The cohomology of $\mathcal{M}_{0,n}$ as an FI-module

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Abstract In this paper we revisit the cohomology groups of the moduli space of $n$-pointed curves of genus zero using the FI-module perspective introduced by Church-Elenberg-Farb. We recover known results about the corresponding representations of the symmetric group.

1 Introduction

Our space of interest is $\mathcal{M}_{0,n}$, the moduli space of $n$-pointed curves of genus zero. It is defined as the quotient

$$\mathcal{M}_{0,n} := \mathcal{F}(\mathbb{P}^1(\mathbb{C}), n)/\text{Aut}(\mathbb{P}^1(\mathbb{C})),$$

where $\mathcal{F}(\mathbb{P}^1(\mathbb{C}), n)$ is the configuration space of $n$-ordered points in the projective line $\mathbb{P}^1(\mathbb{C})$ and the automorphism group of the projective line $\text{Aut}(\mathbb{P}^1(\mathbb{C})) = \text{PGL}_2(\mathbb{C})$ acts componentwise on $\mathcal{F}(\mathbb{P}^1(\mathbb{C}), n)$. For $n \geq 3$, $\mathcal{M}_{0,n}$ is a fine moduli space for the problem of classifying smooth $n$-pointed rational curves up to isomorphism ([16 Proposition 1.1.2]).

The space $\mathcal{M}_{0,n}$ carries a natural action of the symmetric group $S_n$. The cohomology ring of $\mathcal{M}_{0,n}$ is known and the $S_n$-representations $H^i(\mathcal{M}_{0,n}; \mathbb{C})$ are well-understood (see for example [9], [15], [11]).

In this survey, we will consider the sequence of $S_n$-representations $H^i(\mathcal{M}_{0,n}; \mathbb{C})$ as a single object, an FI-module over $\mathbb{C}$. Via this example, we introduce the basics of the FI-module theory developed by Church, Ellenberg and Farb in [3]. We then use a well-known description of the cohomology ring of $\mathcal{M}_{0,n}$ to show in Theorem 4.5 that...
a finite generation property is satisfied which allows us to recover information about the $S_n$-representations in Theorem 5.1. Specifically, we obtain a stability result concerning the decomposition of $H^i \left( M_{0,n}; \mathbb{C} \right)$ into irreducible $S_n$-representations, we exhibit a bound on the lengths of the representations and show that their characters have a highly constrained “polynomial” form.

2 The co-FI-spaces $\mathcal{M}_{0, \bullet}$ and $\mathcal{M}_{0, \bullet+1}$

Let $\text{FI}$ be the category whose objects are natural numbers $n$ and whose morphisms $m \to n$ are injections from $[m] := \{1, \ldots, m\}$ to $[n] := \{1, \ldots, n\}$.

We are interested in the co-FI-space $\mathcal{M}_{0, \bullet}$: the functor from $\text{FI}^{op}$ to the category $\text{Top}$ of topological spaces given by given by $n \mapsto \mathcal{M}_{0,n}$ that assigns to $f : [m] \to [n]$ in $\text{Hom}_{\text{FI}}(m, n)$ the morphism $f^* : \mathcal{M}_{0,n} \to \mathcal{M}_{0,m}$ defined by $f^* \left( \left( [p_1, p_2, \ldots, p_m] \right) \right) = \left( [f(p_1), f(p_2), \ldots, f(p_m)] \right)$. This is a particular case of the co-FI-space $\mathcal{M}_{g, \bullet}$ considered in [12] which is the functor given by $n \mapsto \mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus $g$ with $n$ marked points.

An FI-module over $\mathbb{C}$ is a functor $V$ from $\text{FI}$ to the category of $\mathbb{C}$-vector spaces $\text{Vec}_\mathbb{C}$. Below, we denote $V(n)$ by $V_n$. Church, Ellenberg and Farb used FI-modules in [3] to encode sequences of $S_n$-representations in single algebraic objects and with this added structure significantly strengthened the representation stability theory introduced in [5]. FI-modules translate the representation stability property into a finite generation condition.

By composing the co-FI-space $\mathcal{M}_{0, \bullet}$ with the cohomology functor $H^\bullet(-; \mathbb{C})$, we obtain the FI-module $H^\bullet(\mathcal{M}_{0, \bullet}) := H^\bullet(\mathcal{M}_{0, \bullet}; \mathbb{C})$. We can also consider the graded version $H^\bullet(\mathcal{M}_{0, \bullet}) := H^\bullet(\mathcal{M}_{0, \bullet}; \mathbb{C})$, we call this a graded FI-module over $\mathbb{C}$.

