The cohomology of $\mathcal{M}_{0,n}$ as an FI-module

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Abstract In this paper we revisit the cohomology groups of the moduli space of n-pointed curves of genus zero using the FI-module perspective introduced by Church-Ellenberg-Farb. We recover known results about the corresponding representations of the symmetric group.

1 Introduction

Our space of interest is $\mathcal{M}_{0,n}$, the *moduli space of n-pointed curves of genus zero*. It is defined as the quotient

$$\mathcal{M}_{0,n} := \mathcal{F}\left(\mathbb{P}^1(\mathbb{C}), n\right) / \operatorname{Aut}\left(\mathbb{P}^1(\mathbb{C})\right),$$

where $\mathcal{F}(\mathbb{P}^1(\mathbb{C}), n)$ is the configuration space of n-ordered points in the projective line $\mathbb{P}^1(\mathbb{C})$ and the automorphism group of the projective line $\operatorname{Aut}\left(\mathbb{P}^1(\mathbb{C})\right) = \mathrm{PGL}_2(\mathbb{C})$ acts componentwise on $\mathcal{F}(\mathbb{P}^1(\mathbb{C}), n)$. For $n \geq 3$, $\mathcal{M}_{0,n}$ is a *fine moduli space* for the problem of classifying smooth n-pointed rational curves up to isomorphism ([16, Proposition 1.1.2]).

The space $\mathcal{M}_{0,n}$ carries a natural action of the symmetric group S_n. The cohomology ring of $\mathcal{M}_{0,n}$ is known and the S_n-representations $H^i(\mathcal{M}_{0,n};\mathbb{C})$ are well-understood (see for example [9], [15], [11]).

In this survey, we will consider the sequence of S_n-representations $H^i(\mathcal{M}_{0,n};\mathbb{C})$ as a single object, an *FI-module over* $\mathbb{C}$. Via this example, we introduce the basics of the FI-module theory developed by Church, Ellenberg and Farb in [3]. We then use a well-known description of the cohomology ring of $\mathcal{M}_{0,n}$ to show in Theorem 4.5 that

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a finite generation property is satisfied which allows us to recover information about the S_n-representations in Theorem 5.1. Specifically, we obtain a stability result concerning the decomposition of $H^i(\mathcal{M}_{0,n};\mathbb{C})$ into irreducible S_n-representations, we exhibit a bound on the lengths of the representations and show that their characters have a highly constrained "polynomial" form.

2 The co-FI-spaces $\mathcal{M}_{0,\bullet}$ and $\mathcal{M}_{0,\bullet+1}$

Let **FI** be the category whose objects are natural numbers **n** and whose morphisms $\mathbf{m} \to \mathbf{n}$ are injections from $[m] := \{1,\dots,m\}$ to $[n] := \{1,\dots,n\}$.

We are interested in the *co-FI-space* $\mathcal{M}_{0,\bullet}$: the functor from $\mathbf{FI}^{op}$ to the category **Top** of topological spaces given by given by $\mathbf{n} \mapsto \mathcal{M}_{0,n}$ that assigns to $f : [m] \hookrightarrow [n]$ in $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{m},\mathbf{n})$ the morphism $f^* : \mathcal{M}_{0,n} \to \mathcal{M}_{0,m}$ defined by $f^*\big([(p_1,p_2,\dots,p_n)]\big) = [(p_{f(1)},p_{f(2)},\dots,p_{f(m)})]$. This is a particular case of the co-FI-space $\mathcal{M}_{g,\bullet}$ considered in [12] which is the functor given by $\mathbf{n} \mapsto \mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points.

An *FI-module over* $\mathbb{C}$ is a functor V from **FI** to the category of $\mathbb{C}$-vector spaces $\mathbf{Vec}_{\mathbb{C}}$. Below, we denote $V(\mathbf{n})$ by V_n. Church, Ellenberg and Farb used FI-modules in [3] to encode sequences of S_n-representations in single algebraic objects and with this added structure significantly strengthened the representation stability theory introduced in [5]. FI-modules translate the representation stability property into a finite generation condition.

By composing the co-FI-space $\mathcal{M}_{0,\bullet}$ with the cohomology functor $H^i(-;\mathbb{C})$, we obtain the FI-module $H^i(\mathcal{M}_{0,\bullet}) := H^i(\mathcal{M}_{0,\bullet};\mathbb{C})$. We can also consider the graded version $H^*(\mathcal{M}_{0,\bullet}) := H^*(\mathcal{M}_{0,\bullet};\mathbb{C})$, we call this a *graded FI-module* over $\mathbb{C}$.

