ON THE VIRTUALLY CYCLIC DIMENSION OF NORMALLY POLY-FREE GROUPS

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Abstract. In this note we give an upper bound for the virtually cyclic dimension of any normally poly-free group in terms of its length. In particular, this implies that virtually even Artin groups of FC-type admit a finite dimensional model for the classifying space with respect to the family of virtually cyclic subgroups.

1. Introduction

Given a group G, we say that a collection $\mathcal{F}$ of subgroups of G is a *family* if it is non-empty, closed under conjugation, and under taking subgroups. For a given a family $\mathcal{F}$ of subgroups of G, a G-CW-complex X is a model for the classifying space $E_{\mathcal{F}}G$ if all of its isotropy groups belong to $\mathcal{F}$ and the fixed point set X^H is contractible whenever H belongs to $\mathcal{F}$. It can be shown that a model for the classifying space $E_{\mathcal{F}}G$ always exists and it is unique up to G-homotopy equivalence. In particular, the classifying spaces for the family $\mathcal{F}\text{IN}$ of finite subgroups of G and the family $\mathcal{V}\text{CYC}$ of virtually cyclic subgroups of G, denoted by $\underline{E}G$ and $\underline{\underline{E}}G$ respectively, are relevant due to their connection with the Farrell-Jones and Baum-Connes isomorphism conjectures; see for example [LR05].

The *$\mathcal{F}$-geometric dimension* of G is defined as

$$\mathrm{gd}_{\mathcal{F}}(G) = \min\{n \in \mathbb{N} \mid \text{there is a model for } E_{\mathcal{F}}G \text{ of dimension } n\}.$$

For the trivial family and for the families $\mathcal{F}\text{IN}$ and $\mathcal{V}\text{CYC}$, the number $\mathrm{gd}_{\mathcal{F}}(G)$ is usually denoted by $\mathrm{gd}(G)$, $\underline{\mathrm{gd}}(G)$ and $\underline{\underline{\mathrm{gd}}}(G)$, respectively. The $\mathcal{F}$-geometric dimension has its algebraic counterpart, the *$\mathcal{F}$-cohomological dimension* $\mathrm{cd}_{\mathcal{F}}(G)$, which can be defined in terms of Bredon cohomology. The $\mathcal{F}$-geometric dimension and the $\mathcal{F}$-cohomological dimension satisfy the following inequality:

$$\mathrm{cd}_{\mathcal{F}}(G) \le \mathrm{gd}_{\mathcal{F}}(G) \le \max\{\mathrm{cd}_{\mathcal{F}}(G), 3\}.$$

In this note we study the geometric dimensions $\mathrm{gd}(G)$, $\underline{\mathrm{gd}}(G)$ and $\underline{\underline{\mathrm{gd}}}(G)$ when G is a normally poly-free group. A group G is called *poly-free* if there exists a finite filtration of G by subgroups

$$1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = G$$

such that G_i is normal in G_{i+1}, and the quotient G_{i+1}/G_i is a free group, for $0 \le i \le n-1$. If we have that each G_i is normal in G, we say that G is *normally poly-free*. If there is a filtration such that the free groups G_{i+1}/G_i are of finite rank, we say that G is *poly-f.g.-free*. We define the *length of* G as the minimum $n \in \mathbb{N}$ such that there is a filtration as before. Poly-free groups are torsion-free, locally indicable, have finite asymptotic dimension and satisfy the Baum–Connes Conjecture with coefficients [BKW21, Remark 2]. Furthermore, it has been proved that normally poly-free groups satisfy the Farrell-Jones conjecture [BKW21, Theorem A], see also [AFR00], [BFW23, Theorem 1.1], [JPSSn16, Theorem 2.3.7].

In the literature, there are several examples of poly-free and normally poly-free groups. For instance, free groups and free by infinite cyclic groups are normally poly-free groups of length ≤ 2. Poly-$\mathbb{Z}$ groups are a particular case of poly-free groups and their geometric and virtually cyclic dimensions have been completely characterized in [LW12, Section 5]. Furthermore, pure braid groups of surfaces with nonempty boundary are known to be normally poly-free [AFR00]

Key words and phrases. poly-free groups, classifying spaces, families of subgroups, Bass-Serre Theory.

and so are even Artin groups of FC-type [BGMPP19, Theorem 3.18], [Wu22, Theorem A]. It is an open question [Bes99, Question 2] whether all Artin groups are virtually poly-free.

