# ON THE VIRTUALLY CYCLIC DIMENSION OF NORMALLY POLY-FREE GROUPS

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ABSTRACT. In this note we give an upper bound for the virtually cyclic dimension of any normally poly-free group in terms of its length. In particular, this implies that virtually even Artin groups of FC-type admit a finite dimensional model for the classifying space with respect to the family of virtually cyclic subgroups.

### 1. INTRODUCTION

Given a group G, we say that a collection  $\mathcal{F}$  of subgroups of G is a *family* if it is non-empty and closed under conjugation and taking subgroups. For a given family  $\mathcal{F}$  of subgroups of G, a G-CW-complex X is a model for the classifying space  $E_{\mathcal{F}}G$  if all of its isotropy groups belong to  $\mathcal{F}$  and the fixed point set  $X^H$  is contractible whenever H belongs to  $\mathcal{F}$ . It can be shown that a model for the classifying space  $E_{\mathcal{F}}G$  always exists and it is unique up to G-homotopy equivalence. In particular, the classifying spaces for the family  $F_{IN}$  of finite subgroups of Gand the family  $V_{CYC}$  of virtually cyclic subgroups of G, denoted by  $\underline{E}G$  and  $\underline{E}G$  respectively, are relevant due to their connection with the Farrell-Jones and Baum-Connes isomorphism conjectures; see for example [LR05].

The  $\mathcal{F}$ -geometric dimension of G is defined as

 $\operatorname{gd}_{\mathcal{F}}(G) = \min\{n \in \mathbb{N} | \text{ there is a model for } E_{\mathcal{F}}G \text{ of dimension } n\}.$ 

For the trivial family and for the families  $F_{IN}$  and  $V_{CYC}$ , the number  $gd_{\mathcal{F}}(G)$  is usually denoted by gd(G), gd(G) and gd(G), respectively.

The  $\mathcal{F}$ -geometric dimension has its algebraic counterpart. The orbit category  $\mathcal{O}_{\mathcal{F}}G$  is the category whose objects are *G*-homogeneous spaces G/H with  $H \in \mathcal{F}$  and morphisms are *G*-functions. The category of Bredon modules is the category whose objects are contravariant functors  $M: \mathcal{O}_{\mathcal{F}}G \to Ab$  from the orbit category to the category of abelian groups, and morphisms are natural transformations  $f: M \to N$ . The  $\mathcal{F}$ -cohomological dimension of G, often denoted by  $cd_{\mathcal{F}}(G)$ , is defined as the minimum integer n such that there exists a projective resolution  $P_{\bullet}$  of length n, in in the category of Bredon modules, of the constant Bredon module  $\mathbb{Z}_{\mathcal{F}}$ . The  $\mathcal{F}$ -geometric dimension and the  $\mathcal{F}$ -cohomological dimension satisfy the following inequality (see [LM00, Theorem 0.1]):

$$\operatorname{cd}_{\mathcal{F}}(G) \leq \operatorname{gd}_{\mathcal{F}}(G) \leq \max\{\operatorname{cd}_{\mathcal{F}}(G), 3\}.$$

In this note we study the geometric dimensions gd(G),  $\underline{gd}(G)$  and  $\underline{gd}(G)$  when G is a normally poly-free group. A group G is called *poly-free* if there exists a finite filtration of G by subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$$

such that the quotient  $G_{i+1}/G_i$  is a free group, for  $0 \le i \le n-1$ . If we have that each  $G_i \triangleleft G$ , we say that G is normally poly-free. If there is a filtration such that the free groups  $G_{i+1}/G_i$  are of finite rank, we say that G is poly-f.g.-free. We define the length of G as the minimum  $n \in \mathbb{N}$ such that there is a filtration as before. Poly-free groups are torsion-free, locally indicable, have finite asymptotic dimension (see [Mor19], [Wu22] and references therein) and satisfy the Baum-Connes Conjecture with coefficients [BKW21, Remark 2]. Furthermore, it has been

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proved that normally poly-free groups satisfy the Farrell-Jones conjecture [BKW21, Theorem A], see also [AFR00], [BFW23, Theorem 1.1], [JPSSn16, Theorem 2.3.7].