The co-FI-space $\mathcal{F}(\mathbb{C},\bullet)$ given by $\mathbf{n} \mapsto \mathcal{F}(\mathbb{C},n)$, the configuration space of n ordered points in $\mathbb{C}$, and the corresponding FI-modules $H^i\big(\mathcal{F}(\mathbb{C},\bullet)\big)$ are key in our discussion below. In the expository paper [7], representation stability and FI-modules are motivated mainly through this example. A formal discussion of FI-modules and their properties is given in [3]. In [4] the theory of FI-modules is extended to modules over arbitrary Noetherian rings.

The "shifted" co-FI-space $\mathcal{M}_{0,\bullet+1}$. Consider the functor Ξ_1 from **FI** to **FI** given by $[n] \mapsto [n] \sqcup \{0\}$. Notice that this functor induces the inclusion of groups

$$J_n : S_n = \mathrm{End}_{\mathrm{FI}}[\mathbf{n}] \hookrightarrow \mathrm{End}_{\mathrm{FI}}[\mathbf{n+1}] = S_{n+1}$$

that sends the generator $(i\ i+1)$ of S_n to the transposition $(i+1\ i+2)$ of S_{n+1}. In our discussion below we are interested in the "shifted" co-FI-space $\mathcal{M}_{0,\bullet+1}$ obtained by $\mathcal{M}_{0,\bullet} \circ \Xi_1$. Notice that this co-FI-space is given by $\mathbf{n} \mapsto \mathcal{M}_{0,n+1}$. In the notation from [4, Section 2], this means that the FI-module

$$H^i(\mathcal{M}_{0,\bullet+1}) = S_{+1}\big(H^i(\mathcal{M}_{0,\bullet})\big),$$

where S_{+1} : **FI-Mod** $\to$ **FI-Mod** is the shift functor given by $S_{+1} := - \circ \Xi_1$. The functor S_{+1} performs the restriction, from an S_{n+1}-representation to an S_n-representation, consistently for all n so that the resulting sequence of representations still has the structure of an FI-module. Comparing the S_n-representation $H^i(\mathcal{M}_{0,\bullet+1})_n$ with the S_{n+1}-representation $H^i(\mathcal{M}_{0,\bullet})_{n+1} = H^i(\mathcal{M}_{0,n+1})$ we have an isomorphism of S_n-representations

$$H^i(\mathcal{M}_{0,\bullet+1})_n \cong \mathrm{Res}^{S_{n+1}}_{S_n} H^i(\mathcal{M}_{0,n+1}).$$

3 Relation with the configuration space

We will understand the cohomology ring of $\mathcal{M}_{0,n}$ through its relation with the configuration space $\mathcal{F}(\mathbb{C},n)$ of n ordered points in $\mathbb{C}$.

In our descriptions below, we consider $\mathbb{P}^1(\mathbb{C})$ with coordinates $[t:z]$ and the embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$, given by $z \mapsto [1:z]$ and let $[0:1] = \infty$. We use the brackets to indicate "equivalence class of". Since there is a unique element in $\mathrm{PGL}_2(\mathbb{C})$ that takes any three distinct points in $\mathbb{P}^1(\mathbb{C})$ to $\big([0:1],[1:0],[1:1]\big) = (\infty,0,1)$, every element in $\mathcal{M}_{0,n+1}$ can be written canonically as $\big[\big([0:1],[1:0],[1:1],[t_1:z_1],\cdots,[t_{n-2}:z_{n-2}]\big)\big]$. Hence, $\mathcal{M}_{0,4} \cong \mathbb{P}^1(\mathbb{C})\backslash\{\infty,0,1\}$ and $\mathcal{M}_{0,n+1} \cong \mathcal{F}\big(\mathcal{M}_{0,4}, n-2\big)$.

Define the map $\psi : \mathcal{F}(\mathbb{C},n) \longrightarrow \mathcal{M}_{0,n+1}$ by

$$\psi(z_1,z_2,\ldots,z_n) = \left[\left(\infty,0,1,\frac{z_3-z_1}{z_2-z_1},\frac{z_4-z_1}{z_2-z_1},\cdots,\frac{z_n-z_1}{z_2-z_1}\right)\right].$$

The symmetric group S_n acts on $\mathcal{F}(\mathbb{C},n)$ by permuting the coordinates. Let $(1\;2),(2\;3),\ldots(n-1\;n)$ be transpositions generating S_n and notice that

$$\begin{aligned}
\psi\big((1\;2)\cdot(z_1,z_2,\ldots,z_n)\big) &= \left[\left(\infty,0,1,\tfrac{z_3-z_2}{z_1-z_2},\tfrac{z_4-z_2}{z_1-z_2},\cdots,\tfrac{z_n-z_2}{z_1-z_2}\right)\right]\\
&= \left[\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\cdot\left([0:1],[1:0],[1:1],[1:\tfrac{z_3-z_2}{z_1-z_2}],[1:\tfrac{z_4-z_2}{z_1-z_2}],\cdots,[1:\tfrac{z_n-z_2}{z_1-z_2}]\right)\right]\\
&= \left[\left([0:1],[1:1],[1:0],[1:\tfrac{z_3-z_1}{z_2-z_1}],[1:\tfrac{z_4-z_1}{z_2-z_1}],\cdots,[1:\tfrac{z_n-z_1}{z_2-z_1}]\right)\right]\\
&= (2\;3)\cdot\psi(z_1,z_2,\ldots,z_n)
\end{aligned}$$

