# Stability properties of moduli spaces

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February 1, 2022

## 1 Moduli spaces and stability

*Moduli spaces* are spaces that parametrize topological or geometric data. They often appear in families; for example, the configuration spaces of $n$ points in a fixed manifold, the Grassmannians of linear subspaces of dimension $d$ in $\mathbb{R}^\infty$, and the moduli spaces $\mathcal{M}_g$ of Riemann surfaces of genus $g$. These families are usually indexed by some geometrically defined quantity, such as the number $n$ of points in a configuration, the dimension $d$ of the linear subspaces, or the genus $g$ of a Riemann surface. Understanding the topology of these spaces has been a subject of intense interest for the last 60 years.

For a family of moduli spaces $\{X_n\}_n$ we ask:

*Question* 1.1. How does the topology of the moduli spaces $X_n$ change as the parameter $n$ changes?

For many natural examples of moduli spaces $X_n$, some aspects of the topology get more complicated as the parameter $n$ gets larger. For instance, the dimension of $X_n$ frequently increases with $n$ as well as the number of generators and relations needed to give a presentation of their fundamental groups. But, maybe surprisingly, there are sometimes features of the moduli spaces that 'stabilize' as $n$ increases. In this survey we will describe some forms of stability and some examples of where they arise.

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### 1.1 Homology and cohomology

Algebraic topology is a branch of mathematics that uses tools from abstract algebra to classify and study topological spaces. By constructing *algebraic invariants* of topological spaces, we can translate topological problems into (typically easier) algebraic ones. An algebraic invariant of a space is a quantity or algebraic object, such as a group, that is preserved under homeomorphism or homotopy equivalence. One example is the fundamental group $\pi_1(X, x_0)$ of homotopy classes of loops in a topological space $X$ based at the point $x_0$. Homology and cohomology groups are other examples and are the focus of this article. Their definitions are more subtle than those of homotopy groups like $\pi_1(X, x_0)$, but they are often more computationally tractable and are widely studied.

Given a topological space $X$ and $k \in \mathbb{Z}_{\geq 0}$, we can associate groups $H_k(X; R)$ and $H^k(X; R)$, the $k$th *homology* and *cohomology groups* (with coefficients in $R$), where $R$ is a commutative ring such as $\mathbb{Z}$ or $\mathbb{Q}$. These algebraic invariants define functors from the category of topological spaces to the category of $R$-modules: for any continuous map of topological spaces $f\colon X \to Y$ there are induced $R$-linear maps

$$f_*\colon H_k(X; R) \to H_k(Y; R) \quad \text{(covariant)},$$

$$f^*\colon H^k(Y; R) \to H^k(X; R) \quad \text{(contravariant)}.$$

The cohomology groups $H^*(X; R) = \bigoplus_k H^k(X; R)$ in fact have the structure of a graded $R$-algebra with respect to the *cup product* operation.

The group $H_0(X; \mathbb{Z})$ is the free abelian group on the path components of the topological space $X$ and $H^0(X; \mathbb{Z})$ is its dual. If $X$ is path-connected, $H_1(X; \mathbb{Z})$ is naturally isomorphic to the abelianization of $\pi_1(X, x_0)$ with respect to any basepoint $x_0$, and its elements are certain equivalences classes of (unbased) loops in $X$.

For a topological group $G$ there exists an associated *classifying space* $\mathrm{B}G$ for *principal $G$-bundles*. It is constructed as the quotient of a (weakly) contractible space $\mathrm{E}G$ by a proper free action of $G$. The space $\mathrm{B}G$ is unique up to (weak) homotopy equivalence. If $G$ is a discrete group, then $\mathrm{B}G$ is precisely an *Eilenberg-MacLane space* $K(G,1)$, i.e., a path-connected topological space with $\pi_1(\mathrm{B}G) \cong G$ and trivial higher homotopy groups. For example, up to homotopy equivalence, $\mathrm{B}\mathbb{Z}$ is the circle, $\mathrm{B}\mathbb{Z}_2$ is the infinite-dimensional real projective space $\mathbb{R}\mathrm{P}^\infty$, and the Grassmanian of $d$-dimensional linear subspaces in $\mathbb{R}^\infty$ is $\mathrm{BGL}_d(\mathbb{R})$.

Some motivation to study the cohomology of $\mathrm{B}G$: its cohomology classes define *characteristic classes* of principal $G$-bundles, invariants that measure the 'twistedness' of the bundle. For instance the cohomology algebra $H^*(\mathrm{BGL}_d(\mathbb{R});\mathbb{Z})$ can be described in terms of Pontryagin and Stiefel–Whitney classes.

With $\mathrm{B}G$ we can define the *group homology* and *group cohomology* of a discrete group $G$ by

$$H_k(G;R) := H_k(\mathrm{B}G;R), \quad H^k(G;R) := H^k(\mathrm{B}G;R).$$

We can refine our Question 1.1 to the following:

*Question* 1.2. Given family $\{X_n\}_n$ of moduli spaces or discrete groups, how do the homology and cohomology groups of the $n$th space in the sequence change as the parameter $n$ increases?

In this article we discuss Question 1.2 with a particular focus on the families of configuration spaces and braid groups. For further reading we recommend R. Cohen's survey [Coh09] on stability of moduli spaces.