We give upper bounds for the geometric dimensions $\mathrm{gd}(G)$, $\underline{\mathrm{gd}}(G)$ and $\underline{\underline{\mathrm{gd}}}(G)$ of any normally poly-free group G in terms of its length.

Theorem 1.1. *Let G be a poly-free group of length $n \in \mathbb{N}$.*

a) The geometric dimension $\mathrm{gd}(G) = \underline{\mathrm{gd}}(G)$ is bounded above by n. Furthermore, if G is normally poly-f.g.-free, then $\mathrm{gd}(G) = n$.

b) If G is a normally poly-free group, then the virtually cyclic dimension satisfies

$$\underline{\underline{\mathrm{gd}}}(G) \leq 3(n-1) + 2.$$

By [Lüc00, Theorem 2.4], it follows from Theorem 1.1(b) that any virtually normally poly-free group admits a finite dimensional model for the classifying space with respect to the family of virtually cyclic subgroups. Some examples include:

- Virtually even Artin groups of FC-type.
- The braid group $B_n(S)$ and the pure braid group $P_n(S)$ of n strings on a connected compact surface S with non-empty boundary.

For Artin braid groups $B_n = B_n(\mathbb{D}^2)$ and pure braid groups $P_n = P_n(\mathbb{D}^2)$ the virtually cyclic geometric dimension was explicitly computed in [FGM20]. The existence of finite dimensional models for $\underline{\underline{E}}B_n(S)$ and $\underline{\underline{E}}P_n(S)$ also follows from [NP18, Theorem 1.4] and the Birman exact sequence when the underlying surface S is hyperbolic. However, the upper bounds that we get from Theorem 1.1, when the surface S has non-empty boundary, only depends of n and not of the topology of the underlying surface S.

Remark 1. We don't expect the upper bound obtained in Theorem 1.1 (b) to be optimal. For instance, for $n \geq 2$, if G is a poly-$\mathbb{Z}$ group of length n, then $\underline{\underline{\mathrm{gd}}}(G) \leq n+1$, see [LW12, Theorem 5.13]. Furthermore, the pure braid group P_n is normally poly-free of length $\leq n-1$ and $\underline{\underline{\mathrm{gd}}}(P_n) = n$ for $n \geq 3$, see [FGM20, Corollary 5.9]. When the group G is free-by-cyclic, we prove in Proposition 2.5 below that $\underline{\underline{\mathrm{gd}}}(G) \leq 3$.

Our Theorem 1.1 is proved by an induction argument on the length of the normally poly-free group G. The proof of part (b) uses, as the base for the induction, that the virtually cyclic geometric dimension of a non-abelian free groups is equal to 2. This is known to hold for finitely generated non-abelian free groups [JPL06], and we prove it for general non-abelian free groups in Corollary 2.1. Our argument uses the next result that may be of independent interest.

Theorem 1.2. *Let G be a group such that $\underline{\mathrm{gd}}(G) = 1$, then $\underline{\underline{\mathrm{gd}}}(G) \leq 2$. Moreover, if G is not virtually cyclic and has an element of infinite order, then $\underline{\underline{\mathrm{gd}}}(G) = 2$.*

Remark 2. There are groups G that are not virtually free and that satisfy $\underline{\mathrm{gd}}(G) = 1$, and hence $\underline{\underline{\mathrm{gd}}}(G) \leq 2$ by our Theorem 1.2. If G is the fundamental group of a graph of groups in which all vertex groups are finite and such that the orders of the vertex groups are not uniformly bounded, then $\underline{\mathrm{gd}}(G) = 1$, but the group G is not virtually free; see for instance [SW79, Theorem 7.3] and references therein. An example of such a group is $G = \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2) * (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) * \cdots$, which is not finitely generated. In fact, since such G is not virtually cyclic and has elements of infinite order, Theorem 1.2 implies that $\underline{\underline{\mathrm{gd}}}(G) = 2$.

