Torsion at the threshold for mapping class groups

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Abstract

The mapping class group Γ_g^1 of a closed orientable surface of genus $g \geq 1$ with one marked point can be identified, by the Nielsen action, with a subgroup of the group of orientation preserving homeomorphims of the circle. This inclusion pulls back the powers of the discrete universal Euler class producing classes $\mathrm{E}^n \in H^{2n}(\Gamma_g^1;\mathbb{Z})$ for all $n \geq 1$. In this paper we study the power $n = g$, and prove: E^g is a torsion class which generates a cyclic subgroup of $H^{2g}(\Gamma_g^1;\mathbb{Z})$ whose order is a positive integer multiple of $4g(2g+1)(2g-1)$.

1 Introduction

The mapping class group Γ_g^1 of a closed orientable surface Σ_g of genus $g \geq 1$ with one marked point can be identified, by the Nielsen action, with a subgroup of the group $\mathrm{Homeo}_+ \mathbb{S}^1$ of orientation preserving homeomorphims of the circle. The inclusion $\rho : \Gamma_g^1 \hookrightarrow \mathrm{Homeo}_+ \mathbb{S}^1$ pulls back the powers of the *discrete universal Euler class* producing classes $\mathrm{E}^n \in H^{2n}(\Gamma_g^1;\mathbb{Z})$ for all $n \geq 1$. At the *threshold* $n = g$ we prove the following.

Theorem A. *The cohomology class* E^g *generates a finite cyclic subgroup of* $H^{2g}(\Gamma_g^1;\mathbb{Z})$ *whose order is a positive integer multiple of* $4g(2g+1)(2g-1)$.

The cohomology $H^*(\Gamma_g^1;\mathbb{Z})$ is the ring of characteristic classes of Σ_g-bundles with a section. Some of the first homological calculations for mapping class groups are due to Harer [6] who computed $H_2(\Gamma_g^1;\mathbb{Z})$ for genus $g \geq 5$; see also the work of Korkmaz and Stipsicz [10]. For $g \geq 1$, the *Euler class* E is a generator of $H^2(\Gamma_g^1;\mathbb{Z})$. The existence of torsion in Γ_g^1 implies that the powers E^n are non-trivial classes in $H^{2n}(\Gamma_g^1;\mathbb{Z})$ for $n \geq 1$; see for example [9, Theorem A]. Furthermore, for genus $g \geq 2$, Morita proved in [14, Theorem 7.5] that the classes E^n are torsion-free when $n \leq g/6$. This range was improved to $n \leq g/4$ by Bödigheimer and Tillmann [2, Corollary 1.2].

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On the other hand, the *moduli space* $\mathcal{M}_{g,1}$ of Riemann surfaces of genus g with one marked point is a rational model for the classifying space of the mapping class group Γ_g^1 and $H^*(\mathcal{M}_{g,1};\mathbb{Q}) \cong H^*(\Gamma_g^1;\mathbb{Q})$ when $g \geq 2$. For more about this isomorphism see for example [1], [5] and references therein. The Euler class E corresponds to the restriction of the ψ_1-class of $\mathcal{M}_{g,1}$ from its Deligne-Mumford compatification $\overline{\mathcal{M}}_{g,1}$. The class ψ_1 is a tautological class defined as the first Chern class of the cotangent bundle over $\overline{\mathcal{M}}_{g,1}$ associated to the marked point. Using this algebro-geometric perspective, Looijenga [11] proved that for $n \geq g$ the power E^n vanishes as a cohomology class with rational coefficients. This is also a consequence of a more general result of Ionel [7, Theorem 0.1]. In particular, this implies that the powers E^n are non-trivial torsion classes in $H^{2n}(\Gamma_g^1;\mathbb{Z})$ when $n \geq g$.

The non-triviality of the g-th power of the Euler class E^g, and the fact that its order is a multiple of $4g(2g+1)$ follow from the existence of torsion elements in Γ_g^1 of order $4g$ and $2g+1$; see for example [9, Theorem A]. Our Theorem A gives new information about the order of the g-th power E^g of the Euler class by showing that it is divisible by $2g-1$. This result requires a more involved argument since this torsion in cohomology is not constructed using periodic elements in the mapping class group.

For $g = 1$ Theorem A holds, and the lower bound is sharp since it is known that the Euler class of $\Gamma_1^1 \cong SL(2,\mathbb{Z})$ is a torsion class of order 12. See Example 5.10 below where we work the details of this case to illustrate our approach.

Overview of the proof of Theorem A. All powers of the Euler class E are non-trivial, effectively because Γ_g^1 has non-trivial finite cyclic subgroups; see for instance [9, Theorem A]. So a basic component of the proof of Theorem A is distinguishing finite cyclic from infinite cyclic, or in the language of the Universal Coefficient Theorem, determining whether E^g lies in the *Ext*-term or in the *Hom*-term. Associated to an action of the group Γ_g^1 there is a bi-simplicial set, a double chain complex and a total chain complex which compute the homology and cohomology of the group. On the chains of the complex at the base of the bi-simplicial set we construct a $2g$-chain "dual" to a representative of the class E^g, and we attempt to *lift* it to a $2g$-cycle on the total complex. If it lifts the class E^g is in the *Hom*-term. We show however that there is an obstruction to the lifting, hence the class E^g is in the *Ext*-term.

A special feature of the lifting process is that it is technically difficult to construct and analyze the obstruction in the double chain complex determined by the Nielsen action, but manageable for a modified action which we call the *inversive action.* For the inversive action we consider Γ_g^1 as acting on homotopy classes of un-oriented based loops on a closed surface of genus g by identifying an element of the fundamental group of the surface with its inverse; see Section 3.2. Then, even though Γ_g^1 no longer acts directly on the fundamental group of Σ_g, the formal bi-simplicial constructions still determine the homology and cohomology of Γ_g^1 and there is a cohomology class which pulls back to the class E. It is in this context that we present the discrete Euler class E in Section 4, and study the behavior of E^g.

Lifting using the inversive action is carried out in Section 5. It follows closely the analysis in [8] using a "projective action". It is assumed in [8, Section 10] that the action can be defined on the circle, but the inversive action, which is the correct formulation, cannot be constructed in that way. The projective action leads to a discrepancy in the torsion calculation. We obtain $4g(2g+1)$ as a factor of the torsion whereas [8, Theorem 3a) & Theorem 7] obtains $2g(2g+1)$.

Remark 1.1. One advantage of the inversive action is that the obstruction to the lifting lies within a single summand of the total complex associated to the action where a combinatorial analysis leads to a formula for the cycle. We call the obstruction to the lifting the *transition cycle*, and $2g$ the *threshold dimension*, since we expect the power g of the Euler class of Γ_g^1 marks the transition from infinite cyclic to torsion.

In Section 6 we show that E^g is a non-trivial torsion class in $H^{2g}(\Gamma_g^1;\mathbb{Z})$; see Theorem B and Section 6.4. This can be deduced indirectly from the fact that all the powers E^n of the Euler class are non-trivial in integer cohomology [9, Theorem A], but are known to be trivial in rational cohomology when $n \geq g$ by algebro-geometric techniques [7, Theorem 0.1],[11]. Our intrinsic proof in Sections 6.3 and 6.4 enables us to detect torsion of order $2g-1$. As opposed to torsion of order $4g$ and $2g+1$, it is not constructed using periodic elements in the mapping class group. We expect that additional torsion in cohomology exists and can be discovered by constructing periodic simplicial actions of Γ_g^1.

To carry out the torsion calculation we consider the transition cycle both as an obstruction, and as a boundary in the total complex associated to the inversive action. As an obstruction it naturally determines an infinite cyclic 1-dimensional homology class in a discrete groupoid associated to the inversive action, as we show in Proposition 6.1 and Corollary 6.2. As a boundary it is dual to a representative of the g-th power of the Euler class in the relevant *Ext* term, see Proposition 6.4. We seek classes in the groupoid so that a non-trivial multiple equals the class of the transition cycle. This is technically difficult in the groupoid so we "homotop" the calculation to one in a group, in particular the stabilizer of a chosen base point. The formalities of the homotopy are classical, see Proposition 5.3. How the homotopy applies to enable the calculation is the content of Propositions 6.5 and 6.6. Finally in Section 6.4 we use these results to prove Theorem A.

Organization of the paper. Section 2 deals with preliminaries on bi-simplicial sets and homology. In Section 3 we discuss the Nielsen action and the inversive action of the mapping class group Γ_g^1, and Section 4 presents the discrete Euler class in the context of the constructions in the previous two sections. Lifting using the inversive action is carried out in Section 5. In Section 6 we show that E^g is a non-trivial torsion class and prove Theorem A.

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2 Preliminaries

2.1 Homeomorphisms of the circle and the universal Euler class

Consider the group $\mathrm{Homeo}_+ \mathbb{S}^1$ of orientation preserving homeomorphisms of the circle $\mathbb{S}^1$ with the discrete topology. Let $\mathrm{Homeo}_+(\mathbb{S}^1)_\tau$ denote the same group with the compact-open topology. The starting point for our analysis of the discrete Euler class is the theorem of Mather and Thurston [12, 16], statement c) below. We state some fundamental results.

a) $\mathrm{Homeo}_+(\mathbb{S}^1)_\tau$ is homotopy equivalent to $\mathbb{S}^1$.

b) The classifying space $B\,\mathrm{Homeo}_+(\mathbb{S}^1)_\tau$ is a $K(\mathbb{Z},2)$, and its cohomology is a polynomial algebra over $\mathbb{Z}$ on a generator E_τ of degree 2, called the *universal Euler class.*

c) (Mather–Thurston) The identity $id : \mathrm{Homeo}_+ \mathbb{S}^1 \to \mathrm{Homeo}_+(\mathbb{S}^1)_\tau$ induces an algebra isomorphism $id^* : H^*(B\,\mathrm{Homeo}_+(\mathbb{S}^1)_\tau;\mathbb{Z}) \to H^*(B\,\mathrm{Homeo}_+ \mathbb{S}^1;\mathbb{Z})$.

d) Any inclusion $\iota : \mathbb{Z}_m \hookrightarrow \mathrm{Homeo}_+ \mathbb{S}^1$ induces an epimorphism of polynomial algebras $\iota^* : H^*(B\,\mathrm{Homeo}_+ \mathbb{S}^1;\mathbb{Z}) \to H^*(B\mathbb{Z}_m;\mathbb{Z})$.

A discussion of these results is included in [9] and references therein. See [16] for a general version of c) and Section 4.1 of this paper for a specific proof in the case of homeomorphisms of the circle. In what follows we will not distinguish between the homology of a group and the homology of its classifying space.

Definition 2.1. The *universal discrete Euler class* $\mathbf{E}$ in $H^2(\mathrm{Homeo}_+ \mathbb{S}^1;\mathbb{Z}) \cong \mathbb{Z}$ is the pullback by $id : \mathrm{Homeo}_+ \mathbb{S}^1 \to \mathrm{Homeo}_+(\mathbb{S}^1)_\tau$ of the universal Euler class E_τ.

Notice that the n-th power of the universal discrete Euler class $\mathbf{E}^n$ is a generator of $H^{2n}(\mathrm{Homeo}_+ \mathbb{S}^1;\mathbb{Z}) \cong \mathbb{Z}$ for $n \geq 1$, hence a non-trivial torsion-free cohomology class.

2.2 Homology of a group derived from an action

Suppose a group G acts on a set S. Then it acts on S^∞ which is the infinite simplex on the set. For each $p \geq 0$ the action on the p-simplices S^∞_p gives rise to a groupoid $\Lambda_p G$ whose objects in dimension p are the p-simplices, whose morphisms are $(g,\sigma) \in G \times S^\infty_p$, and whose source and target maps are $s(g,\sigma) = \sigma$ and $t(g,\sigma) = g(\sigma)$. The composition of morphisms (g,σ) and $(f, g(\sigma))$ is (fg,σ). For the simplicial constructions which follow see also [8, Section 4.1].

The action of G gives $\Lambda_p G$ the structure of a simplicial groupoid and extending by nerves in the q direction produces a bi-simplicial set $\Lambda G = \Lambda_{p,q}G$. The horizontal simplicial set for fixed q is the simplical set associated to the simplicial complex $G^q \times S^\infty$. The simplicial complex S^∞ is contractible which implies that the realization of ΛG is homotopically equivalent to BG.