## 1.2 Homological stability

**Definition 1.3.** A sequence of spaces or groups with maps

$$X_0 \xrightarrow{s_0} \dots \xrightarrow{s_{n-2}} X_{n-1} \xrightarrow{s_{n-1}} X_n \xrightarrow{s_n} X_{n+1} \xrightarrow{s_{n+1}} \dots$$

satisfies *homological stability* if, for each $k$, the induced map in degree-$k$ homology

$$(s_n)_* \colon H_k(X_n;\mathbb{Z}) \to H_k(X_{n+1};\mathbb{Z})$$

is an isomorphism for all $n \geq N_k$ for some stability threshold $N_k \in \mathbb{Z}$ depending on $k$. The maps $s_n$ are sometimes called *stabilization maps* and the set $\{(n,k) \in \mathbb{Z}^2 \mid n \geq N_k\}$ is the *stable range.*

If the maps $s_n \colon X_n \to X_{n+1}$ are inclusions we define $X_\infty := \bigcup_{n\geq 1} X_n$ to be the *stable* group or space. Under mild assumptions, if $\{X_n\}_n$ satisfies homological stability, then

$$H_k(X_\infty;\mathbb{Z}) \cong H_k(X_n;\mathbb{Z}) \quad \text{for} \quad n \geq N_k.$$

We call the groups $H_k(X_\infty;\mathbb{Z})$ the *stable homology.*

# 2 An example: configuration spaces and the braid groups

## 2.1 A primer on configuration spaces

**Definition 2.1.** Let $M$ be a topological space, such as a graph or a manifold. The *(ordered) configuration space $F_n(M)$ of $n$ particles on $M$* is the space

$$F_n(M) = \{(x_1,\dots,x_n) \in M^n \mid x_1,\dots,x_n \text{ distinct}\},$$

topologized as a subspace of $M^n$. Notably, $F_0(M)$ is a point and $F_1(M) = M$.

Configuration spaces have a long history of study in connection to topics as broad-ranging as homotopy groups of spheres and robotic motion planning.

One way to conceptualize the configuration space $F_n(M)$ is as the complement of the union of subspaces of $M^n$ defined by equations of the form $x_i = x_j$. In other words, we can construct $F_n(M)$ by deleting the "fat diagonal" of $M^n$, consisting of all $n$-tuples in $M^n$ where two or more components coincide. In the simplest case, when $n = 2$ and $M$ is the interval $[0,1]$, we see that $F_2([0,1])$ consists of two contractible components, as in Figure 1.

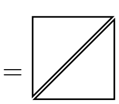
$F_2([0,1]) =$ 

**Figure 1:** The space $F_2([0,1])$ is obtained by deleting the diagonal from the square $[0,1]^2$.

Another way we can conceptualize $F_n(M)$ is as the space of embeddings of the discrete set $\{1, 2, \ldots, n\}$ into $M$, appropriately topologized. We may visualize a point in $F_n(M)$ by labelling $n$ points in $M$, as in Figure 2.

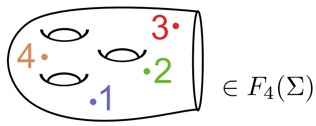



**Figure 2:** A point in the ordered configuration space of an open surface $\Sigma$.

From this perspective, we may reinterpret the path components of $F_2([0,1])$: one component consists of all configurations where particle 1 is to the left of particle 2, and one component has particle 1 on the right. See Figure 3.

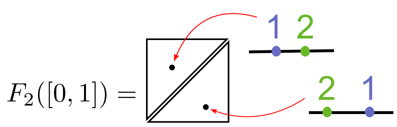



**Figure 3:** The path components of $F_2([0,1])$.

Any path through $[0,1]^2$ that interchanges the relative positions of the two particles must involve a 'collision' of particles, and hence exit the configuration space $F_2([0,1]) \subseteq [0,1]^2$. We encourage the reader to verify that, in general, the configuration space $F_n([0,1])$ is the union of $n!$ contractible path components, indexed by elements of the symmetric group $S_n$. See Figure 4.

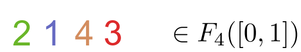



**Figure 4:** A point in $F_4([0,1])$ in the path component indexed by the permutation 2143 in $S_4$.

In contrast, if $M$ is a connected manifold of dimension 2 or more, then $F_n(M)$ is path-connected: given any two configurations, we can construct a path through $M^n$ from one configuration to the other without any 'collisions' of particles. In this case $H_0(F_n(M); \mathbb{Z}) \cong \mathbb{Z}$ for all $n \geq 0$, and this is our first glimpse of stability in these spaces as $n \to \infty$.

For any space $M$, the symmetric group $S_n$ acts freely on $F_n(M)$ by permuting the coordinates of an $n$-tuple $(x_1, \ldots, x_n)$, equivalently, by permuting the labels on a configuration as in Figure 2. The orbit space $C_n(M) = F_n(M)/S_n$ is the *(unordered) configuration space of $n$ particles on $M$*. This is the space of all $n$-element subsets of $M$, topologized as the quotient of $F_n(M)$. The reader may verify that the quotient map (illustrated in Figure 5) is a regular $S_n$-covering space map. In particular, by covering space theory, the quotient map $F_n(M) \to C_n(M)$ induces an injective map on fundamental groups.

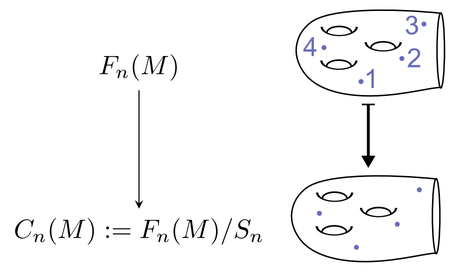



**Figure 5:** The quotient map $F_n(M) \to C_n(M)$

In the case that $M$ is the complex plane $\mathbb{C}$, we can identify $C_n(\mathbb{C})$ with the space of monic degree-$n$ polynomials over $\mathbb{C}$ with distinct roots, by mapping a configuration $\{z_1, \ldots, z_n\}$ to the polynomial $p(x) = (x - z_1) \cdots (x - z_n)$. For this reason the topology of $C_n(\mathbb{C})$ has deep connections to classical problems about finding roots of polynomials.

