Linear representation stable bounds for the integral cohomology of pure mapping class groups

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Abstract

In this paper we study the integral cohomology of pure mapping class groups of surfaces, and other related groups and spaces, as FI-modules. We use recent results from Church, Miller, Nagpal and Reinhold to obtain explicit linear bounds for their presentation degree and to give an inductive description of these FI-modules. Furthermore, we establish new results on representation stability, in the sense of Church and Farb, for the rational cohomology of pure mapping class groups of non-orientable surfaces.

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1 Introduction

Let Σ be a compact connected surface with (possibly empty) boundary and let $P = \{p_1, \dots, p_n\}$ be a set of n distinct points in the interior of Σ, we call these points *punctures*. A compact surface of genus g with r boundary components and n punctures will be denoted by $\Sigma_{g,r}^n$ if it is orientable, or by $N_{g,r}^n$ if it is non-orientable. Let $\mathrm{Diff}(\Sigma, P)$ be the group of all, orientation preserving if Σ is orientable, diffeomorphisms $h : \Sigma \to \Sigma$ such that $h(P) = P$ and h restricts to the identity on the boundary of Σ. The *mapping class group* $\mathrm{Mod}^n(\Sigma)$ is the group of isotopy classes of elements of $\mathrm{Diff}(\Sigma, P)$. The *pure mapping class group* $\mathrm{PMod}^n(\Sigma)$ is the subgroup of $\mathrm{Mod}^n(\Sigma)$ that consists of the isotopy classes of diffeomorphisms fixing each puncture. If $P = \varnothing$, we write $\mathrm{Mod}(\Sigma)$.

In this paper we study the FI-module structure of the integral cohomology of the pure mapping class groups of orientable and non-orientable surfaces and obtain the following result.

Theorem 1.1. *Let Σ be a surface such that:*

$$\Sigma = S^2, \mathbb{T}, \Sigma_1^1 \text{ or } \Sigma = \Sigma_{g,r} \text{ with } 2g + r > 2, \text{ if the surface is orientable, or}$$

$$\Sigma = \mathbb{R}P^2, \mathbb{K} \text{ or } \Sigma = N_{g,r} \text{ with } g \geq 3 \text{ and } r \geq 0 \text{ in the non-orientable case.}$$

Let $k \geq 0$ and $\lambda = 1$ if Σ is orientable and $\lambda = 0$ if Σ is non-orientable.

(a) **(Polynomial Betti numbers growth).** *If $\mathbb{F}$ is a field, then there are polynomials $p_{k,\mathbb{F}}^{\Sigma}$ of degree at most $2k$ such that*

$$\dim_{\mathbb{F}} H^k\big(\mathrm{PMod}^n(\Sigma); \mathbb{F}\big) = p_{k,\mathbb{F}}^{\Sigma}(n)$$

if $n > \max(-1, 16k - 4\lambda - 2)$ (and for all $n \geq 0$ if the surface Σ has nonempty boundary).

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(b) **(Inductive description).** *The natural map*

$$\operatorname{Ind}_{S_{n-1}}^{S_n} H^k\big(\operatorname{PMod}^{n-1}(\Sigma);\mathbb{Z}\big) \to H^k\big(\operatorname{PMod}^{n}(\Sigma);\mathbb{Z}\big)$$

is surjective for $n > \max(0, 18k - 4\lambda - 1)$ (for $n > 2k$ if $\partial\Sigma \neq \varnothing$) and, for $n > \max(0, 34k - 8\lambda - 2)$ (for $n > 2k$ if $\partial\Sigma \neq \varnothing$), the kernel of this map is the image of the difference of the two natural maps[1]

$$\operatorname{Ind}_{S_{n-2}}^{S_n} H^k\big(\operatorname{PMod}^{n-2}(\Sigma);\mathbb{Z}\big) \rightrightarrows \operatorname{Ind}_{S_{n-1}}^{S_n} H^k\big(\operatorname{PMod}^{n-1}(\Sigma);\mathbb{Z}\big).$$

Theorem 1.1 follows from Corollary 4.3 and Propositions 2.3 and 2.1 below. Corollary 4.3 is a particular case of Theorem 4.2, which establishes explicit bounds for the presentation degree of the integral cohomology of classifying spaces $\operatorname{BPDiff}^n(M)$ of pure diffeomorphisms groups of manifolds M of dimension $d \geq 2$. This result is obtained by applying arguments from [CMNR18] to spectral sequences arising from classical fibrations that relate the classifying spaces of groups of pure diffeomorphisms with configuration spaces (see Section 4).

Theorem 1.1(a) is known to be true for all $n \geq 0$ for pure mapping class groups of orientable surfaces with boundary ([JR15, Theorems 1.2]) and in the stable range for orientable closed surfaces ([JR15, Theorem 1.5]). Furthermore, in [BT01, Theorem 1.1] Bödigheimer and Tillmann gave a decomposition of the classifying space associated with the stable pure mapping class groups of orientable surfaces with boundary $\operatorname{PMod}^n(\Sigma_{\infty,r})$. Together with Harer's homological stability theorem [Har85] this implies that for any field $\mathbb{F}$, in cohomological degrees $* \leq g/2$,

$$H^*(\operatorname{PMod}^n(\Sigma_{g,r});\mathbb{F}) \cong H^*(\operatorname{Mod}(\Sigma_{g,r});\mathbb{F}) \otimes \mathbb{F}[x_1,\ldots,x_n],$$

where each x_i has degree 2. The action of the symmetric group S_n on the left hand side corresponds to the action of S_n on the polynomial ring in n variables by permutation of the variables x_i. Therefore, for an orientable surface Σ with non-empty boundary, it follows that for $* \leq g/2$ the polynomials $p^{\Sigma}_{*,\mathbb{F}}(n)$ from Theorem 4.3(a) are given by

$$p^{\Sigma}_{*,\mathbb{F}}(n) = \dim_{\mathbb{F}} H^*(\operatorname{PMod}^n(\Sigma_{g,r});\mathbb{F}) = \begin{cases} \dim_{\mathbb{F}} H^*(\operatorname{Mod}(\Sigma_{g,r});\mathbb{F}) & \text{if } * = 2k+1, \\ \binom{n+k-1}{n-1} + \dim_{\mathbb{F}} H^*(\operatorname{Mod}(\Sigma_{g,r});\mathbb{F}) & \text{if } * = 2k, \end{cases}$$

of degree 0 and k, respectively.

Theorem 1.1(b) gives an inductive description of the FI-module in terms of a explicit finite amount of data. It is equivalent to the statement that the FI-module $H^k(\operatorname{PMod}^\bullet(\Sigma);\mathbb{Z})$ is *centrally stable* at $> \max(0, 34k-8\lambda-2)$, in the sense of Putman's [Put15] original notion of *central stability*, which can be viewed as a reformulation of [CE17, Theorem C] (see for example [Pat17, Prop 6.2]).

