

PERIODICITY OF THE PURE MAPPING CLASS GROUP OF NON-ORIENTABLE SURFACES

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ABSTRACT. We show that the pure mapping class group \mathcal{N}_g^k of a non-orientable closed surface of genus $g \geq 2$ with $k \geq 1$ marked points has p -periodic cohomology for each odd prime p for which \mathcal{N}_g^k has p -torsion. Using the Yagita invariant and cohomology classes obtained from some representations of subgroups of order p , we obtain that the p -period is less or equal than 4 when $g \geq 3$ and $k \geq 1$. Moreover, combining the Nielsen realization theorem and a characterization of the p -period given in terms of normalizers and centralizers of cyclic subgroups of order p , we show that the p -period of \mathcal{N}_g^k is bounded below by 4, whenever \mathcal{N}_g^k has p -periodic cohomology, $g \geq 3$ and $k \geq 0$. These results provide partial answers to questions proposed by G. Hope and U. Tillmann.

1. INTRODUCTION

Let Γ be a group of finite virtual cohomological dimension (vcd) and let p be a prime. The group Γ is called p -periodic if there exists a positive integer d such that the Farrell cohomology groups $\widehat{H}^i(\Gamma; M)$ and $\widehat{H}^{i+d}(\Gamma; M)$ have naturally isomorphic p -primary components for all $i \geq 0$ and for all $\mathbb{Z}\Gamma$ -modules M . The least of such d is called the p -period of Γ and is denoted by $p(\Gamma)$. Farrell cohomology extends Tate cohomology of finite groups to groups of finite vcd and in degrees above the vcd it agrees with the ordinary cohomology of the group. In this paper we study the p -periodicity of the pure mapping class group of a non-orientable closed surface with at least one marked point.

Let Σ be a closed surface and $\{z_1, \dots, z_k\}$ a set of $k \geq 0$ distinct points in Σ , we call them *marked points*. Let $\text{Diff}(\Sigma; k)$ be the group of diffeomorphisms of Σ that preserve the set of marked points and let $\text{PDiff}(\Sigma; k)$ be the subgroup of diffeomorphisms that fix the marked points pointwise. If the surface Σ is orientable, we consider the corresponding subgroups $\text{Diff}^+(\Sigma; k)$ and $\text{PDiff}^+(\Sigma; k)$ of orientation-preserving diffeomorphisms. The *pure mapping class group* $\text{PMod}(\Sigma; k)$ of Σ with k marked points is the group of isotopy classes of $\text{PDiff}(\Sigma; k)$ if Σ is non-orientable and the group of isotopy classes of $\text{PDiff}^+(\Sigma; k)$ when Σ is orientable. We use the notation $\Gamma_g^k := \text{PMod}(S_g; k)$ and $\mathcal{N}_g^k := \text{PMod}(N_g; k)$, where S_g and N_g denote,

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respectively, a closed connected orientable and non-orientable surface of genus g . If the set of marked points is empty, we omit k from the notation.

It is well known that the groups Γ_g^k and \mathcal{N}_g^k have finite vcd and their Farrell cohomology and p -periodicity have been previously studied in the literature. For instance, it is known that for an orientable closed surface of genus $g > 1$ the group Γ_g is never 2-periodic, and for p an odd prime Y. Xia determined in [21] all genera g for which Γ_g is p -periodic. In [8] H.H. Glover, G. Mislin and Y. Xia obtained a formula for the p -period $p(\Gamma_g)$ that holds whenever the group is p -periodic. In contrast, Q. Lu proved in [14] that for an orientable surface of genus $g \geq 1$ with at least one marked point, the group Γ_g^k is always p -periodic with p -period equal to 2. Using these results, Xia [22, 23, 24] and Lu [14] determined the p -primary component of the Farrell cohomology $\widehat{H}^*(\Gamma_{p-1}^k; \mathbb{Z})_{(p)}$, $\widehat{H}^*(\Gamma_{(p-1)/2}^k; \mathbb{Z})_{(p)}$ and $\widehat{H}^*(\Gamma_p; \mathbb{Z})_{(p)}$ where $k \geq 0$ and p is an odd prime. Furthermore, Lu [15] obtained all the p -primary components of the Farrell cohomology of the pure mapping class group Γ_g^k of a surface of low genus $g = 1, 2, 3$, when Γ_g^k has p -torsion, p is an odd prime, and $k \geq 1$.

On the other hand, G. Hope and U. Tillmann investigated in [12] the p -periodicity of the Farrell cohomology of the mapping class group \mathcal{N}_g of a closed non-orientable surface of genus $g \geq 3$. They were able to determine the precise conditions required for this cohomology to exhibit p -periodic behavior. We contribute to the understanding of the p -periodicity of the pure mapping class group \mathcal{N}_g^k of a closed non-orientable surface with at least one marked point by proving the following results (see [12, Question 5.2]).

Theorem 1. *Let $g \geq 2$, $k \geq 1$ and p be an odd prime. The pure mapping class group \mathcal{N}_g^k has p -periodic cohomology whenever \mathcal{N}_g^k contains p -torsion.*

As we observe in Remark 2.4 below, the argument from [12, Proof of Lemma 4.1] actually shows that \mathcal{N}_g^1 is not 2-periodic for $g \geq 3$. Using the Yagita invariant and adapting Lu's methods from [14] to the non-orientable case \mathcal{N}_g^k , we find an upper bound for the p -period in Theorem 3.4, proving that $p(\mathcal{N}_g^k) \leq 4$ whenever the group has p -torsion, $g \geq 3$ and $k \geq 1$. In addition, using a different description of the p -period given in terms of the normalizers and centralizers of subgroups of order p of \mathcal{N}_g^k , we prove in Theorem 4.4 that $p(\mathcal{N}_g^k) \geq 4$ whenever \mathcal{N}_g^k has p -periodic cohomology and $k \geq 0$. Thus, the following result follows.

Theorem 2. *Let $g \geq 3$, $k \geq 1$ and p be an odd prime. If the pure mapping class group \mathcal{N}_g^k contains p -torsion, then its p -period $p(\mathcal{N}_g^k)$ is 4.*

It is worth pointing out that the lower bound $p(\mathcal{N}_g^k) \geq 4$ that we find in Theorem 4.4 applies also to the case without marked points $k = 0$. Combining this result with [12, Theorem 1.1] yields the following result, which partially solves [12, Question 5.1] about finding a lower bound for the p -period of \mathcal{N}_g .