Acknowledgements. We would like to thank Jesús Hernández Hernández for useful discussions. We are specially grateful to Luis Jorge Sánchez Saldaña for several comments on a draft of this paper and for suggesting the argument to prove Lemma 2.3. This paper was partially written while the first author was visiting Northeastern University with funding from the National University of Mexico through a DGAPA-UNAM PASPA sabbatical fellowship. She is grateful to DGAPA-UNAM and thanks the NEU Department of Mathematics and Solomon

Jekel for their hospitality. The second author was supported by a doctoral scholarship of the Mexican Council of Humanities, Science and Technology (CONAHCyT). We are grateful for the financial support of DGAPA-UNAM grant PAPIIT IA106923.

2. Proof of Theorems 1.1 and 1.2

We first prove Theorem 1.1(a) about the geometric dimension of poly-free groups. Notice that since poly-free groups are torsion free, then $\underline{\mathrm{gd}}(G) = \mathrm{gd}(G)$.

Theorem 1.1(a) *If G is a poly-free group of length n, then $\mathrm{gd}(G) \leq n$. Furthermore, if G is normally poly-f.g.-free of lenght n we have that $\mathrm{gd}(G) = n$.*

Proof. The proof is by induction on the length n of the poly-free group. If $n = 1$, then G is a non-trivial free group and $\mathrm{gd}(G) = 1$. Suppose that the claim is true for poly-free groups of length $n \leq k - 1$ and let G be a poly-free group of length k. By definition, there is a filtration of G by subgroups $1 = G_0 \leq G_1 \leq \cdots \leq G_{k-1} \leq G_k = G$ such that G_{i+1}/G_i is a free group for $0 \leq i \leq k - 1$. Consider the following short exact sequence

$$1 \to G_{k-1} \to G \to G/G_{k-1} \to 1.$$

Notice that G/G_{k-1} is a free group with $\mathrm{gd}(G/G_{k-1}) = 1$ and $\mathrm{gd}(G_{k-1}) = k - 1$ by induction hypothesis since G_{k-1} is a poly-free group of length $\leq k - 1$. Then, it follows from [Lüc05, Theorem 5.15] that

$$\mathrm{gd}(G) \leq \mathrm{gd}(G_{k-1}) + \mathrm{gd}(G/G_{k-1}) \leq (k - 1) + 1 = k.$$

If G is a normally poly-f.g.-free of length n, it follows from [Mei80, Theorem 16] that the homological dimension of G over $\mathbb{Q}$ is given by $\mathrm{hd}_{\mathbb{Q}}(G) = n$. Since $\mathrm{gd}(G) \geq \mathrm{cd}(G) \geq \mathrm{hd}_{\mathbb{Q}}(G)$, the furthermore part of the statement follows. □

For $n = 1, 2$, the equality $\mathrm{gd}(G) = n$ holds for examples of normally poly-free groups G of length n that may not be finitely generated. That is the case when G is a free group or a free-by-cyclic group; see Proposition 2.5 (a) below.

The proof of Theorem 1.1 (b) is also done by induction on the length of the poly-free group. We first need some preparatory results. We prove in Corollary 2.1 that the virtually cyclic dimension of non-abelian free groups is equal to 2 and show in Proposition 2.5 that it is at most 3 for free-by-cyclic groups.

In order to include non-finitely generated groups, we prove first Theorem 1.2. Our argument was inspired by [LASSn22, Section 6]. It uses the fact that every virtually cyclic group acting on a simplicial tree T fixes a vertex or acts co-compactly on a unique geodesic line; see [DS99, Lemma 1.1]. Recall that a geodesic line of a tree T is a simplicial embedding of $\mathbb{R}$ in T, where $\mathbb{R}$ has a vertex set $\mathbb{Z}$ and an edge joining any two consecutive integers.

Theorem 1.2 Let G be a group such that $\underline{\mathrm{gd}}(G) = 1$, then $\underline{\underline{\mathrm{gd}}}(G) \leq 2$. Moreover, if G is not virtually cyclic and has an element of infinite order, then $\underline{\underline{\mathrm{gd}}}(G) = 2$.

Proof. Let G be a group with $\underline{\mathrm{gd}}(G) = 1$. Then there is a simplicial tree T which is a model for the classifying space $\underline{E}G$. We promote T to a model for $\underline{\underline{E}}G$ by coning-off on T some geodesics as we now explain.