The bi-simplical set $\Lambda G = \Lambda_{p,q}G$ gives rise to a double chain complex $\mathcal{C} = \mathcal{C}_{p,q}$ which computes the homology of G. Each horizontal simplicial set of ΛG is associated to a simplicial complex so oriented chains can be used to form a double chain group. For each fixed q let the chain complex $\mathcal{C}_{*,q}$ be the classical oriented chain complex of the simplicial complex $G^q \times S^\infty$. Then the free abelian group $\mathcal{C}_{p,q}$ is generated by chains of the form $(f_1, \dots, f_q)[v_0, \dots, v_p]$ where $[v_0, \dots, v_p]$ denotes an oriented p-simplex of S^∞ and $f_1, \dots, f_q \in G$.

This double chain complex determines a total complex $T\mathcal{C}$ given by $T\mathcal{C}_n = \bigoplus_{p+q=n} \mathcal{C}_{p,q}$ with differential $\partial = \partial^h + \partial^v$, where the horizontal and vertical boundary homomorphisms are given on summands by $\partial^h : \mathcal{C}_{p,q} \to \mathcal{C}_{p-1,q}$, $\partial^v : \mathcal{C}_{p,q} \to \mathcal{C}_{p,q-1}$ and satisfy $\partial^h\partial^v + \partial^v\partial^h = 0$. Then the homology of the total complex $H_*(T\mathcal{C})$ is $H_*(G)$.

We define the *orbit chain complex* $H_0^v\mathcal{C}$ *of the action of* G by taking $p \mapsto H_0^v(\mathcal{C}_{p,*})$. It is the chain complex at $q = 0$ in the $\mathcal{E}^1$-term of the spectral sequence obtained by computing the vertical homology of $\mathcal{C}_{p,q}$. Explicitly $H_0^v(\mathcal{C}_{p,*})$ is the free abelian group on the orbits of oriented p-simplices under the action of G.

There is a chain map $\mathrm{q} : T\mathcal{C} \to H_0^v\mathcal{C}$ defined as follows. Given $c = c_0 + c_1 + \cdots + c_p \in T\mathcal{C}_p$, with $c_i \in \mathcal{C}_{p-i,i}$, let

$$\mathrm{q}(c) = [c_0]^v \in H_0^v(\mathcal{C}_{p,*}),$$

where $[c_0]^v$ denotes the vertical homology class of the chain $c_0 \in \mathcal{C}_{p,0}$. We we call q *the orbit chain map of the action of* G.

The homology of a group G computed using the total complex $T\mathcal{C}$ is related to its cohomology by the Universal Coefficient Theorem:

$$0 \to Ext(H_{p-1}(T\mathcal{C}), \mathbb{Z}) \xrightarrow{\alpha} H^p(T\mathcal{C}; \mathbb{Z}) \xrightarrow{\beta} Hom(H_p(T\mathcal{C}), \mathbb{Z}) \to 0. \tag{1}$$

A non-trivial class in $H^p(T\mathcal{C}; \mathbb{Z})$ is either in the image of α, in which case it is finite cyclic, or in the image of β, in which case it is infinite cyclic.

3 The mapping class group and its actions

Consider a closed oriented surface Σ_g of genus $g \geq 1$, and let $z \in \Sigma_g$. The *mapping class group* Γ_g^1 is the group of orientation preserving homeomorphisms of Σ_g which fix z, modulo isotopies which fix z. For an orientation preserving homeomorphism f of Σ_g such that $f(z) = z$, let f_* denote the induced automorphism of $\pi_1(\Sigma_g, z)$. The assignment $[f] \mapsto f_*$ gives a well-defined monomorphism from Γ_g^1 to the automorphism group $Aut(\pi_1(\Sigma_g, z))$, that identifies Γ_g^1 with an index 2 subgroup of $Aut(\pi_1(\Sigma_g, z))$ by the Dehn–Nielsen–Baer Theorem (see for instance [3, Theorem 8.1]).

3.1 The Nielsen action of Γ_g^1

The *Nielsen action of* Γ_g^1 *on* $\mathbb{S}^1$ is given by a monomorphism $\rho : \Gamma_g^1 \hookrightarrow \mathrm{Homeo}_+ \mathbb{S}^1$ that we recall next. For $g = 1$, the group $\Gamma_1^1 \cong \mathrm{SL}(2, \mathbb{Z})$ acts faithfully on rays starting at the origin in the Euclidean plane and ρ is the corresponding monomorphism.

For $g \geq 2$, we choose a hyperbolic metric on Σ_g, so that its universal cover is isometric to the hyperbolic disk $\mathbb{D}$ and the origin in $\mathbb{D}$ is mapped to the marked point $z \in \Sigma_g$. Each non-trivial $\gamma \in \pi_1(\Sigma_g, z)$ acts on $\mathbb{D}$ as a hyperbolic isometry with a unique translation axis with forward endpoint $\gamma_\infty \in \partial\mathbb{D} \approx \mathbb{S}^1$. Since the action of $\pi_1(\Sigma_g, z)$ on $\mathbb{D}$ is cocompact, the set $\Gamma_\infty = \{\gamma_\infty : \gamma \in \pi_1(\Sigma_g, z), \gamma \neq 1\}$ is dense in $\partial\mathbb{D}$. Hence, any automorphism ϕ of $\pi_1(\Sigma_g, z)$ induces a homeomorphism $\partial\phi$ of $\partial\mathbb{D} \approx \mathbb{S}^1$ that takes $\gamma_\infty \in \Gamma_\infty$ to $\phi(\gamma)_\infty$. Precomposing with the action of Γ_g^1 on $\pi_1(\Sigma_g, z)$ gives the homomorphism $\rho : \Gamma_g^1 \to \mathrm{Homeo}_+ \mathbb{S}^1$ which turns out to be injective. See for example [3, Sections 5.5.4 & 8.2.6] for more details.

In particular, for the computations of this paper we consider $\pi_1(\Sigma_g, z)$ presented as the free group on $2g$ elements $\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_3, \mathrm{a}_4, \ldots, \mathrm{a}_{2g-1}$ with the relation

$$\mathrm{a}_0 \cdots \mathrm{a}_{2g-1} = \mathrm{a}_{2g-1} \cdots \mathrm{a}_0. \tag{2}$$

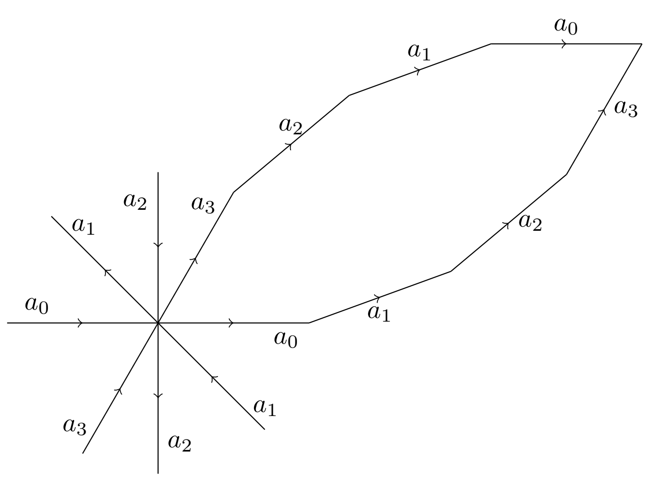

Figure 1: Tiling of the hyperbolic disk $\mathbb{D}$ for genus $g = 2$.

Let us place a $4g$-gon in the plane, with directed edges labeled counterclockwise by $\mathrm{a}_0, \mathrm{a}_1, ..., \mathrm{a}_{2g-1}, \mathrm{a}_0^{-1}, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}^{-1}$ in such a way that the point at the beginning of the edge a_i is at the origin and the polygon generates a tiling of the hyperbolic disk $\mathbb{D}$ for $g > 1$, and of the Euclidean plane when $g = 1$. The resulting tiling will have $4g$ polygons forming a wreath at the origin. There will be $4g$ rays emanating from the origin, $2g$ labeled by the a_i's and $2g$ labeled by their inverses which are antipodal to the a_i's. Figure 1 illustrates the case $g = 2$. We consider the hyperbolic metric on Σ_g induced by this tiling, through the projection $\mathbb{D} \to \Sigma_g$, to define the Nielsen action ρ as described above. For each generator $\mathrm{a_i}$ of $\pi_1(\Sigma_g, z)$, the corresponding translation axis is a geodesic line through the origin of $\mathbb{D}$ that connects two antipodal points of $\partial\mathbb{D} \approx \mathbb{S}^1$. We label the forward endpoint of the translation axis by $\mathrm{a_i}$ and the backward endpoint by $\mathrm{a_i^{-1}}$.

The wreath of polygons at the origin can be organized as follows. Proceeding clockwise from the first polygon, the second polygon has a_0 pointing outward, and a_1, (which precedes it in the relation defining $\pi_1(\Sigma_g, z)$), directed inward. The third has a_1^{-1} directed outward and a_2^{-1} directed inward, and so on ending with a $4g$-polygon, having edges $\mathrm{a_{2g-2}}$ pointing inward and $\mathrm{a_{2g-1}}$ directed outward. After halfway through the process the labelling of the rays by letters repeats, but opposite directions are assigned. It follows that the points $\{\mathrm{a_i}^{\pm 1}\}$ are ordered clockwise in $\mathbb{S}^1$ by

$$\mathrm{a}_0 < \mathrm{a}_1^{-1} < \mathrm{a}_2 < \cdots < \mathrm{a}_{2g-1}^{-1} < \mathrm{a}_0^{-1} < \mathrm{a}_1 < \mathrm{a}_2^{-1} < \cdots < \mathrm{a}_{2g-2}^{-1} < \mathrm{a}_{2g-1}. \tag{3}$$

3.2 The inversive action of Γ_g^1

Let w be an element of $\pi_1(\Sigma_g, z)$. The equivalence relation determined by the relation $w \sim w^{\pm 1}$ divides $\pi_1(\Sigma_g, z)$ into equivalence classes which we refer to as *inversives*. The action of Γ_g^1 on $\pi_1(\Sigma_g, z)$ preserves these equivalence classes and induces an action on the set $\mathbb{I}$ of inversives which we call the *inversive action of* Γ_g^1. We denote the inversive of a_i by A_i.

The Nielsen and inversive actions each determine a bi-simplicial set, a double chain complexes, and a total complex. The group Γ_g^1 acts on the orbit of a point; we choose the point to be a_0 for the Nielsen action and A_0 for the inversive action. We will, when appropriate, denote constructions for the actions by subscripts $\mathbb{N}$ and $\mathbb{I}$. There is a bi-simplicial map of bi-simplicial sets $\nu : \Lambda_{\mathbb{N}}\Gamma_1^g \to \Lambda_{\mathbb{I}}\Gamma_1^g$, induced by the quotient map $\pi_1(\Sigma_g, z) \twoheadrightarrow \mathbb{I}$.

Proposition 3.1. *The map ν induces a chain map from $T\mathcal{C}_{\mathbb{N}}$ to $T\mathcal{C}_{\mathbb{I}}$ which is an isomorphism ν_* on homology and ν^* on cohomology.*

Proof. Each of the bi-simplicial sets realizes as the product of a $K(\Gamma_g^1, 1)$ and an infinite simplex. On the realizations the map induced by ν is the product of the identity and a homotopy equivalence, and therefore a homology equivalence on their associated total complexes. $\square$

3.3 Elementary morphisms

Consider the presentation of $\pi_1(\Sigma_g, z)$ from Section 3.1 and let $\mathrm{a}_{2g}^{-1} = \mathrm{a}_0\mathrm{a}_1\cdots\mathrm{a}_{2g-1}$. As before, we label the forward endpoint by $\mathrm{a}_{2\mathrm{g}}$ and the backward endpoint by $\mathrm{a}_{2\mathrm{g}}^{-1}$ in $\partial\mathbb{D}$ of the line joining the two points. Whether considering the $\mathrm{a}_i^{\pm}$ as elements of $\pi_1(\Sigma_g, z)$ or points in $\mathbb{S}^1$ there exist mapping classes which determine orientation preserving homeomorphisms of the circle, S of order $4g$ and T of order $2g+1$, given by

$$S : \mathrm{a}_0 \to \mathrm{a}_1 \to \cdots \to \mathrm{a}_{2g-1} \to \mathrm{a}_0^{-1} \to \cdots \to \mathrm{a}_{2g-1}^{-1} \to \mathrm{a}_0 \qquad T : \mathrm{a}_0 \to \mathrm{a}_1 \cdots \to \mathrm{a}_{2g-1} \to \mathrm{a}_{2g} \to \mathrm{a}_0.$$

The mapping class $d_{2g} = S^{-1}T$ acts on $\pi_1(\Sigma_g, z)$ by taking a_{2g} to a_{2g-1}^{-1} while fixing all a_i and their inverses for $0 \le i \le 2g-2$. More generally, let $d_{2g-k} := T^{-k}(S^{-1}T)T^k$ for $0 \le k \le 2g$. It takes a_{2g-k} to a_{2g-k-1}^{-1} mod$(2g+1)$, and keeps all a_i and their inverses, with indices other than $2g-k$ and $2g-k-1$, fixed.