The co-FI-space $\mathcal{F}(\mathbb{C}, \bullet)$ given by $n \mapsto \mathcal{F}(\mathbb{C}, n)$, the configuration space of $n$ ordered points in $\mathbb{C}$, and the corresponding FI-modules $H^i(\mathcal{F}(\mathbb{C}, \bullet))$ are key in our discussion below. In the expository paper [7], representation stability and FI-modules are motivated mainly through this example. A formal discussion of FI-modules and their properties is given in [3]. In [4] the theory of FI-modules is extended to modules over arbitrary Noetherian rings.

The “shifted” co-FI-space $\mathcal{M}_{0, \bullet+1}$. Consider the functor $\Xi_1$ from $\text{FI}$ to $\text{FI}$ given by $[n] \mapsto [n] \cup \{0\}$. Notice that this functor induces the inclusion of groups

$$J_n : S_n = \text{End}_{\text{FI}}[n] \hookrightarrow \text{End}_{\text{FI}}[n+1] = S_{n+1}$$

that sends the generator $(i \ i + 1)$ of $S_n$ to the transposition $(i + 1 \ i + 2)$ of $S_{n+1}$. In our discussion below we are interested in the “shifted” co-FI-space $\mathcal{M}_{0, \bullet+1}$ obtained by $\mathcal{M}_{0, \bullet} \circ \Xi_1$. Notice that this co-FI-space is given by $n \mapsto \mathcal{M}_{0,n+1}$. In the notation from [4] Section 2], this means that the FI-module

$$H^i(\mathcal{M}_{0, \bullet+1}) = S_{i+1} \left( H^i(\mathcal{M}_{0, \bullet}) \right),$$
where \( S_{+1} : \text{FI-Mod} \to \text{FI-Mod} \) is the shift functor given by \( S_{+1} := - \circ \mathbb{Z} \). The functor \( S_{+1} \) performs the restriction, from an \( S_{n+1} \)-representation to an \( S_n \)-representation, consistently for all \( n \) so that the resulting sequence of representations still has the structure of an FI-module. Comparing the \( S_n \)-representation \( H^i(M_{0,n+1}) \) with the \( S_{n+1} \)-representation \( H^i(M_{0,n+1}) \) we have an isomorphism of \( S_n \)-representations

\[
H^i(M_{0,n+1}) \cong \text{Res}^{S_{n+1}}_{S_n} H^i(M_{0,n+1}).
\]

## 3 Relation with the configuration space

We will understand the cohomology ring of \( M_{0,n} \) through its relation with the configuration space \( \mathcal{F}(\mathbb{C},n) \) of \( n \) ordered points in \( \mathbb{C} \).

In our descriptions below, we consider \( \mathbb{C}^1(\mathbb{C}) \) with coordinates \([t:z] \) and the embedding \( \mathbb{C} \hookrightarrow \mathbb{C}^1(\mathbb{C}) \), given by \( z \mapsto [1:z] \) and let \([0:1] = \infty \). We use the brackets to indicate “equivalence class of”. Since there is a unique element in \( \text{PGL}_2(\mathbb{C}) \) that takes any three distinct points in \( \mathbb{C}^1(\mathbb{C}) \) to \(([0:1],[1:0],[1:1]) = (\infty,0,1) \), every element in \( M_{0,n+1} \) can be written canonically as \( [(0:1],[1:0],[1:1],[t_1 : z_1],\cdots,[t_{n-2} : z_{n-2}]]) \). Hence, \( M_{0,4} \cong \mathbb{C}^1(\mathbb{C}) \setminus \{\infty,0,1\} \) and \( M_{0,n+1} \cong \mathcal{F}(M_{0,4},n-2) \).

Define the map \( \psi : \mathcal{F}(\mathbb{C},n) \to M_{0,n+1} \) by

\[
\psi(z_1,z_2,\ldots,z_n) = \left( (\infty,0,1,\frac{z_3-z_1}{z_2-z_1},\frac{z_4-z_1}{z_2-z_1},\ldots,\frac{z_n-z_1}{z_2-z_1}) \right).
\]

The symmetric group \( S_n \) acts on \( \mathcal{F}(\mathbb{C},n) \) by permuting the coordinates. Let \((1,2),(2,3),\ldots,(n-1,n)\) be transpositions generating \( S_n \) and notice that

\[
\psi((1,2) \cdot (z_1,z_2,\ldots,z_n)) = \left( (\infty,0,1,\frac{z_3-z_2}{z_1-z_2},\frac{z_4-z_2}{z_1-z_2},\ldots,\frac{z_n-z_2}{z_1-z_2}) \right)
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left( (0:1],[1:0],[1:1],[\frac{z_3-z_1}{z_1-z_2},\frac{z_4-z_1}{z_1-z_2},\ldots,\frac{z_n-z_1}{z_1-z_2}) \right)
\]

\[
= \begin{bmatrix} 0 \cdot 1 & 1 \\ 1 \cdot 1 & 0 \end{bmatrix} \cdot \left( (0:1],[1:0],[1:1],[\frac{z_3-z_1}{z_1-z_2},\frac{z_4-z_1}{z_1-z_2},\ldots,\frac{z_n-z_1}{z_1-z_2}) \right)
\]

\[
= (2,3) \cdot \psi(z_1,z_2,\ldots,z_n)
\]

and in general

\[
\psi((i,i+1) \cdot (z_1,z_2,\ldots,z_n)) = (i+1,i+2) \cdot \psi(z_1,z_2,\ldots,z_n) \quad \text{for} \quad i \geq 2.
\]