In the literature, there are several examples of poly-free and normally poly-free groups. For instance, free groups and free-by-infinite cyclic group are normally poly-free groups of length  $\leq 2$ . Poly-Z groups are a particular case of poly-free groups and their geometric and virtually cyclic dimensions have been completely characterized in [LW12, Section 5]. Furthermore, pure braid groups of surfaces with nonempty boundary are known to be normally poly-free [AFR00] and so are even Artin groups of FC-type [BGMPP19, Theorem 3.18], [Wu22, Theorem A]. It is an open question [Bes99, Question 2] whether all Artin groups are virtually poly-free.

We give upper bounds for the geometric dimensions gd(G),  $\underline{gd}(G)$  and  $\underline{\underline{gd}}(G)$  of any normally poly-free group G in terms of its length.

**Theorem 1.1.** Let G be a poly-free group of length  $n \in \mathbb{N}$ .

- a) The geometric dimension  $gd(G) = \underline{gd}(G)$  is bounded above by n. Furthermore, if G is normally poly-f.g.-free, then gd(G) = n.
- b) If G is a normally poly-free group, then the virtually cyclic dimension satisfies

$$\underline{\mathrm{gd}}(G) \le 3(n-1)+2$$

By [Lüc00, Theorem 2.4], it follows from Theorem 1.1(b) that any virtually normally polyfree group admits a finite dimensional model for the classifying space with respect to the family of virtually cyclic subgroups. Some examples include:

- Virtually even Artin groups of FC-type.
- The braid group  $B_n(S)$  and the pure braid group  $P_n(S)$  of n strings on a connected compact surface S with non-empty boundary.

For Artin braid groups  $B_n = B_n(\mathbb{D}^2)$  and pure braid groups  $P_n = P_n(\mathbb{D}^2)$  the virtually cyclic geometric dimension was explicitly computed in [FGM20]. The existence of finite dimensional models for  $\underline{\underline{E}}B_n(S)$  and  $\underline{\underline{E}}P_n(S)$  also follows from [NP18, Theorem 1.4] and the Birman exact sequence when the underlying surface S is hyperbolic. However, the upper bounds that we get from Theorem 1.1, when the surface S has non-empty boundary, only depends of n and not of the topology of the underlying surface S.

**Remark 1.** We do not expect the upper bound obtained in Theorem 1.1 (b) to be optimal. For instance, for  $n \ge 2$ , if G is a poly- $\mathbb{Z}$  group of length n, then  $\underline{\mathrm{gd}}(G) \le n+1$ , see [LW12, Theorem 5.13]. Furthermore, the pure braid group  $P_n$  is normally poly-free of length  $\le n-1$  and  $\underline{\mathrm{gd}}(P_n) = n$  for  $n \ge 3$ , see [FGM20, Corollary 5.9]. When the group G is free-by-infinite cyclic, we prove in Proposition 2.5 below that  $\mathrm{gd}(G) \le 3$ .

Our Theorem 1.1 is proved by an induction argument on the length of the normally poly-free group G. The proof of part (b) uses, as the base for the induction, that the virtually cyclic geometric dimension of a non-abelian free groups is equal to 2. This is known to hold for finitely generated non-abelian free groups [JPL06], and we prove it for general non-abelian free groups in Corollary 2.1. Our argument uses the next result that may be of independent interest.

**Theorem 1.2.** Let G be a group such that  $\underline{gd}(G) = 1$ , then  $\underline{gd}(G) \leq 2$ . Moreover, if G is not virtually cyclic and has an element of infinite order, then  $\underline{gd}(\overline{G}) = 2$ .