Remark 1.1: In [Put12], a preprint version of [Put15], Putman proved that the homology of pure mapping class groups of manifolds with boundary satisfies central stability. His result [Put12, Theorem D] includes orientable and non-orientable manifolds, not necessarily of finite type, with coefficients in a field (of characteristic bounded below or zero characteristic) and obtains an exponential stable range. In contrast, our approach for dimension 2 works over $\mathbb{Z}$ and gives linear bounds for the stable range. It would be interesting to see whether the central stability techniques used in [Put12] together with results from [CMNR18] could be used to obtain better bounds and include mapping class groups for general manifolds.

[1] See Proposition 2.1 and Remark 2.2 for details.

Furthermore, from [CEF15, Theorem 1.5, 1.13] (see also [CEF14, Proposition 3.9]) it follows that Theorem 4.3 also implies that the rational cohomology of pure mapping class group of surfaces satisfies *representation stability* and have characters that are eventually polynomial. From Proposition 2.7 and the upper bounds for stable degree and the local degree obtained in Corollary 4.3, an explicit stable range is obtained.

Theorem 1.2. *Let $k \geq 0$ and consider Σ a surface as in Theorem 1.1. Let*

$$N = \begin{cases} 4k & \text{if } \partial\Sigma \neq \varnothing, \\ \max(0, 20k - 4\lambda - 1) & \text{otherwise, with } \lambda = 1 \text{ if } \Sigma \text{ is orientable and } \lambda = 0 \text{ if not.} \end{cases}$$

- **i) (Representation stability over $\mathbb{Q}$).** *The decomposition of $H^k\big(\mathrm{PMod}^n(\Sigma); \mathbb{Q}\big)$ into irreducible S_n-representations stabilizes in the sense of uniform representation stability (as defined in [CF13]) for $n \geq N$.*
- **ii) (Polynomial characters).** *For $n \geq N$, the character χ_n of $H^k\big(\mathrm{PMod}^n(\Sigma); \mathbb{Q}\big)$ is of the form $\chi_n = Q_k(Z_1, Z_2, \ldots, Z_r)$, where $\deg(\chi_n) \leq 2k$ and $Q_k \in \mathbb{Q}[Z_1, Z_2, \ldots]$ is a unique polynomial, independent of n, in the class functions $Z_\ell(\sigma) := \#$ cycles of length ℓ in σ, for any $\sigma \in S_n$.*
- **iii) (Stable inner products).** *If $Q \in \mathbb{Q}[Z_1, Z_2, \ldots]$ is any character polynomial, then the inner product $\langle \chi_n, Q\rangle_{S_n}$ is independent of n once $n \geq \max(N, 2k + \deg(Q))$.*
- **iv) ("Twisted" homological stability over $\mathbb{Q}$).** *Let $Q \in \mathbb{Q}[Z_1, Z_2, \ldots]$ be a character polynomial and consider $\{W_n^Q\}_n$ the associated sequence of virtual S_n-representations. Then the dimension $\dim_{\mathbb{Q}}\big(H^k(\mathrm{Mod}^n(\Sigma); W_n^Q)\big)$ is constant for $n \geq N$. In particular, the sequence $\{H^k\big(\mathrm{Mod}^n(\Sigma); \mathbb{Q}\big)\}$ satisfies rational homological stability for $n \geq N$.*

Theorem 1.2 above recovers [JR15, Theorem 1.1] for the pure mapping class group of orientable surfaces and gives new representation stability results for the rational cohomology of pure mapping class groups of non-orientable surfaces and low genus cases that were not worked out explicitly in [JR15] and [JR11]. Using the theory of weight and stability degree from [CEF15] better bounds can be computed for the stable range. However, if the surface Σ has nonempty boundary, from Proposition 2.7 the stable range can be improved to $N = 4k$, which recovers the stable range obtained in [JR15, Theorem 1.1] for orientable surfaces with boundary.

Theorem 1.2(iv) recovers rational homological stability "by punctures" for mapping class groups of surfaces, this was previously obtained in [HW10, Propositon 1.5], with a better linear stable range, for mapping class groups of connected manifolds of dimension ≥ 2 (see also [Han09, Theorems 1.3 & 1.4]).

With the same techniques, we obtain bounds for the presentation degree of the FI-module given by the integral cohomology of the *hyperelliptic mapping class group with marked points* Δ_g^n and representations stability for its rational cohomology groups in Corollary 4.5.

Known results for pure mapping class groups of non-orientable surfaces. Wahl proved in [Wah08, Theorem A] that for $g \geq 4k + 3$, the homology groups $H_k(\mathrm{Mod}(N_{g,r}); \mathbb{Z})$ are independent of g and r. This is the non-orientable version of Harer's stability theorem for mapping class group of orientable surfaces [Har85]. Homological stability for mapping class groups of non-orientable surfaces with marked points was obtained in [Han09, Theorem 1.1]:

for fixed $n \geq 1$ the homology groups $H_k(\mathrm{Mod}^n(N_{g,r}); \mathbb{Z})$ and $H_k(\mathrm{PMod}^n(N_{g,r}); \mathbb{Z})$ are independent of g and r, when $g \geq 4k + 3$.

Wahl and Handbury also study the stable pure mapping class groups of non orientable surfaces with boundary $\mathrm{PMod}^n(N_{\infty,r})$. In particular, [Han09, Theorem 1.2] shows that there exists a homology isomorphism $B\,\mathrm{PMod}^n(N_{\infty,r}) \to B\,\mathrm{PMod}(N_{\infty,r}) \times B(O(2))^n$. This isomorphism can be used to explicitly compute, for $* \geq 4k+3$ the polynomials $p^{\Sigma}_{*,\mathbb{Z}_2}(n)$ from Theorem 1.1(a) for a non-orientable surface Σ with non-empty boundary. On the other hand, Korkmaz [Kor02] and Stukow [Stu10] computed the first homology group for the mapping class group of a non-orientable surfaces. In particular, they proved that

$$H_1(\mathrm{PMod}^n(N_g); \mathbb{Z}) \cong (\mathbb{Z}_2)^{n+1} \text{ if } g \geq 7,$$

see for instance [Kor02, Theorem 5.11]. This computation shows that that homological stability fails integrally for the pure mapping class group with respect to the number of marked points, and suggest a representation stability phenomenon.

For low genus, descriptions of the mod 2 cohomology ring have been obtained for the mapping class group $\mathrm{Mod}^n(\mathbb{K})$ of the Klein bottle $\mathbb{K}$ with marked points ([HX18]), and for the mapping class group $\mathrm{Mod}^n(\mathbb{R}P^2)$ of the projective plane $\mathbb{R}P^2$ with marked points ([MX17]).

Remark 1.2: This paper started from Remark A.14 in [MW18, Appendix A]. In [MW19, Remark 1.4], the authors suggest a method that could possibly improve the liner bounds for pure mapping class groups and classifying spaces of pure diffeomorphisms groups.

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2 Preliminaries on FI-modules

We use the framework of FI-modules to study stability properties of sequences of symmetric group representations. The purpose of this section is mostly of expository nature.

