha_i \cap \alpha_j \subset \Delta$  for distinct *i* and *j*, and
- (2) for each connected component B of the surface obtained by splitting  $F_0$  along  $\alpha_0, \ldots, \alpha_k$ , the Euler characteristic of the double of B along  $\partial B \Delta$  is negative.

The condition (2) ensures that  $\Delta$ -arcs are not null-homotopic (rel  $\Delta$ ), and no two distinct  $\Delta$ -arcs are homotopic (rel  $\Delta$ ).

**Remark 1.** Let us give a detailed explanation of the connected components appearing in condition (2). Consider the surface  $F_0$  and a collection of  $\Delta$ -arcs  $\alpha_0, \ldots, \alpha_k$  such that condition (1) holds. Let  $B^o$  be one of the connected components of the open surface  $F_0 - \bigcup_{i=0}^k \alpha_i$ . Hence  $B^o$  is the interior of a surface B with non-empty boundary, and we have an *attaching map*  $\phi_B: \partial B \to \bigcup_{i=0}^k \alpha_i$ . Since  $\bigcup_{i=0}^k \alpha_i$  is canonically a 1-dimensional CW-complex (with 0-skeleton contained in  $\Delta$ ), we can pull-back that cellular structure to  $\partial B$ . By an abuse of notation we call  $\Delta$  the 0-skeleton of  $\partial B$ . We will be considering each B as a polygon possibly with punctures and every edge will be labeled with the corresponding  $\Delta$ -arc in the the system.

**Remark 2.** Assume s = m, that is we have no punctures and  $F_0 = F$ . If each connected component of  $F - \bigcup_{i=1}^{k} \alpha_i$  is homeomorphic to a disk, then the  $\Delta$ -arcs determine a CW-structure on F.

**Definition 2.1** (Harer's complex of arcs). Let  $\mathcal{A} = \mathcal{A}(\Delta)$  be the simplicial complex that has a k-simplex  $\langle \alpha_0, \ldots, \alpha_k \rangle$  for each rank-k arc system in  $F_0$  and such that  $\langle \beta_0, \ldots, \beta_l \rangle$  is a face of  $\langle \alpha_0, \ldots, \alpha_k \rangle$  if  $\{ [\beta_0], \ldots, [\beta_l] \}$  is contained in  $\{ [\alpha_0], \ldots, [\alpha_k] \}$ .

**Theorem 2.2.** The dimension of  $\mathcal{A}(\Delta)$  is 6g - 7 + 2s + m.

The proof of this theorem follows directly from Lemma 2.4 (lower bound) and Lemma 2.6 (upper bound).

**Lemma 2.3.** Let  $\mathbf{A}$  an arc system in  $F_0$  such that all the pieces of the splitting of  $F_0$  along  $\mathbf{A}$  are triangles except for one of the pieces which is a triangle T with m marked points and n punctures within its interior. Then there is an arc system  $\mathbf{B}$ , obtained by adding 3m + 2n  $\Delta$ -arcs to  $\mathbf{A}$ , that splits  $F_0$  into triangles and once-punctured monogons.

*Proof.* For m = 1 and n = 0, **B** is formed by adding three  $\Delta$ -arcs, splitting T into three triangles. For m = 0 and n = 1, **B** is constructed by adding two  $\Delta$ -arcs and the pieces of the splitting of T along **B** are two triangles and a one-punctured monogon. In the general case, **B** is formed by adding three  $\Delta$ -arcs for each marked point and two  $\Delta$ -arcs for each puncture, resulting in  $3m + 2n \Delta$ -arcs, see Figure 1.



FIGURE 1. Maximal arc system in a triangle.

**Lemma 2.4.** There exists an arc system **A** in  $F_0$  of rank 6g - 7 + 2s + m that splits  $F_0$  into triangles and once-punctured monogons.

*Proof.* For simplicity, let n = s - m denote the number of punctures of  $F_0$ . We will show that there is an arc system which has  $6g - 6 + 3m + 2n \Delta$ -arcs that splits  $F_0$  into triangles and once-punctured monogons. The proof is divided into cases.

Case 1: g = 0.