Corollary 3. *Let p be an odd prime and suppose that the mapping class group \mathcal{N}_g contains p -torsion. Then $p(\mathcal{N}_g) \geq 4$ unless that $g = lp + 2$ for some $l > 0$, and for $0 \leq t < p$ with $l \equiv -t \pmod{p}$ we have that $l + t + 2p > tp$.*

Outline. The paper is organized as follows. In Section 2 we use the connection between the mapping class groups of a non-orientable surface and its orientable double cover, and the result of Q. Lu [14, Theorem 1.7] to prove Theorem 1. In

Section 3 we recall the definition of the Yagita invariant and we use cohomology classes of subgroups of order p to prove that this invariant is lower or equal than 4. This gives us the upper bound for the p -period of \mathcal{N}_g^1 in Theorem 3.3 and a Birman exact sequence argument allows us to obtain the upper bound for general $k \geq 1$ in Theorem 3.4. Finally, in Section 4 we prove that 4 is also a lower bound of the p -period of \mathcal{N}_g^k , when $k \geq 0$; see Theorem 4.4. For this we use Theorem 4.1, a characterization of the p -period given in terms of the index of the normalizers and centralizers of \mathcal{N}_g^k of subgroups of order p over their conjugacy classes.

One of the main ingredients to obtain the lower bound is Theorem 4.2, which states that if $f \in \text{PDiff}(N_g; k)$ is an element of order p , then f and f^{-1} are conjugated in $\text{PHomeo}(N_g; k)$, the group of homeomorphisms of N_g that fix the marked points pointwise. Its proof involves an analysis of the automorphisms of non-Euclidean crystallographic groups (see, for instance, [5, Section 3]). It is based on the classic work of J. Nielsen in [18], in which he introduced the notion of fixed point data of a finite order homeomorphism of an orientable surface and characterized its conjugacy classes using this notion.

2. THE P-PERIODICITY OF THE PURE MAPPING CLASS GROUP

Let p be an odd prime. The main purpose of this section is to show that the pure mapping class group \mathcal{N}_g^k has p -periodic cohomology, whenever \mathcal{N}_g^k contains p -torsion, for $g \geq 2$ and $k \geq 1$. The proof relies on a result of Q. Lu [14, Theorem 1.7], which states that, in the orientable case, the group Γ_g^k has p -periodic cohomology whenever the group contains p -torsion, for $g, k \geq 1$. It also uses the relation between the mapping class group of a non-orientable surface and the mapping class group of its orientable double cover that we recall next.

For N_g a closed connected non-orientable surface of genus g , the non-orientable double cover can be constructed (up to isomorphism) as follows. Let S_{g-1} be a closed orientable surface of genus $g-1$, embedded in \mathbb{R}^3 such that S_{g-1} is invariant under reflections in the xy -, yz -, and xz - planes. Let $\sigma : S_{g-1} \rightarrow S_{g-1}$ be the orientation reversing homeomorphism

$$\sigma(x, y, z) = (-x, -y, -z).$$

Then the quotient $S_{g-1}/\langle \sigma \rangle$ is homeomorphic to N_g and the natural projection $\pi : S_{g-1} \rightarrow N_g$ is a double cover of N_g such that σ is a covering transformation

$$\begin{array}{ccc} S_{g-1} & \xrightarrow{\sigma} & S_{g-1} \\ & \searrow \pi & \swarrow \pi \\ & N_g & \end{array}$$

The following result is well-known (see for example [10, Lemma 2.2]).

Lemma 2.1. *Every diffeomorphism $f : N_g \rightarrow N_g$ admits exactly two liftings $S_{g-1} \rightarrow S_{g-1}$, one of which preserves orientation.*

Furthermore, in the case $f \in \text{Diff}(N_g; k)$, if $\tilde{f} : S_{g-1} \rightarrow S_{g-1}$ is the orientation preserving lifting of π , then $\tilde{f} \in \text{Diff}^+(S_{g-1}; 2k)$. Namely, if $\{z_1, \dots, z_k\} \subset N_g$ is the set of marked points, let $\tilde{z}_1, \dots, \tilde{z}_k \in S_{g-1}$ be such that $\pi^{-1}(z_i) = \{\tilde{z}_i, \sigma(\tilde{z}_i)\}$ and take $\{\tilde{z}_1, \sigma(\tilde{z}_1), \dots, \tilde{z}_k, \sigma(\tilde{z}_k)\}$ as the set of marked points in S_{g-1} . Note that if $f(z_i) = z_j$, then \tilde{f} restricts to a bijection between the fibers $\{\tilde{z}_i, \sigma(\tilde{z}_i)\}$ and

$\{\tilde{z}_j, \sigma(\tilde{z}_j)\}$. Thus, there is a natural way to choose a lift of $f \in \text{Diff}(N_g; k)$ in a continuous manner by taking $\tilde{f} \in \text{Diff}(S_{g-1}; 2k)$ to be orientation preserving. This choice defines a group homomorphism $\tilde{\phi} : \text{Diff}(N_g; k) \rightarrow \text{Diff}^+(S_{g-1}; 2k)$ which induces a homomorphism $\phi : \text{Mod}(N_g; k) \rightarrow \text{Mod}(S_{g-1}; 2k)$ between the corresponding mapping class groups and makes the following diagram commutative

$$\begin{array}{ccc} \text{Diff}(N_g; k) & \xrightarrow{\tilde{\phi}} & \text{Diff}^+(S_{g-1}; 2k) \\ \downarrow & & \downarrow \\ \text{Mod}(N_g; k) & \xrightarrow{\phi} & \text{Mod}(S_{g-1}; 2k). \end{array}$$

For a surface Σ with set of marked points $\{z_1, \dots, z_k\}$, we are using $\text{Mod}(\Sigma; k)$ to denote the group of isotopy classes of elements of $\text{Diff}(\Sigma; k)$ (or $\text{Diff}^+(\Sigma; k)$ if the surface Σ is orientable). The group $\text{Mod}(\Sigma; k)$ acts on the set of marked points by permuting them and the kernel of this action is precisely the pure mapping class group $\text{PMod}(\Sigma; k)$.

The following result was proven in [12, Key Lemma 2.1] and [10, Theorem 1.1].

Proposition 2.2. *Let N_g be a non-orientable surface and let S_{g-1} be its orientable double cover. The homomorphism $\phi : \text{Mod}(N_g; k) \rightarrow \text{Mod}(S_{g-1}; 2k)$ is injective for $g \geq 3$ if $k = 0$ and for all $g \geq 1$ if $k \geq 1$.*

We now prove that, if $g \geq 2$ and $k \geq 1$, the group \mathcal{N}_g^k has p -periodic cohomology whenever \mathcal{N}_g^k has p -torsion.

Proof of the Theorem 1. Suppose that there exist some $g \geq 2$, $k \geq 1$ and some odd prime p such that \mathcal{N}_g^k has p -torsion and is not p -periodic. Then there exists $\mathbb{Z}/p \times \mathbb{Z}/p \leq \mathcal{N}_g^k$ (see [2, Theorem X.6.7]). From Proposition 2.2, it follows that $\mathbb{Z}/p \times \mathbb{Z}/p \cong \phi(\mathbb{Z}/p \times \mathbb{Z}/p) \leq \text{Mod}(S_{g-1}; 2k)$.

