On the non-vanishing of the powers of the Euler class for mapping class groups

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Abstract

The mapping class group of an orientable closed surface with one marked point can be identified, by the Nielsen action, with a subgroup of the group of orientation-preserving homeomorphisms of the circle. This inclusion pulls back the "discrete universal Euler class" producing a non-zero class in the second integral cohomology of the mapping class group. In this largely expository note we determine the non-vanishing behavior of the powers of this class. Our argument relies on restricting the cohomology classes to torsion subgroups of the mapping class group.

1 Introduction

Let Γ_g^k denote the pure mapping class group of a closed orientable surface Σ_g of genus $g \geq 1$ with $k \geq 0$ marked points. Homological properties of the mapping class group of surfaces of finite type have been studied for the last 40 years. For instance, cohomology classes of mapping class groups correspond to characteristic classes of surface bundles. Furthermore, for surfaces of genus $g \geq 2$, the rational cohomology of Γ_g^k coincides with the cohomology of the moduli space $\mathcal{M}_{g,k}$ of Riemann surfaces of genus g with k marked points.

Some of the first homological calculations for mapping class groups are due to Harer. He computed $H_2(\Gamma_g^k; \mathbb{Z})$ for genus $g \geq 5$ in [9] and proved a remarkable homological stability theorem in [10], which was a key result in the proof of Mumford's conjecture for $H^*(\mathcal{M}_g; \mathbb{Q})$ by Madsen and Weiss [19]. Miller [21] and Morita [23] constructed non-trivial cohomology classes in $H^*(\Gamma_g^k; \mathbb{Q})$, while Glover and Mislin [6] used torsion subgroups of the mapping class groups to detect torsion in their cohomology. In the same spirit, we use torsion elements in the mapping class group Γ_g^1 of a surface of genus $g \geq 1$ with one marked point to show the non-vanishing of some classes in $H^*(\Gamma_g^1; \mathbb{Z})$.

For $g \geq 2$, Nielsen defined a faithful action of Γ_g^1 on the circle $\mathbb{S}^1$ which identifies Γ_g^1 with a subgroup of the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of orientation-preserving homeomorphisms of the circle (see for example [8] and Section 4 below). This monomorphism

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$\rho : \Gamma_g^1 \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ pulls back the *discrete universal Euler class* $\mathbf{E}$ and its powers $\mathbf{E}^n$ to Γ_g^1 producing classes $\rho^*(\mathbf{E}^n) =: \mathrm{E}^n \in H^{2n}(\Gamma_g^1;\mathbb{Z})$ for each $n \geq 1$. As we review in Section 2, the nth cup product powers $\mathbf{E}^n$ are known to be non-trivial torsion free cohomology classes in $H^{2n}(\mathrm{Homeo}_+(\mathbb{S}^1);\mathbb{Z})$, for $n \geq 1$. In this note we determine the non-vanishing behavior of the powers $\mathrm{E}^n \in H^{2n}(\Gamma_g^1;\mathbb{Z})$ of the *Euler class* E *for* Γ_g^1.

Theorem A. *For $g \geq 1$ and $n \geq 1$ the cohomology classes $\mathrm{E}^n \in H^{2n}(\Gamma_g^1;\mathbb{Z})$ are nonzero. Furthermore, when $n \geq g$, the subgroup of $H^{2n}(\Gamma_g^1;\mathbb{Z})$ generated by the class E^n is a finite cyclic group of order a multiple of $4g(2g+1)$.*

For genus $g = 1$, Theorem A holds and more is known: the powers of the Euler class for $\Gamma_1^1 \cong \mathrm{SL}(2,\mathbb{Z})$ behave like the pull back of $\mathbf{E}^n$ to a finite cyclic subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ (see Proposition 3.1). The group $\mathrm{SL}(2,\mathbb{Z})$ acts faithfully on rays starting at the origin in the Euclidean plane. The corresponding monomorphism $\rho : \mathrm{SL}(2,\mathbb{Z}) \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ pulls back the class $\mathbf{E}$ to a generator $x := \rho^*(\mathbf{E})$ in $H^2(\mathrm{SL}(2,\mathbb{Z});\mathbb{Z})$ of the cohomology ring

$$H^*(\mathrm{SL}(2,\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}[x]/\langle 12x \rangle.$$

Therefore, the Euler class for $\mathrm{SL}(2,\mathbb{Z})$ and all its powers are non-trivial torsion classes of order 12. In contrast, for genus $g \geq 3$ the Euler class $\mathrm{E} \in H^2(\Gamma_g^1;\mathbb{Z})$ is known to be non-trivial of infinite order. This also follows from our Theorem A and the universal coefficient theorem since the group Γ_g^1 is perfect.