For m = 1 and n = 2, **A** consists of a single  $\Delta$ -arc surrounding one puncture, splitting  $F_0$  into two once-punctured monogons. Note that 6(0) - 6 + 3(1) + 2 = 1, as expected. For m = 1 and  $n \ge 3$ , we take three  $\Delta$ -arcs based on the marked point and each of these surrounding a single puncture. By splitting  $F_0$  along these three  $\Delta$ -arcs, we obtain a triangle with n - 3 punctures and 3 once-punctured monogons. By Lemma 2.3, we add  $2(n - 3) \Delta$ -arcs in the triangle with n - 3 punctures, resulting in the desired arc system **A**, with 3 + 2(n - 3) = 6(0) - 6 + 3(1) + 2n $\Delta$ -arcs. Similar procedures apply to cases where  $m \ge 2$ , by selecting  $\Delta$ -arcs such that one of the pieces resulting from the splitting  $F_0$  along these  $\Delta$ -arcs is a triangle which contain the remaining points of  $\Delta$  and the punctures. Then by Lemma 2.3, we add the remaining  $\Delta$ -arcs.

Case 2:  $g \ge 1$ .

Consider a  $\Delta$ -arc system with 2g arcs, all based in a single point of  $\Delta$ , such that the system splits F in a 4g-polygon. Consider first the arc system  $\mathbf{A}'$  constructed using  $2g \Delta$ -arcs on the sides of the polygon, and  $4g-3 \Delta$ -arcs that connect a vertex of the polygon with the remaining ones. These  $6g-3 \Delta$ -arcs are selected such that the splitting of  $F_0$  along them, only one piece contains the n punctures and the remaining m-1 points of  $\Delta$  in its interior (see Figure 2a). Finally, by Lemma 2.3 we add  $3(m-1)+2n \Delta$ -arcs to the arc system  $\mathbf{A}'$  in the special triangle, obtaining the desired arc system  $\mathbf{A}$ , with  $6g-3+3(m-1)+2n=6g-6+3m+2n \Delta$ -arcs (see Figure 2b).



FIGURE 2. Arc systems  $\mathbf{A}'$  and  $\mathbf{A}$  in a surface  $F_0$ 

**Proposition 2.5** (Characterization of maximal arc systems). Let  $\mathbf{A}$  be an arc system in  $F_0$ . The arc system  $\mathbf{A}$  is maximal if and only if the pieces of the splitting of  $F_0$  along  $\mathbf{A}$  are triangles or once-punctured monogons.

Proof. Suppose that  $\mathbf{A}$  is maximal, and let B be a piece of the splitting of  $F_0$  along  $\mathbf{A}$ . We have that  $B \cong F_{h,b}^n$ , where h is the genus of the surface, b is the number of boundary components, and n is the number of punctures. Let  $\ell$  be the number of marked points on the boundary of B. Note that  $b, \ell \ge 1$ , and there is at least one marked point in each boundary component. We will prove that h = 0, b = 1 (a disk), and either n = 1 and  $\ell = 1$  (a once-punctured monogon), or n = 0 and  $\ell = 3$  (a triangle). If  $h \ge 1$ ,  $b \ge 2$ ,  $n \ge 2$ , or  $\ell \ge 4$ , we can add  $\Delta$ -arcs as shown in Fig. 3a to extend the arc system  $\mathbf{A}$  into a new one, which contradicts the maximality of  $\mathbf{A}$ . Thus,  $h = 0, b = 1, n \le 1$ , and  $\ell \le 3$ . We exclude the case n = 1 and  $\ell = 2, 3$  since we can add new  $\Delta$ -arcs to  $\mathbf{A}$  and obtain a new arc system, contradicting the maximality of  $\mathbf{A}$  (see Fig. 3b). Additionally, we exclude when n = 0 and  $\ell = 1, 2$ , since the boundary of the piece B represents  $\Delta$ -arcs of  $\mathbf{A}$ , and in these cases, the piece B gives us two arcs that are isotopic relative to  $\Delta$  (when  $\ell = 2$ ) or a  $\Delta$ -arc which is isotopic to the marked point (when  $\ell = 1$ ). Therefore B is either a triangle  $(h = 0, b = 1, n = 0, \text{ and } \ell = 3)$  or a once-punctured monogon  $(h = 0, b = 1, n = 1, \text{ and } \ell = 1)$ .