By the Nielsen realization theorem for non-orientable surfaces (see [6, Theorem 5.2]), we can find $f, f' \in \text{PDiff}(N_g; k)$ representing the generators of the subgroup $\mathbb{Z}/p \times \mathbb{Z}/p$ of \mathcal{N}_g^k , such that

$$f^p = 1; \quad f'^p = 1; \quad ff' = f'f.$$

Consider the diffeomorphisms $\tilde{\phi}(f), \tilde{\phi}(f') \in \text{Diff}^+(S_{g-1}; 2k)$. Since f and f' fix the marked points z_i individually, then $\tilde{\phi}(f)$ and $\tilde{\phi}(f')$ restrict to permutations of the fiber $\{\tilde{z}_i, \sigma(\tilde{z}_i)\}$ for $i = 1, \dots, k$. Notice that $\tilde{\phi}(f)$ and $\tilde{\phi}(f')$ have odd order p , therefore they must preserve each of the marked points of S_{g-1} individually, i.e. $\tilde{\phi}(f), \tilde{\phi}(f') \in \text{PDiff}^+(S_{g-1}; 2k)$. From the commutativity of the above diagram, it follows that $\mathbb{Z}/p \times \mathbb{Z}/p \cong \phi(\mathbb{Z}/p \times \mathbb{Z}/p)$ is actually a subgroup of the pure mapping class group Γ_{g-1}^{2k} . This contradicts [14, Theorem 1.7] which states that Γ_{g-1}^{2k} has p -periodic cohomology if $g-1 \geq 1$ and $k \geq 1$. \square

Remark 2.3. The strategy above can be used to prove p -periodicity in \mathcal{N}_g from the cases where Γ_{g-1} is known to be p -periodic (see for instance [21, Theorems 1, 2 and 3]). However, there are cases where \mathcal{N}_g is p -periodic but Γ_{g-1} is not, as shown in [12, Theorem 1.1 and Remark 4.4].

Remark 2.4 (\mathcal{N}_g^1 is not 2-periodic). Take S_{g-1} embedded in \mathbb{R}^3 as before and let $(x_0, 0, 0) \in S_{g-1}$ be a point where the surface intersects with the x -axis. Since

the embedding is symmetric with respect to the reflection by the yz -plane, the point $(-x_0, 0, 0)$ is also in S_{g-1} . Consider the rotations $R_1, R_2 : S_{g-1} \rightarrow S_{g-1}$ given by,

$$R_1(x, y, z) = (-x, -y, z) \quad \text{and} \quad R_2(x, y, z) = (x, -y, -z),$$

as defined in [12, Proof of Lemma 4.1]. These homeomorphisms are involutions and commute with the covering transformation $\sigma : S_{g-1} \rightarrow S_{g-1}$ of the orientable double cover $\pi : S_{g-1} \rightarrow N_g$. Hence, they induce $f_1, f_2 : N_g \rightarrow N_g$ and notice that $f_1, f_2 \in \text{Diff}(N_g; *)$, where the marked point is $* = [(x_0, 0, 0)]$. For genus $g \geq 3$, using the arguments of [12, Proof of Lemma 4.1], we can see that $f_1 f_2 = f_2 f_1$, $f_1^2 = f_2^2 = \text{id}_{N_g}$, f_1 and f_2 are not isotopic to each other relative to $*$ in N_g and their classes $[f_1], [f_2] \in \mathcal{N}_g^1$ are non-trivial. Thus, $\langle [f_1], [f_2] \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \leq \mathcal{N}_g^1$ and therefore \mathcal{N}_g^1 is not 2-periodic.

3. AN UPPER BOUND FOR THE P-PERIOD

Let p be an odd prime. In this section we show that, for $g \geq 3$ and $k \geq 1$, the p -period of the group \mathcal{N}_g^k is bounded above by 4, by adapting the methods of [14] to the non-orientable case. Later on, in Section 4, we will show that the p -period of \mathcal{N}_g^k is greater or equal than 4, proving that $p(\mathcal{N}_g^k) = 4$. By contrast, in the orientable case it was shown in [14, Theorem 1.7] that the p -period of Γ_g^k is equal to 2 if $k \geq 1$ and Γ_g^k contains p -torsion.

We will use the Yagita invariant $Y(\mathcal{N}_g^k, p)$, which can be regarded as a generalization of the p -period if \mathcal{N}_g^k has p -torsion. Also, since we have already proven that \mathcal{N}_g^k is p -periodic for $g \geq 2$, $k \geq 1$, the Yagita invariant $Y(\mathcal{N}_g^k, p)$ coincides with the p -period of \mathcal{N}_g^k by [20, Proposition 4.1.1]; see also [9] for calculations of the Yagita invariant $Y(\Gamma_g, p)$.

Recall the definition of the Yagita invariant as in [17, Section 7]. Let Γ be a group of finite virtual cohomological dimension and $\pi \leq \Gamma$ any subgroup of prime order p . Because π injects into any finite quotient of the form Γ/Δ , where Δ is a torsion-free normal subgroup of finite index in Γ , the image of the restriction map in cohomology $H^i(\Gamma; \mathbb{Z}) \rightarrow H^i(\pi; \mathbb{Z})$ is non-zero for some degree $i > 0$. Reduction mod- p maps $H^*(\pi; \mathbb{Z})$ onto $\mathbb{F}_p[u] \subset H^*(\pi; \mathbb{F}_p)$ with u a generator of $H^2(\pi; \mathbb{F}_p)$. Thus, there exists a maximum value $m = m(\pi, \Gamma)$ such that

$$\text{Im}(H^*(\Gamma; \mathbb{Z}) \rightarrow H^*(\pi; \mathbb{F}_p)) \subset \mathbb{F}_p[u^m] \subset H^*(\pi; \mathbb{F}_p).$$

Moreover, $m(\pi, \Gamma)$ is bounded above by $m(\pi, \Gamma/\Delta)$, where Δ denotes as before a torsion-free normal subgroup of finite index. Since Γ/Δ is finite, it follows from the comments of [25, Section 1] that $m(\pi, \Gamma)$ is bounded by a bound depending only on Γ . The *Yagita invariant* of Γ with respect to the prime p is then defined to be the least common multiple of the values $2m(\pi, \Gamma)$, where π ranges over all subgroups of order p of Γ . It is denoted by $Y(\Gamma, p)$.

First, we prove that for every $g \geq 3$ the p -period of \mathcal{N}_g^1 is bounded above by 4 if \mathcal{N}_g^1 has p -torsion. For simplicity of notation, we write $\text{Diff}(N_g; *)$ instead of the group $\text{Diff}(N_g; 1)$, where $*$ will be thought of as the marked point of N_g .