Notation: An FI-*module* (resp. FB-*module*) is a functor V from the category of finite sets and injections (resp. and bijections) to the category of $\mathbb{Z}-$modules $\mathrm{Mod}_{\mathbb{Z}}$. Given a finite set T we write V_T for $V(T)$ and for every $n \in \mathbb{N}$ we write V_n for $V_{[n]} = V([n])$ where $[n] := \{1, 2, \ldots n\}$. The category of FI-modules $\mathrm{Mod}_{\mathsf{FI}}$ is abelian.

Definition 2.1 Let $\mathbb{F}$ be any ring. An FI-module V is an FI-*module over* $\mathbb{F}$ if the functor $V : \mathrm{Mod}_{\mathsf{FI}} \to \mathrm{Mod}_{\mathbb{Z}}$ factors through $\mathrm{Mod}_{\mathbb{F}} \to \mathrm{Mod}_{\mathbb{Z}}$.

The forgetful functor $\mathcal{F} : \mathrm{Mod}_{\mathsf{FI}} \to \mathrm{Mod}_{\mathsf{FB}}$ has a left adjoint that we denote by $\mathcal{I} : \mathrm{Mod}_{\mathsf{FB}} \to \mathrm{Mod}_{\mathsf{FI}}$. The FI-modules of the form $\mathcal{I}(V)$ are called *induced*. The FI-modules that admit a finite length filtration where the quotients are induced modules are named *semi-induced*.

Definition 2.2 An FI-module V is generated by a set $S \subseteq \sqcup_{n\geq 0} V_n$ if V is the smallest FI-submodule containing S. If V is generated by some finite set, we say that V is *finitely generated*.

Finite generation of FI-modules is preserved when taking subquotients, extensions and tensor products. Furthermore FI-modules over Noetherian rings are locally Noetherian i.e., every sub-FI-module is finitely generated ([CEFN14, Theorem A]). Therefore an FI-module is finitely generated if and only if it is *finitely presented*.

2.1 Generation and presentation degree

Consider the functor $\mathcal{U} : \mathrm{Mod}_{\mathsf{FB}} \to \mathrm{Mod}_{\mathsf{FI}}$ that upgrades each FB-module to an FI-module by declaring that all injections which are not bijections act as the zero map. The left adjoint is called FI-*homology* and it is a right-exact functor denoted by H_0^{FI}. Its ith left-derived functor is denoted by H_i^{FI}, and will be often regarded as an FI-module by post-composing with $\mathcal{U}$.

The *degree* of a non-negatively graded abelian group M is $\deg M = \min\{d \geq -1 : M_n = 0 \text{ for } n > d\}$. In particular, an FI-module or FB-module V gives a non-negatively graded abelian group by evaluating on the sets $[n]$ and $\deg V$ can be considered.

Definition 2.3 For an FI-module or FB-module V let $t_i(V) := \deg H_i^{\mathsf{FI}}(V)$. The *generation degree* of V is $t_0(V)$ and $prd(V) := \max(t_0(V), t_1(V))$ is called the *presentation degree* of V. An FI-module V is *presented in finite degrees* if $t_0(V) < \infty$ and $t_1(V) < \infty$.

Remark 2.1: A finitely generated FI-module is presented in finite degrees. However, being 'finitely generated' is a stronger condition than being presented in finite degrees.

The category of FI-modules presented in finite degrees is abelian. These FI-modules admit the following inductive description.

Proposition 2.1 (Prop 2.4 [CMNR18]). *Let V be an FI-module.*

(1) Then $t_0(V) \leq d$ if and only if $\mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1} \to V_n$ is surjective for $n > d$.

(2) Then $t_1(V) \leq r$ if and only if the kernel of $\mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1} \to V_n$ is the image of the difference of the two natural maps $\mathrm{Ind}_{S_{n-2}}^{S_n} V_{n-2} \rightrightarrows \mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1}$ for $n > r$.

Remark 2.2: Observe that $\mathrm{Ind}_{S_{n-p}}^{S_n} V_{n-p} \cong \bigoplus_{f:[p]\hookrightarrow[n]} V_{[n]\setminus im(f)}$ for $p \geq 1$. The two natural maps $\mathrm{Ind}_{S_{n-2}}^{S_n} V_{n-2} \rightrightarrows \mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1}$ in Proposition 2.1 (2) are given by

$$d_i : \bigoplus_{f:[2]\hookrightarrow[n]} V_{[n]\setminus im(f)} \to \bigoplus_{\bar{f}=f|[2]\setminus\{i\}} V_{[n]\setminus im(\bar{f})}$$

defined by forgetting the element i from the domain $[2]$ of the injective map $f : [2] \hookrightarrow [n]$ and the maps $V_{[n]\setminus im(f)} \to V_{[n]\setminus im(\bar{f})}$ induced by the inclusions of sets $\big([n] \setminus im(f)\big) \hookrightarrow \big([n] \setminus im(\bar{f})\big)$.

Furthermore, if V is an FI-module over a field $\mathbb{F}$ of characteristic zero, then V is finitely generated if and only if the sequence $\{V_n\}_n$ of S_n-representations is *representation stable* in the sense of [CF13] and each V_n is finite-dimensional (see [CEF15, Theorem 1.13]). The notions of *weight* and *stability degree* introduced in [CEF15] can be used to obtain better explicit stable ranges for representation stability over fields of characteristic zero.

2.2 Stable and local degree

In [CMNR18] the authors study the *stable degree* and the *local degree* of an FI-module and show that they are easier to control in spectral sequence arguments than the generation and presentation degree. The precise definitions of these invariants are stated in [CMNR18, Def. 2.8 & 2.12]. We recall here the key properties that will be needed to obtain our main results. Proofs of these statements can be found in [CMNR18, Section 3].

The category FI has a proper self-embedding $\sqcup^\star : \mathsf{FI} \to \mathsf{FI}$ given by $S \mapsto S \sqcup \{\star\}$ and a morphism $f : S \to T$ extends to $f_\star : S \sqcup \{\star\} \to T \sqcup \{\star\}$ with $f_*(\star) = \star$. The shift functor $\mathcal{S} : \mathrm{Mod}_{\mathsf{FI}} \to \mathrm{Mod}_{\mathsf{FI}}$ is given by $V \mapsto V \circ \sqcup^\star$. We denote by $\mathcal{S}^n$ the n-fold iterate of $\mathcal{S}$.

The *local degree* $h^{\max}(V)$ of an FI-module V quantifies how much V needs to be shifted so that $\mathcal{S}^n V$ is semi-induced:

Proposition 2.2 (Cor 2.13[CMNR18]). *Let V be an FI-module presented in finite degrees. Then $\mathcal{S}^n V$ is semi-induced if and only if $n > h^{max}(V)$.*

The natural transformation $\mathrm{id}_{\mathsf{FI}} \to \sqcup^\star$ induces a natural transformation $\iota : \mathrm{id}_{\mathrm{Mod}_{\mathsf{FI}}} \to \mathcal{S}$. This is a natural map of FI-modules $\iota : V \mapsto \mathcal{S}V$ which, for every finite set S, takes V_S to $(\mathcal{S}V)_S = V_{S \cup \{\star\}}$ via the map corresponding to the inclusion $\iota_S : S \hookrightarrow S \cup \{\star\}$. The derivative functor $\Delta : \mathrm{Mod}_{\mathsf{FI}} \to \mathrm{Mod}_{\mathsf{FI}}$ takes an FI-module V to the cokernel $\Delta V := \mathrm{coker}(V \xrightarrow{\iota} \mathcal{S}V)$. We denote by Δ^n is the n-fold iterate of Δ. An element $x \in V(S)$ is *torsion* if there is an injection $f : S \to T$ such that $f_*(x) = 0$. An FI-module V is *torsion* if all its elements are torsion.