We will address Question 1.2 for the families $\{C_n(M)\}_n$ and $\{F_n(M)\}_n$, but we first specialize to the case when $M = \mathbb{C}$. Although the spaces $C_n(\mathbb{C})$ and $F_n(\mathbb{C})$ are path-connected, in contrast to the configuration spaces of $M = [0,1]$, they have rich topological structures: they are classifying spaces for the braid groups and the pure braid groups, respectively, which we now introduce.

## 2.2 A primer on the braid groups

Since $F_n(\mathbb{C})$ is path-connected, as an abstract group its fundamental group is independent of choice of basepoint. For path-connected spaces, we sometimes drop the basepoint from the notation for $\pi_1$.

**Definition 2.2.** The fundamental group $\pi_1(C_n(\mathbb{C}))$ is called the *braid group* $\mathbf{B}_n$ and $\pi_1(F_n(\mathbb{C}))$ is the *pure braid group* $\mathbf{P}_n$.

We can understand $\pi_1(F_n(\mathbb{C}))$ as follows. Choose a basepoint configuration $(z_1, \ldots, z_n)$ in $F_n(\mathbb{C})$, and then we may visualize a loop as a 'movie' where the $n$ particles continuously move around $\mathbb{C}$, eventually returning pointwise to their starting positions. If we represent time by a third spacial dimension, as shown in Figure 6, we can view the particles as tracing out a braid. Note that, up to homeomorphism, we may view $F_n(\mathbb{C})$ as the configuration space of the open 2-disk.

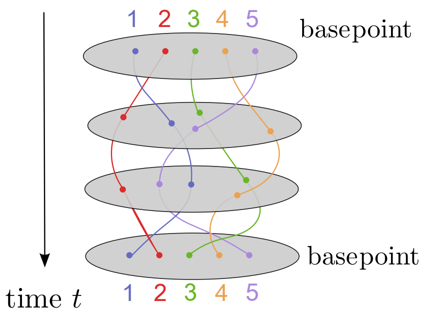



**Figure 6:** A visualization of a loop $\gamma(t)$ in $F_5(\mathbb{C})$ representing an element of $\pi_1(F_5(\mathbb{C})) \cong \mathbf{P}_5$.

Loops in $C_n(\mathbb{C})$ are similar, with the crucial distinction that the $n$ particles are unlabelled and indistinguishable, and so need only return set-wise to their basepoint configuration.

It is traditional to represent elements of the group $\mathbf{B}_n$ and its subgroup $\mathbf{P}_n$ by equivalence classes of *braid diagrams*, as illustrated in Figure 7. These

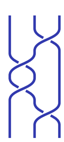


**Figure 7:** A braid on 3 strands

braid diagrams depict $n$ strings (called *strands*) in Euclidean 3-space, anchored at their tops at $n$ distinguished points in a horizontal plane, and anchored at their bottoms at the same $n$ points in a parallel plane. The strands may move in space but may not double back or pass through each other. The group operation is concatenation, as in Figure 8.

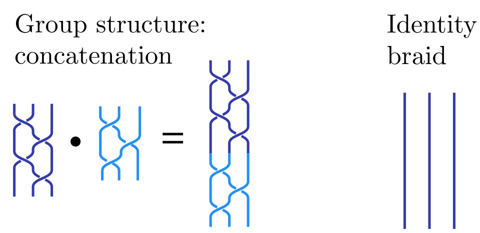



**Figure 8:** The group structure on $\mathbf{B}_n$

The braid groups were defined rigorously by Artin in 1925 [Art25], but the roots of this notion appeared already in the work of Hurwitz, Firckle, and Klein in 1890's, and of Vandermonde in 1771. This topological interpretation of braid groups as the fundamental groups of configuration spaces was formalized in 1962 by Fox and Neuwirth [FN62].

Artin established presentations for the braid group and the pure braid group. His presentation for $\mathbf{B}_n$,

$$\mathbf{B}_n \cong \left\langle \sigma_1, \sigma_2 \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle,$$

uses $(n-1)$ generators $\sigma_i$ corresponding to half-twists of adjacent strands, as in Figure 9.

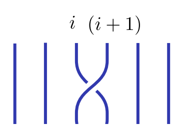



**Figure 9:** Artin's generator $\sigma_i$ for $\mathbf{B}_n$

Artin also gave a finite presentation for $\mathbf{P}_n$. We will not state it in full, but comment that there are $\binom{n}{2}$ generators $T_{ij}$, $(i \neq j,\ i, j \in \{1, 2, \ldots, n\})$ corresponding to full twists of each pair of strands, as in Figure 10.

Corresponding to the regular covering space map $F_n(\mathbb{C}) \to C_n(\mathbb{C})$ of Figure 5, there is a short exact sequence of groups

$$1 \to \mathbf{P}_n \to \mathbf{B}_n \to S_n \to 1.$$

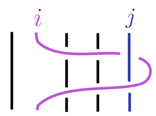



**Figure 10:** Artin's generator $T_{ij} = T_{ji}$ for $\mathbf{P}_n$

The quotient map $\mathbf{B}_n \to S_n$, shown in Figure 11, takes a braid, forgets the $n$ strands, and simply records the permutation induced on their endpoints. The generator $\sigma_i$ maps to the simple transposition $(i\ i+1)$. The kernel is those braids that induce the trivial permutation, i.e., the pure braid group.