Therefore the map \( \psi : \mathcal{F}(\mathbb{C},n) \to M_{0,n+1} \) is equivariant with respect to the inclusion \( J_n : S_n \hookrightarrow S_{n+1} \). In other words, \( \psi : \mathcal{F}(\mathbb{C},\bullet) \to M_{0,\bullet+1} \) is a map of co-FI-spaces.
Relation with the Coxeter arrangement of type $A_{n-1}$. The complement of the complexified Coxeter arrangement of hyperplanes type $A_{n-1}$ is $M(A_{n-1})$, the image of $\mathcal{F}(\mathbb{C}, n)$ under the quotient map $\mathbb{C}^n \to \mathbb{C}^n/N$, where $N = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i = z_j \text{ for } 1 \leq i, j \leq n\}$.

As explained in [9], it turns out that the moduli space $M_{0,n+1}$ is also in bijective correspondence with the projective arrangement

\[
M(dA_{n-1}) := \pi(M(A_{n-1})) \cong M(A_{n-1})/\mathbb{C}^*,
\]

where $\pi : \mathbb{C}^{n-1}\setminus\{0\} \to \mathbb{P}^{n-2}(\mathbb{C})$ is the Hopf bundle projection, which takes $z \in \mathbb{C}^{n-1}\setminus\{0\}$ to $\lambda z$ for $\lambda \in \mathbb{C}^*$. Moreover, the map $\psi$ factors through $M(A_{n-1})$ and $M(dA_{n-1})$.

\[
\begin{array}{ccc}
\mathcal{F}(\mathbb{C}, n) & \xrightarrow{\psi} & M(dA_{n-1}) \\
\downarrow{\pi} & & \downarrow{\cong} \\
M_{0,n+1} & & \end{array}
\]

In [9], Gaiffi extends the natural $S_n$-action on $H^*(M(A_{n-1}); \mathbb{C})$ to an $S_{n+1}$-action using the vertical map in the diagram above and the natural $S_{n+1}$-action on $H^*(M_{0,n+1}; \mathbb{C})$.

The cohomology rings. As proved in [9] Prop. 2.2 & Theorem 3.2, the map $\psi$ allows us to relate the cohomology rings of $M_{0,n+1}$ and $\mathcal{F}(\mathbb{C}, n)$. See also [11, Cor. 3.1].

**Proposition 3.1.** The maps $\psi$ induces an isomorphism of cohomology rings

\[
H^*(\mathcal{F}(\mathbb{C}, n); \mathbb{C}) \cong H^*(M_{0,n+1}; \mathbb{C}) \otimes H^*(\mathbb{C}^*; \mathbb{C})
\]

as $S_n$-modules. The symmetric group $S_n$ acts trivially on $H^*(\mathbb{C}^*; \mathbb{C})$ and acts on $H^*(M_{0,n+1}; \mathbb{C})$ through the inclusion $J_n : S_n \hookrightarrow S_{n+1}$ that sends the generator $(i+1)$ of $S_n$ to the transposition $(i+1)(i+2)$ of $S_{n+1}$.

This means that the map of co-FI-spaces $\psi : \mathcal{F}(\mathbb{C}, \bullet) \to M_{0,\bullet+1}$ induces an isomorphism of graded FI-modules

\[
H^*(\mathcal{F}(\mathbb{C}, \bullet)) \cong H^*(M_{0,\bullet+1}) \otimes H^*(\mathbb{C}^*),
\]

where $H^*(\mathbb{C}^*)$ is the trivial graded FI-module given by $n \mapsto H^*(\mathbb{C}^*; \mathbb{C})$.

Furthermore, Arnol’d obtained a presentation of the cohomology ring of $\mathcal{F}(\mathbb{C}, n)$ in (11).

**Theorem 3.2.** The cohomology ring $H^*(\mathcal{F}(\mathbb{C}, n); \mathbb{C})$ is isomorphic to the $\mathbb{C}$-algebra $\mathcal{R}_n$ generated by 1 and forms $\omega_{i,j} := \frac{d \log(z_j - z_i)}{2\pi i}$, $1 \leq i \neq j \leq n$, with relations...
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\[
\omega_{i,j} = \omega_{j,i}, \quad \omega_{i,j} \omega_{k,j} = -\omega_{k,j} \omega_{i,j} \quad \text{and} \quad \omega_{i,j} \omega_{j,k} + \omega_{j,k} \omega_{k,i} + \omega_{k,i} \omega_{i,j} = 0. \]

The action of \( S_n \) is given by \( \sigma \cdot \omega_{i,j} = \omega_{\sigma(i)\sigma(j)} \) for \( \sigma \in S_n \).