Definition 2.4 The *stable degree* of an FI-module V is $\delta(V) := \min\{n \geq -1 : \Delta^{n+1} V \text{ is torsion}\}$.

Proposition 2.3 (Prop 2.14 [CMNR18]). *Suppose $\mathbb{F}$ is a field, and let V be an FI-module over $\mathbb{F}$ which is presented in finite degrees and with V_n finite dimensional for all n. Then there exists an integer-valued polynomial $p \in \mathbb{Q}[X]$ of degree $\delta(V)$ such that $dim_{\mathbb{F}} V_n = p(n)$ for $n > h^{max}(V)$.*

It turns out that together, the stable degree and the local degree behave well under taking kernels and cokernels and in finite filtrations.

Proposition 2.4 (Prop 3.2 & 3.3 [CMNR18]).

(1) Suppose the FI-module V has a finite filtration $V = F_0 \supset \ldots \supset F_k = 0$ and let $N_i = F_i / F_{i+1}$. Then $\delta(V) = \max_i \delta(N_i)$ and $h^{max}(V) \leq \max_i h^{max}(N_i)$.

(2) Let $f : V \to W$ be a map of FI-modules presented in finite degrees. Then we have the following:

$$\delta(\ker f) \leq \delta(V); \qquad \delta(coker f) \leq \delta(W);$$

$$h^{max}(\ker f), h^{max}(coker f) \leq \max\big(2\delta(V) - 2, h^{max}(V), h^{max}(W)\big).$$

Furthermore, the stable and local degrees are easier to control in spectral sequence arguments. In particular they allow us to obtain linear stable ranges in the “Type A spectral sequence arguments”.

Proposition 2.5 (Prop 4.1 [CMNR18]). *Let $E_r^{p,q}$ be a cohomologically graded first quadrant spectral sequence of FI-modules converging to M^{p+q}. Suppose that for some page d, the FI-modules $E_d^{p,q}$ are presented in finite degrees, and set $D_k = \max_{p+q=k} \delta(E_d^{p,q})$ and $\eta_k = \max_{p+q=k} h^{max}(E_d^{p,q})$. Then we have the following:*

(1) $\delta(M^k) \leq D_k$

(2) $h^{max}(M^k) \leq \max\big(\max_{\ell \leq k+s-d} \eta_\ell, \max_{\ell \leq 2k-d+1}(2D_\ell - 2)\big)$, where $s = \max(k+2, d)$.

Finally, the stable degree and the local degree control the presentation degree of an FI-module and viceversa [CMNR18, Proposition 3.1].

2.3 $\mathsf{FI}\#$-modules

Let $\mathsf{FI}\#$ denote the category with objects finite based sets and with morphisms given by maps of based sets that are injective away from the basepoints: the preimage of all elements except possibly the base point have cardinality at most one. This category is isomorphic to the original one described in [CEF15, Def 4.1.1].

An $\mathsf{FI}\#$-*module* is a functor from $\mathsf{FI}\#$ to $\mathrm{Mod}_{\mathbb{Z}}$. An $\mathsf{FI}\#$-module simultaneously carries an FI- and an FI^{op}-module structure in a compatible way. Church–Ellenberg–Farb [CEF15, Theorem 4.1.5] proved that the restriction of an $\mathsf{FI}\#$-module to FI is an induced FI-module. Hence, for $\mathsf{FI}\#$-modules we have better control of the presentation degree.

Proposition 2.6 (Cor 2.13, Prop 3.1 [CMNR18]). *If V is a semi-induced FI-module, then $h^{\max}(V) = -1$. Therefore, $\delta(V) = t_0(V)$ and $t_1(V) \le \delta(V)$.*

For a finitely generated FI-module V defined over $\mathbb{Q}$, we know that the sequence $\{V_n\}_n$ satisfies representation stability. Moreover we can obtain upper bounds for the stable range in terms of the local and stable degree of V.

Proposition 2.7. *Let V be a finitely generated FI-module defined over $\mathbb{Q}$. Then the sequence $\{V_n\}_n$ satisfies representation stability for $n \ge 2\delta(V) + h^{\max}(V) + 1$. If V is an $\mathsf{FI}\#$-module defined over $\mathbb{Q}$, then the sequence $\{V_n\}_n$ satisfies representation stability for $n \ge 2\delta(V)$.*

Proof. Over $\mathbb{Q}$, semi-induced modules are the same as $\mathsf{FI}\#$-modules (since $\mathsf{FI}\#$-modules are semi-simple). By Proposition 2.2, if $n > h^{\max}(V)$, then $\mathcal{S}^n V$ is semi-induced. Then for $N = h^{\max}(V) + 1$, the shifted FI-module $\mathcal{S}^N V$ is an $\mathsf{FI}\#$-module defined over $\mathbb{Q}$ and by Proposition 2.6 it has generation degree $t_0(\mathcal{S}^N V) = \delta(\mathcal{S}^N V) = \delta(V)$ (see also [CMNR18, Prop 2.9(b)]). Therefore, by [CEF15, Corollary 4.1.8] the sequence $\{(\mathcal{S}^N V)_k\}_k$ is representation stable for $k \ge 2\delta(V)$. Since $\left(\mathcal{S}^N V\right)_k = V_{N+k}$, this means that the sequence $\{V_n\}_n$ satisfies representation stability for $n \ge 2\delta(V) + N = 2\delta(V) + h^{\max}(V) + 1$. ■

3 $\mathsf{FI}[\mathsf{G}]$-modules

In [JR15, Section 5] we introduced the notion on an $\mathsf{FI}[\mathsf{G}]$-module in order to incorporate the action of a group G on our sequences of S_n-representations. These $\mathsf{FI}[\mathsf{G}]$-modules allow us to construct new FI-modules by taking cohomology with twisted coefficients and are key in the spectral sequence arguments below.

Definition 3.1 Let G be a group. An $\mathsf{FI}[\mathsf{G}]$-*module* V is a functor from the category FI to the category $\mathrm{Mod}_{\mathbb{Z}[\mathsf{G}]}$ of G-modules over $\mathbb{Z}$. By forgetting the G-action we get an FI-module and all the notions of degree from the previous section can be considered. We call V an $\mathsf{FI}[\mathsf{G}]$-*module over* $\mathbb{F}$ if the functor $V : \mathrm{Mod}_{\mathsf{FI}} \to \mathrm{Mod}_{\mathbb{Z}[\mathsf{G}]}$ factors through $\mathrm{Mod}_{\mathbb{F}[\mathsf{G}]} \to \mathrm{Mod}_{\mathbb{Z}[\mathsf{G}]}$.