First we prove that the set-wise stabilizer $\mathrm{Stab}_G(\gamma)$ of any geodesic line γ in T is a virtually cyclic group. Consider γ with the simplicial structure induced by T. Note that the group $\mathrm{Aut}(\gamma)$ of simplicial automorphisms of γ is isomorphic to the infinite dihedral group D_∞. Since $\mathrm{Stab}_G(\gamma)$ acts by simplicial automorphisms on γ, then there is a homomorphism of groups $\varphi\colon \mathrm{Stab}_G(\gamma) \to \mathrm{Aut}(\gamma) = D_\infty$. Let us denote by D the image of φ and notice that it is a virtually cyclic group since it is subgroup of a D_∞. On the other hand, $\ker(\varphi)$ fixes point-wise the vertices of γ. Since T is a model for $\underline{E}G$, we have that $\ker(\varphi)$ must be a finite group. It follows from the short exact sequence

$$1 \to \ker(\varphi) \to \mathrm{Stab}_G(\gamma) \xrightarrow{\varphi} D \to 1$$

that $\mathrm{Stab}_G(\gamma)$ is a virtually cyclic group.

Since T is a model for $\underline{E}G$, any infinite virtually cyclic subgroup of G must act co-compactly in a unique geodesic line of T. Let $\mathcal{A}$ be the collection of all the geodesics of T that admit a co-compact action of an infinite virtually cyclic subgroup of G. Consider the space $\hat{T}$ given by the following homotopy G-push-out:

$$\begin{array}{ccc} \bigsqcup_{\gamma\in\mathcal{A}} \gamma & \longrightarrow & T \\ \downarrow & & \downarrow \\ \bigsqcup_{\gamma\in\mathcal{A}} \{*_\gamma\} & \longrightarrow & \hat{T} \end{array}$$

If $H \leq G$ acts co-compactly on the geodesic line γ of T and $g \in G$, then gHg^{-1} acts co-compactly on $g\gamma$. It follows that both $\bigsqcup_{\gamma\in\mathcal{A}} \gamma$ and $\bigsqcup_{\gamma\in\mathcal{A}}\{*_\gamma\}$ are G-CW-complexes, and therefore the space $\hat{T}$ is a G-CW-complex of dimension 2.

We claim that $\hat{T}$ is a model for $\underline{\underline{E}}G$. To show this we need to check the following:

a) For all $x \in \hat{T}$ the isotropy group $\mathrm{Stab}_G(x) \in V\textsc{cyc}$.
b) The fixed point set $\hat{T}^H = \{x \in \hat{T} | \ hx = x, \text{for all } h \in H\}$ is contractible if $H \in V\textsc{cyc}$.

Item a) follows from the construction of $\hat{T}$. Indeed, we have two cases $x \in T$ or $x \in \hat{T} - T$. Observe that in the first case we have that the isotropy group $\mathrm{Stab}_G(x)$ is finite. In the second case, if $x \in \hat{T} - T$ is a conic point, then $\mathrm{Stab}_G(x)$ is infinite virtually cyclic; otherwise the isotropy $\mathrm{Stab}_G(x)$ is contained in the stabilizer of a conic point, hence it is virtually cyclic.

It remains to show that item b) holds. Let $H \in V\textsc{cyc}$ and consider the action of H on the tree T obtained by restricting the action of G on T. It follows that H fixes a vertex of T or it acts co-compactly on a unique geodesic line γ of T.

If H fixes a vertex of T, then H is a finite group since T is a model for $\underline{E}G$. This implies that T^H is a non-empty subtree of T. Therefore $\hat{T}^H$ is obtained from T^H possibly coning-off some geodesics segments, we conclude that $\hat{T}^H$ is contractible.