As a consequence of the isomorphism in Proposition 3.1, we also have a concrete description of the cohomology ring \( H^*(M_{0,n+1}; \mathbb{C}) \). We refer the reader to [11, Cor. 3.1], [9, Theorem 3.4] and references therein.

**Theorem 3.3.** The cohomology ring \( H^*(M_{0,n+1}; \mathbb{C}) \) is isomorphic to the subalgebra of \( \mathcal{R}_n \) generated by 1 and elements \( \theta_{i,j} := \omega_{i,j} - \omega_{12} \) for \( \{i, j\} \neq \{1, 2\} \). The \( S_n \)-action given by \( \sigma \cdot \theta_{i,j} = \theta_{\sigma(i)\sigma(j)} - \theta_{\sigma(1)\sigma(2)} \), for \( \sigma \in S_n \).

### 4 Finite generation

We can use the explicit presentations in Theorems 3.2 and 3.3 to understand the FI-modules \( H^*(\mathcal{F}(\bullet; \mathbb{C})) \) and \( H^*(M_{0,*+1}) \).

An FI-module \( V \) over \( \mathbb{C} \) is said to be **finitely generated in degree** \( \leq m \) if there exist \( v_1, \ldots, v_s \), with each \( v_i \in V_{n_i} \) and \( n_i \leq m \), such that \( V \) is the minimal sub-FI-module of \( V \) containing \( v_1, \ldots, v_s \). Finitely generated FI-modules have strong closure properties: extensions and quotients of finitely generated FI-modules are still finitely generated and finite generation is preserved when taking sub-FI-modules.

Notice that from Theorem 3.2, it follows that \( H^1(\mathcal{F}(n; \mathbb{C}); \mathbb{C}) \) is generated as a \( S_n \)-module by the class \( \omega_{1,2} \). Therefore, the FI-module \( H^1(\mathcal{F}(\bullet; \mathbb{C})) \) is finitely generated in degree 2 by the class \( \omega_{1,2} \) in \( H^1(\mathcal{F}(2; \mathbb{C})) \).

Similarly, from Theorem 3.3, we know that \( H^1(M_{0,n+1}) \) is generated by the \( \theta_{i,j} \) classes and notice that \( \theta_{i,j} = (j 3) \cdot \theta_{1,3} \) for \( j \neq \{1, 2\} \); \( \theta_{2,j} = (j 3) \cdot \theta_{2,3} \) for \( j \neq \{1, 2\} \) and \( \theta_{i,j} = (i 3)(j 4) \cdot \theta_{3,4} \) for \( \{i, j\} \neq \{1, 2\} \). Therefore, \( H^1(M_{0,n+1}) \) is generated by \( \theta_{1,3}, \theta_{2,3} \) and \( \theta_{3,4} \) as an \( S_n \)-module. This means that the FI-module \( H^1(M_{0,*+1}) \) is finitely generated by the classes \( \theta_{1,3}, \theta_{2,3} \) and \( \theta_{3,4} \) in \( H^1(M_{0,*+1}) \), hence in degree 4.

An FI-module \( V \) encodes the information of the sequence \( V_n \) of \( S_n \)-representations. Finite generation of \( V \) puts strong constraints on the decomposition of each \( V_n \) into irreducible representations and its character.

**Notation for representations of** \( S_n \). The irreducible representations of \( S_n \) over \( \mathbb{C} \) are classified by partitions of \( n \). A partition \( \lambda \) of \( n \) is a set of positive integers \( \lambda_1 \geq \cdots \geq \lambda_l \geq 0 \) where \( l \in \mathbb{Z} \) and \( \lambda_1 + \cdots + \lambda_l = n \). We write \( |\lambda| = n \). The corresponding irreducible \( S_n \)-representation will be denoted by \( V_{\lambda} \). Every \( V_{\lambda} \) is defined over \( \mathbb{C} \) and any \( S_n \)-representation decomposes over \( \mathbb{C} \) into a direct sum of irreducibles.

If \( \lambda \) is any partition of \( m \), i.e. \( |\lambda| = m \), then for any \( n \geq |\lambda| + \lambda_1 \) the **padded partition** \( \lambda[n] \) of \( n \) is given by \( n - |\lambda| \geq \lambda_1 \geq \cdots \geq \lambda_l > 0 \). Keeping the notation from [5], we set \( V(\lambda)[n] = V_{\lambda[n]} \) for any \( n \geq |\lambda| + \lambda_1 \). Every irreducible \( S_n \)-representation is of the form \( V(\lambda)[n] \) for a unique partition \( \lambda \). We define the **length** of an irreducible representation of \( S_n \) to be the number of parts in the corresponding partition of \( n \).
The trivial representation has length 1, and the alternating representation has length $n$. We define the length $\ell(V)$ of a finite dimensional representation $V$ of $S_n$ to be the maximum of the lengths of the irreducible constituents.