Remark 3.2. An alternative justification of the clockwise ordering (3) of the points $\{\mathrm{a_i}^{\pm 1}\}$ in $\mathbb{S}^1$ uses the d_i's as follows. The mapping class d_0 takes a_0 to a_{2g}^{-1}, and fixes all the other a's so those two must be consecutive in the cyclic ordering of points on the circle. Whether a_{2g}^{-1}, is the next clockwise entry or counterclockwise entry is a matter of choice, and depends on how the first polygon is constructed relative to the orientation of the circle. The same argument applies to the pair a_{2g}^{-1}, a_{2g-1}, and to the remaining points to complete the determination of their ordering.

Notice d_0 maps a_0 to a_{2g}^{-1} and keeps all other a_i fixed. The ordering (3) must be preserved so, since a_0 is between a_{2g-1} and a_1^{-1} the image of a_0 is either between a_0 and a_{2g-1} or between a_0 and a_1^{-1}. Moreover d_{2g} maps a_{2g}^{-1} to a_{2g-1} keeping all remaining points fixed, so only the first of the two possibilities can hold. From the ordering (3) and this observation we have that the points, including a_{2g}, are ordered clockwise in $\mathbb{S}^1$ as follows.

$$\mathrm{a}_0 < \mathrm{a}_1^{-1} < \mathrm{a}_2 < \cdots < \mathrm{a}_{2g-1}^{-1} < \mathrm{a}_{2g} < \mathrm{a}_0^{-1} < \mathrm{a}_1 < \mathrm{a}_2^{-1} < \cdots < \mathrm{a}_{2g-2}^{-1} < \mathrm{a}_{2g-1} < \mathrm{a}_{2g}^{-1} \tag{4}$$

Definition 3.3. Let A_i denote the inversive represented by a_i. For $0 \le j \le 2g$, we refer to d_j as a *elementary mapping class*. It determines an *elementary morphism* in the double chain complex $\mathcal{C}_{\mathbb{I}}$ of the inversive action, which we write as

$$[A_0, A_1, ..., \hat{A}_{j-1}, A_j, ..., A_{2g}]d_j \in (\mathcal{C}_{\mathbb{I}})_{2g-1,1}.$$

The mapping classes S and T act on the set $\{A_0, ..., A_{2g}\}$ as follows

$$S : A_0 \to A_1 \to \cdots \to A_{2g-1} \to A_0, \qquad T : A_0 \to A_1 \to A_2 \cdots \to A_{2g-1} \to A_{2g} \to A_0.$$

Furthermore, the elementary mapping class d_{2g-k} takes A_{2g-k} to A_{2g-k-1} mod$(2g+1)$, and fixes all the inversives A_i, with indices other than $2g-k$ and $2g-k-1$.

These mapping classes satisfy the following identities in Γ_g^1:

$$\begin{gathered} d_{2g}d_{2g-1}\cdots d_0 = (S^{-1}T)(T^{-1}S^{-1}T^2)\cdots(T^{-2g}S^{-1}T^{2g+1}) = (S^{-1})^{2g+1}T^{2g+1} = S^{-1} \\ (d_{2g}d_{2g-1}\cdots d_0)^{2g} = S^{2g} \quad \& \quad (d_{2g}d_{2g-1}\cdots d_0)^{4g} = 1_{\Gamma_g^1}. \end{gathered} \tag{5}$$

4 The Euler class and its cocycle representatives

4.1 The universal Euler class

The group $G = \mathrm{Homeo}_+ \mathbb{S}^1$ acts on the contractible simplicial complex which is the infinite simplex on $\mathbb{S}^1$. In this section we let $\mathcal{C} = \mathcal{C}_{p,q}$ be the double chain complex of oriented chains associated to the action, with $T\mathcal{C}$ the corresponding total complex and $H_0^v\mathcal{C}$ the orbit chain complex. The following homology computation uses the fact that $Homeo^+\mathbb{R}$ is acyclic, [15, 13], and [4] for a new proof. The essential components of Theorem 4.1 can be found in [8, Section 5], but because the formula for the universal orbit class in Theorem 4.1a) plays a crucial role in the calculations which follow we include the proof here, including some additional material. In particular, we make the application of oriented and ordered simplices more explicit, and we discuss the failure of the Universal Coefficient Theorem in the orbit chain complex and its implications.

Theorem 4.1. *Consider $G = \mathrm{Homeo}_+ \mathbb{S}^1$ acting on the infinite simplex on $\mathbb{S}^1$.*

a) The homology and cohomology of $H_0^v\mathcal{C}$ is $\mathbb{Z}$ in even dimensions and 0 in odd. A generator of $H_{2p}(H_0^v\mathcal{C})$ is represented by the cycle $\mathbf{o}_p := [0, 2p, 1, ..., 2p-1] - [0, 1, 2, ..., 2p]$ in $H_0^v\mathcal{C}$, where $0 < 1 < 2 < \cdots$ is a countable set of clockwise oriented points of the circle.

b) The orbit chain map $\mathrm{q} : T\mathcal{C} \to H_0^v\mathcal{C}$ induces a homology equivalence.

Proof. To prove a) we show that $H_0^v\mathcal{C}$ is given by $\mathbb{Z} \leftarrow \mathbb{Z}_2 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}_2 \leftarrow \cdots$. Any clockwise ordered $(p+1)$-tuple, $(x_0, ..., x_p)$, of points of the circle can be mapped to $\{0, 1, 2, ..., p\}$ in some order. Hence there are at most two orientation classes of p-simplices, one represented by $[0, 1, 2, ..., p]$ and one represented by any odd permutation of the vertices. When p is even, a cyclic permutation of the coordinates of a p-simplex determines a new simplex in the same orientation class, so that there are two orientation classes and the group of oriented p-chains is isomorphic to $\mathbb{Z}$, with $+1$ and -1 representing the classes $[0, 2p, 1, ..., 2p-1]$ and $[0, 1, 2, ..., 2p]$, respectively. Note that $[0, 1, 2, ..., 2p] = -[0, 2p, 1, ..., 2p-1]$. When p is odd a cyclic permutation of the coordinates of a p-simplex determines a simplex in the opposite orientation class, so there is one class of order 2, and the p-th chain group is isomorphic to $\mathbb{Z}_2$. The homology of $H_0^v\mathcal{C}$ is $\mathbb{Z}$ in even dimensions and 0 otherwise, and is therefore generated in dimension $2p$ by the class of the cycle $\mathbf{o}_p = 2[0, 2p, 1, ..., 2p-1]$ or equivalently $[0, 2p, 1, ..., 2p-1] - [0, 1, 2, ..., 2p]$. The cohomology of $H_0^v\mathcal{C}$ is isomorphic to $\mathbb{Z}$ in even dimensions and 0 in odd. This completes the proof of a).

For b) we will use the action of G on ordered simplices as well as on oriented simplices. The former is "right for isotropy" and the latter is "right for orbits". The homomorphism from ordered chains on a simplicial complex to oriented chains is defined by associating to an ordered simplex its orientation class. It induces a map from the double chain complex Δ of the action of G on ordered simplices to the double chain complex $\mathcal{C}$ of the action of G on oriented simplices which is a homology equivalence on each horizontal complex. Therefore there is chain homomorphism $T\Delta \longrightarrow T\mathcal{C}$ which is well defined on orbits and functorially determines a chain homomorphism $H_0^v\Delta \longrightarrow H_0^v\mathcal{C}$. A chain inverse $T\mathcal{C} \longrightarrow T\Delta$ is defined by totally ordering the vertices and choosing for each oriented simplex the ordered simplex given by the total ordering. It too is well defined on orbit complexes so induces a chain inverse $H_0^v\mathcal{C} \longrightarrow H_0^v\Delta$. Consider the commutative diagram of chain complexes

$$\begin{array}{ccc} T\Delta & \longrightarrow & T\mathcal{C} \\ \mathrm{q}_\Delta \downarrow & & \downarrow \mathrm{q} \\ H_0^v\Delta & \longrightarrow & H_0^v\mathcal{C}. \end{array}$$

The chain homomorphisms $T\Delta \longrightarrow T\mathcal{C}$ and $H_0^v\Delta \longrightarrow H_0^v\mathcal{C}$ induce homology equivalences. We show that q_Δ is a homology equivalence, which implies that q is also, and proves b).

The double complex $\Delta_{p,q}$ is the $\mathcal{E}^0$-term of the spectral sequence

$$\mathcal{E}^2_{p,q}(T\Delta) = H_p^h H_q^v(\Delta) \Rightarrow H_{p+q}(T\Delta).$$

We observe that computing homology vertically then horizontally in the the spectral sequence gives

$$\mathcal{E}^2_{p,q} = \begin{cases} H_p(H_0^v\Delta) & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

The homology of each vertical groupoid is isomorphic to the direct sum of the homology of its isotropy subgroups. The isotropy group of each p-simplex is acyclic, for the isotropy group of any point b is isomorphic to the group of orientation preserving homeomorphisms of $\mathbb{S}^1 - \{b\}$, which in turn is isomorphic to $Homeo^+\mathbb{R}$. The isotropy group of a $(p+1)$-element subset of $\mathbb{S}^1$ is isomorphic to a $(p+1)$-fold cartesian product of $Homeo^+\mathbb{R}$, so it too is acyclic. This implies that all the homology groups in $\mathcal{E}^1_{p,q}$ for $q > 0$ are 0.

It remains to show that the isomorphism between $H_p(T\Delta)$ and $H_p(H_0^v\Delta)$ is induced by q_Δ. Let $c_0 \in \Delta_{p,0}$ represent a cycle in $H_p(H_0^v\Delta)$. Because all the vertical groupoids are acyclic it follows that the successive differentials of the spectral sequence are trivial, and c_0 lifts to a p-cycle c in $T\Delta$ (see the introduction to Section 5 for more on the lifting process). Then $\mathrm{q}_\Delta(c) = [c_0]^v$ as required. □

Remark 4.2. The Universal Coefficient Theorem fails for the chain complex $H_0^v\mathcal{C}$ as it is not free. The group $Hom(H_{2p}(H_0^v\mathcal{C}), \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and generated by the homomorphism $\mathbf{o}_p^*$ dual to the *universal orbit cycle*

$$\mathbf{o}_p = [0, 2p, 1, ..., 2p-1] - [0, 1, 2, ..., 2p] = 2[0, 2p, 1, ..., 2p-1].$$

On the other hand, the cohomology group $H^{2p}(H_0^v\mathcal{C}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and it is generated by the cocycle $(\mathbf{o}_p/2)^* \in Hom(H_0^v(\mathcal{C}_{2p,*}), \mathbb{Z})$ dual to the chain

$$(\mathbf{o}_p/2) := [0, 2p, 1, ..., 2p-1].$$

For $p \geq 1$, let us consider the chains

$$(\mathbf{e}_p/2) \coloneqq [0,2p,1,...,2p-1] \in \mathcal{C}_{2p,0} \quad \text{and} \quad \mathbf{e}_p \coloneqq [0,2p,1,...,2p-1] - [0,1,2,...,2p] = 2(\mathbf{e}_p/2).$$

These are chains in the total complex $T\mathcal{C}$ that "lift" $(\mathbf{o}_p/2)$ and the orbit cycle $\mathbf{o}_p$, respectively, in the sense that $\mathrm{q}(\mathbf{e}_p/2) = [\mathbf{e}_p/2]^v = (\mathbf{o}_p/2)$ and $\mathrm{q}(\mathbf{e}_p) = [\mathbf{e}_p]^v = \mathbf{o}_p$.