Otherwise, we have that H acts co-compactly in a unique geodesic line γ on T, then $\gamma \in \mathcal{A}$ and $*_\gamma \in \hat{T}^H$. Notice that for any $x \in \hat{T} - \bigsqcup_{\gamma\in\mathcal{A}}\{*_\gamma\}$ the isotropy group $\mathrm{Stab}_G(x)$ is finite. Indeed, if $x \in T$ this claim follows from the fact that T is a model for $\underline{E}G$. Now if $x \notin T$, then there is some $\gamma \in \mathcal{A}$ such that x lies in the interior of the segment from a point in γ to the coin point $*_\gamma$. In particular, $\mathrm{Stab}_G(x)$ is contained in the stabilizer of this segment given by the intersection of the isotropy groups of the end points, one of which is a finite group. Then $\hat{T}^H \subseteq \bigsqcup_{\gamma\in\mathcal{A}}\{*_\gamma\}$ and by the uniqueness of the geodesic line γ, we conclude that $\hat{T}^H = \{*_\gamma\}$.

We now prove the moreover part of the theorem. Assume that G is not virtually cyclic and let $h \in G$ be an element of infinite order. The cyclic subgroup of G generated by h must act co-compactly in a unique geodesic line γ of T. Since the stabilizer of any geodesic line in T is virtually cyclic and the group G is not, there exists an element $g \in G$ that is not in $\mathrm{Stab}_G(\gamma)$. Notice that the subgroup H generated by h and g cannot be virtually cyclic. Indeed, if H is virtually cyclic, then it must act co-compactly in a unique geodesic line β of T, and so do its cyclic subgroups generated by h and g. By uniqueness of the the geodesic γ stabilized by h, we must have that $\beta = \gamma$. Then H stabilizes γ and, in particular, $g\gamma = \gamma$ which contradicts the fact that we are taking $g \notin \mathrm{Stab}_G(\gamma)$. By [LASSn22, Lemma 2.2] we have that $2 \leq \underline{\underline{\mathrm{gd}}}(H) \leq \underline{\underline{\mathrm{gd}}}(G)$. □

The following result was proved in [JPL06] for finitely generated virtually free groups that are not virtually cyclic.

Corollary 2.1. *Let G be a virtually free group which is not virtually cyclic. Then* $\underline{\underline{\mathrm{gd}}}(G) = 2$.

Proof. Let G be a non-trivial virtually free group. Then G is the fundamental group of a graph of groups in which all vertex groups are finite; see for example [SW79, Theorem 7.3] and notice that this structure result holds for virtually free groups that may not be finitely generated. This implies that $\underline{\mathrm{gd}}(G) = 1$. Since G is not virtually cyclic it follows from Theorem 1.2 that $\underline{\underline{\mathrm{gd}}}(G) = 2$. □

The induction step in the proof to of Theorem 1.1 (b) requires having an upper bound for the virtually cyclic dimension of a free-by-cyclic group. To obtain it in Proposition 2.5 below, we use the following condition introduced by Lück.

Definition 2.2. [Lüc09, Condition 4.1] We say that a group G satisfies *condition* (C) if for every $g, h \in G$ with $|h| = \infty$ and $k, l \in \mathbb{Z}$ we have that $gh^k g^{-1} = h^l$ implies that $|k| = |l|$.

Lemma 2.3. *Consider a short exact sequence of groups*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

such that F is a free group. Then the group G satisfies condition (C).

Proof. Consider the automorphism $\varphi : F \to F$ such that $G \cong F \rtimes_\varphi \mathbb{Z}$.

Let $(x, a), (y, b) \in F \rtimes_\varphi \mathbb{Z}$ with $|(x, a)| = \infty$ and $k, l \in \mathbb{Z}$ such that

$$(y, b)(x, a)^k (y, b)^{-1} = (x, a)^l.$$

Then $b + ka - b = la$; if $a \neq 0$, it follows that $k = l$.

Now assume that $a = 0$ and $x \neq e_F$. Then we have that $(y, b)(x^k, 0)(y, b)^{-1} = (x^l, 0)$, which implies that $y\varphi^b(x^k)y^{-1} = x^l$ in F. In other words,

$$c_y \circ \varphi^b(x^k) = x^l, \tag{1}$$

where $c_y : F \to F$ is the automorphism of F given by conjugation by y.