We will observe, in Sections 3 and 4 below, that the non-triviality of the classes E^n is obtained by restricting the cohomology classes to a torsion subgroup of Γ_g^1 where the corresponding cohomology classes are known to be non-trivial. Our torsion bound in Theorem A comes from the order of specific torsion elements in Γ_g^1. We end this note with Section 5 where we comment on known related results. In particular, Theorems 5.3 and 5.4 give a partial understanding of the behavior of the powers of the "Euler classes" for the pure mapping class groups with one or more marked points.

Further work. In work in progress we use the non-vanishing result Theorem A as a starting point to address the more subtle problem of computing the order of the classes E^n for $n \geq g$ and showing that for $n < g$ the powers E^n are torsion free cohomology classes. In [13] the order of E^n at the threshold dimension $n = g$ was computed by the first author to be $2g(2g+1)$. The difference with the bound obtained in Theorem A is explained in our current work. It arises due to the fact that the computation in [13] was obtained not for the Nielsen action, but for an action that we refer to as the "projective action".

We briefly and informally describe our current research. The approach is similar but more directly geometric than that of [13], and the aim is to provide an elementary, and unified description of the behavior of the powers of the Euler class for the pure mapping class groups with one or more marked points. The group Γ_g^1 acts naturally on the infinite simplex with vertices points of the circle. The action of Γ_g^1 on the circle gives rise to a bi-simplicial set which realizes $B\Gamma_g^1$, and an associated double chain complex which computes its homology. That is the setting for our computations. The elements S and T, described in Section 4 below, allow us to construct in the bi-complex, a $2n$-chain, dual to E^n, for $n \leq g$. We seek to determine how this $2n$-chain transitions from its behavior for $n < g$, where it extends to a cycle (which we know to be non-trivial by Theorem A), to one at the threshold $n = g$ where that fails.

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2 The universal Euler class and its powers

Consider the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of orientation-preserving homeomorphisms and the group of lifts $\widetilde{\mathrm{Homeo}_+(\mathbb{S}^1)}$ with respect to the universal cover $\pi : \mathbb{R} \to \mathbb{S}^1$. There is an epimorphism $p : \widetilde{\mathrm{Homeo}_+(\mathbb{S}^1)} \to \mathrm{Homeo}_+(\mathbb{S}^1)$ with kernel isomorphic to $\mathbb{Z}$ generated by $T : \mathbb{R} \to \mathbb{R}$ the integral translation $T(x) = x + 1$. This defines a non-split central extension of $\mathrm{Homeo}_+(\mathbb{S}^1)$

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}_+(\mathbb{S}^1)} \xrightarrow{p} \mathrm{Homeo}_+(\mathbb{S}^1) \to 1, \tag{1}$$

which is universal, in the sense of [22, Section 5], with kernel isomorphic to the Schur multiplier $H_2(\mathrm{Homeo}_+(\mathbb{S}^1); \mathbb{Z})$. The central extension (1) corresponds to a non-trivial generator $\mathbf{E}$ in $H^2(\mathrm{Homeo}_+(\mathbb{S}^1); \mathbb{Z}) \cong \mathbb{Z}$ that we refer as the *discrete universal Euler class*.

To better understand the powers $\mathbf{E}^n$ of the discrete universal Euler class, let us recall the following classical result due to Mather [20] and Thurston [24]. Given M an orientable manifold, we distinguish the group of orientation-preserving homeomorphisms of M with the discrete topology $\mathrm{Homeo}_+(M)_\delta$ from the topological group $\mathrm{Homeo}_+(M)_\tau$ with the compact-open topology.