and in general

$$\psi\big((i\;i+1)\cdot(z_1,z_2,\ldots,z_n)\big) = (i+1\;i+2)\cdot\psi(z_1,z_2,\ldots,z_n) \text{ for } i \geq 2.$$

Therefore the map $\psi : \mathcal{F}(\mathbb{C},n) \longrightarrow \mathcal{M}_{0,n+1}$ is equivariant with respect to the inclusion $J_n : S_n \hookrightarrow S_{n+1}$. In other words, $\psi : \mathcal{F}(\mathbb{C},\bullet) \longrightarrow \mathcal{M}_{0,\bullet+1}$ is a map of co-FI-spaces.

Relation with the Coxeter arrangement of type A_{n-1}. The *complement of the complexified Coxeter arrangement of hyperplanes type A_{n-1}* is $M(\mathcal{A}_{n-1})$, the image of $\mathcal{F}(\mathbb{C},n)$ under the quotient map $\mathbb{C}^n \to \mathbb{C}^n/N$, where $N = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : z_i = z_j \text{ for } 1 \le i,j \le n\}$.

As explained in [9], it turns out that the moduli space $\mathcal{M}_{0,n+1}$ is also in bijective correspondence with the projective arrangement

$$M(d\mathcal{A}_{n-1}) := \pi\big(M(\mathcal{A}_{n-1})\big) \cong M(\mathcal{A}_{n-1})/\mathbb{C}^*,$$

where $\pi : \mathbb{C}^{n-1}\backslash\{0\} \to \mathbb{P}^{n-2}(\mathbb{C})$ is the Hopf bundle projection, which takes $z \in \mathbb{C}^{n-1}\backslash\{0\}$ to λz for $\lambda \in \mathbb{C}^*$. Moreover, the map ψ factors through $M(\mathcal{A}_{n-1})$ and $M(d\mathcal{A}_{n-1})$.

$$\begin{tikzcd}
\mathcal{F}(\mathbb{C},n) \arrow[r] \arrow[ddr, "\psi"'] & M(\mathcal{A}_{n-1}) \arrow[d, "\pi"] \\
& M(d\mathcal{A}_{n-1}) \arrow[d, "\cong"] \\
& \mathcal{M}_{0,n+1}
\end{tikzcd}$$

In [9], Gaiffi extends the natural S_n-action on $H^*\big(M(\mathcal{A}_{n-1});\mathbb{C}\big)$ to an S_{n+1}-action using the vertical map in the diagram above and the natural S_{n+1}-action on $H^*\big(\mathcal{M}_{0,n+1};\mathbb{C}\big)$.

The cohomology rings. As proved in [9, Prop. 2.2 & Theorem 3.2] , the map ψ allows us to relate the cohomology rings of $\mathcal{M}_{0,n+1}$ and $\mathcal{F}(\mathbb{C};n)$. See also [11, Cor. 3.1].

Proposition 3.1. *The maps ψ induces an isomorphism of cohomology rings*

$$H^*\big(\mathcal{F}(\mathbb{C},n);\mathbb{C}\big) \cong H^*\big(\mathcal{M}_{0,n+1};\mathbb{C}\big) \otimes H^*(\mathbb{C}^*;\mathbb{C})$$

as S_n-modules. The symmetric group S_n acts trivially on $H^*(\mathbb{C}^*;\mathbb{C})$ and acts on $H^*\big(\mathcal{M}_{0,n+1};\mathbb{C}\big)$ through the inclusion $J_n : S_n \hookrightarrow S_{n+1}$ that sends the generator $(i\, i+1)$ of S_n to the transposition $(i+1\ i+2)$ of S_{n+1}.

This means that the map of co-FI-spaces $\psi : \mathcal{F}(\mathbb{C},\bullet) \longrightarrow \mathcal{M}_{0,\bullet+1}$ induces an isomorphism of graded FI-modules

$$H^*\big(\mathcal{F}(\mathbb{C},\bullet)\big) \cong H^*\big(\mathcal{M}_{0,\bullet+1}\big) \otimes H^*(\mathbb{C}^*),$$

where $H^*(\mathbb{C}^*)$ is the trivial graded FI-module given by $\mathbf{n} \mapsto H^*(\mathbb{C}^*;\mathbb{C})$.