It is well known that in a free group, every infinite cyclic subgroup C is contained in a unique maximal cyclic subgroup $C_{\max}$. Consider the infinite cyclic subgroup $C = \langle x \rangle$ of F. From Eq. (1), we have that the automorphism $c_y \circ \varphi^b$ of F sends the subgroup $\langle x^k \rangle$ of C into the subgroup $\langle x^l \rangle$ of C, then it must take C_{max} isomorphically onto C_{max}. Therefore, $c_y \circ \varphi^b|_{C_{max}}$ is an automorphism of the infinite cyclic group C_{max} and hence $c_y \circ \varphi^b|_{C_{max}} = \pm \mathrm{Id}_{C_{max}}$. It follows that $x^l = c_y \circ \varphi^b(x^k) = \pm \mathrm{Id}_{C_{max}}(x^k) = x^{\pm k}$, then $|k| = |l|$. □

Remark 3. More generally, notice that the argument in Lemma 2.3 shows that a semi-direct product $G \cong F \rtimes_\varphi \mathbb{Z}$ satisfies condition (C) whenever the group F is torsion free and satisfies that "any infinite cyclic subgroup C of F is contained in a unique maximal cyclic subgroup C_{max} of F". See [LW12, Section 3] for examples of such groups.

Given H a subgroup of G, we denote the normalizer of H in G by $N_G(H)$ and the corresponding Weyl group by $W_G(H) = N_G(H)/H$.

Lemma 2.4. [Lüc09, Lemma 4.4] *Let n be an integer. Suppose that G satisfies condition (C). Suppose that $\underline{\mathrm{gd}}(G) \leq n$ and for every infinite cyclic subgroup H of G we have $\underline{\mathrm{gd}}(W_G(H)) \leq n$. Then $\underline{\underline{\mathrm{gd}}}(G) \leq n + 1$.*

Proposition 2.5. *Consider a short exact sequence of groups*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

such that F is a free group.

a) The geometric dimension $\mathrm{gd}(G) \leq 2$, *and equality holds if the group G is not free.*

b) The virtually cyclic geometric dimension satisfies $2 \leq \underline{\underline{\mathrm{gd}}}(G) \leq 3$.

Proof. If G is a free group, then $\mathrm{gd}(G) = 1$ and $\underline{\underline{\mathrm{gd}}}(G) \leq 2$ by Corollary 2.1.

Suppose that G is not a free group. First we prove item a). Since G is a poly-free group of length 2, by Theorem 1.1(a) we have that $\mathrm{gd}(G) \leq 2$. On the other hand, since G is non-trivial and non-free, then $\mathrm{cd}(G) \geq 2$ by the Stallings and Swan theorem. Hence $\mathrm{cd}(G) = \mathrm{gd}(G) = 2$.

We prove item b). For the lower bound notice that if $F = \mathbb{Z}$, then G is isomorphic to $\mathbb{Z}^2$ or $\mathbb{Z} \rtimes \mathbb{Z}$, in both cases G is a 2-crystallographic group, it follows from [CFH06] that $\underline{\underline{\mathrm{gd}}}(G) = 3$. If F is not cyclic, then it follows from Corollary 2.1 that $2 = \underline{\underline{\mathrm{gd}}}(F) \leq \underline{\underline{\mathrm{gd}}}(G)$.

We prove next that $\underline{\underline{\mathrm{gd}}}(G) \leq 3$. By Lemma 2.3, a free-by-cyclic group $G \cong F \rtimes \mathbb{Z}$ satisfies condition (C). Therefore, from Lemma 2.4 and item a), it is enough to show that the geometric dimension $\underline{\mathrm{gd}}(W_G(H)) \leq 2$ for any infinite cyclic subgroup H of G.

Let H be a cyclic subgroup of G. From the short exact sequence $1 \to F \to G \xrightarrow{p} \mathbb{Z} \to 1$ we have, by restriction, the short exact sequence

$$1 \to F \cap N_G(H) \to N_G(H) \to Q \to 1, \tag{2}$$

where Q is a subgroup of $\mathbb{Z}$. We need to consider two cases: i) $p(H) = 0$ or ii) $p(H) \neq 0$.

In case i), we assume that $p(H) = 0$, then $H \subseteq F$. Since F is a free group, it follows that $F \cap N_G(H) = N_F(H)$ is a cyclic subgroup of F and $(F \cap N_G(H))/H$ is finite. From the short exact sequence (2) we obtain

$$1 \to (F \cap N_G(H))/H \to W_G(H) \to Q \to 1,$$

and therefore $W_G(H)$ is a virtually cyclic group.