Let V be an $\mathsf{FI}[\mathsf{G}]$-module and consider a path connected space X with fundamental group G. For each integer $p \ge 0$, the *pth*-cohomology $H^p(X; \_\_)$ of X is a covariant functor from the category $\mathrm{Mod}_{\mathsf{FI}[\mathsf{G}]}$ to the category $\mathrm{Mod}_{\mathsf{FI}}$. We now see how the bounds on the stable and local degree of V provide bounds for the $\mathsf{FI}[\mathsf{G}]$-module $H^p(X; \_\_)$.

Proposition 3.1 (Cohomology with coefficients in a $\mathsf{FI}[\mathsf{G}]$-module with finite *prd*). *Let G be the fundamental group of a connected CW complex X and let V be an $\mathsf{FI}[\mathsf{G}]$-module presented* *in finite degrees with $\delta(V) \leq D$ and $h^{\max}(V) \leq \eta$. Then for every $p \geq 0$, the* FI*-module $H^p(X;V)$ is presented in finite degrees with*

$$\delta\big(H^p(X;V)\big) \leq D \quad \text{and} \quad h^{\max}\big(H^p(X;V)\big) \leq \max(2D-2,\eta).$$

Furthermore, if X has finitely many cells in each dimension and V is a finitely generated $\mathsf{FI}[\mathsf{G}]$*-module, then $H^p(X;V)$ is a finitely generated* FI*-module.*

Proof.
Given that $G = \pi_1(X)$, the universal cover $\tilde{X}$ of X has a G-equivariant cellular chain complex $C_*(\tilde{X})$. For each $p \geq 0$, let $\mathcal{A}_p$ be the set of representatives of G-orbits of the p-cells. The group of p-chains $C_p(\tilde{X})$ is a free G-module with a generator for each element in $\mathcal{A}_p$.

To obtain the FI-module $H^p(X;V)$ we consider the complex of FI-modules

$$C^{p-1}(X,V) \xrightarrow{\delta_{p-1}} C^p(X,V) \xrightarrow{\delta_p} C^{p+1}(X,V)$$

where $C^p(X;V)$ is a direct product of $\mathsf{FI}[\mathsf{G}]$-modules $\prod_{\alpha\in\mathcal{A}_p} V$ given by

$$C^p(X;V)_n := \mathrm{Hom}_G(C_p(\tilde{X}),V_n) = \mathrm{Hom}_{\mathbb{Z}[G]}\Big(\bigoplus_{\alpha\in\mathcal{A}_p}\mathbb{Z}[G],V_n\Big) \cong \prod_{\alpha\in\mathcal{A}_p}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],V_n) \cong \prod_{\alpha\in\mathcal{A}_p} V_n.$$

In the category $\mathrm{Mod}_{\mathsf{FI}[\mathsf{G}]}$ direct products preserve exactness (since in $\mathrm{Mod}_{\mathbb{Z}[\mathsf{G}]}$ direct products preserve exactness) and $\mathrm{Ind}_{S_{n-1}}^{S_n}\prod_{\alpha\in\mathcal{A}_p} V \cong \prod_{\alpha\in\mathcal{A}_p}\mathrm{Ind}_{S_{n-1}}^{S_n} V$ (see for example [Lam99, Prop. 4.4]). It follows from the definitions of stable degree and local degree that $\delta(\prod_{\alpha\in\mathcal{A}_p} V) = \delta(V)$ and $h^{\max}(\prod_{\alpha\in\mathcal{A}_p} V) = h^{\max}(V)$ (in the case $\mathcal{A}_p$ is finite, this is just an immediate consequence from Proposition 2.4 (1)). Therefore, for every $p \geq 0$

$$\delta\big(C^p(X;V)\big) \leq D \qquad \text{and} \qquad h^{\max}\big(C^p(X;V)\big) \leq \eta.$$

Since

$$H^p(X;V) = \mathrm{coker}\Big(C^{p-1}(X,V) \to \ker\big(\delta_p : C^p(X,V) \to C^{p+1}(X,V)\big)\Big),$$

the FI-module $H^p(X;V)$ is presented in finite degrees and it follows from Proposition 2.4 (2) that

$$\delta\big(H^p(X;V)\big) \leq \delta\Big(\ker\big(\delta_p : C^p(X,V) \to C^{p+1}(X,V)\big)\Big) \leq \delta\big(C^{p+1}(X,V)\big) = \delta(V) \leq D$$

and

$$\begin{aligned} h^{\max}\big(H^p(X;V)\big) &\leq \max\big(2\delta(C^{p-1}(X,V)) - 2, h^{\max}(C^{p-1}(X,V)), h^{\max}(\ker(\delta_p))\big) \\ &\leq \max\big(2D-2,\eta,\max\big(2D-2,\eta,\eta\big)\big) = \max(2D-2,\eta). \end{aligned}$$

Finally, if X has finitely many cells in each dimension and V is is a finitely generated $\mathsf{FI}[\mathsf{G}]$-module, then $H^p(X;V)$ is a subquotient of the finitely generated FI-modules $C^p(X;V)$. ■

Remark 3.1: If X has finitely many cells in each dimension and V is a finitely generated $\mathsf{FI}[\mathsf{G}]$-module over $\mathbb{Q}$ with weight $\leq m$ and stability degree N, then $H^p(X;V)$ has weight $\leq m$ and stability degree N ([JR15, Prop 5.1]).

3.1 $\mathsf{FI}\,[\mathsf{G}]$-modules and spectral sequences

Let A be an abelian group. and let X be a connected CW-complex, $x \in X$ and suppose that the fundamental group $\pi_1(X, x)$ is G. Consider an FI^{op}-fibration over X: a functor from FI^{op} to the category $\mathsf{Fib}(X)$ of fibrations over X. Let $E : \mathsf{FI}^{op} \to \mathcal{T}op$ be the FI^{op}-space of total spaces and H the FI^{op}-space of fibers over the basepoint x.

The functoriality of the Leray-Serre spectral sequence implies that we have a spectral sequence of FI-modules (see [KM18, Proposition 3.10] for details) $E_*^{p,q} = E_*^{p,q}\big(E \to X\big)$ converging to the graded FI-module $H^*(E; A)$.

The action of the fundamental group G on the cohomology of the fiber gives $H^q(H; A)$ the structure of an $\mathsf{FI}\,[\mathsf{G}]$-module, for all $q \geq 0$. Then, the E_2-page of this spectral sequence is given by the FI-modules

$$E_2^{p,q} = H^p\big(X; H^q(H; A)\big).$$

4 Cohomology of classifying spaces of pure diffeomorphisms groups

In this section we study the FI-module structure of the integral cohomology of classifying spaces of groups of pure diffeomorphisms. We consider classical fibrations that allow us to relate these cohomology groups with the cohomology of configuration spaces. Using the spectral sequence arguments from [CMNR18] together with their bounds on stable degree and local degree for cohomology of configuration spaces (see Lemma 4.1 below) and our Proposition 3.1 we obtain the corresponding bounds for classifying spaces of groups of pure diffeomorphisms in Theorem 4.2.