The main idea of the proof is based on [14, Theorem 1.4 and Theorem 1.7] and is outlined below. Given a subgroup $\pi \leq \mathcal{N}_g^1$ of order p , one can use Nielsen's realization theorem to obtain a lift $\tilde{\pi}$ in $\text{Diff}(N_g; *)$. On the other hand, the action of $\text{Diff}(N_g; *)$ on N_g induces a representation $\rho : \text{Diff}(N_g; *) \rightarrow GL_2(\mathbb{R})$ given by

sending a diffeomorphism $f : N_g \rightarrow N_g$ to its differential in $*$, $df_* : T_*N_g \rightarrow T_*N_g$, and which restricts to a faithful representation $\tilde{\rho} : \tilde{\pi} \rightarrow GL_2^+(\mathbb{R})$. The induced map at the level of classifying spaces $B\tilde{\rho} : B\tilde{\pi} \rightarrow BGL_2^+(\mathbb{R})$ satisfies that there exists a class $c_1 \in H^2(BGL_2^+(\mathbb{R}); \mathbb{Z})$, essentially the first Chern class, such that $(B\tilde{\rho})^*(c_1) \neq 0$. By diagram chasing we then obtain a class in $H^4(\mathcal{N}_g^1; \mathbb{Z})$, related to the first Potryagin class in $H^4(BGL_2(\mathbb{R}); \mathbb{Z})$, which restricts to the non-zero element in $H^4(B\pi; \mathbb{Z})$ corresponding to c_1^2 . Therefore, we obtain $m(\pi, \mathcal{N}_g^1) \leq 2$ and the result for the p -period of \mathcal{N}_g^1 follows. Finally, we use the Birman short exact sequence to obtain the result for any $k \geq 1$ by an induction argument on k .

We start by stating a few technical results. The following was proven in [14, Proof of Theorem 1.4].

Lemma 3.1. *Let $\rho : G \rightarrow GL_2^+(\mathbb{R}) \simeq SO(2)$ be a faithful representation of a non-trivial cyclic group G . Then there exists $c_1 \in H^2(BGL_2^+(\mathbb{R}); \mathbb{Z}) \cong H^2(BSO(2); \mathbb{Z})$ such that*

$$(B\rho)^*(c_1) \neq 0 \quad \text{in} \quad H^2(BG; \mathbb{Z}).$$

Lemma 3.2. *There exists a non-zero element $p_4 \in H^4(BO(2); \mathbb{Z})$ such that*

$$(B\iota)^*(p_4) = c_1^2$$

where ι is the canonical inclusion $SO(2) \hookrightarrow O(2)$.

Proof. Consider the following commutative triangle

$$\begin{array}{ccc} SO(2) & & \\ \downarrow \iota & \searrow i & \\ O(2) & \xrightarrow{\rho} & SO(3) \end{array}$$

where $\rho : O(2) \rightarrow SO(3)$ is the homomorphism given by

$$\rho(A) = \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix}.$$

and $i : SO(2) \rightarrow SO(3)$ is the natural inclusion. Consider the induced map $Bi : BSO(2) \rightarrow BSO(3)$ of classifying spaces, this map is a fibration with fiber S^2 . Recall from [1, Theorem 1.5], [7, Theorem 1] that the integral cohomology of $BSO(3)$ is given as a graded algebra by

$$H^*(BSO(3); \mathbb{Z}) \cong \mathbb{Z}[v_3, p'_4] / \langle 2v_3 \rangle,$$

where the subscripts indicate the degree of each generator. On the other hand, the cohomology of $BSO(2)$ is given by

$$H^*(BSO(2); \mathbb{Z}) = H^*(BU(1); \mathbb{Z}) = \mathbb{Z}[c_1],$$

where c_1 denotes the first Chern class. By a straightforward analysis of the Serre spectral sequence, we can see that the induced homomorphism

$$(Bi)^* : H^4(BSO(3); \mathbb{Z}) \rightarrow H^4(BSO(2); \mathbb{Z})$$

is an isomorphism. Thus, there exists $p''_4 \in H^4(BSO(3); \mathbb{Z})$ such that $(Bi)^*(p''_4) = c_1^2$. Therefore, $p_4 := (B\rho)^*(p''_4) \in H^4(BO(2); \mathbb{Z})$ is the desired cohomology class. \square

We now proceed to prove our main result of the section.

Theorem 3.3 (Upper bound for the p -period of \mathcal{N}_g^1). *Let $g \geq 3$, and p be an odd prime. If \mathcal{N}_g^1 contains p -torsion, then the p -period of \mathcal{N}_g^1 is lower or equal than 4, i.e., $p(\mathcal{N}_g^1) \leq 4$.*

Proof. We will show that the Yagita invariant $Y(\mathcal{N}_g^1, p) \leq 4$. Let $\pi \leq \mathcal{N}_g^1$ be a subgroup of order p . By the Nielsen realization theorem for non-orientable surfaces (see [6, Theorem 5.2]), there exists a subgroup $\tilde{\pi} \leq \text{Diff}(N_g; *)$ such that $\tilde{\pi} \cong \pi$. Thus there is a commutative diagram

$$\begin{array}{ccc} \tilde{\pi} & \xrightarrow{\tilde{i}} & \text{Diff}(N_g; *) \\ \cong \downarrow & & \downarrow \\ \pi & \xrightarrow{i} & \mathcal{N}_g^1. \end{array}$$

Hence, $\tilde{\pi}$ acts on the surface N_g as a group of diffeomorphisms that fix $*$. We obtain the following representation which arises from letting a diffeomorphism of $(N_g, *)$ act on T_*N_g , the unoriented tangent space of N_g at $*$

$$\rho : \text{Diff}(N_g, *) \longrightarrow GL_2(\mathbb{R}), \quad f \longmapsto df_*.$$

Consider the representation of $\tilde{\pi}$ given by the composition

$$\rho \circ \tilde{i} : \tilde{\pi} \xrightarrow{\tilde{i}} \text{Diff}(N_g; *) \xrightarrow{\rho} GL_2(\mathbb{R}).$$

Since $\tilde{\pi}$ is a cyclic group of odd order p , we have $\text{Im}(\rho \circ \tilde{i}) \subset GL_2^+(\mathbb{R})$. Denote by $\tilde{\rho} : \tilde{\pi} \rightarrow GL_2^+(\mathbb{R})$ the resulting representation by restricting the image. Being N_g non-orientable, it can be given a dianalytic structure of a Klein surface on which $\tilde{\pi}$ acts as a group of rotations on a neighborhood of $*$ with respect to this structure. Thus, $\tilde{\rho}$ is a faithful representation. By Lemma 3.1 there exists a class $c_1 \in H^2(BGL_2^+; \mathbb{Z})$ such that

$$(1) \quad v := (B\tilde{\rho})^*(c_1) \neq 0 \quad \text{in} \quad H^2(\tilde{\pi}; \mathbb{Z}).$$

This information can be summarized in the following commutative diagram.