Now, consider an arc system  $\mathbf{A}$  that splits  $F_0$  into triangles or once-punctured monogons. Suppose  $\mathbf{A}$  is not maximal. Thus, there exists an arc system  $\mathbf{A}'$  such that  $\mathbf{A} \subsetneq \mathbf{A}'$ . Let  $\alpha \in \mathbf{A}' - \mathbf{A}$ . Then,  $\alpha$  lies in a piece B of the splitting of  $F_0$  along  $\mathbf{A}$ , which is either a triangle or a once-punctured monogon. Then,  $\alpha$  is isotopic to a  $\Delta$ -arc of  $\mathbf{A}$  or a marked point. However, this contradicts the assumption that  $\mathbf{A}'$  forms an arc system. Therefore,  $\mathbf{A}$  is maximal.  $\Box$ 



FIGURE 3.  $\Delta$ -arcs for extending the arc system

**Lemma 2.6.** Let A be a maximal arc system in  $F_0$ . Then the rank of A is 6g - 7 + 2s + m.

Proof. Let **A** be a maximal arc system in  $F_0$ . We have two cases m = s or  $m \neq s$ . In the first case, we have by Proposition 2.5 and Remark 2 that **A** induces a CW-structure in F. Denote by V the number of vertices, E the number of edges, and T the number of triangles. Then  $\chi(F) = V - E + T = m - E + T$ . Note that each arc labels exactly two edges of the triangles, hence we have  $E = \frac{3T}{2}$ . Thus  $\chi(F) = 2 - 2g = m - E + \frac{2E}{3}$ , it follows that E = 6g - 6 + 3m. Therefore **A** has rank 6g - 7 + 3m.

Now suppose that  $s \neq m$ . Note that any two once-punctured monogons in the splitting of  $F_0$  along **A** cannot share a side unless the surface is homeomorphic to the sphere with two punctures and one marked point. If  $F_0$  is the sphere with two punctures and one marked point the claim is clear, then assume that we are not in this case. Therefore any once-punctured monogon shares a side with a triangle.

Note that in the splitting of  $F_0$ , there are exactly s - m pieces that are once-punctured monogons. Additionally the closed surface resulting from  $F_0$  by capping the punctures is F. Hence, by the previous observation, when we cap the punctures in  $F_0$  we need to remove exactly 2(s-m) arcs to obtain an arc systems  $\mathbf{A}'$  in F. By the closed case we have rank $(\mathbf{A}') = 6g - 7 + 3m$ . It follows that rank $(\mathbf{A}) = 6g - 7 + 2s + m$ .

**Proposition 2.7.** Any arc system  $\mathbf{A} = \langle \alpha_0, \dots, \alpha_k \rangle$  in  $F_0$  is contained in a maximal arc system. In particular, an arc system of rank 6g - 7 + 2s + m is maximal.

*Proof.* Suppose that **A** is not maximal. By Proposition 2.5, the splitting of  $F_0$  along the arc system **A** yields at least one piece that is neither a triangle nor a once-punctured monogon. As in the proof of Proposition 2.5, we can add a  $\Delta$ -arc  $\alpha_{k+1}$  as described in Fig. 3 extending the arc system **A** into a new system **A'**. By Lemma 2.6 we add a finite number of  $\Delta$ -arcs to obtain a maximal arc system which contains **A**.