In the case ii), we have $p(H) \neq 0$ and $Q/p(H)$ is a finite cyclic group. From the short exact sequence (2) we get

$$1 \to F \cap N_G(H) \to W_G(H) \to Q/p(H) \to 1,$$

where $F \cap N_G(H)$ is a free group. Hence $W_G(H)$ is a virtually free group.

In both cases i) and ii) we have that $\underline{\mathrm{gd}}(W_G(H)) \leq 1$ and hence $\underline{\underline{\mathrm{gd}}}(G) \leq 3$. □

Remark 4. If we assume in Proposition 2.5 that G is a countable group, then the result can be deduced from [Deg17, Corollary 3]. See also [DP14, Theorem B] for a criterion for more general groups that fit into an extension with torsion-free quotient to admit a finite-dimensional classifying space with virtually cyclic stabilizers.

The following result of Lück and Weiermann allows us to relate the geometric dimensions associated to nested families of subgroups.

Lemma 2.6. [LW12, Proposition 5.1 (i)] *Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of a group G. Let $n \geq 0$ be an integer such that for any $H \in \mathcal{G}$, there is an n-dimensional model for $E_{\mathcal{F} \cap H}(H)$. Then $\mathrm{gd}_{\mathcal{F}}(G) \leq \mathrm{gd}_{\mathcal{G}}(G) + n$.*

We are ready to prove Theorem 1.1 (b).

Theorem 1.1 (b) Let G be a normally poly-free group of length n. Then $\underline{\underline{\mathrm{gd}}}(G) \leq 3(n-1)+2$.

Proof. The proof is done by induction on n. If $n = 1$, the statement follows from Corollary 2.1. Suppose that the result holds for normally poly-free groups of length $n \leq k - 1$.

Let G be a normally poly-free group of length k. Then there is a filtration of G by subgroups $1 = G_0 < G_1 < \cdots < G_{k-1} < G_k = G$ such that G_i is normal in G, and the quotient G_{i+1}/G_i is a free group. We consider the following short exact sequence

$$1 \to G_1 \to G \xrightarrow{p} G/G_1 \to 1.$$

Note that G/G_1 is a normally poly-free group of length $\leq k - 1$. Consider the family pull-back $p^*(\mathcal{G})$ of the family $\mathcal{G}$ of virtually cyclic subgroups of G/G_1, i.e. $p^*(\mathcal{G})$ is the family of subgroups of G generated by

$$\{p^{-1}(L) : L \text{ is a virtually cyclic subgroup of } G/G_1\}.$$

Note that a model X of $E_{\mathcal{G}}(G/G_1)$ is a model of $E_{p^*(\mathcal{G})}G$ via the action given by projection p, then $\mathrm{gd}_{p^*(\mathcal{G})}(G) \leq \mathrm{gd}_{\mathcal{G}}(G/G_1) = \underline{\underline{\mathrm{gd}}}(G/G_1)$. Let $\mathcal{VCYC}$ denote the family of virtually cyclic subgroups of G and notice that $\mathcal{VCYC} \subseteq p^*(\mathcal{G})$. By Lemma 2.6 we have

$$\underline{\underline{\mathrm{gd}}}(G) \leq \underline{\underline{\mathrm{gd}}}(G/G_1) + \max\{\mathrm{gd}_{\mathcal{VCYC}\cap p^{-1}(L)}(p^{-1}(L)) : \ L \in \mathcal{G}\}. \tag{3}$$

We show that $\mathrm{gd}_{\mathcal{VCYC}\cap p^{-1}(L)}(p^{-1}(L)) \leq 3$ for any $L \in \mathcal{G}$.

If L is the trivial group, then $p^{-1}(L) \cong G_1$ is a free group and $\mathrm{gd}_{\mathcal{VCYC}\cap p^{-1}(L)}(p^{-1}(L)) = \underline{\underline{\mathrm{gd}}}(p^{-1}(L)) = 2$ by Corollary 2.1. Otherwise, L is an infinite cyclic subgroup of the torsion free group G/G_1. From the short exact sequence

$$1 \to G_1 \to p^{-1}(L) \to L \to 1,$$

we see that $p^{-1}(L)$ is a free-by-cyclic group. By Proposition 2.5 we have that

$$\mathrm{gd}_{\mathcal{VCYC}\cap p^{-1}(L)}(p^{-1}(L)) = \underline{\underline{\mathrm{gd}}}(p^{-1}(L)) \leq 3.$$

From Eq. (3) and the induction hypothesis we have $\underline{\underline{\mathrm{gd}}}(G) \leq \underline{\underline{\mathrm{gd}}}(G/G_1)+3 \leq 3(k-1)+2$. □

References

[AFR00] C. S. Aravinda, F. T. Farrell, and S. K. Roushon. Algebraic K-theory of pure braid groups. *Asian J. Math.*, 4(2):337–343, 2000.