$$\begin{array}{ccccc} & & GL_2^+(\mathbb{R}) & \xrightarrow{\iota} & GL_2(\mathbb{R}) & & df_* : T_*N_g \rightarrow T_*N_g \\ \text{Faithful} & & \uparrow \tilde{\rho} & & \uparrow \rho & & \uparrow \\ \text{rep.} & & \tilde{\pi} & \xrightarrow{\tilde{i}} & \text{Diff}(N_g; *) & & f \\ & & \cong \downarrow & & \downarrow & & \downarrow \\ & & \pi & \xrightarrow{i} & \mathcal{N}_g^1 & & \end{array}$$

Notice that the natural inclusions induce a homotopy commutative diagram at the level of classifying spaces

$$\begin{array}{ccc} BSO(2) & \xrightarrow{B\iota} & BO(2) \\ \simeq \downarrow & & \downarrow \simeq \\ BGL_2^+(\mathbb{R}) & \xrightarrow{B\iota} & BGL_2(\mathbb{R}) \end{array}$$

where the vertical maps are homotopy equivalences. Thus, by the Lemma 3.2 there exists a class $p_4 \in H^4(BGL_2(\mathbb{R}); \mathbb{Z})$ such that

$$(2) \quad (B\iota)^*(p_4) = c_1^2.$$

Passing to cohomology of classifying spaces

$$\begin{array}{ccccc}
 & & & \longleftarrow & \\
 & & & & \longleftarrow p_4 \\
 & & & & \uparrow \\
 & & & & H^* BGL_2(\mathbb{R}) \\
 & & & \xleftarrow{(B\iota)^*} & \\
 & & & H^* BGL_2^+(\mathbb{R}) & \\
 & & & \downarrow c_1^2 & \\
 & & & H^*(B\tilde{\pi}) & \xleftarrow{(\tilde{B}i)^*} H^* B\text{Diff}(N_g; *) \\
 & & & \downarrow \cong & \downarrow \cong \\
 & & & H^*(B\pi) & \xleftarrow{(Bi)^*} H^*(BN_g^1) \\
 & & & \downarrow \cong & \downarrow \cong \\
 & & & 0 \neq \bullet & \bullet \\
 & & & \longleftarrow & \\
 & & & & \longleftarrow
 \end{array}$$

$v \neq 0$

Since the identity component of the group $\text{Diff}(N_g; *)$ for $g \geq 3$ is contractible by [11, Prop. 2 and Thm. 2], it follows that the induced map in classifying spaces $B\text{Diff}(N_g; *) \rightarrow BN_g^1$ is a homotopy equivalence. Thus, the vertical right arrow in the bottom square is an isomorphism. This argument exhibits a class in $H^4(BN_g^1; \mathbb{Z}) = H^4(\mathcal{N}_g^1; \mathbb{Z})$ (namely the image of p_4) that maps to a non-zero class under the restriction

$$H^4(\mathcal{N}_g^1; \mathbb{Z}) \rightarrow H^4(\pi; \mathbb{Z}) \xrightarrow{\text{mod } p} \mathbb{F}_p[u].$$

Therefore, $m(\pi; \mathcal{N}_g^1) \leq 2$ and thus $p(\mathcal{N}_g^1) \leq 4$. \square

Finally, we use the Birman exact sequence (see [11, Proposition 1 and Lemma 1] and [13, Theorem 2.1]) to generalize the previous result from \mathcal{N}_g^1 to \mathcal{N}_g^k for any $k \geq 1$ and $g \geq 3$.

Theorem 3.4 (Upper bound for the p -period of \mathcal{N}_g^k). *Let $k \geq 1$, $g \geq 3$ and p an odd prime. If the group \mathcal{N}_g^k contains p -torsion, then the p -period of \mathcal{N}_g^k is $p(\mathcal{N}_g^k) \leq 4$.*

Proof. The proof is by induction on the number of marked points. The case $k = 1$ is precisely Theorem 3.3. Assume that the result holds for the case of $k \geq 1$ and suppose that \mathcal{N}_g^{k+1} has p -torsion. Since $g \geq 3$ we can consider the Birman exact sequence

$$1 \rightarrow \pi_1(N_g^k) \rightarrow \mathcal{N}_g^{k+1} \rightarrow \mathcal{N}_g^k \rightarrow 1,$$

where N_g^k denotes the surface obtained from N_g by removing the k marked points. Since $\pi_1(N_g^k)$ is a free group, it follows that \mathcal{N}_g^k must contain p -torsion. Thus, from Theorem 1 we see that \mathcal{N}_g^k has p -periodic cohomology and by the induction hypothesis we have $p(\mathcal{N}_g^k) \leq 4$. Since $\text{vcd}(\mathcal{N}_g^k)$ and $\text{cd}(\pi_1(N_g^k))$ are finite, it follows from [14, Lemma 1.1] that \mathcal{N}_g^{k+1} has p -periodic cohomology and $p(\mathcal{N}_g^{k+1}) \mid p(\mathcal{N}_g^k)$. Consequently, $p(\mathcal{N}_g^{k+1}) \leq 4$. \square

4. A LOWER BOUND FOR THE P-PERIOD

In this part, we find a lower bound for the p -period of \mathcal{N}_g^k . Our approach uses the following result, which can be deduced by [8, Lemma 3.1] and the Brown decomposition Theorem [2, Corollary X.7.4].

Theorem 4.1. *Suppose Γ has finite vcd and p -periodic cohomology. Moreover, assume that Γ contains only finitely many conjugacy classes of subgroups of order p . Then the p -period of Γ is given by*

$$p(\Gamma) = 2 \cdot \text{lcm}\{[N_\Gamma(\mathbb{Z}/p) : C_\Gamma(\mathbb{Z}/p)] \mid \mathbb{Z}/p \in S\} \cdot p^d$$

for some integer $d \geq 0$, where S is a set of representatives of the conjugacy classes of subgroups of Γ of order p , $N_\Gamma(\mathbb{Z}/p)$ and $C_\Gamma(\mathbb{Z}/p)$ are the normalizer and centralizer of \mathbb{Z}/p in Γ respectively.

To apply the above result, first observe that if $\mathbb{Z}/p = \langle \alpha \rangle$, then for each $\beta \in N_\Gamma(\mathbb{Z}/p)$ there exists $0 < m_\beta < p$ such that $\beta\alpha\beta^{-1} = \alpha^{m_\beta}$. This allows us to define a bijection

$$\begin{aligned} N_\Gamma(\mathbb{Z}/p)/C_\Gamma(\mathbb{Z}/p) &\rightarrow \{m \in \{1, \dots, p-1\} \mid \alpha^m \text{ is conjugate to } \alpha\} \\ [\beta] &\mapsto m_\beta. \end{aligned}$$

Therefore, if we determine all the powers to which one of the possible generators of the subgroup \mathbb{Z}/p is conjugated, we can obtain information about the p -period of the group Γ . According to the previous results, in the case of \mathcal{N}_g^k where $k \geq 1$, it suffices to find at least one nontrivial power α^m such that α and α^m are conjugated in \mathcal{N}_g^k to conclude that $p(\mathcal{N}_g^k) = 4$. In the case $k = 0$, this nontrivial power gives us a lower bound for the p -period $p(\mathcal{N}_g) \geq 4$, for the genera $g \geq 3$ when the group \mathcal{N}_g has p -periodic Farrell cohomology. However, this power is difficult to find directly in the mapping class group \mathcal{N}_g^k . Fortunately, the Nielsen realization theorem allows us to work with elements in $\text{PDiff}(N_g; k)$ rather than mapping classes in \mathcal{N}_g^k .