### 3. HARER'S SPINE AND ITS DIMENSION

In this section we recall the definition of Harer's spine  $\mathcal{Y}$  from [Har86, Section 2] and compute its dimension proving Theorem 1.1. Let  $1 \leq m \leq s$ ,  $\Delta = \{p_1, \ldots, p_m\}$ ,  $P = \{p_{m+1}, \ldots, p_s\}$  and  $F_0 = F - P$  as in Section 2. The decorated Teichmüller space  $\mathscr{T}_g^s(\Delta)$  is the space of all pairs  $(R, \lambda)$  where R is a point of  $\mathscr{T}_g^s$  and  $\lambda$  is a projective class of a collection of positive weights on the m points of  $\Delta$ . Topologically  $\mathscr{T}_g^s(\Delta)$  is homeomorphic to the product of  $\mathscr{T}_g^s$  and a open simplex of dimension m-1. There is a natural diagonal action of  $\operatorname{PMod}_g^s$  on  $\mathscr{T}_g^s(\Delta)$  when  $m \leq s$ , and only when m = s it actually admits an action of  $\operatorname{Mod}_g^s$ .

We say an arc system  $\mathbf{A}$  fills up  $F_0$  provided every point in  $\Delta$  is the initial or final point of an arc in  $\mathbf{A}$ , and each connected component B of the splitting of  $F_0$  along  $\mathbf{A}$  is either homeomorphic to a disk or a once-puntured disk. As a direct consequence of Proposition 2.5 every maximal arc system fills up  $F_0$ . Note that when P is empty, that is we have no punctures,  $\mathbf{A}$  determines a CW-structure on F such that  $\Delta$  is the 0-skeleton and the union of the arcs in  $\mathbf{A}$  form the 1-skeleton.

Let  $\mathcal{A}_{\infty}(\Delta)$  be the codimension 2 subcomplex of  $\mathcal{A}(\Delta)$  formed by all the arc systems that do not fill up  $F_0$ . From the definition we can see that  $\mathrm{PMod}_g^s$  acts simplicially on  $\mathcal{A}(\Delta)$  and preserves the subcomplex  $\mathcal{A}_{\infty}(\Delta)$ . Notice that it admits an action of  $\mathrm{Mod}_g^s$  only when s = m. The following result states that  $\mathcal{A}(\Delta) - \mathcal{A}_{\infty}(\Delta)$  defines an ideal triangulation of the decorated Teichmüller space  $\mathscr{T}_q^s(\Delta)$ .

**Theorem 3.1.** [Har86, Theorem 1.3] There is a  $PMod_q^s$ -equivariant homeomorphism

$$\phi: \mathscr{T}_g^s(\Delta) \longrightarrow (\mathcal{A}(\Delta) - \mathcal{A}_\infty(\Delta)).$$

Harer's proof of this theorem in [Har86] is in the conformal category and is due to Mumford. In [Har88, Chapter 2] Harer attributes to W. Thurston the original construction of this ideal triangulation. A proof using hyperbolic geometry was provided by B.H. Bowditch and D.B. A. Epstein [BE88]. See also the work of R. C. Penner [Pen87] for an alternative proof. In the statement of Theorem 3.1 we can replace  $PMod_q^s$  by  $Mod_q^s$  when s = m.

Let us denote by  $\mathcal{A}^0$  and by  $\mathcal{A}^0_{\infty}$  the barycentric subdivisions of the complexes  $\mathcal{A}(\Delta)$  and  $\mathcal{A}_{\infty}(\Delta)$ , respectively.

**Definition 3.2** (Harer's spine). Let  $\mathcal{Y}(=\mathcal{Y}_g^{s,m})$  be the subcomplex of  $\mathcal{A}^0$  spanned by the arc systems that fill up  $F_0$ . Therefore, the vertices of  $\mathcal{Y}$  consist of the set of arc systems that fill up  $F_0$  and an  $\ell$ -simplex of  $\mathcal{Y}$  is given by a chain of arc systems

$$\mathbf{A}_0 \subset \cdots \subset \mathbf{A}_{\ell},$$

where all the inclusions are strict and all the  $\mathbf{A}_i$  fill up  $F_0$ . We call *Harer's spine* both the complex  $\mathcal{Y}$  and the subspace  $\phi^{-1}(\mathcal{Y}) \subset \mathscr{T}_q^s(\Delta)$ .

**Remark 3.** We are mostly following the notation from [Har86]. However, instead of using  $\mathcal{Y}$ , Harer uses the notation  $Y^0$  for the spine, since (as he explains) it coincides with the first barycentric subdivision of a complex Y given by the dual of  $\mathcal{A}$ .