**Remark 2.** There are groups G that are not virtually free and that satisfy  $\underline{\mathrm{gd}}(G) = 1$ , and hence  $\underline{\mathrm{gd}}(G) \leq 2$  by our Theorem 1.2. If G is the fundamental group of a graph of groups in which all vertex groups are finite and such that the orders of the vertex groups are not uniformly bounded, then  $\underline{\mathrm{gd}}(G) = 1$ , but the group G is not virtually free; see for instance [SW79, Theorem 7.3] and references therein. An example of such a group is  $G = \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2) * (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) * \cdots$ , which is not finitely generated. In fact, since such G is not virtually cyclic and has elements of infinite order, Theorem 1.2 implies that  $\mathrm{gd}(G) = 2$ .

For a group G which is not virtually cyclic, M. Fluch proved that  $\underline{gd}(G) = 2$  when G is a Gromov-hyperbolic group with  $\underline{gd}(G) \leq 2$  [Flu10, Proposition 4.9], this includes the case when G is the fundamental group of a finite graph of finite groups [Flu10, Proposition 4.12].

### 2. Proof of Theorems 1.1 and 1.2

We first prove Theorem 1.1(a) about the geometric dimension of poly-free groups. Notice that since poly-free groups are torsion free, then gd(G) = gd(G).

**Theorem 1.1(a)** If G is a poly-free group of length n, then  $gd(G) \le n$ . Furthermore, if G is normally poly-f.g.-free of length n we have that gd(G) = n.

*Proof.* The proof is by induction on the length n of the poly-free group. If n = 1, then G is a non-trivial free group and gd(G) = 1. Suppose that the claim is true for poly-free groups of length  $n \leq k - 1$  and let G be a poly-free group of length k. By definition, there is a filtration of G by subgroups  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$  such that  $G_{i+1}/G_i$  is a free group for  $0 \leq i \leq k - 1$ . Consider the following short exact sequence

$$1 \to G_{k-1} \to G \to G/G_{k-1} \to 1.$$

Notice that  $G/G_{k-1}$  is a free group with  $gd(G/G_{k-1}) = 1$  and  $gd(G_{k-1}) = k-1$  by induction hypothesis since  $G_{k-1}$  is a poly-free group of length k-1. Then, it follows from [Lüc05, Theorem 5.15] that

$$gd(G) \le gd(G_{k-1}) + gd(G/G_{k-1}) \le (k-1) + 1 = k.$$

If G is a normally poly-f.g.-free of length n, it follows from [Mei80, Theorem 16] that the homological dimension of G over  $\mathbb{Q}$  is given by  $hd_{\mathbb{Q}}(G) = n$ . On the other hand

$$\operatorname{gd}(G) \ge \operatorname{cd}(G) \ge \operatorname{cd}_{\mathbb{Q}}(G) \ge \operatorname{hd}_{\mathbb{Q}}(G).$$

For the last inequality see for example [Bie81, Theorem 4.6]. Therefore the furthermore part of the statement follows.  $\hfill \Box$ 

For n = 1, 2, the equality gd(G) = n holds for example of normally poly-free groups G of length n that may not be finitely generated. That is the case when G is a free group or a free-by-infinite cyclic group; see Proposition 2.5 (a) below. The equality gd(G) = n also holds for poly-Z groups G of lenght n [Lüc05, Example 5.26].

The proof of Theorem 1.1 (b) is also done by induction on the length of the poly-free group. We first need some preparatory results. We prove in Corollary 2.1 that the virtually cyclic dimension of non-abelian free groups is equal to 2 and show in Proposition 2.5 that it is at most 3 for free-by-infinite cyclic groups.

In order to include non-finitely generated groups, we prove first Theorem 1.2. Our argument was inspired by [LASSn22, Section 6]. It uses the fact that every virtually cyclic group acting on a simplicial tree T fixes a vertex or acts co-compactly on a unique geodesic line; see [DS99, Lemma 1.1]. Recall that a geodesic line of a tree T is a simplicial embedding of  $\mathbb{R}$  in T, where  $\mathbb{R}$  has a vertex set  $\mathbb{Z}$  and an edge joining any two consecutive integers.