Definition 4.3. We refer to $(\mathbf{o}_p/2)^*$ as the *p-th universal orbit cocycle* and to the pull-back cocycle $(\mathbf{e}_p/2)^* \coloneqq \mathrm{q}^*(\mathbf{o}_p/2)^*$ as the *p-th universal Euler cocycle.*

Proposition 4.4. *The p-th universal Euler cocycle $(\mathbf{e}_p/2)^*$ satisfies*

$$\langle(\mathbf{e}_p/2)^*, (\mathbf{e}_p/2)\rangle = 1 \quad \textit{and} \quad \langle(\mathbf{e}_p/2)^*, \mathbf{e}_p\rangle = 2,$$

and it represents $2\mathbf{E}^p$ in the total complex $T\mathcal{C}$.

Proof. Since the class $\mathbf{E}^p$ generates the cohomology group $H^{2p}(G;\mathbb{Z}) \cong \mathbb{Z}$, the cocycle $(\mathbf{e}_p/2)^*$ represents an integral multiple of $\mathbf{E}^p$. The total complex $T\mathcal{C}$ satisfies the Universal Coefficient Theorem (1) and we know from Theorem 4.1 that q induces an isomorphism in homology. Under the composition of isomorphisms

$$H^{2p}(T\mathcal{C};\mathbb{Z}) \xrightarrow{\beta} Hom(H_{2p}(T\mathcal{C}),\mathbb{Z}) \xrightarrow{(\mathrm{q}^*)^{-1}} Hom(H_{2p}(H_0^v\mathcal{C}),\mathbb{Z})$$

the cohomology class $\mathbf{E}^p$ corresponds to $\mathbf{o}_p^*$ the homomorphism dual to the orbit cycle $\mathbf{o}_p$, which generates $Hom(H_{2p}(H_0^v\mathcal{C}),\mathbb{Z}) \cong \mathbb{Z}$. On the other hand,

$$\langle(\mathbf{e}_p/2)^*, \mathbf{e}_p\rangle = \langle \mathrm{q}^*(\mathbf{o}_p/2)^*, \mathbf{e}_p\rangle = \langle(\mathbf{o}_p/2)^*, \mathrm{q}(\mathbf{e}_p)\rangle = \langle(\mathbf{o}_p/2)^*, \mathbf{o}_p\rangle = 2.$$

Similarly, we obtain $\langle(\mathbf{e}_p/2)^*, (\mathbf{e}_p/2)\rangle = 1$. Therefore the composition $(\mathrm{q}^*)^{-1} \circ \beta$ takes the cohomology class represented by the cocycle $(\mathbf{e}_p/2)^*$ to $2\mathbf{o}_p^*$. It follows that the cocycle $(\mathbf{e}_p/2)^*$ corresponds to $2\mathbf{E}^p$. □

4.2 The Euler class of Γ_g^1

We now consider the Nielsen and inversive actions of Γ_g^1 to study the behavior of the g-the power of the Euler class of Γ_g^1.

Definition 4.5. The Nielsen action $\rho : \Gamma_g^1 \hookrightarrow \mathrm{Homeo}_+ \mathbb{S}^1$ pulls-back the powers $\mathbf{E}^p$ producing classes $\mathrm{E}^p \in H^{2p}(\Gamma_g^1;\mathbb{Z})$ for all $p \geq 1$. We refer to E as the *Euler class of* Γ_g^1.

Consider the following diagram of chain complexes:

$$\begin{array}{ccccc} T\mathcal{C}_{\mathbb{I}} & \leftarrow & T\mathcal{C}_{\mathbb{N}} & \rightarrow & T\mathcal{C} \\ & & \downarrow & & \downarrow \\ & & H_0^v\mathcal{C}_{\mathbb{N}} & \rightarrow & H_0^v\mathcal{C} \end{array} \tag{6}$$

The vertical arrows are the corresponding orbit chain maps $\mathrm{q}_{\mathbb{N}}$ and q; the horizontal arrows on the right are induced by the Nielsen action ρ; the horizontal arrow on the left is induced by the map ν of Proposition 3.1. We use this diagram and the universal cocycles to define cocycles in the total complexes associated to the Nielsen and inversive actions.

With the Nielsen action ρ and the universal cocycle we can define the cocycles

$$(o_g/2)^*_{\mathbb{N}} := \rho^*((\mathbf{o}_g/2)^*) \quad \text{and} \quad (e_g/2)^*_{\mathbb{N}} := \rho^*((\mathbf{e}_g/2)^*).$$

From the commutativity of diagram 6, these cocyles are related as follows

$$(e_g/2)^*_{\mathbb{N}} = \rho^*((\mathbf{e}_g/2)^*) = \rho^*(\mathrm{q}^*(\mathbf{o}_g/2)^*) = \mathrm{q}^*_{\mathbb{N}}(\rho^*(\mathbf{o}_g/2)^*) = \mathrm{q}^*_{\mathbb{N}}((o_g/2)^*_{\mathbb{N}}).$$

Furthermore, from the definition of the Euler class, the cocycle $(e_g/2)^*_{\mathbb{N}}$ represents the cohomology class $2\,\mathrm{E}^g \in H^{2g}(\Gamma^1_g;\mathbb{Z})$ in the total complex of the Nielsen action.

Consider the *orbit chains* in the complex $H^v_0\mathcal{C}_{\mathbb{N}}$

$$(o_g/2)_{\mathbb{N}} := [\mathrm{a}_0^{-1}, \mathrm{a}_{2g}^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}] \quad \text{and}$$

$$(o_g)_{\mathbb{N}} := [\mathrm{a}_0^{-1}, \mathrm{a}_{2g}^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}] - [\mathrm{a}_0^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}, \mathrm{a}_{2g}^{-1}] = 2(o_g/2)_{\mathbb{N}},$$

and the corresponding "lifts" to the total complex $T\mathcal{C}_{\mathbb{N}}$

$$(e_g/2)_{\mathbb{N}} := [\mathrm{a}_0^{-1}, \mathrm{a}_{2g}^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}] \quad \text{and}$$

$$(e_g)_{\mathbb{N}} := [\mathrm{a}_0^{-1}, \mathrm{a}_{2g}^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}] - [\mathrm{a}_0^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}, \mathrm{a}_{2g}^{-1}] = 2(e_g/2)_{\mathbb{N}},$$

which are chains in $(\mathcal{C}_{\mathbb{N}})_{2g,0}$. Since the elements $\{\mathrm{a}_i^{\pm 1}\}$ are cyclically ordered as indicated in (4), these chains satisfy $\rho_*((o_g)_{\mathbb{N}}) = \mathbf{o}_g$ and $\rho_*((e_g)_{\mathbb{N}}) = \mathbf{e}_g$.

Proposition 4.6. *The cocycle $(e_g/2)^*_{\mathbb{N}}$ has the following evaluations:*

$\langle (e_g/2)^*_{\mathbb{N}}, (e_g/2)_{\mathbb{N}} \rangle = 1,$

$\langle (e_g/2)^*_{\mathbb{N}}, (e_g)_{\mathbb{N}} \rangle = 2,$ *and*

$\langle (e_g/2)^*_{\mathbb{N}}, c_i \rangle = 0,$ *for $c_i \in (\mathcal{C}_{\mathbb{N}})_{2g-i,i}$ with $i \geq 1$.*

Proof. The proof of the second evaluation is similar to the first. For the first evaluation

$$\langle (e_g/2)^*_{\mathbb{N}}, (e_g)_{\mathbb{N}} \rangle = \langle \rho^*((\mathbf{e}_g/2)^*), (e_g)_{\mathbb{N}} \rangle = \langle (\mathbf{e}_g/2)^*, \rho_*((e_g)_{\mathbb{N}}) \rangle = \langle (\mathbf{e}_g/2)^*, \mathbf{e}_g \rangle = 2,$$

where the last equality follows from Proposition 4.4.

For the third evaluation, let $c_i \in (\mathcal{C}_{\mathbb{N}})_{2g-i,i}$ with $i \geq 1$. Since $\mathrm{q}_{\mathbb{N}}(c_i) = 0$, it follows that

$$\langle (e_g/2)^*_{\mathbb{N}}, c_i \rangle = \langle \mathrm{q}^*_{\mathbb{N}}((o_g/2)^*_{\mathbb{N}}), c_i \rangle = \langle (o_g/2)^*_{\mathbb{N}}, \mathrm{q}_{\mathbb{N}}(c_i) \rangle = 0.$$

□

We turn our attention to the inversive action. Recall that from Proposition 3.1 we have a chain map $\nu : T\mathcal{C}_{\mathbb{N}} \to T\mathcal{C}_{\mathbb{I}}$ which induces an isomorphism in homology and cohomology. So there exists a class in $H^*(T\mathcal{C}_{\mathbb{I}};\mathbb{Z})$ which is mapped by ν^* to the class of the Euler cocycle for the Nielsen action $(e_g/2)^*_{\mathbb{N}}$ in $H^*(T\mathcal{C}_{\mathbb{N}};\mathbb{Z})$. We construct a particular cocycle representing this class.

A chain map j_* which induces an inverse to ν_* in homology, can be found as follows. For each inversive we choose a representative. This determines a map from the vertices of the infinite simplex on which Γ^1_g acts inversively to the vertices of the infinite simplex on which Γ^1_g acts by the Nielsen action. So the map extends to one between the infinite simplices. Furthermore, since Γ^1_g is the group acting in both cases it extends to a map between the bi-complexes of the action, hence between the corresponding double chain complexes. We have $\nu_* j_*$ induces the identity in homology and since ν_* is an isomorphism in homology so is j_*. We can choose j_* on vertices arbitrarily, so we send A_i to a_i^{-1} when i is even and A_i to a_i when i is odd. This chain map is j_*.

Definition 4.7. The *g-th Euler cocycle for the inversive action* is $(e_g/2)^*_{\mathbb{I}} := j^*((e_g/2)^*_{\mathbb{N}})$.

By construction, $\nu^*((e_g/2)^*_{\mathbb{I}}) = (e_g/2)^*_{\mathbb{N}}$. We define chains in the total complex $T\mathcal{C}_{\mathbb{I}}$, specifically in $(\mathcal{C}_{\mathbb{I}})_{2g,0}$, as follows:

$$(e_g/2)_{\mathbb{I}} := [A_0, A_{2g}, A_1, \ldots, A_{2g-1}] \quad \text{and}$$

$$(e_g)_{\mathbb{I}} := [A_0, A_{2g}, A_1, \ldots, A_{2g-1}] - [A_0, A_1, A_2, \ldots, A_{2g-1}, A_{2g}] = 2(e_g/2)_{\mathbb{I}}.$$

The arrows in diagram (6) relate the cocycles and chains that we have constructed in the total complexes and the orbit chain complexes as follows:

$$\begin{array}{ccccc} (e_g/2)^*_{\mathbb{I}} & \mapsto & (e_g/2)^*_{\mathbb{N}} & \mapsfrom & (\mathbf{e}_g/2)^* \\ & & \uparrow & & \uparrow \\ & & (o_g/2)^*_{\mathbb{N}} & \mapsfrom & (\mathbf{o}_g/2)^* \end{array} \qquad \begin{array}{ccccc} (e_g)_{\mathbb{I}} & \mapsfrom & (e_g)_{\mathbb{N}} & \mapsto & \mathbf{e}_g \\ & & \downarrow & & \downarrow \\ & & (o_g)_{\mathbb{N}} & \mapsto & \mathbf{o}_g \end{array}$$

Proposition 4.8. *The g-th Euler cocycle for the inversive action $(e_g/2)^*_{\mathbb{I}}$ represents, in the total complex $T\mathcal{C}_{\mathbb{I}}$, the cohomology class $2\,\mathrm{E}^g \in H^{2g}(\Gamma^1_g;\mathbb{Z})$, and it satisfies*

$$\langle (e_g/2)^*_{\mathbb{I}}, (e_g/2)_{\mathbb{I}} \rangle = 1 \quad \textit{and} \quad \langle (e_g/2)^*_{\mathbb{I}}, (e_g)_{\mathbb{I}} \rangle = 2.$$

Furthermore, a chain $c_i \in (\mathcal{C}_{\mathbb{I}})_{2g-i,i}$ with $i \geq 1$, satisfies $\langle (e_g/2)^*_{\mathbb{I}}, c_i \rangle = 0$.