Harer states the following result:

**Theorem 3.3.** [Har86, Theorem 2.1] The complex  $\mathcal{A}^0$  equivariantly deformation retracts onto the 4g - 5 + s + m dimensional complex  $\mathcal{Y}$ . This retraction provides a  $\mathrm{PMod}_g^s$ -equivariant homotopy equivalence between  $\mathcal{A}^0 - \mathcal{A}_\infty^0$  and  $\mathcal{Y}$ .

Harer's computation of the dimension of  $\mathcal{Y}$  is correct when m = s, however is off by 1 when m < s as the following example and our proof of Theorem 1.1 below show.

**Example 3.4** (Case  $g \ge 1$ , s = 2, m = 1). Let F be a closed surface of genus  $g \ge 1$  with s = 2 marked points  $\{p_1, p_2\}$  and we take  $\Delta = \{p_1\}$  and  $P = \{p_2\}$ . From [Har86, Theorem 2.1] the dimension of Harer's spine  $\mathcal{Y}$  is equal to  $4g - 2 = \operatorname{vcd}(\operatorname{Mod}_g^2)$ . However, let us exhibit a (4g - 1)-simplex of  $\mathcal{Y}$ .

Consider a CW-structure of the surface F given by the 0-cell  $p_1$  and 1-cells  $\alpha_0, \ldots, \alpha_{2g-1}$  that cut the surface into a 4g-gon; see figure 4a. Notice that the arc system  $\mathbf{A}_0 = \langle \alpha_0, \ldots, \alpha_{2g-1} \rangle$ cuts the surface  $F_0 = F - \{p_2\}$  into a 4g-gon with a puncture in P, hence it fills up  $F_0$  and it gives a vertex of  $\mathcal{Y}$  of minimal weight= 2g - 1. By adding the 4g - 3 orange arcs  $\alpha_{2g}, \ldots \alpha_{6g-4}$ and the 2 violet arcs  $\alpha_{6g-3}$  and  $\alpha_{6g-2}$  from figure 4b, we obtain an arc system  $\mathbf{A}_{4g-1}$  of rank 6g - 2 which fills up the surface  $F_0$  and gives a vertex of  $\mathcal{Y}$  of maximal weight= 6g - 2. Hence we obtain the chain

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_{4g-1},$$

by adding an arc at a time, which defines a (4g-1)-simplex of  $\mathcal{Y}$ .



FIGURE 4. Arcs that give the arc systems  $\mathbf{A}_0$  of minimal weight and  $\mathbf{A}_{4g-1}$  of maximal weight.

Proof of Theorem 1.1. Note that, whenever  $\mathbf{A}_1 \subset \mathbf{A}_2$  and  $\mathbf{A}_1$  fills up  $F_0$ , we have that  $\mathbf{A}_2$  also fills up. Recall that every maximal arc system in  $F_0$  fills up the surface and is of rank 6g - 7 + 2s + m (Lemma 2.6), and every arc system is contained in a maximal one (Proposition 2.7). Hence to compute the dimension of  $\mathcal{Y}$  it is enough to determine the rank of a minimal arc system that fills up  $F_0$ . Now we split the proof into two cases.

**Case 1:** m < s. In this case the set P is non-empty and  $F_0$  is a punctured surface.

We claim an arc system  $\mathbf{A}$  is minimal among the arc systems that fill up  $F_0$  if and only if it splits  $F_0$  into pieces all of which are once-punctured discs. First assume that  $\mathbf{A}$  fills up  $F_0$ , it is minimal and there is a connected component B of corresponding splitting that is homeomorphic to a disc. Then, by connectedness of  $F_0$ , an edge of B must have the same label as an edge of another connected component (this connected component exists because P is nonempty). Thus removing the corresponding arc we get a strictly smaller arc system that fills up  $F_0$ . For the converse, assume that  $\mathbf{A}$  splits  $F_0$  into once-punctured discs and let B be a connected component of the corresponding splitting. Then removing one arc from B will have the effect of either gluing two different pieces or gluing two edges of the same piece, in either case the resulting arc system do not fill up  $F_0$ .