Furthermore, Arnol'd obtained a presentation of the cohomology ring of $\mathcal{F}(\mathbb{C},n)$ in ([1]).

Theorem 3.2. *The cohomology ring $H^*\big(\mathcal{F}(\mathbb{C},n);\mathbb{C}\big)$ is isomorphic to the $\mathbb{C}$-algebra $\mathcal{R}_n$ generated by 1 and forms $\omega_{i,j} := \frac{d\log(z_j - z_i)}{2\pi i}$, $1 \le i \ne j \le n$, with relations*

$\omega_{i,j} = \omega_{j,i}$, $\omega_{i,j}\omega_{k,l} = -\omega_{k,l}\omega_{i,j}$ and $\omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0$. The action S_n is given by $\sigma \cdot \omega_{i,j} = \omega_{\sigma(i)\sigma(j)}$ for $\sigma \in S_n$.

As a consequence of the isomorphism in Proposition 3.1, we also have a concrete description of the cohomology ring $H^*\big(\mathcal{M}_{0,n+1};\mathbb{C}\big)$. We refer the reader to [11, Cor. 3.1], [9, Theorem 3.4] and references therein.

Theorem 3.3. *The cohomology ring $H^*\big(\mathcal{M}_{0,n+1};\mathbb{C}\big)$ is isomorphic to the subalgebra of $\mathcal{R}_n$ generated by 1 and elements $\theta_{i,j} := \omega_{i,j} - \omega_{12}$ for $\{i,j\} \neq \{1,2\}$. The S_n-action given by $\sigma \cdot \theta_{i,j} = \theta_{\sigma(i),\sigma(j)} - \theta_{\sigma(1),\sigma(2)}$, for $\sigma \in S_n$.*

4 Finite generation

We can use the explicit presentations in Theorems 3.2 and 3.3 to understand the FI-modules $H^i\big(\mathcal{F}(\bullet,\mathbb{C})\big)$ and $H^i(\mathcal{M}_{0,\bullet+1})$.

An FI-module V over $\mathbb{C}$ is said to be *finitely generated in degree* $\leq m$ if there exist $v_1,\dots,v_s$, with each $v_i \in V_{n_i}$ and $n_i \leq m$, such that V is the minimal sub-FI-module of V containing $v_1,\dots,v_s$. Finitely generated FI-modules have strong closure properties: extensions and quotients of finitely generated FI-modules are still finitely generated and finite generation is preserved when taking sub-FI-modules.

Notice that from Theorem 3.2 it follows that $H^1\big(\mathcal{F}(n;\mathbb{C});\mathbb{C}\big)$ is generated as an S_n-module by the class $\omega_{1,2}$. Therefore, the FI-module $H^1\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ is finitely generated in degree 2 by the class $\omega_{1,2}$ in $H^1\big(\mathcal{F}(2;\mathbb{C})\big)$.

Similarly, from Theorem 3.3 we know that $H^1(\mathcal{M}_{0,n+1})$ is generated by the $\theta_{i,j}$ classes and notice that $\theta_{1,j} = (j\ 3)\cdot\theta_{1,3}$ for $j \neq \{1,2\}$; $\theta_{2,j} = (j\ 3)\cdot\theta_{2,3}$ for $j \neq \{1,2\}$ and $\theta_{i,j} = (i\ 3)(j\ 4)\cdot\theta_{3,4}$ for $\{i,j\} \neq \{1,2\}$. Therefore, $H^1(\mathcal{M}_{0,n+1})$ is generated by $\theta_{1,3}$, $\theta_{2,3}$ and $\theta_{3,4}$ as an S_n-module. This means that the FI-module $H^1(\mathcal{M}_{0,\bullet+1})$ is finitely generated by the classes $\theta_{1,3}$, $\theta_{2,3}$ and $\theta_{3,4}$ in $H^1(\mathcal{M}_{0,\bullet+1})_4$, hence in degree 4.

An FI-module V encodes the information of the sequence V_n of S_n-representations. Finite generation of V puts strong constraints on the decomposition of each V_n into irreducible representations and its character.

Notation for representations of S_n. The irreducible representations of S_n over $\mathbb{C}$ are classified by partitions of n. A partition λ of n is a set of positive integers $\lambda_1 \geq \cdots \geq \lambda_l > 0$ where $l \in \mathbb{Z}$ and $\lambda_1 + \cdots + \lambda_l = n$. We write $|\lambda| = n$. The corresponding irreducible S_n-representation will be denoted by V_λ. Every V_λ is defined over $\mathbb{C}$ and any S_n-representation decomposes over $\mathbb{C}$ into a direct sum of irreducibles.