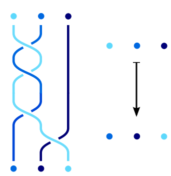


**Figure 11:** The quotient map $\mathbf{B}_n \to S_n$

## 2.3 Homological stability for the braid groups

Arnold [Arn70] calculated some homology groups of $\mathbf{B}_n$ in low degree (Table 1).

| $n$ \ $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | | | | | |
| 1 | $\mathbb{Z}$ | | | | | |
| 2 | $\mathbb{Z}$ | $\mathbb{Z}$ | | | | |
| 3 | $\mathbb{Z}$ | $\mathbb{Z}$ | | | | |
| 4 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | | | |
| 5 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | | | |
| 6 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | |
| 7 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | |
| 8 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_6$ | $\mathbb{Z}_3$ |
| 9 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_6$ | $\mathbb{Z}_3$ |

Table 1: The homology groups $H_k(\mathbf{B}_n;\mathbb{Z})$. Empty spaces are zero groups. Stable groups are shaded.

The $k = 0$ column follows from the fact that $C_n(\mathbb{R}^2)$ is path-connected and the $k = 1$ column can be obtained by abelianizing Artin's presentation of $\mathbf{B}_n$. Even the low-degree calculations in Table 1 suggest a pattern: the homology of $\mathbf{B}_n$ in a fixed degree $k$ becomes independent of $n$ as $n$ increases.

Arnold [Arn70] proved the following stability result, in terms of the stabilization map $s_n\colon \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ defined by adding an unbraided $(n+1)^{st}$ strand as in Figure 12.

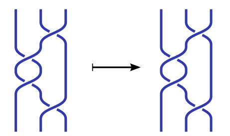


**Figure 12:** The stabilization map $s_3\colon \mathbf{B}_3 \hookrightarrow \mathbf{B}_4$

**Theorem 2.3** (Arnold [Arn70])**.** *For each $k \geq 0$, the induced map*

$$(s_n)_*\colon H_k(\mathbf{B}_n;\mathbb{Z}) \to H_k(\mathbf{B}_{n+1};\mathbb{Z})$$

*is an isomorphism for $n \geq 2k$.*

The family $\{C_n(\mathbb{C})\}_n$ therefore satisfies homological stability. Arnold [Arn70] in fact proved the result for cohomology, and Theorem 2.3 follows from the universal coefficients theorem.

May [May72] and Segal [Seg73] proved that the stable braid group $\mathbf{B}_\infty$ has the same homology as the path component of the trivial loop in the double loop space $\Omega^2 S^2$. F. Cohen [Coh76] and Vaĭnšteĭn [Vaĭ78]. computed the cohomology ring with coefficients in $\mathbb{F}_p$ ($p$ an odd prime), and described $H^k(\mathbf{B}_n;\mathbb{Z})$ in terms of the groups $H^{k-1}(\mathbf{B}_n;\mathbb{F}_p)$ ($p$ prime) for $k \geq 2$.

## 2.4 Homological stability for configuration spaces

For a $d$-manifold $M$, it is possible to visualize homology classes in $F_n(M)$ and $C_n(M)$ concretely. Consider Figure 13. This figure shows a 2-parameter family of configurations in $F_n(M)$; in fact (because the two loops do not intersect) it shows an embedded torus $S^1 \times S^1 \hookrightarrow F_5(M)$. Thus, up to sign, this figure represents an element of $H_2(F_5(M))$. In a sense, the loop traced out by particle 3 arises from the homology of the surface $M$, and the loop traced out

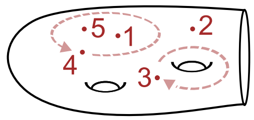



**Figure 13:** A class in $H_2(F_5(M))$

by particle 4 arises from the homology of $F_n(\mathbb{R}^d)$. From the homology of $M$ and $F_n(\mathbb{R}^d)$, it is possible to generate lots of examples of homology classes in $F_n(M)$. The problem of understanding additive relations among these classes, however, is subtle, and the groups $H_k(F_n(M);\mathbb{Z})$ are unknown in most cases.

When $M$ is (punctured) Euclidean space, the (co)homology groups of $F_n(M)$ were computed by Arnold and Cohen. However, even in the case that $M$ is a genus-$g$ surface, we currently do not know the Betti numbers $\beta_k = \mathrm{rank}(H_k(F_n(M);\mathbb{Z}))$. Recently Pagaria [Pag20, Corollary 2.9] computed the asymptotic growth rate of the Betti numbers in the case $M$ is a torus. In the case of *unordered* configuration spaces, in 2016 Drummond-Cole and Knudsen [DCK17] computed the Betti numbers of $C_n(M)$ for $M$ a surface of finite type.

Even though the (co)homology groups of configurations spaces remain largely mysterious, the tools of homological stability give us a different approach to understanding their structure.

Theorem 2.3 on stability for braid groups raises the question of whether the unordered configurations spaces $\{C_n(M)\}_n$ satisfy homological stability for a larger class of topological spaces $M$. Let $M$ be a connected manifold. To generalize Theorem 2.3 we must define stabilization maps

$$\begin{aligned} C_n(M) &\longrightarrow C_{n+1}(M) \\ \{x_1,\dots,x_n\} &\longmapsto \{x_1,\dots,x_n,x_{n+1}\} \end{aligned}$$

Unfortunately, in general there is no way to choose a distinct particle $x_{n+1}$ continuously in the inputs $\{x_1,\dots,x_n\}$, and no continuous map of this form exists. To define the stabilization maps, we must assume extra structure on $M$, for example, assume that $M$ is the interior of a manifold with nonempty boundary. Then, if we choose a boundary component, it is possible to define the stabilization map $s_n\colon C_n(M)\to C_{n+1}(M)$ by placing the new particle in a sufficiently small collar neighbourhood of the boundary component. This procedure (illustrated in Figure 14) is informally described as 'adding a particle at infinity.'