**Theorem 1.2** Let G be a group such that  $\underline{gd}(G) = 1$ , then  $\underline{gd}(G) \leq 2$ . Moreover, if G is not virtually cyclic and has an element of infinite order, then  $\underline{gd}(\overline{G}) = 2$ .

*Proof.* Let G be a group with  $\underline{gd}(G) = 1$ . Then there is a simplicial tree T which is a model for the classifying space  $\underline{E}G$ . We promote T to a model for  $\underline{\underline{E}}G$  by coning-off on T some geodesics as we now explain, see Fig. 1.

First we prove that the set-wise stabilizer  $\operatorname{Stab}_G(\gamma)$  of any geodesic line  $\gamma$  in T is a virtually cyclic group. Consider  $\gamma$  with the simplicial structure induced by T. Note that the group  $\operatorname{Aut}(\gamma)$  of simplicial automorphisms of  $\gamma$  is isomorphic to the infinite dihedral group  $D_{\infty}$ . Since  $\operatorname{Stab}_G(\gamma)$  acts by simplicial automorphisms on  $\gamma$ , then there is a homomorphism of groups  $\varphi$ :  $\operatorname{Stab}_G(\gamma) \to \operatorname{Aut}(\gamma) = D_{\infty}$ . Let us denote by D the image of  $\varphi$  and notice that it is a



FIGURE 1. Promoting T to  $\hat{T}$ .

virtually cyclic group since it is subgroup of a  $D_{\infty}$ . On the other hand, ker( $\varphi$ ) fixes point-wise the vertices of  $\gamma$ . Since T is a model for  $\underline{E}G$ , we have that ker( $\varphi$ ) must be a finite group. It follows from the short exact sequence

$$1 \to \ker(\varphi) \to \operatorname{Stab}_G(\gamma) \xrightarrow{\varphi} D \to 1$$

that  $\operatorname{Stab}_G(\gamma)$  is a virtually cyclic group.

Since T is a model for  $\underline{E}G$  we have that any infinite virtually cyclic subgroup of G does not fix a point in T. Hence, it follows from [DS99, Lemma 1.1] that any infinite virtually cyclic subgroup of G must act co-compactly in a unique geodesic line of T. Let  $\mathcal{A}$  be the collection of all the geodesics of T that admit a co-compact action of an infinite virtually cyclic subgroup of G. Consider the space  $\hat{T}$  given by the following homotopy G-push-out:



If  $H \leq G$  acts co-compactly on the geodesic line  $\gamma$  of T and  $g \in G$ , then  $gHg^{-1}$  acts cocompactly on  $g\gamma$ . It follows that both  $\bigsqcup_{\gamma \in \mathcal{A}} \gamma$  and  $\bigsqcup_{\gamma \in \mathcal{A}} \{*_{\gamma}\}$  are G-CW-complexes, and therefore the space  $\hat{T}$  is a G-CW-complex of dimension 2.

We claim that  $\hat{T}$  is a model for <u>E</u>G. To show this we need to check the following:

a) For all  $x \in \hat{T}$  the isotropy group  $\operatorname{Stab}_G(x) \in V \operatorname{CYC}$ .

b) The fixed point set  $\hat{T}^H = \{x \in \hat{T} \mid hx = x, \text{ for all } h \in H\}$  is contractible if  $H \in V CYC$ .

Item a) follows from the construction of  $\hat{T}$ . Indeed, we have two cases  $x \in T$  or  $x \in \hat{T} - T$ . Observe that in the first case we have that the isotropy group  $\operatorname{Stab}_G(x)$  is finite. In the second case, if  $x \in \hat{T} - T$  is a conic point, then  $\operatorname{Stab}_G(x)$  is infinite virtually cyclic; otherwise the isotropy  $\operatorname{Stab}_G(x)$  is contained in the stabilizer of a conic point, hence it is virtually cyclic.