Theorem 2.1 (Thurston–Mather)**.** *Let M be any orientable manifold. The identity map $id : \mathrm{Homeo}_+(M)_\delta \to \mathrm{Homeo}_+(M)_\tau$ induces a continuous function between classifying spaces*

$$\eta : \mathrm{BHomeo}_+(M)_\delta \to \mathrm{BHomeo}_+(M)_\tau$$

which is a homology equivalence.

The subgroup of rotations $SO(2, \mathbb{R})$ is canonically identified with $\mathbb{S}^1$ and the inclusion $\mathbb{S}^1 \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)_\tau$ is a homotopy equivalence; see for example [5, Prop. 4.2]. Then $\mathrm{BHomeo}_+(\mathbb{S}^1)_\tau \simeq \mathrm{B}\,\mathbb{S}^1 \simeq \mathbb{CP}^\infty$ and its cohomology ring is given by

$$H^*(\mathrm{BHomeo}_+(\mathbb{S}^1)_\tau; \mathbb{Z}) \cong \mathbb{Z}[x],$$

where x is a generator of $H^2(\mathrm{BHomeo}_+(\mathbb{S}^1)_\tau; \mathbb{Z}) \cong \mathbb{Z}$. Hence, for the circle $\mathbb{S}^1$, the Thurston–Mather theorem and the universal coefficient theorem imply the following result.

Corollary 2.2. *The cohomology ring of the discrete group* $\mathrm{Homeo}_+(\mathbb{S}^1)$ *is a polynomial ring generated by the universal Euler class* $\mathbf{E} \in H^2(\mathrm{Homeo}_+(\mathbb{S}^1); \mathbb{Z})$*, i.e.*

$$H^*(\mathrm{Homeo}_+(\mathbb{S}^1); \mathbb{Z}) = H^*(\mathrm{BHomeo}_+(\mathbb{S}^1)_\delta; \mathbb{Z}) \cong \mathbb{Z}[\mathbf{E}].$$

Therefore, all the powers $\mathbf{E}^n$ are non-trivial torsion-free cohomology classes. See also Section 5 in [13] for a different proof.

3 Torsion and non-triviality of the powers of the Euler class

We observe next that, when restricted to a finite cyclic subgroup $\mathbb{Z}/k\,\mathbb{Z}$, the powers of the universal discrete Euler class pull back to non-trivial torsion classes. Recall that the cohomology ring of $\mathbb{Z}/k\,\mathbb{Z}$ is known to be

$$H^*(\mathbb{Z}/k\,\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}[x]/\langle kx\rangle, \qquad \text{where } x \text{ is a generator of } H^2(\mathbb{Z}/k\,\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/k\,\mathbb{Z}\,.$$

Proposition 3.1. *For any monomorphism* $\phi : \mathbb{Z}/k\mathbb{Z} \hookrightarrow \mathrm{Homeo}_+ \mathbb{S}^1$*, the pull-back of the discrete universal Euler class* $\phi^*(\mathbf{E}) \in H^2(\mathbb{Z}/k\mathbb{Z}; \mathbb{Z})$ *is a generator of the cohomology ring* $H^*(\mathbb{Z}/k\mathbb{Z}; \mathbb{Z})$*. In particular, all the powers* $(\phi^*(\mathbf{E}))^n = \phi^*(\mathbf{E}^n)$ *are non-trivial torsion classes in* $H^{2n}(\mathbb{Z}/k\mathbb{Z}; \mathbb{Z})$ *of order* k*.*

Proof. Consider first the finite cyclic group $\mathbb{Z}/k\mathbb{Z}$ acting faithfully on the circle by a rotation of angle $2\pi/k$. Then the pull back of the universal central extension (1) of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is the non-split central extension

$$0 \to \mathbb{Z} \xrightarrow{\times k} \mathbb{Z} \to \mathbb{Z}/k\,\mathbb{Z} \to 1$$

of $\mathbb{Z}/k\,\mathbb{Z}$ by $\mathbb{Z}$ which corresponds to $\phi^*(\mathbf{E})$ and is a generator of $H^2(\mathbb{Z}/k\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/k\,\mathbb{Z}$; see for example [18, Ch. IV.7]. The pull back of the powers $\mathbf{E}^n$ correspond to generators of $H^{2n}(\mathbb{Z}/k\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/k\,\mathbb{Z}$ by functoriality of the cup product.