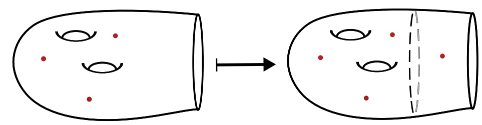


**Figure 14:** Stabilization map $s_3\colon C_3(M)\to C_4(M)$

In the 1970's McDuff proved that the sequence $\{C_n(M)\}_n$ satisfies homological stability and Segal gave explicit stable ranges.

**Theorem 2.4** (McDuff [McD75]; Segal [Seg79])**.** *Let $M$ be the interior of a compact connected manifold with nonempty boundary. For each $k\geq 0$ the maps*

$$(s_n)_*\colon H_k(C_n(M);\mathbb{Z}) \longrightarrow H_k(C_{n+1}(M);\mathbb{Z})$$

*are isomorphisms for $n\geq 2k$.*

Concretely, this theorem states that degree-$k$ homology classes arise from subconfigurations on at most $2k$ particles. Heuristically, these homology classes have the form of Figure 15.

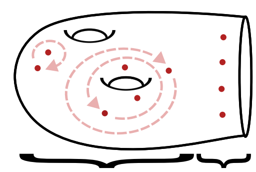



**Figure 15:** A homology class after stabilizing by the addition of $n-2k$ particles.

Moreover, McDuff related the homology of the stable space $C_\infty(M)$ to the homology of $\Gamma(M)$, the space of compactly-supported smooth sections of the bundle over $M$ obtained by taking the fibrewise one-point compactification of the tangent bundle of $M$.

## 3 Other stable families

We briefly describe some other significant families satisfying (co)homological stability.

**Symmetric groups.** Nakaoka [Nak61] proved that the symmetric groups $\{S_n\}_n$ satisfy homological stability with respect to the inclusions $S_n \hookrightarrow S_{n+1}$ (see also [Ker05]). The Barratt–Priddy–Quillen theorem [BP72] states that the infinite symmetric group $S_\infty = \bigcup_n S_n$ has the same homology of $\Omega_0^\infty S^\infty$, the path-connected component of the identity in the infinite loop space $\Omega^\infty S^\infty$.

**General linear groups.** Let $R$ be a ring. Consider the sequence of general linear groups $\{\mathrm{GL}_n(R)\}_n$ with the inclusions $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ given by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

In the 1970's Quillen studied the homology of these groups when $R$ is a finite field $\mathbb{F}_q$ of characteristic $p$ in his seminal work [Qui72] on the $K$-theory of finite fields. He computes $H^*(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_\ell)$ for prime $\ell \neq p$ and determines a vanishing range for $\ell = p$.

Charney [Cha80] proved homological stability when $R$ is a Dedekind domain. Van der Kallen [vdK80], building on work of Maazen [Maa79], proved the case that $R$ is an associative ring satisfying Bass's "stable rank condition"; this arguably includes any naturally-arising ring.

These results are part of a large stability literature on classical groups that warrants its own survey. It includes work by Betley, Cathelineau, Charney, Collinet, Dwyer, Essert, Friedlander, Friedrich Galatius, Guin, Hutchinson, Kupers, Miller, Mirzaii, Nesterenko, Panin, Patzt, Randal-Williams, Sah, Sprehn, Suslin, Tao, Vaseršteĭn, Vogtmann, and Wahl [Vas69, Fri76, Cha80, Dwy80, Vog81, Sus84, Sah86, Gui87, Pan87, NS89, Bet90, MvdK02, Mir05, Cat07, HT10, Col11, Ess13, Fri17, GKRW18b, KMP18, SW20], among others. Homological stability is known to hold for special linear groups, orthogonal groups, unitary groups, and other families of classical groups. There is ongoing work to study (co)homology with twisted coefficients, and sharpen the stable ranges.

**Mapping class groups and moduli space of Riemann surfaces.** Let $\Sigma_{g,1}$ be an oriented surface of genus $g$ with one boundary component and let the *mapping class group*

$$\mathrm{Mod}(\Sigma_{g,1}) := \pi_0(\mathrm{Diff}^+(\Sigma_{g,1} \text{ rel } \partial))$$

be the group of isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ fixing a collar neighbourhood of the boundary. There is a map $t_g\colon \mathrm{Mod}(\Sigma_{g,1}) \hookrightarrow \mathrm{Mod}(\Sigma_{g+1,1})$ induced by the inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$ by extending a diffeomorphism by the identity on the complement $\Sigma_{g+1,1} \setminus \Sigma_{g,1}$, as in Figure 16.

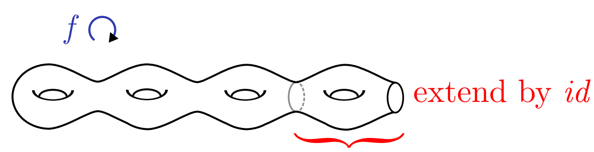



**Figure 16:** The map $\mathrm{Mod}(\Sigma_{3,1}) \to \mathrm{Mod}(\Sigma_{4,1})$ is induced by the inclusion $\Sigma_{3,1} \hookrightarrow \Sigma_{4,1}$

There is also a map *cap*: $\mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_g)$ induced by gluing a disk on the boundary component of $\Sigma_{g,1}$. Harer [Har85] proved that the sequence $\{\mathrm{Mod}_{g,1}\}_g$ satisfies homological stability with respect to the inclusions $t_g$ and that for large $g$ the map *cap* induces isomorphisms on homology. The proof and the stable ranges have been improved by work of Ivanov, Boldsen and others [Iva93, Bol12]. Madsen and Weiss [MW07] computed the stable homology by identifying the homology of mapping class groups, in the stable range, with the homology of a certain infinite loop space.