It remains to show that item b) holds. Let  $H \in V CYC$  and consider the action of H on the tree T obtained by restricting the action of G on T. It follows that H fixes a vertex of T or it acts co-compactly on a unique geodesic line  $\gamma$  of T.

If H fixes a vertex of T, then H is a finite group since T is a model for  $\underline{E}G$ . This implies that  $T^H$  is a non-empty subtree of T. Therefore  $\hat{T}^H$  is obtained from  $T^H$  possibly coning-off some geodesics segments, and we conclude that  $\hat{T}^H$  is contractible.

Otherwise, we have that H acts co-compactly in a unique geodesic line  $\gamma$  on T, then  $\gamma \in \mathcal{A}$ and  $*_{\gamma} \in \hat{T}^{H}$ . Notice that for any  $x \in \hat{T} - \bigsqcup_{\gamma \in \mathcal{A}} \{*_{\gamma}\}$  the isotropy group  $\operatorname{Stab}_{G}(x)$  is finite. Indeed, if  $x \in T$  this claim follows from the fact that T is a model for  $\underline{E}G$ . Now if  $x \notin T$ , then there is some  $\gamma \in \mathcal{A}$  such that x lies in the interior of the segment from a point in  $\gamma$  to the coin point  $*_{\gamma}$ . In particular,  $\operatorname{Stab}_{G}(x)$  is contained in the stabilizer of this segment given by the intersection of the isotropy groups of the end points, one of which is a finite group. Then  $\hat{T}^{H} \subseteq \bigsqcup_{\gamma \in \mathcal{A}} \{*_{\gamma}\}$  and by the uniqueness of the geodesic line  $\gamma$ , and we conclude that  $\hat{T}^{H} = \{*_{\gamma}\}$ .

We now prove the second part of the theorem. Assume that G is not virtually cyclic and let  $h \in G$  be an element of infinite order. The cyclic subgroup of G generated by h must act co-compactly in a unique geodesic line  $\gamma$  of T. Since the stabilizer of any geodesic line in T is virtually cyclic and the group G is not, there exists an element  $g \in G$  that is not in  $\operatorname{Stab}_G(\gamma)$ . Notice that the subgroup H generated by h and g cannot be virtually cyclic. Indeed, if H is virtually cyclic, then it must act co-compactly in a unique geodesic line  $\beta$  of T, and so do its cyclic subgroups generated by h and g. By uniqueness of the geodesic  $\gamma$  stabilized by h, we must have that  $\beta = \gamma$ . Then H stabilizes  $\gamma$  and, in particular,  $g\gamma = \gamma$  which contradicts the fact that we are taking  $g \notin \operatorname{Stab}_G(\gamma)$ . By [LASSn22, Lemma 2.2] we have that  $2 \leq \operatorname{gd}(H) \leq \operatorname{gd}(G)$ .  $\Box$ 

The following result was proved in [JPL06] for finitely generated virtually free groups that are not virtually cyclic.

**Corollary 2.1.** Let G be a virtually free group which is not virtually cyclic. Then gd(G) = 2.

*Proof.* Let G be a non-trivial virtually free group. Then G is the fundamental group of a graph of groups in which all vertex groups are finite; see for example [SW79, Theorem 7.3] and notice that this structure result holds for virtually free groups that may not be finitely generated. Therefore, the Bass-Serre tree associated with the graph of groups is a model for  $\underline{E}G$  and gd(G) = 1. Since G is not virtually cyclic it follows from Theorem 1.2 that gd(G) = 2.

The induction step in the proof to of Theorem 1.1 (b) requires having an upper bound for the virtually cyclic dimension of a free-by-infinite cyclic group. To obtain it in Proposition 2.5 below, we use the following condition introduced by Lück.