The proof of the following result is postponed to the end of this section, since new tools need to be introduced.

Theorem 4.2. *Let $g \geq 3$ and $k \geq 0$. If $f \in \text{PDiff}(N_g; k)$ is of order p , then f and f^{-1} are conjugated in $\text{PHomeo}(N_g; k)$.*

Remark 4.3. Notice that when the surface is orientable, a result such as Theorem 4.2 does not hold in general. For instance, if there is at least one marked point, from Theorem 4.1 and [14, Theorem 1.4] it follows that such a result can never occur.

Having this result, we can prove one of our main results.

Theorem 4.4 (Lower bound for the p -period of \mathcal{N}_g^k). *Let $k \geq 0$, $g \geq 3$ and p be an odd prime. If the group \mathcal{N}_g^k has p -periodic Farrell cohomology, then the p -period is bounded below by 4, that is, $p(\mathcal{N}_g^k) \geq 4$.*

Proof. Let $\mathbb{Z}/p = \langle \alpha \rangle$ be a subgroup of prime order p in \mathcal{N}_g^k . By the above discussion, we have a bijection

$$N_{\mathcal{N}_g^k}(\mathbb{Z}/p)/C_{\mathcal{N}_g^k}(\mathbb{Z}/p) \rightarrow \{m \in \{1, \dots, p-1\} \mid \alpha^m \text{ is conjugate to } \alpha\}.$$

By the Nielsen realization theorem there exists a diffeomorphism $f \in \text{PDiff}(N_g; k)$ such that $f \in \alpha$ and $f^p = 1$. By Theorem 4.2 we have that f and f^{-1} are conjugated in $\text{PHomeo}(N_g; k)$. Thus, there exists $s \in \text{PHomeo}(N_g; k)$ such that $sf s^{-1} = f^{-1}$. Let β denote the image of s under the canonical projection $\text{PHomeo}(N_g; k) \rightarrow \mathcal{N}_g^k$, then $\beta\alpha\beta^{-1} = \alpha^{-1}$. Thus, α is conjugated to α^{-1} in \mathcal{N}_g^k , which implies that $[N_{\mathcal{N}_g^k}(\mathbb{Z}/p) : C_{\mathcal{N}_g^k}(\mathbb{Z}/p)] \geq 2$.

By Theorem 4.1, we have the following expression of the p -period

$$p(\mathcal{N}_g^k) = 2 \cdot \text{lcm}\{[N_{\mathcal{N}_g^k}(\mathbb{Z}/p) : C_{\mathcal{N}_g^k}(\mathbb{Z}/p)] \mid \mathbb{Z}/p \in S\} \cdot p^d,$$

where S is a set of representatives of the conjugation classes of subgroups of order p and $d \geq 0$. Thus, the formula entails

$$p(\mathcal{N}_g^k) \geq 4 \cdot p^d \geq 4,$$

which proves the result. \square

Proof of the Theorem 4.2.

In this section, we complete the proof of our main theorem by proving that f and f^{-1} are conjugated in $\text{PHomeo}(N_g; k)$, whenever $f \in \text{PDiff}(N_g; k)$ is of order p . For this purpose, we will use non-Euclidean crystallographic groups (*NEC groups*, for short), which are similar to Fuchsian groups, but orientation-reversing isometries of the hyperbolic plane \mathbb{H}^2 are allowed. For a general discussion of this topic, we refer the reader to [4, Section 0.2] and [3, Section 1]. The surface N_g will be uniformized by a NEC group K such that $f : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/K$ is a dianalytic map. To find a homeomorphism that conjugates the diffeomorphisms, f and f^{-1} , we construct an automorphism of K that connects f and its inverse. The advantage of using NEC groups lies in the fact that every automorphism of a NEC group can be realized geometrically (see [16, Theorem 3]). In this way, we can find a homeomorphism $\tau : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that with respect to the universal cover $q_K : \mathbb{H}^2 \rightarrow \mathbb{H}^2/K$, this induces a homeomorphism $s : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/K$ with the desired property $s \circ f \circ s^{-1} = f^{-1}$.

Surface-kernel epimorphism. Let $f \in \text{PDiff}(N_g; k)$ be an element of order p . By [19, Proposition 1], the lifting $\tilde{f} := \tilde{\phi}(f) \in \text{Diff}(S_{g-1}; 2k)$ has a finite number of fixed points, thus f also has a finite number of fixed points. Denote the fixed points of f by $z_1, \dots, z_t \in N_g$, with the convention that the first k points are the marked points. We can endow the surface N_g with a dianalytic structure \mathfrak{X} such that $\langle f \rangle$ is a group of automorphisms of the Klein surface $(N_g; \mathfrak{X})$ or, in other words, the mapping

$$f : (N_g; \mathfrak{X}) \rightarrow (N_g; \mathfrak{X})$$

is dianalytic. By the uniformization theorem of Klein surfaces, there exists a non-Euclidean crystallographic group K isomorphic to the fundamental group of N_g such that the quotient surface \mathbb{H}^2/K is isomorphic to N_g as Klein surfaces. Let $\gamma : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the lifting of the diffeomorphism $f : N_g \rightarrow N_g$ to the universal cover $q_K : \mathbb{H}^2 \rightarrow \mathbb{H}^2/K$. Since f is dianalytic, then $\gamma : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is an isometry and this allows us to define the NEC group

$$\Gamma := \langle K, \gamma \rangle.$$

Therefore the quotient space \mathbb{H}^2/Γ is homeomorphic to $N_g/\langle f \rangle$ which in turn is homeomorphic to N_h , where h satisfies the Riemann-Hurwitz equation,

$$g - 2 = p(h - 2) + t(p - 1),$$

for $h \geq 1$ and $t \geq k$. On the other hand, it is not hard to see that $K \triangleleft \Gamma$, thus, we have a ramified covering $q : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/\Gamma$ and for every element $u \in \Gamma$ there exists an induced homeomorphism $\hat{u} : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/K$ defined for each $\zeta \in \mathbb{H}^2$ as

$$\zeta \cdot K \mapsto u(\zeta) \cdot K,$$

which makes the following diagram commutative

$$\begin{array}{ccc} \mathbb{H}^2 & \xrightarrow{u} & \mathbb{H}^2 \\ \downarrow & & \downarrow \\ N_g \cong \mathbb{H}^2/K & \xrightarrow{\hat{u}} & N_g \cong \mathbb{H}^2/K. \end{array}$$

We can see that $\hat{u} : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/K$ is a covering transformation for the ramified cover $q : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/\Gamma$. We now define the epimorphism $\theta : \Gamma \rightarrow \langle f \rangle \cong \mathbb{Z}/p$ given by

$$\theta(u) = \hat{u} \quad \text{for all } u \in \Gamma,$$

and in this way, a short exact sequence is obtained

$$1 \rightarrow K \rightarrow \Gamma \xrightarrow{\theta} \langle f \rangle \rightarrow 1.$$

In the literature, the epimorphism θ is called *smooth* or *surface-kernel epimorphism*; see for example [3, Section 1.4]. From now on, we will consider the surface N_g as the quotient \mathbb{H}^2/K and its elements are represented by $\zeta \cdot K$.