If λ is any partition of m, i.e. $|\lambda| = m$, then for any $n \geq |\lambda| + \lambda_1$ the *padded partition* $\lambda[n]$ of n is given by $n - |\lambda| \geq \lambda_1 \geq \cdots \geq \lambda_l > 0$. Keeping the notation from [5], we set $V(\lambda)_n = V_{\lambda[n]}$ for any $n \geq |\lambda| + \lambda_1$. Every irreducible S_n-representation is of the form $V(\lambda)_n$ for a unique partition λ. We define the *length* of an irreducible representation of S_n to be the number of parts in the corresponding partition of n.

The trivial representation has length 1, and the alternating representation has length n. We define the *length* $\ell(V)$ of a finite dimensional representation V of S_n to be the maximum of the lengths of the irreducible constituents.

We say that an FI-module V over $\mathbb{C}$ has *weight* $\leq d$ if for every $n \geq 0$ and every irreducible constituent $V(\lambda)_n$ we have $|\lambda| \leq d$. The degree of generation of an FI-module V gives an upper bound for the weight ([3, Prop. 3.2.5]). The weight of an FI-module is closed under subquotients and extensions. Moreover, if a finitely generated FI-module V has weight $\leq d$, by definition, $\ell(V_n) \leq d+1$ for all n and the alternating representation cannot not appear in the decomposition into irreducibles of V_n once $n > d+1$.

Notice that the FI-module $H^1\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ has weight at most 2 and so does $H^1(\mathcal{M}_{0,\bullet+1})$, since it is a sub-FI-module of $H^1\big(\mathcal{F}(\bullet;\mathbb{C})\big)$.

An FI-module V has *stability degree* $\leq N$, if for every $a \geq 0$ and $n \geq N+a$ the map of coinvariants

$$(I_n)_* : (V_n)_{S_{n-a}} \to (V_{n+1})_{S_{(n+1)-a}} \tag{1}$$

induced by the standard inclusion $I_n : \{1,\dots n\} \hookrightarrow \{1,\dots,n,n+1\}$, is an isomorphism of S_a-modules (see [3, Definition 3.1.3] for a more general definition). Here, S_{n-a} is the subgroup of S_n that permutes $\{a+1,\dots,n\}$ and acts trivially on $\{1,2,\dots,a\}$. The coinvariant quotient $(V_n)_{S_{n-a}}$ is the S_a-module $V_n \otimes_{\mathbb{C}[S_{n-a}]} \mathbb{C}$, the largest quotient of V_n on which S_{n-a} acts trivially.

The finite generation properties of the FI-modules $H^i\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ have already been discussed in [3, Example 5.1.A].

Proposition 4.1. *The FI-module $H^i\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ is finitely generated with weight $\leq 2i$ and has stability degree $\leq 2i$*

Proof. From Theorem 3.2 the graded FI-module $H^*\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ is generated by the FI-module $H^1\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ that has weight ≤ 2. It follows by [3, Theorem 4.2.3] that $H^i\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ is finitely generated with weight $\leq 2i$. Moreover, in [3] it is shown that $H^i\big(\mathcal{F}(\bullet;\mathbb{C})\big)$ has the additional structure of what [3] calls an FI#-module, which implies that it has stability degree bounded above by the weight (see proof of [3, Cor. 4.1.8]).

Finite generation for the FI-modules $H^i(\mathcal{M}_{0,\bullet+1})$ follows from Theorem 3.3 and Proposition 4.1.

Theorem 4.2. *The FI-module $H^i(\mathcal{M}_{0,\bullet+1})$ is finitely generated in degree $\leq 4i$, with weight $\leq 2i$ and has stability degree $\leq 2i$.*

Proof. By Theorem 3.3 the graded FI-module $H^*(\mathcal{M}_{0,\bullet+1})$ is generated by the FI-module $H^1(\mathcal{M}_{0,\bullet+1})$, which is finitely generated in degree ≤ 4 and has weight $\leq$ 2. It follows from [3, Proposition 2.3.6] that the FI-module $H^i(\mathcal{M}_{0,\bullet+1})$ is finitely generated in degree $\leq 4i$. By [3, Corolary 4.2.A] it has weight $\leq 2i$.

Moreover, from [3, Lemma 3.1.6] we have that the stability degree of $H^i(\mathcal{M}_{0,\bullet+1})$ is bounded above by the stability degree of $H^i\big(\mathcal{F}(\bullet;\mathbb{C})\big)$.

From $H^i(\mathcal{M}_{0,\bullet+1})$ to the FI-module $H^i(\mathcal{M}_{0,\bullet})$. The relation between the degree of generation of an FI-module V and its "shift" $S_{+1}V$ was established in [4, Cor. 2.13]. We can also relate the weights and stability degrees using the classical branching rule (see e.g. [8]).