**Definition 2.2.** [Lüc09, Condition 4.1] We say that a group G satisfies condition (C) if for every  $g, h \in G$  with  $|h| = \infty$  and  $k, l \in \mathbb{Z}$  we have that  $gh^k g^{-1} = h^l$  implies that |k| = |l|.

There are many collections of groups that satisfy condition (C), including mapping class groups [JPTN16, Proposition 4.1], any discrete group which acts properly and isometrically on a complete proper CAT(0)-space [Lüc09, Proof of Theorem 0.1], and outer automorphism groups of finitely generated free groups [GHS23, Proposition 3.1].

Lemma 2.3. Consider a short exact sequence of groups

$$1 \to F \to G \to \mathbb{Z} \to 1$$

such that F is a free group. Then the group G satisfies condition (C).

*Proof.* Consider the automorphism  $\varphi: F \to F$  such that  $G \cong F \rtimes_{\varphi} \mathbb{Z}$ .

Let  $(x, a), (y, b) \in F \rtimes_{\varphi} \mathbb{Z}$  with  $|(x, a)| = \infty$  and  $k, l \in \mathbb{Z}$  such that

$$(y,b)(x,a)^k(y,b)^{-1} = (x,a)^l$$

Then b + ka - b = la; if  $a \neq 0$ , it follows that k = l.

Now assume that a = 0 and  $x \neq e_F$ . Then we have that  $(y, b)(x^k, 0)(y, b)^{-1} = (x^l, 0)$ , which implies that

(1) 
$$x^{l} = y\varphi^{b}(x^{k})y^{-1} = c_{y} \circ \varphi^{b}(x^{k}),$$

where  $c_y: F \to F$  is the automorphism of F given by conjugation by y.

Consider the infinite cyclic subgroup  $C = \langle x \rangle$  of F and let  $C_{\max}$  be the unique maximal cyclic subgroup of F containing it. From Eq. (1), we have that the automorphism  $c_y \circ \varphi^b$  of F must

take  $C_{\max}$  isomorphically onto  $C_{\max}$ . It follows that  $x^l = c_y \circ \varphi^b(x^k) = \pm \mathrm{Id}_{C_{\max}}(x^k) = x^{\pm k}$ , therefore |k| = |l|.

**Remark 3.** More generally, notice that the argument in Lemma 2.3 shows that a semi-direct product  $G \cong F \rtimes_{\varphi} \mathbb{Z}$  satisfies condition (C) whenever the group F is torsion free and satisfies that "any infinite cyclic subgroup C of F is contained in a unique maximal cyclic subgroup  $C_{max}$  of F". See [LW12, Section 3] for examples of such groups.

Given H a subgroup of G, we denote the normalizer of H in G by  $N_G(H)$  and the corresponding Weyl group by  $W_G(H) = N_G(H)/H$ .

**Lemma 2.4.** [Lüc09, Lemma 4.4] Let n be an integer. Suppose that G satisfies condition (C). Suppose that  $\underline{\mathrm{gd}}(G) \leq n$  and for every infinite cyclic subgroup H of G we have  $\underline{\mathrm{gd}}(W_G(H)) \leq n$ . Then  $\underline{\mathrm{gd}}(G) \leq n+1$ .

**Proposition 2.5.** Consider a short exact sequence of groups

$$1 \to F \to G \to \mathbb{Z} \to 1$$

such that F is a free group.

a) The geometric dimension  $gd(G) \leq 2$ , and equality holds if the group G is not free.

b) The virtually cyclic geometric dimension satisfies  $2 \leq \operatorname{gd}(G) \leq 3$ .

*Proof.* If G is a free group, then gd(G) = 1 and  $gd(G) \le 2$  by Corollary 2.1.

Suppose that G is not a free group. First we prove item a). Since G is a poly-free group of length 2, by Theorem 1.1(a) we have that  $gd(G) \leq 2$ . On the other hand, since G is non-trivial and non-free, then  $cd(G) \geq 2$  by the Stallings and Swan theorem. Hence cd(G) = gd(G) = 2.