Canonical presentation of the NEC group Γ . Since the canonical projection $q_\Gamma : \mathbb{H}^2 \rightarrow \mathbb{H}^2/\Gamma$ has t branched points corresponding to the fixed points of f , whose ramification index is equal to p and \mathbb{H}^2/Γ is homeomorphic to N_h , then Γ has an algebraic presentation given by (see [3, Proposition 1.1.4])

$$(3) \quad \langle x_1, \dots, x_t, d_1, \dots, d_h \mid x_1 \dots x_t \cdot d_1^2 \cdot \dots \cdot d_h^2 = x_1^p = \dots = x_t^p = 1 \rangle,$$

where x_1, \dots, x_t are elliptic elements of $\text{Aut}(\mathbb{H}^2)$ and d_1, \dots, d_h are glide reflections of $\text{Aut}(\mathbb{H}^2)$. The above presentation will be called a *canonical presentation* of Γ and the generators will also be called *canonical generators*. With this presentation, notice that the elliptic generators contain, in some sense, the information about the fixed points of f .

Remark 4.5. Consider the canonical presentation (3) of Γ . Let $\zeta_1, \dots, \zeta_t \in \mathbb{H}^2$ be the fixed points of the elliptic canonical generators x_1, \dots, x_t , respectively. By definition of $\theta : \Gamma \rightarrow \langle f \rangle$, we have that

$$\theta(x_i)(\zeta_i \cdot K) := x_i(\zeta_i) \cdot K = \zeta_i \cdot K$$

and $\theta(x_i) = f^{m_i}$ for some $1 \leq m_i \leq p - 1$. Since $(m_i, p) = 1$, we have that $\zeta_i \cdot K$ is also a fixed point of f , for each $i = 1, \dots, t$.

Remark 4.6. Suppose that $\zeta \cdot K \in \mathbb{H}^2/K$ is a fixed point of f . Then for each $u \in \Gamma$ we have that

$$u(\zeta) \cdot K = \zeta \cdot K.$$

Auxiliary epimorphisms θ_1 and θ_2 . Given an element $u \in \Gamma$ it is clear that

$$\theta(u) = f^{m_u} = (f^{-1})^{-m_u}, \text{ for some } 1 \leq m_u \leq p.$$

We will consider m_u as an element of \mathbb{Z}/p , to avoid problems with the range from which we can select m_u . We define the epimorphisms $\theta_1 : \Gamma \rightarrow \mathbb{Z}/p$ and $\theta_2 : \Gamma \rightarrow \mathbb{Z}/p$ given by $\theta_1(u) = m_u$ and $\theta_2(u) = -m_u$. Observe that, by definition

$$\theta(u) = f^{\theta_1(u)} \quad \text{and} \quad \theta(u) = (f^{-1})^{\theta_2(u)}$$

where, abusing the notation $\theta_1(u)$ and $\theta_2(u)$ are thought of as integers and not as classes of \mathbb{Z}/p . With this observation, we can see that the epimorphisms θ_1 and θ_2 give us a distinction on how to choose a preferred generator from the group of covering transformations of $q : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/\Gamma$. Namely, for θ_1 we take f as the preferred generator while for θ_2 we take f^{-1} .

The following result connects the two epimorphisms θ_1 and θ_2 by an isomorphism $\psi : \Gamma \rightarrow \Gamma$. This result, combined with the previous discussion that θ_1 and θ_2 contain information on the choice of a preferred generator of \mathbb{Z}/p (which are f and f^{-1}), gives us the guideline to prove that f and f^{-1} are conjugate, as will be seen in later results.

Lemma 4.7. *For the epimorphisms $\theta_1 : \Gamma \rightarrow \mathbb{Z}/p$ and $\theta_2 : \Gamma \rightarrow \mathbb{Z}/p$ there exists an isomorphism $\psi : \Gamma \rightarrow \Gamma$ such that the following diagram is commutative:*

$$(4) \quad \begin{array}{ccc} \Gamma & & \\ \downarrow \psi & \searrow \theta_1 & \\ \Gamma & & \mathbb{Z}/p \\ & \nearrow \theta_2 & \end{array}$$

Moreover, if $\zeta_i \in \mathbb{H}^2$ is the fixed point of the elliptic generator x_i , then there exists $u_i \in \Gamma$ such that the elliptic generator $\psi(x_i)$ has $u_i(\zeta_i) \in \mathbb{H}^2$ as a fixed point.

Proof. Consider the canonical presentation (3) of Γ and define the following elements of Γ :

$$\begin{aligned} \eta &= x_1 \cdot \dots \cdot x_t \cdot d_1, \\ \chi_i &= x_{i+1} \cdot x_{i+2} \cdot \dots \cdot x_t & \text{for } i = 1, \dots, t-1, & \quad \chi_t = 1, \\ \delta_j &= d_{j+1}^2 \cdot d_{j+2}^2 \cdot \dots \cdot d_h^2 & \text{for } j = 1, \dots, h-1, & \quad \delta_h = 1. \end{aligned}$$

We define the function $\psi : \Gamma \rightarrow \Gamma$ at the level of generators as follows:

$$\begin{aligned} x_i &\mapsto \eta \cdot \chi_i \cdot x_i^{-1} \cdot \chi_i^{-1} \cdot \eta^{-1} & \text{for } i = 1, \dots, t \\ d_1 &\mapsto \eta^2 \cdot \delta_1 \cdot \eta^{-1} \\ \delta_j &\mapsto \eta \cdot \delta_j \cdot d_j^{-1} \cdot \delta_j^{-1} \cdot \eta^{-1} & \text{for } j = 2, \dots, h. \end{aligned}$$

Since the relation of the group Γ is preserved by ψ , it follows that ψ defines a group homomorphism from Γ to Γ . Moreover, it can be checked that ψ is an isomorphism.