Proposition 4.3. *Let λ be a partition of $n+1$ and V_λ the corresponding irreducible S_{n+1}-representation, then as S_n-representations we have the decomposition*

$$Res_{S_n}^{S_n+1} V_\lambda \cong \bigoplus_{\nu} V_\nu$$

over those partitions ν of n obtained from λ by removing one box from one of the columns of the corresponding Young diagram.

Theorem 4.4 (Finite generation and "shifted" FI-modules). *Let V be a finitely generated FI-module generated in degree $\leq d$, then $S_{+1}V$ is finitely generated in degree $\leq d$. Conversely, if the FI-module $S_{+1}V$ is finitely generated in degree $\leq d$, then V is finitely generated in degree $\leq d+1$.*

Furthermore, if $S_{+1}V$ has weight $\leq M$ and stability degree $\leq N$, then V has weight $\leq M+1$ and stability degree $\leq N+1$. Conversely, if V has weight $\leq M$ and stability degree $\leq N$, then $S_{+1}V$ has weight $\leq M$ and stability degree $\leq N$.

Proof. If V has weight $\leq M$, then for all $n \geq 0$, the irreducible components $V(\mu)_{n+1}$ of V_{n+1} have $|\mu| \leq M$. From Proposition 4.3 it follows that $\mathrm{Res}_{S_n}^{S_n+1} V(\mu)_{n+1}$ will be a direct sum of irreducibles $V(\lambda)_n$, with $|\lambda| \leq |\mu| \leq M$. Conversely, if $S_{+1}V$ has weight $\leq M$, then each irreducible component $V(\lambda)_n$ of $S_{+1}V_n$ has $|\lambda| \leq M$. By Proposition 4.3, it comes from the restriction of some $V(\mu)_{n+1}$ with $|\mu| \leq |\lambda|+1 \leq M+1$.

On the other hand, the functor Ξ_1 sends $\{1,\ldots,a\}$ into $\{2,\ldots,a+1\}$ and $\{a+1,\ldots,n\}$ into $\{a+2,\ldots,n+1\}$. Therefore, the inclusion $J_n : S_n \hookrightarrow S_{n+1}$ maps the subgroup S_{n-a} of S_n onto the subgroup $S_{(n+1)-(a+1)}$ of S_{n+1} and we have that

$$(V_{n+1})_{S_{(n+1)-(a+1)}} = V_{n+1} \otimes_{\mathbb{C}[S_{(n+1)-(a+1)}]} \mathbb{C} = S_{+1}(V)_n \otimes_{\mathbb{C}[S_{n-a}]} \mathbb{C} = \big(S_{+1}(V)_n\big)_{S_{n-a}},$$

which implies the statement about stability degrees.

In [12] we proved finite generation for the FI-modules $H^i(\mathcal{M}_{g,\bullet})$ when $g \geq 2$. The case when $g = 0$ follows from Theorem 4.4 and Theorem 4.2.

Theorem 4.5. *The FI-module $H^i(\mathcal{M}_{0,\bullet})$ is finitely generated with weight $\leq 2i+1$ and has stability degree $\leq 4i$.*

The first cohomology group. Recall that $H^1(\mathcal{M}_{0,\bullet+1})_n$ is generated by the classes $\theta_{i,j} = \omega_{i,j} - \omega_{1,2}$ and it is a subrepresentation of $H^1(\mathcal{F}(n,\mathbb{C}))$ which has a basis given by the classes $\omega_{i,j}$. In particular, notice that $\dim H^1(\mathcal{M}_{0,\bullet+1})_n = \dim H^1(\mathcal{F}(n,\mathbb{C})) - 1$. Moreover for $n \geq 4$, we have the decomposition

$$H^1(\mathcal{F}(n,\mathbb{C})) = V(0)_n \oplus V(1)_n \oplus V(2)_n.$$

Then, for $n \geq 4$ the S_n-representation

$$H^1(\mathcal{M}_{0,\bullet+1})_n = V(1)_n \oplus V(2)_n \cong \mathrm{Res}^{S_{n+1}}_{S_n} H^1(\mathcal{M}_{0,n+1}).$$

Proposition 4.3 implies that for $n \geq 4$, we have that $H^1(\mathcal{M}_{0,n+1}) = V(2)_{n+1}$ as a representation of S_{n+1}. Moreover, notice that $H^1(\mathcal{M}_{0,n+1})$ is finitely generated by the classes $\theta_{1,3}$, $\theta_{2,3}$ and $\theta_{3,4}$ in $H^1(\mathcal{M}_{0,5})$ not only an an S_n-module, but also as an S_{n+1}-module. Therefore, the FI-module $H^1(\mathcal{M}_{0,\bullet})$ is finitely generated in degree≤ 5 and has weight ≤ 2.