We prove item b). For the lower bound notice that if  $F = \mathbb{Z}$ , then G is isomorphic to  $\mathbb{Z}^2$  or  $\mathbb{Z} \rtimes \mathbb{Z}$ , in both cases G is a 2-crystallographic group, it follows from [CFH06] that  $\underline{\mathrm{gd}}(G) = 3$ . If F is not cyclic, then it follows from Corollary 2.1 that  $2 = \underline{\mathrm{gd}}(F) \leq \underline{\mathrm{gd}}(G)$ .

We prove next that  $\underline{\mathrm{gd}}(G) \leq 3$ . By Lemma 2.3, a free-by-cyclic group  $G \cong F \rtimes \mathbb{Z}$  satisfies condition (C). Therefore, from Lemma 2.4 and item a), it is enough to show that the geometric dimension  $\mathrm{gd}(W_G(H)) \leq 2$  for any infinite cyclic subgroup H of G.

Let *H* be a cyclic subgroup of *G*. From the short exact sequence  $1 \to F \to G \xrightarrow{p} \mathbb{Z} \to 1$  we have, by restriction, the short exact sequence

(2) 
$$1 \to F \cap N_G(H) \to N_G(H) \to Q \to 1,$$

where Q is a subgroup of Z. We need to consider two cases: i) p(H) = 0 or ii)  $p(H) \neq 0$ .

In case i), we assume that p(H) = 0, then  $H \subseteq F$ . Since F is a free group, it follows that  $F \cap N_G(H) = N_F(H)$  is a cyclic subgroup of F and  $(F \cap N_G(H))/H$  is finite. From the short exact sequence (2) we obtain

$$1 \to (F \cap N_G(H))/H \to W_G(H) \to Q \to 1,$$

and therefore  $W_G(H)$  is a virtually cyclic group.

In the case *ii*), we have  $p(H) \neq 0$  and Q/p(H) is a finite cyclic group. From the short exact sequence (2) we get

$$1 \to F \cap N_G(H) \to W_G(H) \to Q/p(H) \to 1,$$

where  $F \cap N_G(H)$  is a free group. Hence  $W_G(H)$  is a virtually free group.

In both cases i) and ii) we have that 
$$\underline{\mathrm{gd}}(W_G(H)) \leq 1$$
 and hence  $\underline{\mathrm{gd}}(G) \leq 3$ .

**Remark 4.** If we assume in Proposition 2.5 that G is a countable group, then the result can be deduced from [Deg17, Corollary 3]. See also [DP14, Theorem B] for a criterion for more general groups that fit into an extension with torsion-free quotient to admit a finite-dimensional classifying space with virtually cyclic stabilizers.

The following result of Lück and Weiermann allows us to relate the geometric dimensions associated to nested families of subgroups.

**Lemma 2.6.** [LW12, Proposition 5.1 (i)] Let  $\mathcal{F} \subseteq \mathcal{G}$  be two families of subgroups of a group G. Let  $n \geq 0$  be an integer such that for any  $H \in \mathcal{G}$ , there is an n-dimensional model for  $E_{\mathcal{F} \cap H}(H)$ . Then  $\mathrm{gd}_{\mathcal{F}}(G) \leq \mathrm{gd}_{\mathcal{G}}(G) + n$ .

We are ready to prove Theorem 1.1 (b).

**Theorem 1.1 (b)** Let G be a normally poly-free group of length n. Then  $gd(G) \leq 3(n-1)+2$ .

*Proof.* The proof is done by induction on n. If n = 1, the statement follows from Corollary 2.1. Suppose that the result holds for normally poly-free groups of length  $n \le k - 1$ .