The rational homology of the mapping class group $\mathrm{Mod}(\Sigma_g)$ is the same as that of the *moduli space* $\mathcal{M}_g$ *of Riemann surfaces* of genus $g \geq 2$. This moduli space parametrizes:

- isometry classes of hyperbolic structures on $\Sigma_g$,
- conformal classes of Riemannian metrics on $\Sigma_g$,
- biholomorphism classes of complex structures on the surface $\Sigma_g$,

- isomorphism classes of smooth algebraic curves homeomorphic to $\Sigma_g$.

One consequence of Harer's stability theorem and the Madsen–Weiss's theorem is their proof of *Mumford's conjecture* [Mum83]: the rational cohomology of $\mathcal{M}_g$ is a polynomial algebra on generators $\kappa_i$ of degree $2i$, the so-called *Mumford–Morita–Miller classes*, in a stable range depending on $g$. See Tillman's survey [Til13] and Wahl's survey [Wah13].

Homological stability was established for mapping class groups of non-orientable surfaces by Wahl [Wah08], for mapping class groups of some 3-manifolds by Hatcher–Wahl [HW10] and framed, Spin, and Pin mapping class groups by Randal-Williams [RW14].

**Automorphism groups of free groups.** Let $F_n$ denote the free group of rank $n$. Hatcher and Vogtmann [HV98] proved that the sequence $\{\mathrm{Aut}(F_n)\}_n$ is homologically stable with respect to inclusions $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_{n+1})$. Galatius [Gal11] computed the stable homology by proving that $H_*(\mathrm{Aut}(F_\infty);\mathbb{Z}) \cong H_*(\Omega_0^\infty S^\infty;\mathbb{Z}) \cong H_*(S_\infty;\mathbb{Z})$. In particular, for $n > 2k+1$,

$$H_k(\mathrm{Aut}(F_n);\mathbb{Q}) \cong H_k(\mathrm{Aut}(F_\infty);\mathbb{Q}) = 0.$$

**Moduli spaces of high-dimensional manifolds.** Let $M$ be a smooth compact manifold. The moduli space $\mathcal{M}(M)$ of manifolds of type $M$ is the classifying space $\mathrm{BDiff}(M \text{ rel } \partial)$. In the last few years Galatius and Randal-Williams [GRW18] proved homological stability for $\mathcal{M}(M)$ for simply connected manifolds $M$ of dimension $2d > 4$, with respect to the $n$-fold connected sum with $S^d \times S^d$. This generalizes Harer's result to higher-dimensional manifolds. They also obtained a generalized Madsen–Weiss theorem for simply connected manifolds of dimension $2d > 4$ [GRW17]. Homological stability with respect to connected sum with $S^p \times S^q$, for $p < q < 2p-2$ was obtained by Perlmutter [Per16].

# 4 A proof strategy

There is a well-established strategy for proving homological stability that traces back to unpublished work by Quillen in the 1970's [Qui74]. We describe a simplified version of Quillen's argument for a family of discrete groups with inclusions.

Recall that a *$p$-simplex* $\Delta^p$ is a $p$-dimensional polytope defined as the convex hull of $(p+1)$ points in $\mathbb{R}^p$ in general position, called its *vertices*. For example, a 0-simplex is a point, a 1-simplex is a closed line segment, and a 2-simplex is triangle. A *face* of a simplex is the convex hull of a subset of its vertices. A map $f\colon \Delta^p \to \Delta^q$ is *simplicial* if it maps vertices to vertices, and takes the form

$$f\colon \sum_{i=0}^{p} t_i v_i \mapsto \sum_{i=0}^{p} t_i f(v_i)$$

with $v_0, \dots, v_p$ the vertices of $\Delta^p$ and $0 \le t_i \le 1$, $\sum_i t_i = 1$.

A *triangulation* of a topological space $W$ is a decomposition of $W$ as a union of simplices, such that the intersection $\sigma \cap \tau$ of any pair of simplices $\sigma, \tau$ in $W$ is either empty or equal to a single common face of $\sigma$ and $\tau$. A triangulated space is called a *simplicial complex*. A map $f$ of simplicial complexes is *simplicial* if it maps simplices to simplices and its restriction to each simplex is simplicial.

A simplicial complex $W$ is called $(-1)$-*connected* if it is nonempty, 0-*connected* if it is path-connected, and 1-*connected* if it is simply connected. More generally, a nonempty simplicial complex $W$ is called $d$-*connected* if its homotopy groups $\pi_i(W)$ vanish for all $0 \le i \le d$. By the Hurewicz theorem, $W$ is $d$-connected ($d \ge 2$) if and only if $W$ is simply connected and $H_i(X) = 0$ for all $2 \le i \le d$.

With this terminology, we can now describe Quillen's argument. The following formulation of Theorem 4.1 is due to Hatcher–Wahl [HW10, Theorem 5.1].