In sections 4.3 and 4.4 we describe how this gives us bounds for the presentation degree of the integral cohomology of pure mapping class groups of orientable and non-orientable surfaces (Corollaries 4.3 & 4.4) and hyperelliptic mapping class groups with marked points (Corollary 4.5).

4.1 Configuration spaces

For a topological space M, following the notation in [CMNR18], we denote by $\mathrm{PConf}(M)$ the FI^{op}-space that sends the set S to the space of embeddings of S into M. For a given inclusion $f : S \hookrightarrow T$ in $\mathrm{Hom}_{\mathsf{FI}}(S, T)$ the corresponding restriction $f_* : \mathrm{PConf}_S(M) \to \mathrm{PConf}_T(M)$ is given by precomposition. In particular, for the set $[n] = \{1, ..., n\}$ we obtain the configuration space $\mathrm{PConf}_n(M) = \mathrm{Emb}([n], M)$ of ordered n-tuples of distinct points on M. By taking integral cohomology we obtain an FI-module $H^k(\mathrm{PConf}(M); \mathbb{Z})$. The following Lemma 4.1 is one of the main applications in [CMNR18] of Proposition 2.5 and is a key ingredient in the proof of Theorem 4.2.

Lemma 4.1 (Theorem 4.3 [CMNR18] Linear ranges for configuration spaces). *Let M be a connected manifold of dimension $d \geq 2$, and set*

$$\mu = \begin{cases} 2 & \text{if } d = 2 \\ 1 & \text{if } d \geq 3 \end{cases} \qquad \lambda = \begin{cases} 0 & \text{if } M \text{ is non-orientable} \\ 1 & \text{if } M \text{ is orientable} \end{cases}$$

Let A be an abelian group. Then we have:

(1) $\delta\big(H^q(PConf(M); A)\big) \leq \mu q$,

(2) $h^{\max}\big(H^q(PConf(M); A)\big) \leq \max(-1, 4\mu q - 2\mu\lambda - 2)$,

(3) $t_0\big(H^q(PConf(M; A)\big) \leq \max(\mu q, 5\mu q - 2\mu\lambda - 1)$,

(4) $t_1\big(H^q(PConf(M); A)\big) \leq \max(\mu q, 9\mu q - 4\mu\lambda - 2)$.

4.2 Classifying spaces of pure diffeomorphisms groups

Let M be a connected, smooth manifold with (possibly empty) boundary of dimension $d \geq 2$ and $\mathfrak{p} \in \mathrm{PConf}_n(\mathring{M})$, in other words $\big(\mathfrak{p}(1), \mathfrak{p}(2), \ldots, \mathfrak{p}(n)\big)$ is an ordered configuration of n points in the interior $\mathring{M}$ of M. We denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M, with the compact-open topology. By abusing notation, we use $\mathrm{Diff}(M)$ instead of $\mathrm{Diff}(M \text{ rel } \partial M)$ if $\partial M \neq \varnothing$ and if M is orientable, we can restrict to orientation-preserving diffeomorphisms.

Let $\mathrm{Diff}^{\mathfrak{p}}(M) = \mathrm{Diff}(M, P)$ be the subgroup of $\mathrm{Diff}(M)$ of diffeomorphisms that leave invariant the set $P = \mathfrak{p}([n])$ and $\mathrm{PDiff}^{\mathfrak{p}}(M) = \mathrm{PDiff}(M, P)$ is the subgroup of $\mathrm{Diff}(M)$ that consists of the diffeomorphims that fix each point in P. Since M is connected, if $\mathfrak{p}, \mathfrak{q} \in \mathrm{PConf}_n(\mathring{M})$, then $\mathrm{Diff}^{\mathfrak{p}}(M) \approx \mathrm{Diff}^{\mathfrak{q}}(M)$ and $\mathrm{PDiff}^{\mathfrak{p}}(M) \approx \mathrm{PDiff}^{\mathfrak{q}}(M)$. We denote them by $\mathrm{Diff}^n(M)$ and $\mathrm{PDiff}^n(M)$, respectively. An inclusion $f : [m] \hookrightarrow [n]$ induces a restriction $\mathrm{PDiff}^{\mathfrak{p}}(M) \to \mathrm{PDiff}^{\mathfrak{p}\circ f}(M)$.

We consider $\mathrm{BPDiff}^{\bullet}(M) : \mathsf{FI}^{op} \to \mathcal{T}op$ to be the FI^{op}-space given by $[n] \mapsto \mathrm{BPDiff}^n(M)$, where $\mathrm{BPDiff}^n(M)$ is the classifying space of the group $\mathrm{PDiff}^n(M)$. There is a fiber bundle

$$\mathrm{BPDiff}^n(M) \longrightarrow \mathrm{BDiff}(M) \tag{1}$$

where the fiber is $\mathrm{Diff}(M)/\,\mathrm{PDiff}^n(M) \approx \mathrm{PConf}_n(\mathring{M})$, the configuration space of n ordered points in the interior of M. This gives us a functor from FI^{op} to the category $\mathsf{Fib}\big(\mathrm{BDiff}(M)\big)$. We now have all the ingredients to obtain linear bounds on the presentation degree of the FI-modules $H^k(\mathrm{BPDiff}^{\bullet}(\Sigma); A)$, for any abelian group A.

Theorem 4.2. *Let M be a connected real manifold with (possibly empty) boundary of dimension $d \geq 2$ and set*

$$\mu = \begin{cases} 2 & \text{if } d = 2 \\ 1 & \text{if } d \geq 3 \end{cases} \qquad\qquad \lambda = \begin{cases} 0 & \text{if } M \text{ is non-orientable} \\ 1 & \text{if } M \text{ is orientable} \end{cases}$$

Let A be any abelian group, then for $k \geq 0$, the FI-module $H^k(BPDiff^{\bullet}(M); A)$ is presented in finite degrees and

(1) $\delta\big(H^k\big(BPDiff^{\bullet}(M); A\big)\big) \leq \mu k,$

(2) $h^{\max}\big(H^k\big(BPDiff^{\bullet}(M); A\big)\big) \leq \max(-1, 8\mu k - 2\mu\lambda - 2),$

(3) $t_0\big(H^k\big(BPDiff^{\bullet}(M); A\big)\big) \leq \max(0, 9\mu k - 2\mu\lambda - 1),$

(4) $t_1\big(H^k\big(BPDiff^{\bullet}(M); A\big)\big) \leq \max(0, 17\mu k - 4\mu\lambda - 2).$

Furthermore, if $\mathring{M}$ has the homotopy type of a CW-complex of finite type and $BDiff(M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension, then $H^k(BPDiff^{\bullet}(M); A)$ is finitely generated.