Let **A** a minimal arc system in  $\mathcal{Y}$ , we claim the rank r of **A** is 2g + s - 3. Then **A** induces a CW-structure on F with m 0-cells, r + 1 1-cells, and s - m 2-cells (one for each once-punctured disc in the splitting of  $F_0$ ). Hence, as F is a closed surface of genus g, the Euler characteristic of F leads to

$$2 - 2q = m - (r + 1) + (s - m) = -r + s - 1.$$

hence r = 2g + s - 3. Therefore

$$\dim(\mathcal{Y}) = (6g - 7 + 2s + m) - (2g + s - 3) = 4g - 4 + s + m.$$

**Case 2:** m = s. In this case the set of punctures is empty, and any arc system that fills up the surface determines a cellular structure on  $F = F_0$ . We claim an arc system is minimal among the arc systems that fill up if and only if it splits F into exactly one disc. The proof of this claim is similar to the analogous claim in the previous case. A similar Euler characteristic computation as in the previous case implies that the rank r of a minimal arc system that fills up F satisfies

$$2 - 2g = s - (r+1) + 1 = s - r,$$

hence r = 2g + s - 2. Therefore

$$\dim(\mathcal{Y}) = (6g - 7 + 2s + m) - (2g + s - 2) = 4g - 5 + s + m.$$

We end by explaining how the dimension of Harer's spine when s = 1, and a Briman exact sequence argument, imply the existence of a model for  $\underline{E} \operatorname{PMod}_g^s$  of minimal dimension for  $s \geq 2$ . In [Mis10, Theorem 4.2], G. Mislin also used a Birman exact sequence argument to show the existence of co-compact models for  $\underline{E} \operatorname{PMod}_g^s$ ; see also the argument in [AMP14, Remark 4.5] which applies for pure mapping class groups.

Recall that the proper geometric dimension of a group G, denoted  $\underline{gd}(G)$ , is the minimum n for which there exists an n-dimensional model for  $\underline{E}G$ . If G is virtually torsion free, then  $vcd(G) \leq gd G$  ([Lüc05, Theorem 5.23]).

**Corollary 3.5.** For  $s \ge 1$  and 2g + s > 2, there exists a model for  $\underline{E} \operatorname{PMod}_g^s$  of dimension equal to  $\operatorname{vcd}(\operatorname{PMod}_g^s)$ .

*Proof.* From Theorem 1.1, for g = 0 and  $s \ge 3$ , the dimension of Harer's spine  $\mathcal{Y}$  is  $vcd(PMod_0^s) = s - 3$ , and for  $g \ge 1$  and s = 1, we have that  $\dim \mathcal{Y} = vcd(PMod_g^1) = 4g - 3$ . Therefore, in these cases Harer's spine  $\mathcal{Y}$  is a model of  $\underline{E} PMod_g^s$  of minimal dimension.

Now let  $s \ge 2$  and  $g \ge 1$ . Then  $\pi_1(F_g^{s-1})$  is a nonabelian free group and recall that, due to a well-known theorem by Dunwoody, any finite extension  $\Delta$  of a non trivial finitely generated free group has  $\underline{gd}(\Delta) = 1$ . Hence, from [Lüc05, Theorem 5.16] applied to the Birman short exact sequence

$$1 \to \pi_1(F_g^{s-1}) \to \operatorname{PMod}_g^s \to \operatorname{PMod}_g^{s-1} \to 1$$

it follows from an inductive argument that

 $\underline{\mathrm{gd}}(\mathrm{PMod}_g^s) \leq \underline{\mathrm{gd}}(\mathrm{PMod}_g^{s-1}) + 1 \leq \underline{\mathrm{gd}}(\mathrm{PMod}_g^1) + (s-1) = \mathrm{vcd}(\mathrm{PMod}_g^1) + (s-1) = \mathrm{vcd}(\mathrm{PMod}_g^s).$ 