**Theorem 4.1** (**Quillen's argument for homological stability**)**.** *Let $0 \hookrightarrow G_1 \hookrightarrow \ldots \hookrightarrow G_n \hookrightarrow \ldots$ be a sequence of discrete groups. For each $n$ let $W_n$ be a simplicial complex with a simplicial action of $G_n$ satisfying the following properties:*

*(i) The simplicial complexes $W_n$ are $\left(\frac{n-2}{2}\right)$-connected.*

*(ii)* *For each $p \geq 0$, the group $G_n$ acts transitively on the set of $p$-simplices.*

*(iii)* *For each simplex $\sigma_p$ in $W_n$, the stabilizer $stab(\sigma_p)$ fixes $\sigma_p$ pointwise.*

*(iv)* *The stabilizer $stab(\sigma_p)$ of a $p$-simplex $\sigma_p$ is conjugate in $G_n$ to the subgroup $G_{n-p-1} \subseteq G_n$. (By convention $G_n = 0$ if $n \leq 0$.)*

*(v)* *For each edge $[v_0, v_1]$ in $W_n$, there exists $g \in G_n$ such that $g \cdot v_0 = v_1$ and $g$ commutes with all elements of $G_n$ that fix $[v_0, v_1]$ pointwise.*

*Then the sequence $\{G_n\}_n$ is homologically stable. Specifically, the inclusion $G_n \hookrightarrow G_{n+1}$ induces an isomorphism on degree-$k$ homology for $n \geq 2k+1$ and a surjection for $n = 2k$.*

Theorem 4.1 follows from a formal algebraic argument involving a sequence of spectral sequences associated to the complexes $W_n$. We remark, for the readers familiar with spectral sequences, that for each $n$ we obtain a homology spectral sequence by using $W_n \times_{G_n} \mathrm{E}G_n$ to build an approximation to $\mathrm{B}G_n$ from the spaces $\mathrm{B}G_{n-p}$ for $p > 0$. The $n$th spectral sequence has $E^1$ page

$$E^1_{p,q} \cong H_q(stab(\sigma_p); \mathbb{Z}) \cong H_q(G_{n-p-1}; \mathbb{Z}),$$
$$E^1_{-1,q} \cong H_q(G_n; \mathbb{Z}),$$

and $E^1_{p,q} = 0$ for $p < -1$.

The assumption that the complexes $W_n$ are highly connected implies that the spectral sequence converges to 0 for $p + q \leq \frac{n-1}{2}$. The differential

$$d^1 \colon E^1_{0,i} = H_i(G_{n-1}; \mathbb{Z}) \longrightarrow E^1_{-1,i} = H_i(G_n; \mathbb{Z})$$

is the map induced by the inclusion $G_{n-1} \hookrightarrow G_n$. Under the hypotheses of the theorem, we can argue by induction on $i$ that this map is an isomorphism (respectively, a surjection) in the desired range, to complete the proof of Theorem 4.1.

In practice, given Theorem 4.1, the most difficult step in a proof of homological stability is usually the proof that the complexes $W_n$ are highly connected.

In recent years, the argument that we just outlined has been axiomatized by Randal-Williams and Wahl [RWW17] and Krannich [Kra19] to give a very general framework to prove homological stability results, including (co)homology with twisted abelian and polynomial coefficients. Another axiomatization is due to Hepworth [Hep20].

## 4.1 An example: the braid group $\mathbf{B}_n$

Let $\mathbb{D}^2$ be the closed disk. Fix $n$ marked points in its interior and a distinguished point $* \in \partial\mathbb{D}^2$. Associated to the braid group $\mathbf{B}_n$ is an $(n-1)$-dimensional simplicial complex $W_n$ called the *arc complex* which we define combinatorially.

- **vertices:** $W_n$ has a vertex for each isotopy class of embedded arcs in $\mathbb{D}^2$ joining $*$ with one of the marked points.

- **$p$-simplices:** A set of $(p+1)$ vertices spans a $p$-simplex if the corresponding isotopy classes can be represented by arcs that are pairwise disjoint except at their starting point $*$.

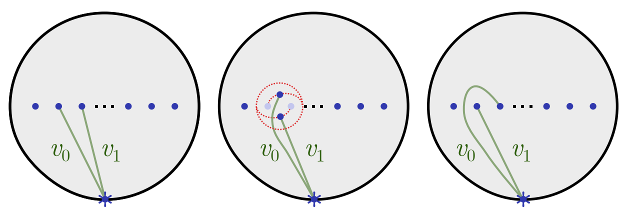



**Figure 17:** The action of $\sigma_2 \in \mathbf{B}_n$ on a 1-simplex $\{v_0, v_1\}$ of the arc complex $W_n$.

Hatcher and Wahl [HW10] proved that $W_n$ is $\left(\frac{n-2}{2}\right)$-connected (it is in fact contractible, see [Dam13]).

The braid group $\mathbf{B}_n$ is isomorphic to the group $\mathrm{Mod}^n(\mathbb{D}^2)$ of isotopy classes of diffeomorphisms of the closed disk that stabilize the set of marked points and restrict to the identity on $\partial\mathbb{D}^2$. Thus $\mathbf{B}_n$ has an action on $W_n$ that is simplicial and satisfies conditions $(i)$-$(v)$. See Figure 17. Theorem 4.1 gives a modern proof of homological stability for $\mathbf{B}_n$ (Theorem 2.3), a result originally due to Arnold.

# 5 Representation stability

## 5.1 Configuration spaces revisited

Let us address Question 1.2 for the ordered configuration spaces $\{F_n(M)\}_n$ when $M$ is the interior of a compact connected manifold with nonempty boundary. As with the unordered configuration spaces, given a choice of boundary component, we can define a stabilization map $F_n(M) \to F_{n+1}(M)$ that continuously introduces a new particle 'at infinity'. See Figure 18.

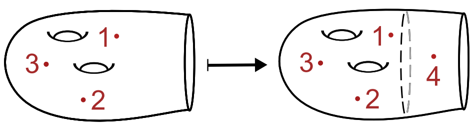



**Figure 18:** Stabilization map $F_3(M) \to F_4(M)$

This suggests the question: for a fixed manifold $M$, do the spaces $\{F_n(M)\}_n$ satisfy homological stability? The answer is, in contrast to $\{C_n(M)\}_n$, they do not, as we will verify directly.