Up to conjugacy, the rotation group $SO(2, \mathbb{R})$ is the only maximal compact subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$; see for instance [5, Prop. 4.1]. As a consequence, any finite subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is conjugate to a cyclic group of rotations, and the proposition is then true for any monomorphism $\phi : \mathbb{Z}/k\mathbb{Z} \hookrightarrow \mathrm{Homeo}_+ \mathbb{S}^1$. □

Corollary 3.2. *Let* Γ *be a discrete group acting faithfully on* $\mathbb{S}^1$ *by orientation-preserving homeomorphisms. Suppose that* Γ *has a torsion element of order* k*.*

a) For all $n \geq 1$*, the pull back of* $\mathbf{E}^n$ *is a non-trivial class in* $H^{2n}(\Gamma; \mathbb{Z})$*.*

b) If the pull back of $\mathbf{E}^n$ *is a torsion class, it must have order a multiple of* k*.*

Proof. By hypothesis we have monomorphisms

$$\mathbb{Z}/k\mathbb{Z} \xhookrightarrow{\iota} \Gamma \xhookrightarrow{\psi} \mathrm{Homeo}_+(\mathbb{S}^1).$$

By Proposition 3.1, for any $n \geq 1$ the pull back

$$(\psi \circ \iota)^*(\mathbf{E}^n) = \iota^*(\psi^*(\mathbf{E}^n)) = \iota^*((\psi^*(\mathbf{E}))^n)$$

is a non-trivial class in $H^{2n}(\mathbb{Z}/k\mathbb{Z};\mathbb{Z})$ of order k. Therefore, the class $(\psi^*(\mathbf{E}))^n$ is a non-trivial class in $H^{2n}(\Gamma;\mathbb{Z})$. If $(\psi^*(\mathbf{E}))^n$ is a torsion class, then it must have order a multiple of k since ι^* takes it to a torsion class of order k. □

4 Powers of the Euler class for mapping class groups

Let Σ_g denote the closed orientable surface of genus $g \geq 1$ and $z \in \Sigma_g$. The *mapping class group* Γ_g^1 *with one marked point* is the group of orientation-preserving homeomorphisms of Σ_g modulo isotopy, where the point z is required to stay fixed under isotopies.

Consider the presentation of the fundamental group

$$\pi_1(\Sigma_g, z) = \langle a_1, a_2, \ldots, a_{2g} |\ a_1 \cdots a_{2g} \cdot a_1^{-1} \cdots a_{2g}^{-1} = 1\rangle.$$

The Dehn-Nielsen-Baer theorem identifies Γ_g^1 with an index 2 subgroup of the automorphism group $\mathrm{Aut}\,(\pi_1(\Sigma_g, z))$ (see for example [4, Ch. 8]). Under this identification, the automorphisms of $\pi_1(\Sigma_g, z)$

$$S: \begin{array}{l} a_1 \mapsto a_2 \\ a_1 \mapsto a_3 \\ \quad\vdots \\ a_{2g} \mapsto a_1^{-1} \end{array} \qquad\qquad T: \begin{array}{l} a_1 \mapsto a_2 \\ a_1 \mapsto a_3 \\ \quad\vdots \\ a_{2g} \mapsto a_{2g}^{-1} \cdots a_2^{-1} a_1^{-1} \end{array}$$

represent torsion elements of Γ_g^1 or order $4g$ and $2g+1$, respectively. Geometrically they are related to the $4g$-gon and $2(2g+1)$-gon symmetries of the surface Σ_g.

For $g = 1$, the Dehn-Nielsen-Baer theorem implies $\Gamma_1^1 \cong \mathrm{SL}(2,\mathbb{Z})$ and we have that

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

of orders 4 and 3, respectively.