Proof. We verify the first formula on the left. The proof of the one on its right is similar.

$$\begin{aligned} 1 = \langle (e_g/2)^*_{\mathbb{N}}, (e_g/2)_{\mathbb{N}} \rangle &= \langle (e_g/2)^*_{\mathbb{N}}, [\mathrm{a}_0^{-1}, \mathrm{a}_{2g}^{-1}, \mathrm{a}_1, \mathrm{a}_2^{-1}, ..., \mathrm{a}_{2g-1}] \rangle \\ &= \langle (e_g/2)^*_{\mathbb{N}}, j_*[A_0, A_{2g}, A_1, ..., A_{2g-1}] \rangle \\ &= \langle j^*((e_g/2)^*_{\mathbb{N}}), [A_0, A_{2g}, A_1, ..., A_{2g-1}] \rangle \\ &= \langle (e_g/2)^*_{\mathbb{I}}, (e_g/2)_{\mathbb{I}} \rangle. \end{aligned}$$

Now let $c_i \in (\mathcal{C}_{\mathbb{I}})_{2g-i,i}$ with $i \geq 1$. To show that $\langle (e_g/2)^*_{\mathbb{I}}, c_i\rangle = 0$, write $c_i = \nu_* \tilde{c}_i$ where $\tilde{c}_i \in (\mathcal{C}_{\mathbb{N}})_{2g-i,i}$. Then, by Proposition 4.6 we have

$$\langle (e_g/2)^*_{\mathbb{I}}, c_i\rangle = \langle (e_g/2)^*_{\mathbb{I}}, \nu_* \tilde{c}_i\rangle = \langle \nu^*(e_g/2)^*_{\mathbb{I}}, \tilde{c}_i\rangle = \langle (e_g/2)^*_{\mathbb{N}}, \tilde{c}_i\rangle = 0.$$

□

5 The transition cycle

We describe a process in which we attempt to *lift* a chain in $\mathcal{C}_{n,0}$ to a cycle in $T\mathcal{C}$. Generally, consider an n-chain $c_0 \in \mathcal{C}_{n,0}$. The horizontal boundary $b_0 \in \mathcal{C}_{n-1,0}$ of c_0 is a horizontal cycle. Suppose it bounds vertically; let $c_1 \in \mathcal{C}_{n-1,1}$ be the vertical boundary of b_0. Continuing in this manner, construct $b_k \in \mathcal{C}_{n-k,k-1}$ and, if possible, $c_k \in \mathcal{C}_{n-k,k}$. If the lifting process leads to the construction of $c_n \in \mathcal{C}_{0,n}$, then $c = c_0 + c_1 + \cdots + c_n$ is an n-cycle in the total chain complex $T\mathcal{C}$. If the lifting terminates at an earlier stage k, that is b_k fails to lift further, then the construction produces an n-boundary $b_0 + b_1 + \cdots + b_k$, and $c_0 + c_1 + \cdots + c_k \in T\mathcal{C}_n$ is not a cycle.

For example, consider the chain $\mathbf{e}_g = [0, 2g, 1, ..., 2g-1] - [0, 1, 2, ..., 2g]$ in $\mathcal{C}_{2g,0}$ in the universal setting . It lifts to a $2g$-cycle c in $T\mathcal{C}$, since by Theorem 4.1 the orbit chain map q induces a homology isomorphism, and a generator of $H_{2g}(H_0^v\mathcal{C})$ is represented by the orbit cycle $\mathbf{o}_g = \mathrm{q}(\mathbf{e}_g)$. In contrast, the chain $(\mathbf{e}_g/2) = [0, 2g, 1, ..., g-1] \in \mathcal{C}_{2g,0}$, cannot be lifted to even c_1, for its chain boundary is a sum of an odd number of distinct faces so orbits cannot cancel.

In this section we consider the inversive action of the mapping class group Γ_g^1 and attempt to lift the $2g$-chain $(e_g)_{\mathbb{I}} = [A_0, A_{2g}, A_1, \ldots, A_{2g-1}] - [A_0, A_1, A_2, \ldots, A_{2g-1}, A_{2g}]$ in $(\mathcal{C}_{2g,0})_{\mathbb{I}}$ to a $2g$-cycle in $T\mathcal{C}_{\mathbb{I}}$. We will show that the lifting will terminate with an element $b_1 \in (\mathcal{C}_{\mathbb{I}})_{2g-2,1}$ which we call the *transition cycle t*. In Section 5.1 we construct the cycle t and analyze it combinatorially. This will lead us to a proof in Section 6.1 that t is an obstruction to the lifting and then in Section 6.2 to a calculation of the torsion of E^g .

After this point we suppress the subscript $\mathbb{I}$ when the inversive context is clear.

5.1 Construction of the transition cycle

We apply the lifting process to $c_0 = e_g = [A_0, A_{2g}, A_1, \ldots, A_{2g-1}] - [A_0, A_1, A_2, \ldots, A_{2g-1}, A_{2g}]$.

Proposition 5.1. *There is a chain $c_1 \in \mathcal{C}_{2g-1,1}$ so that $\partial^v(c_1) = \partial^h(e_g)$.*

Proof. We group the horizontal face maps of e_g into pairs that lie in the same orbit, but have opposite signs. Let $Q = [A_0, A_{2g}, A_1, ..., A_{2g-1}]$ and $R = [A_0, A_1, A_2, ..., A_{2g}]$, then, by definition, $e_g = Q - R$. Notice that $\partial_0 Q - \partial_1 Q = [\hat{A}_0, A_{2g}, A_1, ..., A_{2g-1}] - [A_0, \hat{A}_{2g}, ..., A_{2g-1}]$.

The two simplices on the right hand side are in the same orbit since the mapping class d_0 maps $[A_0, A_1, ..., A_{2g-1}]$ to $[A_{2g}, A_1, ..., A_{2g-1}]$. Then $\partial_0 Q - \partial_1 Q = 0$ in oriented homology.

For $3 \le k \le 2g-1$, and k odd consider

$$\partial_{k-1}Q - \partial_k Q = [A_0, A_{2g}, A_1, ..., \hat{A}_{k-1}, ..., A_{2g-1}] - [A_0, A_{2g}, A_1, A_2, ..., \hat{A}_{k-2}, ..., A_{2g-1}].$$

The two simplices are in the same orbit of d_k and $\partial_{k-1}Q - \partial_k Q = 0$. There is one face of Q remaining which is $\partial_{2g}Q = [A_0, A_{2g}, 1, ..., \hat{A}_{2g-1}]$. Now consider R, and notice that the difference $\partial_{k-1}R - \partial_k R = [A_0, A_1, ..., \hat{A}_k, ..., A_{2g-1}, A_{2g}] - [A_0, A_1, A_2, ..., \hat{A}_{k-1}, ..., A_{2g}]$ consists of simplices in the same orbit of d_{k-1} for $1 \le k \le 2g-1$, and k odd. Then $\partial_{k-1}R - \partial_k R = 0$. The remaining face is $\partial_{2g}R = [A_0, A_1, ..., A_{2g-1}, \hat{A}_{2g}]$. The two remaining faces of e_g, $[A_0, A_{2g}, A_1, ..., \hat{A}_{2g-1}] - [A_0, A_1, ..., A_{2g-1}, \hat{A}_{2g}]$, are in the orbit of d_{2g}, which can be seen by cycling A_{2g} to the last slot. So $\partial^h(e_g)$ bounds vertically. □

Explicitly: $c_1 = \sum_{i=0}^{2g}(-1)^i[A_0, ..., \hat{A}_i, A_{i+1}, ..., A_{2g}]d_{i+1},$ where $i \equiv n \operatorname{mod}(2g+1)$.

Definition 5.2. The *transition cycle* is the chain $t = \partial^h(c_1) \in \mathcal{C}_{2g-2,1}$. It is a cycle, vertically and horizontally, and a boundary horizontally. It breaks into two parts as follows:

$$t = \sum_{i>j}(-1)^{i+j}[A_0, ..., \hat{A}_j, ..., \hat{A}_i, ..., A_{2g}]d_{i+1} + \sum_{i<j}(-1)^{i+j-1}[A_0, ..., \hat{A}_i, ..., \hat{A}_j, ..., A_{2g}]d_{i+1}. \quad (7)$$

In summary, we have constructed a chain $c_0 + c_1$ that lifts e_g for the inversive action, and the transition cycle t is its boundary. Next we analyze the combinatorics of t and use it to show, in Corollary 6.3 below, that the lifting process terminates.

5.2 Holonomy of the transition cycle

The transition cycle t is a 1-chain in a groupoid. In order to determine torsion arising from t considered as an obstruction it seems necessary, or at least desirable, to view it as a 1-chain in a group. The natural choice is the isotropy or stabilizer group of a point, for the classifying space of a connected, discrete groupoid is homotopy equivalent to the classifying space of the isotropy group of any base point, as we show below. Connected means there is a morphism from any object to a single base point.

Proposition 5.3. *Consider a connected discrete groupoid Λ and the isotropy group Λ_{b} of a base point* b. *The inclusion $\Lambda_{\mathrm{b}} \to \Lambda$ induces a homotopy equivalence $B\Lambda_{\mathrm{b}} \to B\Lambda$.*

Proof. For each object y of Λ choose a morphism m_{y} from y to b. Define the *holonomy functor* F from $\Lambda \to \Lambda_{\mathrm{b}}$, considering both Λ and Λ_{b} as categories, by $f \mapsto \mathrm{m}_{t(f)} \circ f \circ \mathrm{m}_{s(f)}^{-1}$. The assignment $\mathrm{y} \mapsto \mathrm{m}_{\mathrm{y}}$ is a natural transformation from the functor F to the identity functor. Natural transformations of functors induce homotopy equivalences on classifying spaces. Therefore F determines a homotopy inverse to the inclusion $\Lambda_{\mathrm{b}} \to \Lambda$. □

Remark 5.4. The holonomy functor induces an isomorphism on homology which is independent of the choice of morphisms m_{y}.

If f is a morphism in the isotropy group of an object y and it is conjugate in the groupoid Λ to a morphism in the isotropy group of b, then the 1-cycles they determine represent the same homology class in $H_1(B\Lambda)$.

In what follows we consider the vertical groupoids $\Lambda_p := \Lambda_p\Gamma_g^1$ of the bi-simplicial set constructed from the inversive action of Γ_g^1 on oriented simplices of inversives. The groupoid Λ of interest is the component of Λ_{2g-2} containing the object $[A] := [A_0, A_1, ..., A_{2g-2}]$, and we denote by $\Lambda_{[A]}$ the isotropy group of that object. In Proposition 5.8 of the next section we compute the image, under the holonomy functor, of the transition cycle t, considered as a 1-cycle in the groupoid Λ. We find it to be $2g(2g+1)[A_0, \dots, A_{2g-2}]d_{2g} + [A_0, \dots, A_{2g-2}]S^{2g}$. It will facilitate the calculation of torsion when the obstruction to lifting is represented by a 1-cycle in a group rather than in a groupoid.

5.3 Combinatorial structure of the transition cycle

Notice that in the dimension of interest, an object of Λ_{2g-2} is an orientation class of a $(2g-1)$-tuple of inversives and a morphism is a mapping class in Γ_g^1 acting "inversively" on objects. The vertical homology of the double complex $\mathcal{C}$ in dimension $2g-2$ is precisely the homology of Λ_{2g-2}.

We decompose the transition cycle t into an alternating sum of $g+1$ distinct vertical 1-cycles, $L_0, ..., L_g$ in the groupoid Λ_{2g-2}.

$$t = L_0 - L_1 + L_2 + \cdots + (-1)^g L_g \tag{8}$$

The chains L_0 and L_g will each be a sum of $2g+1$ terms of (7). The remaining chains will each be a sum of $2(2g+1)$ terms. Together they have $2g(2g+1)$ terms. In the formulas below integers are taken mod $2g+1$.

$$L_0 = \sum_{i=0}^{2g}[A_0, ..., \hat{A}_i, \hat{A}_{i+1}, ..., A_{2g}]d_{i+1} \qquad L_g = \sum_{i=0}^{2g}[A_0, ..., \hat{A}_i, ..., \hat{A}_{i+g}, ..., A_{2g}]d_{i+g+1}$$

$$L_k = \sum_{i=0}^{2g}[A_0, ..., \hat{A}_i, ..., \hat{A}_{i+k}, ..., A_{2g}]d_{i+k+1} + [0, ..., \hat{A}_i, ..., \hat{A}_{i+k+1}, ..., A_{2g}]d_{i+1}, \;\; 1 \le k \le g-1.$$

Proposition 5.5. *For each $0 \le k \le g$ the chain L_k, is a vertical 1-cycle.*

Proof. First consider any of the $2g+1$ terms of L_0. Each is, on its own, a vertical cycle, because, generally, the support of d_{n+1} acting on $\{A_i\}$ is $\{A_n, A_{n+1}\}$.