On the other hand, notice that ψ satisfies the following properties:

$$\theta_2(\psi(x_i)) = \theta_2(\eta \cdot \chi_i \cdot x_i^{-1} \cdot \chi_i^{-1} \cdot \eta^{-1}) = -\theta_2(x_i) \quad \text{for } i = 1, \dots, t,$$

$$\theta_2(\psi(d_1)) = \theta_2(\eta^2 \cdot \delta_1 \cdot \eta^{-1}) = \theta_2(d_1) + \sum_{i=1}^t \theta_2(x_i) + 2 \cdot \sum_{j=2}^h \theta_2(d_j)$$

$$\theta_2(\psi(d_j)) = \theta_2(\eta \cdot \delta_j \cdot d_j^{-1} \cdot \delta_j^{-1} \cdot \eta^{-1}) = -\theta_2(d_j) \quad \text{for } j = 2, \dots, h.$$

Since the generators of the group Γ satisfy the relation $x_1 \cdot \dots \cdot x_t \cdot d_1^2 \cdot \dots \cdot d_h^2 = 1$, it follows that

$$-\theta_2(d_1) = \theta_2(d_1) + \sum_{i=1}^t \theta_2(x_i) + 2 \cdot \sum_{j=2}^h \theta_2(d_j),$$

which implies that

$$\begin{aligned} \theta_2(\psi(x_i)) &= -\theta_2(x_i) = \theta_1(x_i) & \text{for all } i = 1, \dots, t, \\ \theta_2(\psi(d_j)) &= -\theta_2(d_j) = \theta_1(d_j) & \text{for all } j = 1, \dots, h, \end{aligned}$$

since, by the definition of $\theta_2 : \Gamma \rightarrow \mathbb{Z}/p$, we have $\theta_1(u) = -\theta_2(u)$ for all $u \in \Gamma$. It follows that the condition holds for all $u \in \Gamma$, that is, $\theta_2(\psi(u)) = \theta_1(u)$. Therefore, $\psi : \Gamma \rightarrow \Gamma$ is the desired isomorphism.

Finally, by the definition of ψ , for each $i = 1, \dots, t$, if ζ_i is the fixed point of the elliptic generator x_i , then $u_i(\zeta_i) \in \mathbb{H}^2$ is the fixed point of $\psi(x_i)$, where $u_i = \eta \cdot \chi_i$. This completes the proof. \square

We now proceed to prove that if $f \in \text{PDiff}(N_g; k)$ is of order p , then f and f^{-1} are conjugated in $\text{PHomeo}(N_g; k)$.

Proof of the Theorem 4.2. Let K be the NEC surface group such that $f : \mathbb{H}^2/K \rightarrow \mathbb{H}^2/K$ is an isometry and $\gamma : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ the lifting of f to the universal cover $q_K : \mathbb{H}^2 \rightarrow \mathbb{H}^2/K$. Consider the NEC group $\Gamma = \langle K, \gamma \rangle$, the surface kernel epimorphism $\theta : \Gamma \rightarrow \langle f \rangle$ and the two auxiliary epimorphisms $\theta_1, \theta_2 : \Gamma \rightarrow \mathbb{Z}/p$ defined above. By Lemma 4.7, we can construct an isomorphism $\psi : \Gamma \rightarrow \Gamma$ such that the diagram (4) is commutative and if the fixed points of the elliptic generators are $\zeta_i \in \mathbb{H}^2$, then the fixed points of $\psi(x_i)$ are equal to $u_i(\zeta_i) \in \mathbb{H}^2$, for some $u_i \in \Gamma$. By [16, Theorem 3], the isomorphism $\psi : \Gamma \rightarrow \Gamma$ is realized geometrically, this means that there exists a homeomorphism $\tau : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that

$$(5) \quad \psi(u) = \tau u \tau^{-1} \quad \text{for all } u \in \Gamma.$$

Now, by the commutativity of diagram (4), we can see that $\psi(\ker(\theta_1)) = \ker(\theta_2)$. But $K = \ker(\theta_1) = \ker(\theta_2)$, which implies that $\psi|_K : K \rightarrow K$ is an automorphism of K . Thus, $\tau : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ induces the following homeomorphisms

$$\begin{aligned} \widehat{s} : \mathbb{H}^2/\Gamma &\rightarrow \mathbb{H}^2/\Gamma & s : \mathbb{H}^2/K &\rightarrow \mathbb{H}^2/K \\ \zeta \cdot \Gamma &\mapsto \tau(\zeta) \cdot \Gamma & \zeta \cdot K &\mapsto \tau(\zeta) \cdot K, \end{aligned}$$

and these are such that the following diagram is commutative

$$\begin{array}{ccc}
\mathbb{H}^2 & \xrightarrow{\tau} & \mathbb{H}^2 \\
\downarrow & & \downarrow \\
\mathbb{H}^2/K & \xrightarrow{s} & \mathbb{H}^2/K \\
\downarrow & & \downarrow \\
\mathbb{H}^2/\Gamma & \xrightarrow{\hat{s}} & \mathbb{H}^2/\Gamma.
\end{array}$$

By definition of the epimorphism $\theta : \Gamma \rightarrow \langle f \rangle$ and using the above diagram, it follows that $\theta(\tau\gamma\tau^{-1}) = s \circ f \circ s^{-1}$. On the other hand, by definition of $\theta_2 : \Gamma \rightarrow \mathbb{Z}/p$, we have that $\theta(\tau\gamma\tau^{-1}) = (f^{-1})^{\theta_2(\tau\gamma\tau^{-1})}$, but from diagram (4) and equation (5) it follows that $\theta_2(\tau\gamma\tau^{-1}) = \theta_2(\psi(\gamma)) = \theta_1(\gamma) = 1$, therefore

$$s \circ f \circ s^{-1} = f^{-1}.$$

It remains to prove that $s \in \text{PHomeo}(N_g; k)$. According to the Lemma 4.7, for each $i = 1, \dots, t$ if $\zeta_i \in \mathbb{H}^2$ is the fixed point of the elliptic generator x_i , then the fixed point of $\psi(x_i)$ is $u_i(\zeta_i)$ for some $u_i \in \Gamma$. Applying ψ to each of the elliptic generators and by equation (5) we have that $\psi(x_i) = \tau x_i \tau^{-1}$. Thus, $\tau(\zeta_i)$ is a fixed point of $\psi(x_i)$. Since the elliptic transformation $\psi(x_i)$ only has one fixed point in \mathbb{H}^2 , it follows that $\tau(\zeta_i) = u_i(\zeta_i)$. This implies, by Remark 4.6 and the definition of s that

$$s(\zeta_i \cdot K) = \tau(\zeta_i) \cdot K = u_i(\zeta_i) \cdot K = \zeta_i \cdot K.$$

Moreover, as we pointed out in Remark 4.5, the points $\zeta_i \cdot K$ are the fixed points of f . Hence, the marked points of N_g remain fixed by s . Therefore $s \in \text{PHomeo}(N_g; k)$, which completes the proof. \square

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