We say that an FI-module $V$ over $\mathbb{C}$ has weight $\leq d$ if for every $n \geq 0$ and every irreducible constituent $V(\lambda)_n$ we have $|\lambda| \leq d$. The degree of generation of an FI-module $V$ gives an upper bound for the weight (\cite[Prop. 3.2.5]{3}). The weight of an FI-module is closed under subquotients and extensions. Moreover, if a finitely generated FI-module $V$ has weight $\leq d$, by definition, $\ell(V_n) \leq d + 1$ for all $n$ and the alternating representation cannot appear in the decomposition into irreducibles of $V_n$ once $n > d + 1$.

Notice that the FI-module $H^1(\mathcal{F}(\bullet; \mathbb{C}))$ has weight at most 2 and so does $H^1(M_{0,*+1})$, since it is a sub-FI-module of $H^1(\mathcal{F}(\bullet; \mathbb{C}))$.

An FI-module $V$ has stability degree $\leq N$, if for every $a \geq 0$ and $n \geq N + a$ the map of coinvariants

$$(I_n)_*: (V_n)_{S_n - a} \rightarrow (V_{n+1})_{S_{n+1} - a}$$

(1)

induced by the standard inclusion $I_n : \{1,\ldots,n\} \rightarrow \{1,\ldots,n,n+1\}$, is an isomorphism of $S_n$-modules (see \cite[Definition 3.1.3]{3} for a more general definition). Here, $S_{n-a}$ is the subgroup of $S_n$ that permutes $\{a+1,\ldots,n\}$ and acts trivially on $\{1,2,\ldots,a\}$. The coinvariant quotient $(V_n)_{S_n-a}$ is the $S_n$-module $V_n \otimes_{\mathbb{C}[S_n]} \mathbb{C}$, the largest quotient of $V_n$ on which $S_{n-a}$ acts trivially.

The finite generation properties of the FI-modules $H^i(\mathcal{F}(\bullet; \mathbb{C}))$ have already been discussed in \cite[Example 5.1.A]{3}.

**Proposition 4.1.** The FI-module $H^i(\mathcal{F}(\bullet; \mathbb{C}))$ is finitely generated with weight $\leq 2i$ and has stability degree $\leq 2i$

**Proof.** From Theorem 3.2 the graded FI-module $H^i(\mathcal{F}(\bullet; \mathbb{C}))$ is generated by the FI-module $H^i(\mathcal{F}(\bullet; \mathbb{C}))$ that has weight $\leq 2$. It follows by \cite[Theorem 4.2.3]{3} that $H^i(\mathcal{F}(\bullet; \mathbb{C}))$ is finitely generated with weight $\leq 2i$. Moreover, in \cite{3} it is shown that $H^i(\mathcal{F}(\bullet; \mathbb{C}))$ has the additional structure of what \cite{3} calls an FI#-module, which implies that it has stability degree bounded above by the weight (see proof of \cite[Cor. 4.1.8]{3}).

Finite generation for the FI-modules $H^i(M_{0,*+1})$ follows from Theorem 3.3 and Proposition 4.1.

**Theorem 4.2.** The FI-module $H^i(M_{0,*+1})$ is finitely generated in degree $\leq 4i$, with weight $\leq 2i$ and has stability degree $\leq 2i$.

**Proof.** By Theorem 3.3 the graded FI-module $H^i(M_{0,*+1})$ is generated by the FI-module $H^i(M_{0,*+1})$, which is finitely generated in degree $\leq 4$ and has weight $\leq 2$. It follows from \cite[Proposition 2.3.6]{3} that the FI-module $H^i(M_{0,*+1})$ is finitely generated in degree $\leq 4i$. By \cite[Corollary 4.2.A]{3} it has weight $\leq 2i$.

Moreover, from \cite[Lemma 3.1.6]{3} we have that the stability degree of $H^i(M_{0,*+1})$ is bounded above by the stability degree of $H^i(\mathcal{F}(\bullet; \mathbb{C}))$. 

From $H^i(M_{0,n+1})$ to the FI-module $H^i(M_{0,*})$. The relation between the degree of generation of an FI-module $V$ and its “shift” $S_+V$ was established in [4 Cor. 2.13]. We can also relate the weights and stability degrees using the classical branching rule (see e.g. [3]).