Let G be a normally poly-free group of length k. Then there is a filtration of G by subgroups  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$  such that  $G_i \triangleleft G$ , and the quotient  $G_{i+1}/G_i$  is a free group. We consider the following short exact sequence

$$1 \to G_1 \to G \xrightarrow{p} G/G_1 \to 1.$$

Note that  $G/G_1$  is a normally poly-free group of length  $\leq k-1$ . Consider the family pull-back  $p^*(\mathcal{G})$  of the family  $\mathcal{G}$  of virtually cyclic subgroups of  $G/G_1$ , i.e.  $p^*(\mathcal{G})$  is the family of subgroups of G generated by

 $\{p^{-1}(L): L \text{ is a virtually cyclic subgroup of } G/G_1\}.$ 

A model X of  $E_{\mathcal{G}}(G/G_1)$  is a model of  $E_{p^*(\mathcal{G})}G$  via the action given by the projection p. For the sake of completeness we give a proof, by checking the following:

a) For all  $x \in X$  the isotropy group  $\operatorname{Stab}_G(x) \in p^*(\mathcal{G})$ .

b) The fixed point set  $X^H = \{x \in X \mid hx = x, \text{ for all } h \in H\}$  is contractible if  $H \in p^*(\mathcal{G})$ .

For item a) note that  $\operatorname{Stab}_G(x) = \{g \in G \mid p(g)x = x\} \subseteq p^{-1}(\operatorname{Stab}_{G/G_1}(x))$ , since  $\operatorname{Stab}_{G/G_1}(x) \in \mathcal{G}$  it follows that  $\operatorname{Stab}_G(x) \in p^*(\mathcal{G})$ .

For item b) let  $H \in p^*(\mathcal{G})$ . Then there is  $L \in \mathcal{G}$  such that  $H < p^{-1}(L)$  and it follows that p(H) is virtually cyclic. Note that  $X^H = \{x \in X \mid p(h)x = x, \text{ for all } h \in H\} = X^{p(H)}$ , which is contractible since X is a model for  $E_{\mathcal{G}}(G/G_1)$ .

For the preceding paragraph, we have  $\operatorname{gd}_{p^*(\mathcal{G})}(G) \leq \operatorname{gd}_{\mathcal{G}}(G/G_1) = \operatorname{gd}(G/G_1)$ . Let  $V \operatorname{cyc}$  denote the family of virtually cyclic subgroups of G and notice that  $\overline{V} \operatorname{cyc} \subseteq p^*(\mathcal{G})$ . By Lemma 2.6 we have

(3) 
$$\underline{\underline{\mathrm{gd}}}(G) \leq \underline{\underline{\mathrm{gd}}}(G/G_1) + \max\{\underline{\mathrm{gd}}_{VCYC\cap p^{-1}(L)}(p^{-1}(L)): \ L \in \mathcal{G}\}.$$

We show that  $\operatorname{gd}_{VCYC\cap p^{-1}(L)}(p^{-1}(L)) \leq 3$  for any  $L \in \mathcal{G}$ .

If L is the trivial group, then  $p^{-1}(L) \cong G_1$  is a free group and  $\operatorname{gd}_{VCYC\cap p^{-1}(L)}(p^{-1}(L)) = \underline{\operatorname{gd}}(p^{-1}(L)) = 2$  by Corollary 2.1. Otherwise, L is an infinite cyclic subgroup of the torsion free group  $G/G_1$ . From the short exact sequence

$$1 \to G_1 \to p^{-1}(L) \to L \to 1,$$

we see that  $p^{-1}(L)$  is a free-by-cyclic group. By Proposition 2.5 we have that

$$\operatorname{gd}_{VCYC\cap p^{-1}(L)}(p^{-1}(L)) = \underline{\operatorname{gd}}(p^{-1}(L)) \le 3.$$

From Eq. (3) and the induction hypothesis we have  $\underline{\mathrm{gd}}(G) \leq \underline{\mathrm{gd}}(G/G_1) + 3 \leq 3(k-1) + 2$ .  $\Box$ 

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#### DECLARATIONS

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