5 The S_n-representations $H^i(\mathcal{M}_{0,n};\mathbb{C})$

At this point we can apply the theory of FI-modules to the finitely generated FI-modules $H^i(\mathcal{M}_{0,\bullet})$ and $H^i(\mathcal{M}_{0,\bullet+1})$ to obtain information about the corresponding sequences of S_n-representations and their characters. The following result is a direct consequence from [3, Prop. 3.3.3 and Theorem 3.3.4] and Theorems 4.2 and 4.5.

Theorem 5.1. *Let $i \geq 0$. For $n \geq 4i+2$, the sequence $\{H^i(\mathcal{M}_{0,n})\}$ of representations of S_n and the sequence $\{H^i(\mathcal{M}_{0,\bullet+1})_{n-1}\}$ of S_{n-1}-representations satisfy the following:*

(a) *The decomposition into irreducibles of $H^i(\mathcal{M}_{0,n};\mathbb{C})$ and of $H^i(\mathcal{M}_{0,\bullet+1};\mathbb{C})_{n-1}$ stabilize in the sense of uniform representation stability ([5]) with stable range $n \geq 4i+2$.*

(b) *The length of $H^i(\mathcal{M}_{0,\bullet+1};\mathbb{C})_{n-1}$ is bounded above by $2i$ and the length of $H^i(\mathcal{M}_{0,n};\mathbb{C})$ is bounded above by $2i+1$.*

(c) *The sequence of characters of the representations $H^i(\mathcal{M}_{0,\bullet+1};\mathbb{C})_{n-1}$ and $H^i(\mathcal{M}_{0,n};\mathbb{C})$ are* eventually polynomial*, in the sense that there exist* character polynomials *$P_i(X_1,X_2,\dots,X_r)$ and $Q_i(X_1,X_2,\dots,X_s)$ in the cycle-counting functions $X_k(\sigma)$:=(number of k-cycles in σ) such that for all $n \geq 4i+2$:*

$$\chi_{H^i(\mathcal{M}_{0,\bullet+1};\mathbb{C})_{n-1}}(\sigma) = P_i(X_1,X_2,\dots,X_r)(\sigma) \quad \text{for all} \quad \sigma \in S_{n-1}, \quad \text{and}$$

$$\chi_{H^i(\mathcal{M}_{0,n};\mathbb{C})}(\sigma) = Q_i(X_1,X_2,\dots,X_s)(\sigma) \quad \text{for all} \quad \sigma \in S_n.$$

Moreover, the degree of P_i is $\leq 2i$ and the degree of Q_i is $\leq 2i+1$, where we take $\deg X_k = k$. In particular, $r \leq 2i$ and $s \leq 2i+1$.

If $e \in S_{n-1}$ is the identity element, from Theorem 5.1(c), we obtain that the dimensions

$$\dim_{\mathbb{C}}\left(H^i(\mathcal{M}_{0,n};\mathbb{C})\right) = \chi_{H^i(\mathcal{M}_{0,\bullet+1};\mathbb{C})_{n-1}}(e) = P_i(X_1(e),\dots,X_r(e)) = P_i(n-1,\dots,0)$$

are polynomials in n of degree $\leq 2i$. This agrees with the known Poincaré polynomial of $\mathcal{M}_{0,n}$ (see[15, Cor. 2.10] and also [11, 5.5(8)]).

From Theorem 4.5 and the definition of weight, we recover the fact that the alternating representation does not appear in the cohomology of $\mathcal{M}_{0,n}$ ([15, Prop. 2.16]).

Theorem 5.1(a) implies that the dimensions of the vector spaces $H^i(\mathcal{M}_{0,n}/S_n;\mathbb{C})$ and $H^i(\mathcal{M}_{0,n+1}/S_n;\mathbb{C})$ are constant. For the sequence $\{\mathcal{M}_{0,n}/S_n\}$, this is actually a trivial consequence from the fact that $\mathcal{M}_{0,n}/S_n$ has the cohomology of a point as shown in [15, Theorem 2.3].

Recursive relation for characters. In [9, Theorem 4.1], Gaiffi obtained a recursive formula that connects the characters of the S_n-representations $H^*(\mathcal{M}_{0,\bullet+1})_n$ and $H^*(\mathcal{M}_{0,n})$ as follows

$$\chi_{H^i(\mathcal{M}_{0,\bullet+1})_n} = \chi_{H^i(\mathcal{M}_{0,n})} + \big(X_1 - 1\big)\cdot \chi_{H^{i-1}(\mathcal{M}_{0,n})} \quad \text{for } n \geq 3. \tag{2}$$

In particular, we know that $\chi_{H^1(\mathcal{M}_{0,\bullet+1})_n} = \chi_{H^1(\mathcal{F}(n,\mathbb{C}))} - 1 = \binom{X_1}{2} + X_2 - 1$ when $n \geq 4$. Therefore, for $i = 1$, the recursive formula (2) gives us the character polynomial of degree 2

$$\chi_{H^1(\mathcal{M}_{0,n})} = \chi_{H^1(\mathcal{M}_{0,\bullet+1})_n} - \big(X_1 - 1\big)\cdot \chi_{H^0(\mathcal{M}_{0,n})} = \binom{X_1}{2} + X_2 - X_1 = \chi_{V(2)}$$

as expected since $H^1(\mathcal{M}_{0,n}) = V(2)_n$.