#### References

- [AJPTN18] J. Aramayona, D. Juan-Pineda, and A. Trujillo-Negrete. On the virtually cyclic dimension of mapping class groups of punctured spheres. *Algebr. Geom. Topol.*, 18(4):2471–2495, 2018.
- [AMP14] J. Aramayona and C. Martínez-Pérez. The proper geometric dimension of the mapping class group. Algebr. Geom. Topol., 14(1):217–227, 2014.
- [BE88] B. H. Bowditch and D. B. A. Epstein. Natural triangulations associated to a surface. *Topology*, 27(1):91–117, 1988.
- [Bro90] S. A. Broughton. The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups. *Topology Appl.*, 37(2):101–113, 1990.
- [BV06] M. R. Bridson and K. Vogtmann. Automorphism groups of free groups, surface groups and free abelian groups. In Problems on mapping class groups and related topics, volume 74 of Proc. Sympos. Pure Math., pages 301–316. Amer. Math. Soc., Providence, RI, 2006.
- [CJRLASS] N. Colin, R. Jiménez Rolland, P. L. León Álvarez, and L. J. Sánchez Saldaña. A spine for Teichmüller space of nonorientable punctured surfaces. *In preparation.*
- [FB24] M. Fortier Bourque. The dimension of Thurston's spine. Int. Math. Res. Not. IMRN, (4):3487– 3496, 2024.
- [FGM20] R. Flores and J. González-Meneses. Classifying spaces for the family of virtually cyclic subgroups of braid groups. Int. Math. Res. Not., (5):1575–1600, 2020.
- [Har86] J. L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Invent. Math.*, 84(1):157–176, 1986.
- [Har88] J. L. Harer. The cohomology of the moduli space of curves. In *Theory of moduli (Montecatini Terme, 1985)*, volume 1337 of *Lecture Notes in Math.*, pages 138–221. Springer, Berlin, 1988.

- [HOP14] S. Hensel, D. Osajda, and P. Przytycki. Realisation and dismantlability. *Geom. Topol.*, 18(4):2079–2126, 2014.
- [Irm24] I. Irmer. The Differential Topology of the Thurston Spine of Teichmüller Space. *arXiv e-prints*, page arXiv:2211.03429, January 2024.
- [Ji14] L. Ji. Well-rounded equivariant deformation retracts of Teichmüller spaces. *Enseign. Math.*, 60(1-2):109–129, 2014.
- [JPTN16] D. Juan-Pineda and A. Trujillo-Negrete. On classifying spaces for the family of virtually cyclic subgroups in mapping class groups. *Pure Appl. Math. Q.*, 12(2):261–292, 2016.
- [JRLASS24] R. Jiménez Rolland, P. L. León Álvarez, and L. J. Sánchez Saldaña. Commensurators of abelian subgroups and the virtually abelian dimension of mapping class groups. J. Pure Appl. Algebra, 228(6):Paper No. 107566, 22, 2024.
- [JW10] L. Ji and S. A. Wolpert. A cofinite universal space for proper actions for mapping class groups. In In the tradition of Ahlfors-Bers. V, volume 510 of Contemp. Math., pages 151–163. Amer. Math. Soc., Providence, RI, 2010.
- [Lüc05] W. Lück. Survey on classifying spaces for families of subgroups. In Infinite groups: geometric, combinatorial and dynamical aspects, volume 248 of Progr. Math., pages 269–322. Birkhäuser, Basel, 2005.
- [Mis10] G. Mislin. Classifying spaces for proper actions of mapping class groups. *Münster J. Math.*, 3:263–272, 2010.
- [NP18] B. Nucinkis and N. Petrosyan. Hierarchically cocompact classifying spaces for mapping class groups of surfaces. Bull. Lond. Math. Soc., 50(4):569–582, 2018.
- [Pen87] R. C. Penner. The decorated Teichmüller space of punctured surfaces. Comm. Math. Phys., 113(2):299–339, 1987.
- [Thu85] W. Thurston. A spine for Teichmüller space. *Preprint*, 1985.

Instituto de Matemáticas, Universidad Nacional Autónoma de México. Oaxaca de Juárez, Oaxaca, México 68000

Email address: rita@im.unam.mx Email address: ncolin@im.unam.mx

Email address: porfirio.leon@im.unam.mx

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México

Email address: luisjorge@ciencias.unam.mx