Now consider L_k for $1 \le k \le g-1$, and the simplex $[\hat{A}_0, \dots, \hat{A}_k, \dots, A_{2g}]$ which is the coefficient of the first term in the sum. To prove the Proposition we construct, for each k, a sequence of composable elementary morphisms, starting and finishing at the object $[\hat{A}_0, ..., \hat{A}_k, ..., A_{2g}]$. By construction these are 1-cycles in the groupoid Λ_{2g-2}, hence vertical cycles. Then we show that each sequence represents the 1-chain L_k. In the following formula, for simplicity, we suppress the coefficients of each elementary morphism, for they are determined by the initial object. The composite

$$(d_0 d_k)(d_{2g} d_{k-1})\cdots(d_{2g-k+2} d_1)(d_{2g-k+1} d_0)(d_{2g-k} d_{2g})\cdots(d_2 d_{k+2})(d_1 d_{k+1}) \tag{9}$$

maps the simplex $[\hat{A}_0, \dots, \hat{A}_k, ..., A_{2g}]$ component-wise to $[A_{2g-1}, A_{2g}, \hat{A}_0, ..., \hat{A}_k, ..., A_{2g-2}]$, as we now observe. Note the two simplices are in the same orientation class. The pair of elementary homeomorphisms within each set of parentheses corresponds, in the definition of L_k, to the composite of two elementary morphisms with a fixed index i, the first one of which comes from the sum on the left and the second from the sum on the right in the expression for L_k. The first pair of elementary morphisms corresponds to $i = 0$ and maps $[\hat{A}_0, ..., \hat{A}_k, ..., A_{2g}]$ to $[A_0, \hat{A}_1, ..., \hat{A}_{k+1}, ..., A_{2g}]$. Continuing in this manner, we see that the $2(2g+1)$ morphisms in (9) match up with the terms in L_k, as claimed.

When $k = g$ the composite $d_0(d_g d_{2g})(d_{g-1} d_{2g-1})\cdots(d_2 d_{g+2})(d_1 d_{g+1})$ maps A_0 to A_g and A_g back to A_0. This requires $2g+1$ transformations, so in this case the $(2g-1)$-tuple $[\hat{A}_0, ..., \hat{A}_{g+1}, ..., A_{2g}]$ is restored in $2g+1$ steps, and shows that L_g is a vertical cycle. □

So each $L_i \in \mathcal{C}_{2g-2,1}$ is a vertical 1-cycle and therefore can be represented by a *directed loop* in Λ_{2g-2} using the sequence of transformations (9). Below we use the notation $a \sim_v b$ to denote that two vertical 1-cycles a and b are homologous in the groupoid Λ_{2g-2}.

Lemma 5.6. $L_1 \sim_v -L_2 \sim_v \cdots \sim_v (-1)^g L_{g-1} \sim_v 2(-1)^{g+1} L_g$.

Proof. Consider neighboring chains L_k and $-L_{k+1}$, $1 \le k \le g-2$ each oriented by the direction of the loops representing them. The loop L_k, starting at $[\hat{A}_0, ..., \hat{A}_{k+1}, ..., A_{2g}]$, is formed by a sequence of morphisms using the composite $(d_{k+1} d_0)(d_k d_{2g})\cdots(d_{k+3} d_2)(d_{k+2} d_1)$. If we flip the transformations within each set of parentheses, $(d_0 d_{k+1})\cdots(d_2 d_{k+3})(d_1 d_{k+2})$, we obtain the loop L_{k+1} starting at $[\hat{A}_0, ..., \hat{A}_{k+1}, ..., A_{2g}]$. Any square formed by pairs of consecutive morphisms in the two loops has the form

$$\begin{array}{ccc} & \xrightarrow{d_m} & \\ d_n \uparrow & & \uparrow d_n \\ & \xrightarrow[d_m]{} & \end{array}$$

where horizontal followed by vertical is in L_k and vertical followed by horizontal is in L_{k+1}.

For each square the "separation" $|m-n|$, between vertices indexed by m and those by n, is greater than 1. In that case the elementary mapping classes d_m and d_n commute since the commutator $d_n^{-1} d_m^{-1} d_n d_m$ fixes all the a_i's. Equivalently, each commutator defines a null-homotopic loop in Λ_{2g-2}. All loops L_k and $-L_{k+1}$, for $k \ne 0, k \ne g-1$, are therefore freely homotopic. In the exceptional case recall L_g is a sum $2g+1$ morphisms, which is half the number of morphisms of L_{g-1}, so that L_{g-1} is freely homotopic, with a change in sign, to twice L_g. Free homotopy of a pair of loops implies the loops are homologous. □

Recall S^{2g} has order 2 in Γ_g^1, since it is in the isotropy group $[A_0,\dots,A_{2g-2}]$, it determines a vertical 1-cycle $[A_0,\dots,A_{2g-2}]S^{2g}$ in Λ_{2g-1}. By a slight abuse of notation, in what follows we abbreviate $[A_0,\dots,A_{2g-2}]S^{2g}$ by S^{2g}.

Lemma 5.7. $L_0 + L_1 + \cdots + L_g \sim_v S^{2g}$.

Proof. The set of morphisms in the sum $L_0 + L_1 + \cdots + L_g$ is equal to the set of morphisms forming the composite morphism $[A_0,\dots,A_{2g-2}](d_{2g}d_{2g-1}d_0)^{2g}$. The composite is S^{2g} in Γ_g^1 by (5). Since there is a composite in the groupoid of all the morphisms which is S^{2g} the sum is homologous to S^{2g}. □

Proposition 5.8. $t \sim_v (2g)(2g+1)[A_0,A_1,\dots,A_{2g-2}]d_{2g} + S^{2g}$

Proof. Using Lemma 5.6 we can write each of $L_1, L_2, ..., L_{g-1}$ in terms of L_g. Then Lemma 5.7 determines L_g in terms of L_0 and S^{2g}, as shown in the following formula:

$$L_g \sim_v (-1)^g \; L_0 + S^{2g}. \tag{10}$$

Applying Lemma 5.6 to the expression (8) for t gives

$$t \sim_v L_0 + (-1)^g(2g-1)L_g. \tag{11}$$

After eliminating L_g from (10) using (11) we obtain $t \sim_v (2g)\; L_0 + (2g-1)S^{2g}$. Now L_0 is a sum of $2g+1$ distinct terms, each of which is a cycle conjugate to $[A_0,A_1,\dots,A_{2g-2}]d_{2g}$ in the groupoid Λ_{2g-1}, for any homotopy equivalence from Λ_{2g-2} to the isotropy group of $[A_0,A_1,\dots,A_{2g-2}]$ obtained by means of Proposition 5.3 will map each of the cycles L_i to $[A_0,A_1,\dots,A_{2g-2}]d_{2g}$ by conjugation. This gives the desired formula for t. Note, S^{2g} has order 2, so an odd multiple is homologous to it. □

Since S^{2g} has order 2 in Γ_g^1, Proposition 5.8 implies the following:

Corollary 5.9. $2t \sim_v (4g)(2g+1)[A_0,A_1,\dots,A_{2g-2}]d_{2g}$.

Example 5.10. We illustrate the constructions of this section with the genus $g=1$ case. The mapping class group Γ_1^1 is $SL(2,\mathbb{Z})$. A elementary simplex, together with its associated elementary mapping classes, is given by the data

$$\mathrm{a}_0 = (1,0) \qquad \mathrm{a}_1 = (0,1) \qquad \mathrm{a}_2 = (-1,-1)/\sqrt{2}$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad d_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad d_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \quad d_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We consider $SL(2,\mathbb{Z})$ acting on inversives. Notice that in this case inversives are in one to one correspondence with lines in the plane that pass through the origin, and the action of $SL(2,\mathbb{Z})$ factors through $PSL(2,\mathbb{Z})$.

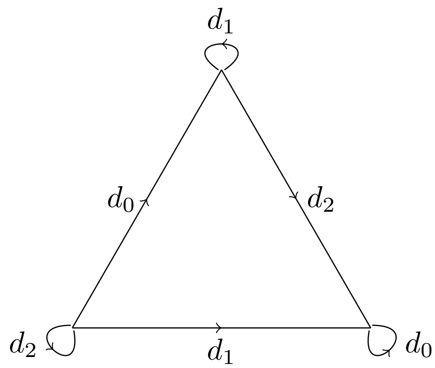

Figure 2: The transition cycle t for genus $g = 1$

Note, to obtain $2t$ we composed the six morphisms shown in Figure 2 twice. We will see that E^g is represented in Ext by a homomorphism taking the boundary $2t$ to 2. That representation will enable us to determine torsion behavior of E^g.

The chain c_1 is $[A_1, A_2]d_1 - [A_0, A_2]d_2 + [A_0, A_1]d_0$.

The transition cycle t is $([A_2] - [A_1])d_1 + ([A_0] - [A_2])d_2 + ([A_1] - [A_0])d_0$.

The terms with a positive coefficient correspond to elementary morphisms which fix an object. The three with a negative coefficient move A_1 to A_0, A_2 to A_1, and A_0 to A_2. The transition cycle is illustrated in Figure 2, where the loops stand for the elementary morphisms which fix objects, and arrows for the elementary morphisms that move objects. We can write the transition cycle t as $t = L_0 - L_1$ where $L_0 = [A_2]d_1 + [A_0]d_2 + [A_1]d_0$ and $L_1 = [A_1]d_1 + [A_2]d_2 + [A_0]d_0$.

The elementary morphisms have explicit formulas in $SL(2, \mathbb{Z})$. We can therefore compute directly the element in the isotropy group of $[A_0]$ determined by $2t$.

$$(d_0^{-1}d_1d_2^{-1}d_0d_1^{-1}d_2)^2 = \begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix}^2 = -12d_2.$$

6 Torsion and the order of E^g

In this section we show that E^g is a torsion class, and prove Theorem A.

6.1 Termination of the lifting

As before, let Λ be the component of Λ_{2g-2} containing the object $[A] := [A_0, A_1, ..., A_{2g-2}]$ and denote by $\Lambda_{[A]}$ the isotropy group of that object. By Proposition 5.3, the holonomy homomorphism induces an isomorphism $H_1(\Lambda) \to H_1(\Lambda_{[A]})$.

Notice that $\Lambda_{[A]}$ is a subgroup of Γ_g^1 that can also be identified as a subgroupoid of Λ. The transition cycle t is a 1-cycle in Λ and $[A_0, A_1, \dots, A_{2g-2}] \cdot d_{2g}$ is a 1-cycle in $\Lambda_{[A]} \subset \Lambda$. By Proposition 5.8 the cycles t and $2g(2g+1)[A_0, A_1, \dots, A_{2g-2}] \cdot d_{2g} + S^{2g}$ are homologous in Λ.

Proposition 6.1. *The elementary mapping class d_{2g} represents an element of infinite order in $H_1(\Lambda_{[A]})$.*

Proof. Consider $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$ generated by $\mathrm{a}_0, \dots, \mathrm{a}_{2g-1}$. The action of the mapping class group Γ_g^1 on $H_1(\Sigma_g)$ gives, with respect to this generating set, a group homomorphism $\psi : \Gamma_g^1 \to SL(2g, \mathbb{Z})$ defined by $f \mapsto f_*$. We will find an abelian subquotient $\mathcal{K}/\mathcal{T}$ of $SL(2g, \mathbb{Z})$ so that ψ maps $\Lambda_{[A]}$ to $\mathcal{K}/\mathcal{T}$ and takes d_{2g} to an element of infinite order. The map then factors through the abelianization, which implies the class of d_{2g} has infinite order in $H_1(\Lambda_{[A]})$.