**Proposition 4.3.** Let $\lambda$ be a partition of $n+1$ and $V_\lambda$ the corresponding irreducible $S_{n+1}$-representation, then as $S_n$-representations we have the decomposition

$$\text{Res}_{S_n}^{S_{n+1}}V_\lambda \cong \bigoplus_v V_v$$

over those partitions $v$ of $n$ obtained from $\lambda$ by removing one box from one of the columns of the corresponding Young diagram.

**Theorem 4.4 (Finite generation and “shifted” FI-modules).** Let $V$ be a finitely generated FI-module generated in degree $\leq d$, then $S_+V$ is finitely generated in degree $\leq d$. Conversely, if the FI-module $S_+V$ is finitely generated in degree $\leq d$, then $V$ is finitely generated in degree $\leq d+1$.

Furthermore, if $S_+V$ has weight $\leq M$ and stability degree $\leq N$, then $V$ has weight $\leq M+1$ and stability degree $\leq N+1$. Conversely, if $V$ has weight $\leq M$ and stability degree $\leq N$, then $S_+V$ has weight $\leq M$ and stability degree $\leq N$.

**Proof.** If $V$ has weight $\leq M$, then for all $n \geq 0$, the irreducible components $V(\mu)_{n+1}$ of $V_{n+1}$ have $|\mu| \leq M$. From Proposition 4.3 it follows that $\text{Res}_{S_n}^{S_{n+1}}V(\mu)_{n+1}$ will be a direct sum of irreducibles $V(\lambda)_{n}$, with $|\lambda| \leq |\mu| \leq M$. Conversely, if $S_+V$ has weight $\leq M$, then each irreducible component $V(\lambda)_{n}$ of $S_+V$ has $|\lambda| \leq M$. By Proposition 4.3, it comes from the restriction of some $V(\mu)_{n+1}$ with $|\mu| \leq |\lambda| + 1 \leq M+1$.

On the other hand, the functor $\Xi_1$ sends $\{1, \ldots, a\}$ into $\{2, \ldots, a+1\}$ and $\{a+1, \ldots, n\}$ into $\{a+2, \ldots, n+1\}$. Therefore, the inclusion $J_n : S_n \hookrightarrow S_{n+1}$ maps the subgroup $S_n$ of $S_{n+1}$ onto the subgroup $S_{n+1}(\bullet_{n+1})$ of $S_{n+1}$ and we have that

$$(V_{n+1})_{S_n \hookrightarrow S_{n+1}(\bullet_{n+1})} = V_{n+1} \otimes C[S_{n+1}(\bullet_{n+1})] C = S_{n+1}(V)_{n} \otimes C[S_{a-n}] C = (S_{n+1}(V)_{n})_{S_{n-a}},$$

which implies the statement about stability degrees.

In [12] we proved finite generation for the FI-modules $H^i(M_{g,*})$ when $g \geq 2$. The case when $g = 0$ follows from Theorem 4.4 and Theorem 4.2.

**Theorem 4.5.** The FI-module $H^i(M_{0,*})$ is finitely generated with weight $\leq 2i$ and has stability degree $\leq 4i$.

**The first cohomology group.** Recall that $H^1(M_{0,*})$ is generated by the classes $\theta_{i,j} = \omega_{i,j} - \omega_{i,2}$ and it is a subrepresentation of $H^1(\mathcal{F}(n, \mathbb{C}))$ which has a basis given by the classes $\omega_n$. In particular, notice that $\dim H^1(M_{0,*}) = \dim H^1(\mathcal{F}(n, \mathbb{C})) - 1$. Moreover for $n \geq 4$, we have the decomposition

$$H^1(\mathcal{F}(n, \mathbb{C})) = V(0)_{n} \oplus V(1)_{n} \oplus V(2)_{n}. $$
Then, for \( n \geq 4 \) the \( S_n \)-representation

\[
H^1(M_{0,\bullet+1})_n = V(1)_n \oplus V(2)_n \cong \text{Re}^{S_{n+1}}_n H^1(M_{0,n+1}).
\]

Proposition 4.3 implies that for \( n \geq 4 \), we have that \( H^1(M_{0,n+1}) = V(2)_{n+1} \) as a representation of \( S_{n+1} \). Moreover, notice that \( H^1(M_{0,n+1}) \) is finitely generated by the classes \( \theta_{1,3}, \theta_{2,3} \) and \( \theta_{3,4} \) in \( H^1(M_{0,5}) \) not only as an \( S_n \)-module, but also as an \( S_{n+1} \)-module. Therefore, the FI-module \( H^1(M_{0,\bullet}) \) is finitely generated in degree \( \leq 5 \) and has weight \( \leq 2 \).

5 The \( S_n \)-representations \( H^i(M_{0,n}; \mathbb{C}) \)

At this point we can apply the theory of FI-modules to the finitely generated FI-modules \( H^i(M_{0,\bullet}) \) and \( H^i(M_{0,\bullet+1}) \) to obtain information about the corresponding sequences of \( S_n \)-representations and their characters. The following result is a direct consequence from [3, Prop. 3.3.3 and Theorem 3.3.4] and Theorems 4.2 and 4.5.