For $g \geq 2$, Nielsen defined a faithful action of Γ_g^1 on $\mathbb{S}^1$ by orientation-preserving homeomorphisms that we briefly describe next; see [8] and [4] for a more detailed discussion. Fixing a hyperbolic metric on Σ_g, its universal cover can be identified with the hyperbolic disk $\mathbb{D}^2$, which has a natural compactification to a closed disc. Let $\tilde{z} \in \mathbb{D}^2$ be a distinguished lift of the marked point $z \in \Sigma_g$. For $f \in \mathrm{Homeo}_+(\Sigma_g)$ fixing the marked point z let $\tilde{f}$ denote the unique lift of f to $\mathbb{D}^2$ that fixes $\tilde{z}$. It can be shown that the action of $\tilde{f}$ on $\mathbb{D}^2$ extends to a homeomorphism of the boundary circle, which depends only on the isotopy class of f. This procedure gives a well-defined monomorphism

$$\rho : \Gamma_g^1 \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1),$$

which is the *Nielsen action* of Γ_g^1 on $\mathbb{S}^1$.

For genus $g \geq 2$, the Gromov boundary of $\pi_1(\Sigma_g, z)$ is a topological circle $\mathbb{S}^1$, on which the group of automorphisms $\mathrm{Aut}(\pi_1(\Sigma_g, z))$ acts faithfully by homeomorphisms. With the Dehn-Nielsen-Baer identification, this boundary action is conjugate to the geometric action that we just described.

The *Euler class for* Γ_g^1 is defined as $\mathrm{E} := \rho^*(\mathbf{E}) \in H^2(\Gamma_g^1; \mathbb{Z})$, the pull back of the discrete universal Euler class under the standard action. This cohomology class corresponds to the central extension

$$1 \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma_g^1 \to 1,$$

where $\Gamma_{g,1}$ denotes the mapping class group of an orientable surface $\Sigma_{g,1}$ of genus g with one boundary component (find more details in [4, Ch 5.5] and [7]). The epimorphism above is induced from the inclusion $\Sigma_{g,1} \simeq \Sigma_g - N_\epsilon(z) \hookrightarrow \Sigma_g^1$, where $N_\epsilon(z) = \{x \in \Sigma_g^1 : d(x, z) < \epsilon\}$ for a small $\epsilon > 0$. The kernel is generated by a Dehn twist around the boundary component, which is the simple loop $\partial N_\epsilon(z)$ around the marked point z.

Proof of Theorem A. The mapping classes S and T generate torsion cyclic subgroups in Γ_g^1 of order $k = 4g$ and $k = 2g + 1$, respectively. For $g = 1$, the group $\Gamma_1^1 \cong \mathrm{SL}(2, \mathbb{Z})$ acts faithfully on rays starting at the origin in the Euclidean plane, and for genus $g \geq 2$ the Nielsen action is faithful. It follows from Corollary 3.2 a) that the powers E^n are non-trivial for all $n \geq 1$. For $n \geq g$ the powers E^n are known to vanish over the rationals (Theorem 5.3), hence the classes E^n are torsion and by Corollary 3.2 b) must have order a common multiple of $k = 4g$ and $k = 2g + 1$. $\square$

Remark: We can apply Corollary 3.2 a) to any finite cyclic subgroup of Γ_g^1. The order of a finite cyclic subgroup of Γ_g^1 is known to be at most $4g + 2$ and this upper bound is attained for $g \geq 2$ (see for example [4, Cor.7.6]). Since $4g$ and $2g + 1$ are relatively prime, the subgroups generated by S and T that we consider in the proof of Theorem A give us the largest lower bound $4g(2g + 1)$ for torsion that we could find with these elements.

If other torsion elements of Γ_g^1 are known, one can try to look for a better lower bound. For instance, if $g = p - 1$ with p a prime number, then Γ_g^1 has p-torsion [16, Thm 2.7] and from Corollary 3.2 b) we obtain that the cohomology classes E^n have order a common multiple of $4g$, $2g + 1$ and $g + 1$, when $n \geq g$.