If $f \in \Lambda_{[A]}$, then f is a mapping class that permutes the inversives $A_0, A_1, \dots, A_{2g-2}$ preserving the orientation class $[A_0, A_1, \dots, A_{2g-2}]$. Moreover, the mapping class f must preserve the cyclic ordering (3). Then f_* acts on the generators $\mathrm{a}_0, \dots, \mathrm{a}_{2g-2}$ by $\pm P$, where P is a $(2g-1) \times (2g-1)$ permutation matrix with $\det P = 1$. The image $K = \psi(f)$ has the form $K = QT$ below, where I is the identity $(2g-1) \times (2g-1)$ matrix and $n_i \in \mathbb{Z}$ for $0 \le i \le 2g-2$. In particular d_{2g} maps to the matrix with $n_i = 1$ for all i, and $P = I$.

$$K = \pm \begin{pmatrix} & \cdots & & n_0 \\ & \cdots & & n_1 \\ & P & & \vdots \\ & \cdots & & n_{2g-2} \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} & \cdots & & n_0 \\ & \cdots & & n_1 \\ & I & & \vdots \\ & \cdots & & n_{2g-2} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} & \cdots & & 0 \\ & \cdots & & 0 \\ & \pm P & & \vdots \\ & \cdots & & 0 \\ 0 & \cdots & 0 & \pm 1 \end{pmatrix}$$

Matrices of the form K and Q are closed under products and inverses so they determine subgroups $\mathcal{K}$ and $\mathcal{Q}$ of $SL(2g, \mathbb{Z})$. The subset $\mathcal{T}$ of $\mathcal{K}$ which consists of matrices of the form K with $n_0 + \cdots + n_{2g-2} = 0$ is a normal subgroup of $\mathcal{K}$.

We claim that the quotient group $\mathcal{K}/\mathcal{T}$ is abelian. Indeed, consider the composite $\mathcal{Q} \hookrightarrow \mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{T}$. The image of this homomorphism is the subgroup of cosets represented by $Q \in \mathcal{Q}$. The product decomposition $K = QT$ shows that every coset in $\mathcal{K}/\mathcal{T}$ has a representative in $\mathcal{Q}$. Therefore, the homomorphism $\mathcal{Q} \to \mathcal{K}/\mathcal{T}$ is onto. It follows that $\mathcal{K}/\mathcal{T}$ is isomorphic to a quotient of $\mathcal{Q}$ which must be abelian since $\mathcal{Q}$ is abelian.

Hence, the composite $\Lambda_{[A]} \xrightarrow{\psi} \mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{T}$ factors through the abelianization $H_1(\Lambda_{[A]})$. Moreover, the mapping class d_{2g} maps under this composite to an element in $\mathcal{K}/\mathcal{T}$ of infinite order, namely the coset represented by the matrix Q in $\mathcal{Q}$ with $n_i = 1$ for $i = 0, \dots, 2g-2$. So the elementary mapping class d_{2g} has infinite order in $H_1(\Lambda_{[A]})$. □

Corollary 6.2. *The transition cycle t is a 1-cycle in the groupoid Λ that represents a class of infinite order in $H_1(\Lambda)$.*

Proof. By Proposition 6.1 the elementary mapping class d_{2g} represents a class of infinite order in $H_1(\Lambda_{[A]})$, so $4g(2g+1)d_{2g}$ does as well. By Proposition 5.8 the 1-cycle $4g(2g+1)[A_0, A_1, \dots, A_{2g-2}]d_{2g}$ and $2t$ represent the same class in $H_1(\Lambda)$. By Proposition 5.3 the holonomy functor $\Lambda \to \Lambda_{[A]}$ induces an isomorphism in homology, then $2t$ and hence t represent elements of infinite order in $H_1(\Lambda)$. □

We are now prepared to show that the transition cycle t gives an obstruction to the existence of a cycle in the total complex that lifts any non-zero multiple of $(e_g/2)$.

Corollary 6.3. *Neither the chain $(e_g/2)$, nor any multiple of it, lifts to a cycle in the total complex of the inversive action. That is, there does not exist a cycle $c = c_0 + \dots + c_{2g}$, where $c_i \in \mathcal{C}_{2g-i,i}$ such that $c_0 = k(e_g/2)$, for any $k \in \mathbb{Z} - \{0\}$.*

Proof. We attempt to lift the chain $k(e_g/2)$ in $\mathcal{C}_{2g,0}$ to a cycle in the total complex $T\mathcal{C}$, as explained in the introduction of Section 5. If k is odd, then there is no chain $c_1 \in \mathcal{C}_{2g-1,1}$ such that $\partial^v(c_1) = \partial^h(k(e_g/2))$, for its chain boundary is a sum of an odd number of distinct faces.

If $k = 2m$ is even, we can continue the lifting process of by $k(e_g/2)$ taking

$$c_1 = m\sum_{i=0}^{2g}(-1)^i[A_0, ..., \hat{A}_i, A_{i+1}, ..., A_{2g}]d_{i+1} \in \mathcal{C}_{2g-1,1},$$

where $i \equiv n \bmod (2g+1)$. If a further extension were to exist, there would be a chain c_2 satisfying $-\partial^v c_2 = \partial^h c_1 = mt$. Since c_2 is a 2-chain in the groupoid Λ_{2g-2}, this implies that a non-trivial multiple of t bounds vertically in Λ_{2g-2}. However, by Corollary 6.2, the transition cycle t has infinite order in $H_1(\Lambda)$, and hence in $H_1(\Lambda_{2g-2})$. Therefore, no such chain c_2 can exist, the lifting terminates, and there is no cycle $c = c_0 + c_1 + c_2 + ... + c_{2g}$ such that $c_0 = k(e_g/2)$. □

6.2 The cohomology class E^g in Ext

We use our approach to directly prove that E^g is a torsion class. This can be deduced from results on the rational cohomology of moduli spaces obtained by algebro-geometric techniques [7, 11], but Theorem B provides a self-contained, intrinsic proof.

Consider the homomorphism $\beta : H^{2g}(T\mathcal{C};\mathbb{Z}) \to Hom(H_{2g}(T\mathcal{C}), \mathbb{Z})$ of the Universal Coefficient Theorem (1).

Theorem B. *The homomorphism β maps E^g to 0, hence E^g is a torsion class.*

Proof. By Proposition 4.8 the class $2\,\mathrm{E}^g$ is represented by the cocycle $(e_g/2)^*$ satisfying $\langle (e_g/2)^*, (e_g/2)\rangle = 1$. Since $(e_g/2)^*$ is a homomorphism from $T\mathcal{C}_{2g}$ onto $\mathbb{Z}$, it splits the $2g$-chains $T\mathcal{C}_{2g}$ into a summand C generated by the chain $(e_g/2)$ and a summand C' on which $(e_g/2)^*$ is trivial. The homomorphism $\beta(2\,\mathrm{E}^g) \in Hom(H_{2g}(T\mathcal{C}), \mathbb{Z})$ is represented by the restriction of the cocycle $(e_g/2)^*$ to $2g$-cycles $\mathcal{K}_{2g}$ in the total complex $T\mathcal{C}$.

If $\beta(2\,\mathrm{E}^g)$ were non-trivial in $Hom(H_{2p}(T\mathcal{C}),\mathbb{Z})$ there would exist a $2g$-cycle z with the property that $(e_g/2)^*(z) \neq 0$. Therefore, when the cycle z is written in terms of a basis determined by the splitting of $T\mathcal{C}_{2g}$ as $C \oplus C'$, it must have a non-trivial term in C. However, by Corollary 6.3, neither the chain $(e_g/2)$ nor any non-zero multiple of it, can be extended, by adding terms in C', to a cycle in the total complex. Consequently z must be an element of the subgroup C' in which case we obtain $(e_g/2)^*$ is the zero homomorphism when restricted to $\mathcal{K}_{2g}$. We conclude that β maps $2\,\mathrm{E}^g$ to 0, which implies that β maps E^g to 0 as well. □

Now let us recall the definition of $\alpha : Ext(H_n(T\mathcal{C}),\mathbb{Z}) \to H^{n+1}(T\mathcal{C};\mathbb{Z})$ in the Universal Coefficient Theorem (1). Let $\mathcal{K}_n$ denote the n-cycles and $\mathcal{B}_n$ the n-boundaries of the total complex $T\mathcal{C}$. The group $Ext(H_n(T\mathcal{C}),\mathbb{Z})$ is isomorphic to $Hom(\mathcal{B}_n,\mathbb{Z})/\iota(Hom(\mathcal{K}_n,\mathbb{Z}))$, where ι is the homomorphism which restricts an element of $Hom(\mathcal{K}_n,\mathbb{Z})$ to $\mathcal{B}_n$. Given $f \in Hom(\mathcal{B}_n,\mathbb{Z})$ representing a class in $Ext(H_n(T\mathcal{C}),\mathbb{Z})$, the element $\alpha(f) \in Hom((T\mathcal{C})_{n+1},\mathbb{Z})$ representing a class in $H^{n+1}(T\mathcal{C};\mathbb{Z})$ is defined by

$$\langle \alpha(f), c\rangle = \langle f, \partial c\rangle, \text{ for any } c \in (T\mathcal{C})_{n+1}.$$

Proposition 6.4. *There is a well-defined homomorphism $\chi \in Hom(\mathcal{B}_{2g-1},\mathbb{Z})$ which represents $2\,\mathrm{E}^g$ in $Ext(H_{2g-1}(T\mathcal{C}),\mathbb{Z})$ and satisfies $\langle \chi, t\rangle = 2$. Furthermore, the image of χ is contained in $2\mathbb{Z}$.*

Proof. Consider the chains $c_0 = e_g$, c_1 and the transition cycle $t = \partial(c_0 + c_1)$, as in Section 5.1. By Theorem B the cohomology class $2\,\mathrm{E}^g$, represented by the cocycle $(e_g/2)^*$, is a torsion class. Hence, the Universal Coefficient Theorem (1) implies that there exists $\chi \in Hom(\mathcal{B}_{2g-1},\mathbb{Z})$ such that $\alpha(\chi) = (e_g/2)^*$. It satisfies

$$\langle \chi, t\rangle = \langle \chi, \partial(c_0 + c_1)\rangle = \langle \alpha(\chi), c_0 + c_1\rangle = \langle (e_g/2)^*, c_0 + c_1\rangle.$$

On the other hand, by the evaluations of Proposition 4.8, we have $\langle (e_g/2)^*, c_1\rangle = 0$ and

$$\langle (e_g/2)^*, c_0 + c_1\rangle = \langle (e_g/2)^*, c_0\rangle = \langle (e_g/2)^*, e_g\rangle = 2,$$

which completes the proof of the first part of the proposition.

For the furthermore part of the proposition, consider the evaluation of χ on a boundary $b \in \mathcal{B}_{2g-1}$. If the evaluation is non-trivial then $b = \partial(b_0 + b_1)$ with $0 \neq b_0 \in \mathcal{C}_{2g,0}$ and $0 \neq b_1 \in \mathcal{C}_{2g-1,1}$. The chain b_0 can be written as a sum of oriented objects in the form $\sum \pm[x_0, ..., x_{2g}]$. The simplicial boundary of each term $\pm[x_0, ..., x_{2g}]$ is formally a sum of $2g+1$ objects, some with a coefficient of $+1$ and some with a coefficient -1. If b_0 were the sum of an odd number of signed objects, then it's boundary would also be a sum of an odd number of signed objects, and the number of objects with a positive coefficient would not match the number of objects with a negative coefficient. On the other hand, for ∂b_0 to lift to b_1 there must exist for every object in the sum ∂b_0 with a positive coefficient an object with a negative coefficient in order for there to be a morphism between the two. Consequently there must be an even number of terms in b_0 when written as a sum of signed objects. In that case the evaluation of $(e_g/2)^*$ on b_0, which is the same as the evaluation of χ on b, would be even as claimed. $\square$

6.3 Detecting the order of torsion at the threshold

We use the following results to show in Section 6.4 the non-triviality of E^g and derive information about its order.

In this section's statements we assume there exists $\mathrm{d} \in \Lambda_{[A]} \subset \Lambda$ and $\lambda > 1$ such that $2t \sim_v 2\lambda\mathrm{d}$. Therefore we have $2\lambda\mathrm{d} - 2t = \partial^v c_2$ for some chain $c_2 \in \mathcal{C}_{2g-2,2}$. It follows that $\partial(2c_0 + 2c_1 + c_2) = 2\lambda\mathrm{d} + R$, where $R := \partial^h c_2 \in \mathcal{C}_{2g-3,2}$.