Proof. Associated to the fibrations (1) we have a cohomologically graded first quadrant Leray-Serre spectral sequence of FI-modules converging to $H^{p+q}(\mathrm{BPDiff}^{\bullet}(M); A)$ with E_2-term:

$$E_2^{p,q} = H^p\big(\mathrm{BDiff}(M); H^q(\mathrm{PConf}(\mathring{M}); A)\big).$$

From the linear bounds from Lemma 4.1 and Proposition 3.1, we obtain

$$\delta(E_2^{p,q}) \leq \mu q \qquad \text{and} \qquad h^{\max}(E_2^{p,q}) \leq \max(-1, 4\mu q - 2\mu\lambda - 2).$$

Then
$$D_k = \max_{p+q=k} \delta(E_2^{p,q}) \leq \mu k;\ \eta_k = \max_{p+q=k} h^{\max}(E_d^{p,q}) \leq \max(-1, 4\mu k - 2\mu\lambda - 2).$$
Therefore, from Proposition 2.5 we obtain
$$\delta\big(H^k(\mathrm{BPDiff}^\bullet(M); A)\big) \leq \mu k;$$
$$h^{\max}\big(H^k(\mathrm{BPDiff}^\bullet(M); \mathbb{Z})\big) \leq \max\big(\max_{\ell\leq 2k} \eta_\ell, \max_{\ell \leq 2k-1}(2\mu\ell - 2)\big)$$
$$\leq \max\big(\max(-1, 4\mu(2k) - 2\mu\lambda - 2), 2\mu(2k-1) - 2\big) \leq \max(-1, 8\mu k - 2\mu\lambda - 2),$$
which are the bounds in (1) and (2). From [CMNR18, Proposition 3.1] we obtain
$$t_0\big(H^k\big(\mathrm{BPDiff}^\bullet(M); A\big)\big) \leq \mu k + \max(-1, 8\mu k - 2\mu\lambda - 2) + 1 = \max(0, 9\mu k - 2\mu\lambda - 1) \quad \text{and}$$
$$t_1\big(H^k\big(\mathrm{BPDiff}^\bullet(M); A\big)\big) \leq \mu k + 2\big(\max(-1, 8\mu k - 2\mu\lambda - 2)\big) + 2 = \max(0, 17\mu k - 4\mu\lambda - 2).$$

If $\mathring{M}$ has the homotopy type of a CW-complex of finite type, then the FI-modules $H^q(\mathrm{PConf}(\mathring{M}); A)$ are finitely generated. Therefore if $\mathrm{BDiff}(M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension, then we get finite generation from Proposition 3.1. ■

Remarks 4.1:

- For dimension $d \geq 3$, Theorem 4.2 is an integral version of [JR15, Theorem 6.6] that includes the case of M being non-orientable. In the statement of [JR15, Theorem 6.6] the hypothesis of M being orientable was not explicitly stated, but was necessary at that time since [CEF15, Theorem 4] was only proven for orientable manifolds. Better bounds can presumably be obtained using the improvement in the linear stable ranges obtained in [Bah18] and [MW19] for configuration spaces.

- If M is an irreducible compact connected orientable 3-manifold with non-empty boundary, then the classifying space $\mathrm{BDiff}^+(M \text{ rel } \partial M)$ has the homotopy type of a finite aspherical CW-complex [HM97], and from Theorem 4.2 it follows that $H^k(\mathrm{BPDiff}^\bullet(M); A)$ is finitely generated. If M is a closed 2-connected oriented smooth manifold of dimension $d \neq 4, 5, 7$, then $\mathrm{BDiff}^+(M)$ is *homologically finite type* [Kup19, Corollary C] and, from the fiber bundle (1) and [Kup19, Lemma 2.5], so it is $\mathrm{BPDiff}^n(M)$. Therefore, from Theorem 4.2 and Proposition 2.3 it follows that, for any field $\mathbb{F}$, the Betti numbers $\dim_{\mathbb{F}} H^k(\mathrm{BPDiff}^n(M); \mathbb{F})$ are eventually given by a polynomial on n of degree at most μk.

4.3 Pure mapping class groups of surfaces

Let M be a connected, smooth manifold with (possibly empty) boundary of dimension $d \geq 2$. The mapping group of M with marked points $P = \mathfrak{p}([n]) \in \mathrm{PConf}_n(\mathring{M})$ is defined as $\mathrm{Mod}^n(M) = \pi_0\big(\mathrm{Diff}^n(M)\big)$. The pure mapping class group is given by $\mathrm{PMod}^n(M) = \pi_0\big(\mathrm{PDiff}^n(M)\big)$. When no marked points are considered, we just write $\mathrm{Mod}(M)$. Following the notation in [JR11], we consider the FI^{op}-group $\mathrm{PMod}^\bullet(M)$ given by $[n] \mapsto \mathrm{PMod}^n(M)$ and for an inclusion $f : [m] \hookrightarrow [n]$, the associated homomorphism $f^* : \mathrm{PMod}^n(M) \to \mathrm{PMod}^m(M)$ is induced from the restriction $\mathrm{PDiff}^{\mathfrak{p}}(M) \to \mathrm{PDiff}^{\mathfrak{p}\circ f}(M)$. By taking integral cohomology we obtain an FI-module $H^k(\mathrm{PMod}^\bullet(M); \mathbb{Z})$ for $k \geq 0$.

When M is a closed manifold, the diagonal action of $\mathrm{Diff}(M)$ on $\mathrm{PConf}_n(M)$ gives rise to the Borel fiber bundle
$$\mathrm{PConf}_n(M) \longrightarrow \mathrm{EDiff}(M) \times_{\mathrm{Diff}(M)} \mathrm{PConf}_n(M) \longrightarrow \mathrm{BDiff}(M), \qquad (2)$$

where

$$\mathrm{EDiff}(M)\times_{\mathrm{Diff}(M)}\mathrm{PConf}_n(M)\approx \mathrm{EDiff}(M)\times_{\mathrm{Diff}(M)}\frac{\mathrm{Diff}(M)}{\mathrm{PDiff}^n(M)}=\frac{\mathrm{EDiff}(M)}{\mathrm{PDiff}^n(M)}\approx \mathrm{BPDiff}^n(M).$$

Then the fiber bundle (2) can be identified with (1) (see for example [Coh10, Theorem 3.2]). For surfaces $M=\Sigma$, these Borel constructions frequently give $K(\pi,1)'s$ with fundamental group

$$\pi_1\big(\mathrm{EDiff}(\Sigma)\times_{\mathrm{Diff}(\Sigma)}\mathrm{PConf}_n(\Sigma)\big)=\pi_1\big(\mathrm{BPDiff}^n(\Sigma)\big)=\mathrm{PMod}^n(\Sigma)$$

and the base space of the fiber bundle (2) has the homotopy type of a finite CW-complex. This is the case for the sphere S^2 with $n\geq 3$ and the torus $\mathbb{T}$ with $n\geq 2$ (see for example [Coh10, Section 9 & 10] and references therein); the projective plane $\mathbb{R}P^2$ with $n\geq 2$ ([EE69], [Wan02, Corollary 2.6]), and the Klein bottle $\mathbb{K}$ with $n\geq 1$ [HX17, Theorems 6.1 & 7.1].