5 Related results

For $g \geq 0$ and $k \geq 1$, consider $z_1, z_2, ..., z_k \in \Sigma_g$. The *pure mapping class group* Γ_g^k of Σ_g is the group of orientation-preserving homeomorphisms of Σ_g, modulo isotopy, where the points z_i are required to stay fixed under isotopies. For $g \geq 2$ and $k \geq 0$, the moduli space $\mathcal{M}_{g,k}$ of genus g Riemann surfaces with k marked points is a rational model for the classifying space $B\Gamma_g^k$; therefore $H^*(\mathcal{M}_{g,k}; \mathbb{Q}) \cong H^*(\Gamma_g^k; \mathbb{Q})$. For $k = 0$, we use the notation Γ_g and $\mathcal{M}_g$. Research in the last 40 years has been motivated by the following general problem:

Problem 5.1. Compute the groups $H^i(\Gamma_g^k; \mathbb{K})$ for all g, k and i and understand the ring structure of $H^*(\Gamma_g^k; \mathbb{K})$, for coefficients $\mathbb{K} = \mathbb{Q}$ and $\mathbb{Z}$.

We comment briefly on some of the most remarkable results towards the answer of Problem 5.1. Harer proved in [10] that the cohomology groups $H^i(\Gamma_g^k; \mathbb{Z})$ are independent of the genus g and k in degrees small relative to i. The range where this happens is called the "stable range" and has been improved over the years. Mumford conjectured, and Madsen and Weiss proved in [19], that in low cohomological degrees $H^*(\mathcal{M}_g; \mathbb{Q})$ is a polynomial algebra in classes κ_i of degree $2i$, giving a complete picture of the cohomology ring $H^*(\mathcal{M}_g; \mathbb{Q})$ in the stable range. Outside of the stable range, a few of the cohomology groups are known, even rationally. For instance, there are complete computations for genus $g \leq 4$ and for low cohomology degrees; see for example [1] for computations of the cohomology ring of Γ_2.

The study of the cohomology groups and their ring structure in the "unstable range" is an active area of research. The κ-classes mentioned before are examples of tautological classes of $\mathcal{M}_{g,k}$, cohomology classes "naturally coming from geometry". An important direction of research is given by the Faber conjectures [3] which describe the structure of the ring generated by the tautological classes.

The "Euler classes" for the pure mapping class group. For $k \geq 1$, once the marked points $z_1, ..., z_k$ are fixed in Σ_g we will distinguish the Γ_g^1's by writing $\Gamma_g^{z_i}$ for the mapping class group with the single marked point z_i. For $1 \leq i \leq k$, each $\Gamma_g^{z_i}$ is a distinct quotient group of Γ_g^k: two elements of Γ_g^k determine the same element of $\Gamma_g^{z_i}$ if they are isotopic by an isotopy which keeps z_i fixed. Let $p_i : \Gamma_g^k \to \Gamma_g^{z_i}$ denote the quotient homomorphism. As already mentioned in Section 4, the $\Gamma_g^{z_i}$ can be identified with a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$. The composite

$$\rho_i : \Gamma_g^k \xrightarrow{p_i} \Gamma_g^{z_i} \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1) \tag{2}$$

pulls back the powers $\mathbf{E}^n$ producing the classes $\mathrm{E}_i^n \in H^{2n}(\Gamma_g^k; \mathbb{Z})$ for all $1 \leq i \leq k$ and $n \geq 1$. In trying to contribute to a partial answer to Problem 5.1, our work focuses on the following problem:

Problem 5.2. Understand the behavior of the "Euler classes" E_i and their powers $\mathrm{E}_i^n \in H^{2n}(\Gamma_g^k; \mathbb{Z})$ for $k \geq 1$ and $n \geq 1$.

Our Theorem A and Theorems 5.3 and 5.4 below partially answer this problem.

Harer computed in [9] the second integral homology group of Γ_g^k for $g \geq 5$. It is known that $H_2(\Gamma_{g,r}^k;\mathbb{Z}) \cong H_2(\Gamma_{g,r+k};\mathbb{Z}) \oplus \mathbb{Z}^k$ for $g \geq 3$ and $r, k \geq 0$ [14, Prop. 1.4]. The summand $\mathbb{Z}^k$ of $H^2(\Gamma_g^k;\mathbb{Z})$ is generated by the cohomology classes $\mathrm{E}_1, \mathrm{E}_2 \ldots, \mathrm{E}_k$.