Proposition 6.5. *The homomorphism $\chi \in Hom(\mathcal{B}_{2g-1}, \mathbb{Z})$ satisfies $\langle\chi, 2\lambda\mathrm{d} + R\rangle = 4$.*

Proof. Recall that $c_0 = e_g$, $c_1 \in \mathcal{C}_{2g-1,1}$ and $c_2 \in \mathcal{C}_{2g-2,2}$. Since $\alpha(\chi) = (e_g/2)^*$, it follows by Proposition 4.8 that

$$\langle\chi, 2\lambda\mathrm{d} + R\rangle = \langle\chi, \partial(2c_0 + 2c_1 + c_2)\rangle = \langle\alpha(\chi), 2c_0 + 2c_1 + c_2\rangle = 2\langle\alpha(\chi), c_0\rangle + \langle\alpha(\chi), 2c_1 + c_2\rangle = 4.$$

$\square$

Proposition 6.6. *Let $m > 0$ be the order of $2\,\mathrm{E}^g$. The homomorphism $m\chi$ extends to $Hom((T\mathcal{C})_{2g-1}, \mathbb{Z})$ and the extension satisfies $2m = \langle m\chi, \lambda\mathrm{d}\rangle = \lambda\langle m\chi, \mathrm{d}\rangle$. Furthermore, λ divides m.*

Proof. By Proposition 6.4, the class $2\,\mathrm{E}^g$ can be represented by $\chi \in Hom(\mathcal{B}_{2g-1}, \mathbb{Z})$ in $Ext(H_{2g-1}(T\mathcal{C}), \mathbb{Z})$. Since $2\,\mathrm{E}^g$ has order m, the multiple $m\chi$ can be extended to an element of $Hom(\mathcal{K}_{2g-1}, \mathbb{Z})$. Cycles are a direct summand of chains, then $m\chi$ can be extended to an element of $Hom((T\mathcal{C})_{2g-1}, \mathbb{Z})$ by defining to be 0 on the complement of $\mathcal{K}_{2g-1}$. Therefore, the evaluation of $m\chi$ on the chain $\lambda\mathrm{d} + R$ can be computed as

$$\langle m\chi, 2\lambda\mathrm{d}\rangle + \langle m\chi, R\rangle = \langle m\chi, 2\lambda\mathrm{d} + R\rangle = m\langle\chi, 2\lambda\mathrm{d} + R\rangle = 4m,$$

where the last equality follows from Proposition 6.5. In order to prove that $\langle m\chi, \lambda d\rangle = 2m$, or equivalently $\langle m\chi, 2\lambda d\rangle = 4m$, we show below that $\langle m\chi, R\rangle = 0$.

Consider the differential $\partial = \partial^h + \partial^v$ of the total complex $T\mathcal{C}$, restricted to $\mathcal{C}_{2g-2,2}$, and the following short exact sequences:

$$0 \longleftarrow H \xleftarrow{\partial^h} \mathcal{C}_{2g-2,2} \longleftarrow ker\partial^h \longleftarrow 0, \quad H \subset \mathcal{C}_{2g-3,2}$$

$$0 \longleftarrow V \xleftarrow{\partial^v} \mathcal{C}_{2g-2,2} \longleftarrow ker\partial^v \longleftarrow 0, \quad V \subset \mathcal{C}_{2g-2,1}$$

$$0 \longleftarrow H \oplus V \xleftarrow{\partial^h + \partial^v} \mathcal{C}_{2g-2,2} \longleftarrow ker(\partial^h + \partial^v) \longleftarrow 0.$$

The groups H and V are free so the homomorphisms ∂^h, ∂^v and $\partial = \partial^h + \partial^v$ split. Let σ denote the splitting homomorphism for ∂. We claim $\sigma(H \oplus V) = \sigma|_H(H) \oplus \sigma|_V(V)$. Indeed, since σ is $1-1$ the image of σ is a subgroup isomorphic to $H \oplus V$. An element α in the image of σ has the form $\alpha = \sigma(h+v)$ for unique elements $h \in H$, $v \in V$. Then also $\alpha = \sigma(h) + \sigma(v)$ for unique elements $h \in H$, $v \in V$, which verifies the claim.

Denote $\sigma|_H$ and $\sigma|_V$ by σ^h and σ^v. Note that σ^h splits ∂^h and σ^v splits ∂^v. Note also that $\sigma^h(H) = \sigma(H,0)$ and $\sigma^v(V) = \sigma(0,V)$ have only the 0-element in common.

Now $\mathcal{C}_{2g-2,2} = \sigma^h(H) \oplus ker\partial^h = \sigma^v(V) \oplus ker\partial^v$, hence $\sigma^h(H) \subset ker\partial^v$. Then $\partial^v\sigma^h = 0$, and in particular $\partial^v\sigma^h(R) = 0$. By definition of the splitting $\partial^h\sigma^h(R) = R$ so the chain $\sigma^h(R)$ bounds R, i.e. $\partial(\sigma^h(R)) = R$. Therefore,

$$\langle m\chi, R\rangle = \langle m\chi, \partial(\sigma^h(R))\rangle = \langle \alpha(m\chi), \sigma^h(R)\rangle = \langle m(e_g/2)^*, \sigma^h(R)\rangle = 0.$$

The last term is 0 since $\sigma^h(R) \in \mathcal{C}_{2g-2,2}$.

From $\langle m\chi, \lambda d\rangle = \lambda\langle m\chi, d\rangle = 2m$, it follows that λ divides $2m$. We claim that $\langle m\chi, \mathrm{d}\rangle$ is an even integer, so λ actually divides m. By Proposition 6.5, the image of χ on boundaries is contained in $2\mathbb{Z}$. Therefore $m\chi$, which extends to cycles and to chains, has image contained $2m\mathbb{Z}$. Hence $\langle m\chi, \mathrm{d}\rangle \in 2m\mathbb{Z}$ as claimed

□

Corollary 6.7. *If $\lambda > 1$, then E^g is a non-trivial torsion class of order divisible by 2λ.*

Proof. Let m be the order of $2\,\mathrm{E}^g$. By Proposition 6.6, we have that λ divides m. Since $\lambda > 1$, this implies that $m > 1$. Therefore $2\,\mathrm{E}^g$ is a non-trivial torsion class, and hence so it is E^g and it has order $2m$. □

6.4 Proof of Theorem A

In this section we use the setting developed in the paper to obtain information about the order of E^g. Our approach allow us to detect torsion of order $2g-1$ in our proof of Theorem A, which, as opposed to torsion of order $4g$ and $2g+1$, is not obtained using periodic elements in the mapping class group.

To find torsion of order $2g-1$ in cohomology, we consider a $w \in Aut(\pi_1(\Sigma_g, z))$ defined on the generators $\{\mathrm{a}_0, ..., \mathrm{a}_{2g-2}, \mathrm{a}_{2g-1}\}$ of $\pi_1(\Sigma_g, z)$ as follows:

$$\mathrm{a}_0 \to \mathrm{a}_1 \to \mathrm{a}_2 \to \cdots \to \mathrm{a}_{2g-2} \to \mathrm{a}_0 \qquad \mathrm{a}_{2g-1} \to \mathrm{a}_0^{-1}\mathrm{a}_{2g-1}\mathrm{a}_0^{-1}.$$

It can be seen that w is a well-defined element of $Aut(\pi_1(\Sigma_g, z))$ by checking that the defining relation (2) is preserved.

Proposition 6.8. *The automorphism w is a mapping class in Γ_g^1 such that $w^{2g-1} = d_{2g}^2$.*

Proof. The automorphism w^{2g-1} maps a_{2g-1} to $(\mathrm{a}_{2g-2}^{-1}\cdots\mathrm{a}_0^{-1})\mathrm{a}_{2g-1}(\mathrm{a}_0^{-1}\cdots\mathrm{a}_{2g-2}^{-1})$ and fixes all the remaining a's. On the other hand, recall d_{2g}^{-1} maps a_{2g-1} to $\mathrm{a}_{2g}^{-1} = \mathrm{a}_0\cdots\mathrm{a}_{2g-1} = \mathrm{a}_{2g-1}\cdots\mathrm{a}_0$ (see Section 3.3). Composing w^{2g-1} with d_{2g}^{-1} on the left and using the defining relation gives $\mathrm{a}_{2g-1} \to \mathrm{a}_{2g-1}(\mathrm{a}_0^{-1}\cdots\mathrm{a}_{2g-2}^{-1})$ and composing again gives $d_{2g}^{-2}w^{2g-1} : \mathrm{a}_{2g-1} \to \mathrm{a}_{2g-1}$ so that $d_{2g}^{-2}w^{2g-1}$ is the identity automorphism. Note w is orientation preserving since $w^{2g-1} = d_{2g}^2$ is and $2g-1$ is odd. ☐

Corollary 6.9. $2t \sim_v (2g)(2g+1)(2g-1)[A_0, \dots, A_{2g-2}]w$.

Proof. Since w acts by an even cyclic permutation on the generators $\{\mathrm{a}_0, ..., \mathrm{a}_{2g-2}\}$, it fixes $[A] = [A_0, \dots, A_{2g-2}]$. Then $w \in \Lambda_{[A]}$ and Proposition 6.8 implies that $2d_{2g} = (2g-1)w$ holds in $H_1(\Lambda_{[A]})$. Hence, from the isomorphism in homology induced by the holonomy homomorphim $\Lambda \to \Lambda_{[A]}$, we have that $2[A_0, \dots, A_{2g-2}]d_{2g} \sim_v (2g-1)[A_0, \dots, A_{2g-2}]w$. The statement follows from Corollary 5.9. ☐

Theorem A. The power E^g is a non-trivial torsion class, and its order is a positive multiple of $4g(2g+1)(2g-1)$.

Proof. By Theorem B we know that E^g is a torsion class in $H^{2g}(\Gamma_g^1; \mathbb{Z})$. From Corollary 6.9, we have that $2t \sim_v \lambda\mathrm{d}$, where $\lambda = (2g)(2g+1)(2g-1)$ and $\mathrm{d} = [A_0, \dots, A_{2g-2}]w \in \Lambda_{[A]} \subset \Lambda$. Then Corollary 6.7 implies that the class E^g is non-trivial and its order is divisible by $2\lambda = (4g)(2g+1)(2g-1)$ as claimed. ☐

References

[1] C.F. Bödigheimer, R. Hain, Editors, *Mapping class groups and moduli spaces for Riemann surfaces*, Contemporary Mathematics 150, A.M.S. Proceedings, (1991).

[2] C.F. Bödigheimer and U. Tillmann. *Stripping and splitting decorated mapping class groups.* Cohomological methods in homotopy theory (Bellaterra, 1998), 47–57, Progr. Math., 196, Birkhäuser, Basel, 2001.

[3] B. Farb and D. Margalit. *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton University Press, (2012).

[4] F. Fournier-Facio, N. Monod and S. Nariman. *The bounded cohomology of transformation groups of Euclidean spaces and discs*, arXiv preprint:2405.20395v1 (2024).

[5] R. Hain and E. Looijenga. *Mapping class groups and moduli spaces of curves.* Algebraic Geometry-Santa Cruz, Proc. Sympos. Pure Math., AMS **62** (1997), 97–142.

[6] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1983), 221–239.

[7] E. N. Ionel. *Topological recursive relations in* $H^{2g}(\mathcal{M}_{g,n})$, Invent. Math. **148** (2002), 627–658.

[8] S. Jekel. *Powers of the Euler class*, Adv. Math. **229** (2012), 1949–1975.

[9] S. Jekel and R. Jiménez Rolland. *On the non-vanishing of powers of the Euler class of mapping class groups*, Arnold J. of Math. 7, 159-168 (2021).

[10] M. Korkmaz, and A. I. Stipsicz. The second homology group of mapping class groups of oriented surfaces. Math. Proc. Cambridge Philos. Soc. **134** (2003), 479–489.

[11] E. Looijenga. *On the tautological ring of* $\mathcal{M}_g$, Invent. Math **121** (1995), 411–419.

[12] J. N. Mather. *The vanishing of the homology of certain groups of homeomorphisms*, Topology **10** (1971), 297–298.

[13] D. McDuff. *The homology of some groups of diffeomorphisms*, Comment. Math. Helv. 55 (1980), no. 1, 97-129.

[14] S. Morita. *Characteristic classes of surface bundles*, Invent. Math. 90 (1987), no. 3, 551–577.

[15] G. Segal. *Classifying spaces related to foliations,* Topology **17** (1978), 367–382.

[16] W. Thurston. *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 304–307.

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