**Theorem 5.1.** Let \( i \geq 0 \). For \( n \geq 4i+2 \), the sequence \( \{H^i(M_{0,n})\} \) of representations of \( S_n \) and the sequence \( \{H^i(M_{0,\bullet+1})_{n-1}\} \) of \( S_{n-1} \)-representations satisfy the following:

(a) The decomposition into irreducibles of \( H^i(M_{0,n}; \mathbb{C}) \) and of \( H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1} \) stabilize in the sense of uniform representation stability ([5]) with stable range \( n \geq 4i+2 \).

(b) The length of \( H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1} \) is bounded above by \( 2i \) and the length of \( H^i(M_{0,n}; \mathbb{C}) \) is bounded above by \( 2i+1 \).

(c) The sequence of characters of the representations \( H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1} \) and \( H^i(M_{0,n}; \mathbb{C}) \) are eventually polynomial, in the sense that there exist character polynomials \( P_i(X_1, X_2, \ldots, X_r) \) and \( Q_i(X_1, X_2, \ldots, X_s) \) in the cycle-counting functions \( X_k(\sigma) := (\text{number of } k\text{-cycles in } \sigma) \) such that for all \( n \geq 4i+2 \):

\[
\chi_{H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1}}(\sigma) = P_i(X_1, X_2, \ldots, X_r)(\sigma) \quad \text{for all } \sigma \in S_{n-1}, \quad \text{and}
\]

\[
\chi_{H^i(M_{0,n}; \mathbb{C})}(\sigma) = Q_i(X_1, X_2, \ldots, X_s)(\sigma) \quad \text{for all } \sigma \in S_n.
\]

Moreover, the degree of \( P_i \) is \( \leq 2i \) and the degree of \( Q_i \) is \( \leq 2i+1 \), where we take \( \text{deg } X_k = k \). In particular, \( r \leq 2i \) and \( s \leq 2i+1 \).

If \( e \in S_{n-1} \) is the identity element, from Theorem 5.1(c), we obtain that the dimensions

\[
\dim_{\mathbb{C}}(H^i(M_{0,n}; \mathbb{C})) = \chi_{H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1}}(e) = P_i(X_1(e), \ldots, X_r(e)) = P_i(n-1, \ldots, 0)
\]
are polynomials in \( n \) of degree \( \leq 2i \). This agrees with the known Poincaré polynomial of \( \mathcal{M}_{0,n} \) (see [15] Cor. 2.10 and also [11] 5.5(8)).

From Theorem 4.1 and the definition of weight, we recover the fact that the alternating representation does not appear in the cohomology of \( \mathcal{M}_{0,n} \) ([15] Prop. 2.16)].

Theorem 5.1(a) implies that the dimensions of the vector spaces \( H^i(\mathcal{M}_{0,n}/S_n; \mathbb{C}) \) and \( H^0(\mathcal{M}_{0,n+1}/S_n; \mathbb{C}) \) are constant. For the sequence \( \{\mathcal{M}_{0,n}/S_n\} \), this is actually a trivial consequence from the fact that \( \mathcal{M}_{0,n}/S_n \) has the cohomology of a point as shown in [15] Theorem 2.3].

Recursive relation for characters. In [9] Theorem 4.1], Gaiffi obtained a recursive formula that connects the characters of the \( \mathcal{S}_n \)-representations \( H^i(\mathcal{M}_{0,n+1}; \mathbb{C}) \) and \( H^i(\mathcal{M}_{0,n}; \mathbb{C}) \) as follows

\[
\chi_{H^i(\mathcal{M}_{0,n+1}; \mathbb{C})} = \chi_{H^i(\mathcal{M}_{0,n}; \mathbb{C})} + (X_1 - 1) \cdot \chi_{H^{i-1}(\mathcal{M}_{0,n}; \mathbb{C})} \quad \text{for } n \geq 3. \tag{2}
\]

In particular, we know that \( \chi_{H^1(\mathcal{M}_{0,n+1}; \mathbb{C})} = \chi_{H^1(\mathcal{S}(n, \mathbb{C}))} - 1 = \left( \frac{X_1}{2} \right) + X_2 - 1 \) when \( n \geq 4 \). Therefore, for \( i = 1 \), the recursive formula (2) gives us the character polynomial of degree 2

\[
\chi_{H^1(\mathcal{M}_{0,n}; \mathbb{C})} = \chi_{H^1(\mathcal{M}_{0,n+1}; \mathbb{C})} - (X_1 - 1) \cdot \chi_{H^0(\mathcal{M}_{0,n}; \mathbb{C})} = \left( \frac{X_1}{2} \right) + X_2 - X_1 = \chi_{V(2)}
\]

as expected since \( H^1(\mathcal{M}_{0,n}; \mathbb{C}) = V(2) \).