For $k \geq 1$, the class $\mathrm{E}_i \in H^2(\Gamma_g^k;\mathbb{Q})$ corresponds to the restriction of the ψ_i-class to $\mathcal{M}_{g,k}$ from its Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,k}$. The class ψ_i is the first Chern class of the cotangent bundle bundle over $\overline{\mathcal{M}}_{g,k}$ associated to the marked point z_i; see for example [7]. The ψ-classes are also tautological classes of $\mathcal{M}_{g,k}$.

From the algebro-geometric perspective, a result by Ionel [11, Thm 0.1] establishes that any monomial in the tautological classes of degree at least g vanishes when restricted to $H^*(\mathcal{M}_{g,k};\mathbb{Q})$ when $g \geq 2$ and $k \geq 1$. As a particular case, we have the following vanishing result over the rationals (which also follows from the work of Looijenga [15] for $k = 1$).

Theorem 5.3 (Vanishing over $\mathbb{Q}$ [11])**.** *For any $1 \leq i \leq k$, the powers $\mathrm{E}_i^n \in H^{2n}(\Gamma_g^k;\mathbb{Q})$ vanish for $n \geq g$.*

Vanishing results in [12] and [13] for $n \geq g$ apply to the nth power of the Euler class of Γ_g^1 considered as an element of $\mathrm{Hom}(H_{2n}(\Gamma_g^1;\mathbb{Z});\mathbb{Z})$ under the universal coefficient theorem, agreeing with the the results in Theorem 5.3 for $k = 1$.

For genus $g \geq 2$ and $k \geq 1$, from Morita's result [23, Thm 7.5] it follows that $\mathrm{E}_1^n, \mathrm{E}_2^n, \ldots \mathrm{E}_k^n$ generate a summand of $H^{2n}(\Gamma_g^k;\mathbb{Q})$ isomorphic to $\mathbb{Q}^k$ for $2n \leq g/3$. This range has been improved to $2n \leq g/2$ in [2, Cor 1.2]. Hence, there is a non-vanishing result for $g \geq 2$:

Theorem 5.4 (Non-vanishing and independence [23], [2])**.** *For any $1 \leq i \leq k$, the powers $\mathrm{E}_i^n \in H^{2n}(\Gamma_g^k;\mathbb{Z})$ are non-trivial torsion free independent classes for $n \leq g/4$.*

For $k = 1$, Theorem A improves on Theorem 5.4 by showing non-vanishing of E^n for all $n \geq 1$. For $g \geq 1$, $k \geq 2$ and $1 \leq i \leq k$ the composite ρ_i described in (2) has torsion free kernel and any torsion subgroup contained in Γ_g^k injects in $\mathrm{Homeo}_+(\mathbb{S}^1)$ through ρ_i. Hence, the strategy used in the proof of Theorem A can be used to show non-vanishing of the classes E_i^n, as long as the pure mapping class group Γ_g^k has non-trivial torsion subgroups. The lower bound for the order of E_i^n, when $n \geq g$, depends on the order of these torsion subgroups.

For genus $g = 1, 2, 3$ and $k \geq 1$, Lu [17, Section 1] investigated the p-torsion in Γ_g^k, for p a prime number. From her computations, then non-vanishing of the classes E_i^n and specific lower bounds for their order, when $n \geq g$ and $g = 2, 3$, can be obtained as long as $k < 2g + 3$. On the other hand, from [16, Thm 2.7], the pure mapping class group Γ_g^2 contains p-torsion if and only if Γ_g^1 has p-torsion. Therefore, the cohomology classes $\mathrm{E}_1^n, \mathrm{E}_2^n \in H^{2n}(\Gamma_g^2;\mathbb{Z})$ have order a multiple of $2g(4g + 1)$.

In contrast, from [17, Lemma 1.1(i)] it follows that the group Γ_g^k is torsion free for $k \geq 2g + 3$. As a consequence, the torsion in cohomology disappears in high degree and the integral classes E_i^n eventually vanish.

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