Furthermore, if P_i and Q_i are the character polynomials of $H^i(\mathcal{M}_{0,\bullet+1})_n$ and $H^i(\mathcal{M}_{0,n})$ from Theorem 5.1 (c) for $n \geq 4i+2$, then formula (2) can be written as $Q_i = P_i - (X_1 - 1)\cdot Q_{i-1}$ and $\deg Q_i \leq max\big(\deg P_i, 1 + \deg Q_{i-1}\big) \leq 2i$. As a consequence of this and Theorem 5.1 (c) we have that, for $n \geq 4i+2$, the values of $\chi_{H^i(\mathcal{M}_{0,\bullet+1};\mathbb{C})_n}(\sigma)$ and $\chi_{H^i(\mathcal{M}_{0,n};\mathbb{C})}(\sigma)$ depend only on "short cycles", i.e. cycles on σ of length $\leq 2i$.

More is known about the S_n-representations. In this paper we were mainly interested in highlighting the methods, since more precise information about the characters of the S_n-representations is known. The moduli space $\mathcal{M}_{0,n}$ can be represented by a finite type $\mathbb{Z}$-scheme and the manifold $\mathcal{M}_{0,n}(\mathbb{C})$ of $\mathbb{C}$-points of this scheme corresponds to the definition in Section 1. In [15] Kisin and Lehrer used an equivariant comparison theorem in ℓ-adic cohomology and the Grothendieck-Lefschetz's fixed point formula to obtain explicit descriptions of the graded character of the S_n-action on the cohomology of $\mathcal{M}_{0,n}(\mathbb{C})$ via counts of number of points of varieties over finite fields. With their techniques they obtain the Poincaré polynomial of a permutation in S_n of a specific cycle type acting on $H^*(\mathcal{M}_{0,n};\mathbb{C})$ ([15, Theorem 2.9]) and a description of the top cohomology $H^{n-3}(\mathcal{M}_{0,n};\mathbb{C})$ [15, Proposition 2.18]. Furthermore, Getzler uses the language of operads in [11] to obtain formulas for the characters of the S_n-modules $H^i(\mathcal{M}_{0,n};\mathbb{C})$.

The cohomology of $\overline{\mathcal{M}}_{0,n}$. A related space of interest is $\overline{\mathcal{M}}_{0,n}$, the *Deligne-Mumford compactification* of $\mathcal{M}_{0,n}$. It is a *fine moduli space* for stable n-pointed rational curves for $n \geq 3$ (see [16, Chapter 1] and reference therein). It can also be constructed from $M(d\mathcal{A}_{n-1})$ using the theory of wonderful models of hyperplanes arrangements developed by De Concini and Procesi (see for example [10, Chapter 2]). The space $\overline{\mathcal{M}}_{0,n}$ also carries a natural action of the symmetric group S_n. Hence, a natural question to ask is whether the FI-module theory could tell us something about its cohomology groups as S_n-representations.

Explicit presentations of the cohomology ring of the manifold of complex points $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ have been obtained by Keel [14] and Yuzvinsky [19]. Moreover, several recursive and generating formulas for the Poincaré polynomials have been computed (for instance see [19],[11], [17], [2]). The sequence $H^i(\overline{\mathcal{M}}_{0,n}(\mathbb{C});\mathbb{C})$ has the structure of an FI-module, however, the Betti numbers of $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ grow exponentially in n, which precludes finite generation. Therefore an analogue of Theorem 5.1 cannot be obtained for this space.

On the other hand, as observed in [6], the manifold $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of real points of $\overline{\mathcal{M}}_{0,n}$ is topologically similar to $\mathcal{F}(\mathbb{C}, n-1)$, the configuration space of $n-1$ ordered points in $\mathbb{C}$, in the sense that both are $K(\pi,1)$-spaces, have Poincaré polynomials with a simple factorization and Betti numbers that grow polynomially in n. The cohomology ring of the real locus $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ was completely determined in [6] and an explicit formula for the graded character of the S_n-action was obtained in [18]. The presentation of the cohomology ring given in [6] can be used to prove finite generation for the FI-modules $H^i(\overline{\mathcal{M}}_{0,n}(\mathbb{R});\mathbb{C})$ and to obtain an analogue of Theorem 5.1 for this space (see [13]).

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