Furthermore, if \( P_i \) and \( Q_i \) are the character polynomials of \( H^i(\mathcal{M}_{0,n+1}; \mathbb{C}) \) and \( H^i(\mathcal{M}_{0,n}; \mathbb{C}) \) from Theorem 5.1(c) for \( n \geq 4i + 2 \), then formula (2) can be written as \( Q_i = P_i - (X_1 - 1) \cdot Q_{i-1} \) and \( \deg Q_i \leq \max(\deg P_i, 1 + \deg Q_{i-1}) \leq 2i \). As a consequence of this and Theorem 5.1(c) we have that, for \( n \geq 4i + 2 \), the values of \( \chi_{H^i(\mathcal{M}_{0,n+1}; \mathbb{C})}(\sigma) \) and \( \chi_{H^i(\mathcal{M}_{0,n}; \mathbb{C})}(\sigma) \) depend only on “short cycles”, i.e. cycles on \( \sigma \) of length \( \leq 2i \).

More is known about the \( \mathcal{S}_n \)-representations. In this paper we were mainly interested in highlighting the methods, since more precise information about the characters of the \( \mathcal{S}_n \)-representations is known. The moduli space \( \mathcal{M}_{0,n} \) can be represented by a finite type \( \mathbb{Z} \)-scheme and the manifold \( \mathcal{M}_{0,n}(\mathbb{C}) \) of \( \mathbb{C} \)-points of this scheme corresponds to the definition in Section 1. In [15] Kisin and Lehrer used an equivariant comparison theorem in \( \ell \)-adic cohomology and the Grothendieck-Lefschetz’s fixed point formula to obtain explicit descriptions of the graded character of the \( \mathcal{S}_n \)-action on the cohomology of \( \mathcal{M}_{0,n}(\mathbb{C}) \) via counts of number of points of varieties over finite fields. With their techniques they obtain the Poincaré polynomial of a permutation in \( \mathcal{S}_n \) of a specific cycle type acting on \( H^*(\mathcal{M}_{0,n}; \mathbb{C}) \) ([15] Theorem 2.9]) and a description of the top cohomology \( H^{n-3}(\mathcal{M}_{0,n}; \mathbb{C}) \) [15] Proposition 2.18]. Furthermore, Getzler uses the language of operads in [11] to obtain formulas for the characters of the \( \mathcal{S}_n \)-modules \( H^*(\mathcal{M}_{0,n}; \mathbb{C}) \).
The cohomology of $\overline{M}_{0,n}$. A related space of interest is $\overline{M}_{0,n}$, the Deligne-Mumford compactification of $M_{0,n}$. It is a fine moduli space for stable $n$-pointed rational curves for $n \geq 3$ (see [16, Chapter 1] and reference therein). It can also be constructed from $M(dA_{n-1})$ using the theory of wonderful models of hyperplanes arrangements developed by De Concini and Procesi (see for example [10, Chapter 2]). The space $\overline{M}_{0,n}$ also carries a natural action of the symmetric group $S_n$. Hence, a natural question to ask is whether the FI-module theory could tell us something about its cohomology groups as $S_n$-representations.

Explicit presentations of the cohomology ring of the manifold of complex points $\overline{M}_{0,n}(\mathbb{C})$ have been obtained by Keel [14] and Yuzvinsky [19]. Moreover, several recursive and generating formulas for the Poincaré polynomials have been computed (for instance see [19], [11], [17], [2]). The sequence $H^*(\overline{M}_{0,n}(\mathbb{C}); \mathbb{C})$ has the structure of an FI-module, however, the Betti numbers of $\overline{M}_{0,n}(\mathbb{C})$ grow exponentially in $n$, which precludes finite generation. Therefore an analogue of Theorem 5.1 cannot be obtained for this space.

On the other hand, as observed in [6], the manifold $\overline{M}_{0,n}(\mathbb{R})$ of real points of $\overline{M}_{0,n}$ is topologically similar to $\mathcal{F}(\mathbb{C}, n-1)$, the configuration space of $n-1$ ordered points in $\mathbb{C}$, in the sense that both are $K(\pi,1)$-spaces, have Poincaré polynomials with a simple factorization and Betti numbers that grow polynomially in $n$. The cohomology ring of the real locus $\overline{M}_{0,n}(\mathbb{R})$ was completely determined in [6] and an explicit formula for the graded character of the $S_n$-action was obtained in [13]. The presentation of the cohomology ring given in [6] can be used to prove finite generation for the FI-modules $H^*(\overline{M}_{0,n}(\mathbb{R}); \mathbb{C})$ and to obtain an analogue of Theorem 5.1 for this space (see [13]).

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References

The cohomology of $\mathcal{M}_{0,n}$ as an FI-module