[Bes99] M. Bestvina. Non-positively curved aspects of Artin groups of finite type. *Geom. Topol.*, 3:269–302, 1999.

[BFW23] M. Bestvina, K. Fujiwara, and D. Wigglesworth. The Farrell-Jones conjecture for hyperbolic-by-cyclic groups. *Int. Math. Res. Not. IMRN*, (7):5887–5904, 2023.

[BGMPP19] R. Blasco-García, C. Martínez-Pérez, and L. Paris. Poly-freeness of even Artin groups of FC type. *Groups Geom. Dyn.*, 13(1):309–325, 2019.

[BKW21] B. Brück, D. Kielak, and X. Wu. The Farrell-Jones conjecture for normally poly-free groups. *Proc. Amer. Math. Soc.*, 149(6):2349–2356, 2021.

[CFH06] F. Connolly, B. Fehrman, and M. Hartglass. On the dimension of the virtually cyclic classifying space of a crystallographic group. *arXiv e-prints*, page arXiv:math/0610387, 2006.

[Deg17] D. Degrijse. A cohomological characterization of locally virtually cyclic groups. *Adv. Math.*, 305:935–952, 2017.

[DP14] D. Degrijse and N. Petrosyan. Geometric dimension of groups for the family of virtually cyclic subgroups. *J. Topol.*, 7(3):697–726, 2014.

[DS99] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.*, 135(1):25–44, 1999.

[FGM20] R. Flores and J. González-Meneses. Classifying spaces for the family of virtually cyclic subgroups of braid groups. *Int. Math. Res. Not.*, (5):1575–1600, 2020.

[JPL06] D. Juan-Pineda and I. J. Leary. On classifying spaces for the family of virtually cyclic subgroups. In *Recent developments in algebraic topology*, volume 407 of *Contemp. Math.*, pages 135–145. Amer. Math. Soc., Providence, RI, 2006.

[JPSSn16] D. Juan-Pineda and L. J. Sánchez Saldaña. The K and L theoretic Farrell-Jones isomorphism conjecture for braid groups. In *Topology and geometric group theory*, volume 184 of *Springer Proc. Math. Stat.*, pages 33–43. Springer, [Cham], 2016.

[LASSn22] P. L. León Álvarez and L. J. Sánchez Saldaña. Classifying spaces for the family of virtually abelian subgroups of orientable 3-manifold groups. *Forum Math.*, 34(5):1277–1296, 2022.

[LR05] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.

[Lüc00] W. Lück. The type of the classifying space for a family of subgroups. *J. Pure Appl. Algebra*, 149(2):177–203, 2000.

[Lüc05] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[Lüc09] W. Lück. On the classifying space of the family of virtually cyclic subgroups for CAT(0)-groups. *Münster J. Math.*, 2:201–214, 2009.

[LW12] W. Lück and M. Weiermann. On the classifying space of the family of virtually cyclic subgroups. *Pure Appl. Math. Q.*, 8(2):497–555, 2012.

[Mei80] D. Meier. On the homological dimension of poly-locally free groups. *J. London Math. Soc. (2)*, 22(3):449–459, 1980.

[NP18] B. Nucinkis and N. Petrosyan. Hierarchically cocompact classifying spaces for mapping class groups of surfaces. *Bull. Lond. Math. Soc.*, 50(4):569–582, 2018.

[SW79] P. Scott and T. Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser*, pages pp 137–203. Cambridge Univ. Press, Cambridge-New York, 1979.

[Wu22] X. Wu. Poly-freeness of Artin groups and the Farrell-Jones conjecture. *J. Group Theory*, 25(1):11–24, 2022.

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