For the punctured torus Σ_1^1 there is also a Borel construction

$$\mathrm{PConf}_n(\Sigma_1^1)\to \mathrm{ESL}(2,\mathbb{Z})\times_{\mathrm{SL}(2,\mathbb{Z})}\mathrm{PConf}_n(\Sigma_1^1)\to \mathrm{BSL}(2,\mathbb{Z})$$

where the total space $\mathrm{ESL}(2,\mathbb{Z})\times_{\mathrm{SL}(2,\mathbb{Z})}\mathrm{PConf}_n(\Sigma_1^1)$ is a $K(\mathrm{PMod}^n(\Sigma_1^1),1)$ ([Coh10, Theorem 10.6]) for all $n\geq 1$ and conclusions from Theorem 4.2 also apply for this case.

For $\Sigma=\Sigma_g^r$ with $2g+r>2$ or $\Sigma=N_g^r$ with $g\geq 3$, we have that $\mathrm{BDiff}(\Sigma)$ is a $K(\pi,1)$ (since $\mathrm{BDiff}(\Sigma)\simeq \mathrm{BHomeo}(\Sigma)$ and the components of $\mathrm{Homeo}(\Sigma)$ are contractible [Ham66, Theorems 5.1, 5.2, 5.3]), [EE69]). On the other hand, $\mathrm{PConf}_n(\mathring{\Sigma})$ is also also aspherical (this follows from the Fadell–Neuwirth fibrations [FN62] by an inductive argument). Hence all occurring spaces in the fibration (1) are aspherical and $\mathrm{BPDiff}^n(\Sigma)$ is a $K(\mathrm{PMod}^n(\Sigma),1)$. The long exact sequence in homotopy groups for the fibration (1) results in the Birman exact sequence ([Bir74], [Kor02, Theorem 2.1])

$$1\to\pi_1(\mathrm{PConf}_n(\mathring{\Sigma}))\to \mathrm{PMod}^n(\Sigma)\to \mathrm{Mod}(\Sigma)\to 1. \tag{3}$$

Furthermore, the groups $\mathrm{Mod}(\Sigma)$ are of type FP_∞ (see for example [Iva87, Section 6]).

Hence, for the surfaces Σ considered in this section, $H^k\big(\mathrm{BPDiff}^\bullet(\Sigma);A\big)\cong H^k\big(\mathrm{PMod}^\bullet(\Sigma);A\big)$ and Theorem 4.2 gives us the following.

Corollary 4.3 (Linear ranges for pure mapping class groups). *Let Σ be a surface such that:*

$\Sigma=S^2,\mathbb{T},\Sigma_1^1$ or $\Sigma=\Sigma_{g,r}$ with $2g+r>2$, if the surface is orientable, or

$\Sigma=\mathbb{R}P^2,\mathbb{K}$ or $\Sigma=N_{g,r}$ with $g\geq 3$ and $r\geq 0$ in the non-orientable case.

and let A be any abelian group. Then for $k\geq 0$, the FI-module $H^k\big(\mathrm{PMod}^\bullet(\Sigma);A\big)$ is finitely presented with upper bounds for the stable, local, generation and presentation degrees as in Theorem 4.2 (with $\mu=2$).

Surfaces with boundary. If Σ has nonempty boundary, then $H^k(\mathrm{PMod}^\bullet(\Sigma);A)$ has an $\mathsf{FI}\#$-module structure for $k\geq 0$ (see for example [JR15, Prop 6.3]). Therefore, by Proposition 2.6, we obtain better bounds for the presentation degree.

Corollary 4.4 (Surfaces with boundary). *Let $\Sigma=\Sigma_{g,r}$ with $g\geq 1$ and $r\geq 1$ or $\Sigma=N_{g,r}$ with $g\geq 3$ and $r\geq 1$. Then for any abelian group A and $k\geq 0$ the cohomology $H^k\big(\mathrm{PMod}^\bullet(\Sigma);A\big)$ is a finitely presented $\mathsf{FI}\#$-module with local degree $=-1$ and with stable degree, generation degree and presentation degree bounded above by $2k$.*

4.4 Hyperelliptic mapping class groups

For a surface Σ_g the *hyperelliptic diffeomorphism* is the order two map with $2g+2$ fixed points and which acts as a rotation of π around a central axis of Σ_g. We denote the mapping class of this diffeomorphism in $\mathrm{Mod}(\Sigma_g)$ by ι.

Definition 4.1 The *hyperelliptic mapping class group* Δ_g is the normalizer of $\langle\iota\rangle \cong \mathbb{Z}_2$ in $\mathrm{Mod}(\Sigma_g)$. The group Δ_g^n, the *hyperelliptic mapping class group with marked points*, is defined as the preimage of Δ_g under the forgetful map $\mathrm{PMod}^n(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$.

For $g = 1, 2$, we have that $\Delta_g = \mathrm{Mod}(\Sigma_g)$, but for $g \geq 3$, Δ_g is neither normal nor of finite index in $\mathrm{Mod}(\Sigma_g)$. Moreover, there exists a non split central extension (see [BH71])

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Delta_g \to \mathrm{Mod}^{2g+2}(\Sigma_0) \to 1. \tag{4}$$

The cohomology ring of Δ_g has been studied before, see for example [BCP01] and [Coh93]. In particular, the map $\Delta_g \to \mathrm{Mod}^{2g+2}(\Sigma_0)$ gives a cohomology isomorphism with coefficients in a ring R containing $1/2$. For $g \geq 3$, by restricting the short exact sequence (3), we obtain a Birman exact sequence for hyperelliptic mapping class groups :

$$1 \to \pi_1(\mathrm{PConf}_n(\Sigma_g)) \to \Delta_g^n \to \Delta_g \to 1. \tag{5}$$

We can consider the FI^{op}-group $\Delta_g^\bullet$ given by $[n] \mapsto \Delta_g^n$ and for each inclusion $f : [m] \hookrightarrow [n]$, we restrict the homomorphism $f^* : \mathrm{PMod}^n(\Sigma_g) \to \mathrm{PMod}^m(\Sigma_g)$ to $f^* : \Delta_g^n \to \Delta_g^m$. Therefore, for any abelian group A and $k \geq 0$, we obtain FI-modules $H^k\big(\Delta_g^\bullet; A\big)$. From (4) it follows that Δ_g is of type FP_∞. The Lyndon–Hochschild–Serre spectral sequences associated to the extensions (5) give us a first quadrant spectral sequence of FI-modules converging to the graded FI-module $H^*(\Delta_g^\bullet; A)$. Therefore we can follow the proof of Theorem 4.2 to obtain finite presentation and the same explicit upper bounds. Moreover, with $\mathbb{Q}$-coefficients, the proof of [JR15, Theorem 6.1] will give the same upper bounds for the weight and stability degree.

Corollary 4.5. *Let $g \geq 3$ and $k \geq 0$ and A be any abelian group, then the cohomology $H^k\big(\Delta_g^\bullet; A\big)$ is a finitely presented FI-module with upper bounds for the stable, local, generation and presentation degrees as in Theorem 4.2 (with $\lambda = 1$ and $\mu = 2$). Furthermore, over $\mathbb{Q}$ the FI-module $H^k\big(\Delta_g^\bullet; \mathbb{Q}\big)$ has weight$\leq 2k$ and stability degree $\leq 4k$ and the sequence $\{H^k\big(\Delta_g^n; \mathbb{Q}\big)\}_n$ is representation stable for $n \geq 6k$.*

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