Let $M = \mathbb{C}$, so the homology $H_1(F_n(\mathbb{C}); \mathbb{Z})$ in degree 1 is the abelianization of the pure braid group $\mathbf{P}_n$. Artin's presentation implies that $\mathbf{P}_n^{ab} \cong \mathbb{Z}^{\binom{n}{2}}$ is free abelian on the images $\alpha_{ij}$ of the $\binom{n}{2}$ generators $T_{ij}$ of Figure 10. Viewed as a homology class in $F_n(\mathbb{C})$, we can represent $\alpha_{ij}$ by the loop illustrated in Figure 19. Hence, $\mathrm{rank}(H_1(\mathbf{P}_n; \mathbb{Z}))$ grows quadratically in $n$, and homological stability fails.

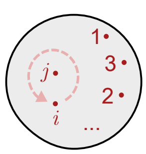



**Figure 19:** The homology class $\alpha_{ij} \in H_1(F_n(\mathbb{C}))$

Church and Farb [CF13], however, proposed a new paradigm for stability in spaces like the ordered configuration spaces $F_n(M)$ of a manifold $M$. Because (co)homology is functorial, the $S_n$-action on $F_n(M)$ induces an action of $S_n$ on the (co)homology groups. Even though the (co)homology does not stabilize as a sequence of abelian groups, they proposed, it does stabilize as a sequence of $S_n$-representations.

There are several ways to formalize the idea of stability for a sequence of $S_n$–representations. One way, which was initially the primary focus of Church and Farb, is to consider the multiplicities of irreducible representations in the rational (co)homology groups. Suppose $V$ is a finite-dimensional rational $S_n$-representation. Because $S_n$ is a finite group, $V$ is *semisimple*: it decomposes as a direct sum of irreducible subrepresentations. The multiplicities of the irreducible components are uniquely defined and determine $V$ up to isomorphism.

The irreducible rational $S_n$-representations are classified, and are in canonical bijection with partitions of $n$. A *partition* $\lambda$ of a positive integer $n$ is a set of positive integers (called the *parts* of $\lambda$) that sum to $n$. It is traditionally encoded by a *Young diagram*, a collection of $n$ boxes arranged into rows of decreasing lengths equal to the parts of $\lambda$. For example, the Young diagram corresponds to the partition $3+2$ of 5. If $\lambda$ is a partition of $n$ (equivalently, a Young diagram of size $n$), we write $V_\lambda$ to denote the irreducible $S_n$-representation associated to $\lambda$.

Church and Farb observed a pattern in the homology of $F_n(\mathbb{C})$, which we illustrate in Figure 20 in homological degree 1.

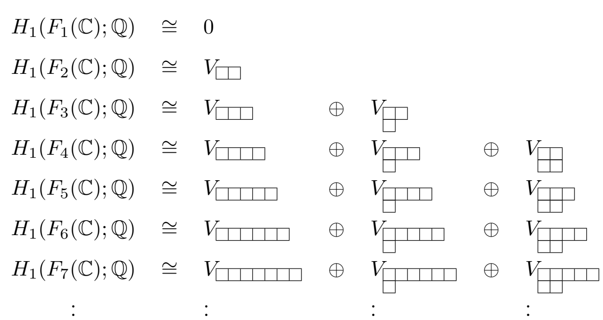



**Figure 20:** The decomposition of the homology groups $H_1(F_n(\mathbb{C}); \mathbb{Q})$ for some small values of $n$.

For $n \geq 4k$, we can recover the decomposition of $H_k(F_n(\mathbb{C}); \mathbb{Q})$ into irreducible components simply by taking the decomposition of $H_k(F_{n-1}(\mathbb{C}); \mathbb{Q})$ and adding a single box to the top row of each Young diagram. They showed that this pattern holds for all $k$, and Church [Chu12] later proved that it holds for the cohomology groups $H^k(F_n(M);\mathbb{Q})$ of the ordered configuration space of a connected oriented manifold of finite type.

Church, Farb, and others observed the same patterns in the (co)homology of a number of other families of groups and spaces. These results raise the question,

*Question* 5.1. What underlying structure is responsible for these patterns?

Church, Ellenberg, Farb, Nagpal, Putman, and Sam answered this question by developing an algebraic framework that brought their work into a broader field, now called the field of *representation stability*. See, for example, [CF13, CEF15, CEFN14, Put15, CE17, PS17]. Other pioneers of the field, who approached it from different perspectives, include Sam, Snowden, Djament, Pirashvili, Vespa, Gan, and Li. Some selected references are [Pir00, DV10, SS12, Sno13, SS15, GL15, SS16, Dja16, SS17, DV19].

## 5.2 FI-modules

The key to answering Question 5.1 is the concept of an FI-module. The theory of FI–modules gives a conceptual framework that explains the ubiquity of the patterns observed in so many naturally-arising sequences of $S_n$-representations, and it also provides algebraic machinery to prove stronger results with streamlined arguments.

**Definition 5.2.** Let FI be the category whose objects are finite sets (including $\varnothing$), and whose morphisms are all injective maps. Given a commutative ring $R$ (typically $\mathbb{Z}$ or $\mathbb{Q}$), an FI-*module* $V$ over $R$ is a functor from FI to the category of $R$-modules.

To describe an FI-module $V$, it is enough to consider the "standard" finite sets in FI,

$$[0] = \varnothing \quad \text{and} \quad [n] = \{1, 2, \ldots, n\}.$$

For $n \geq 0$, we write $V_n$ to denote the image of $V$ on $[n]$. The endomorphisms of $[n]$ in FI are the symmetric group $S_n$, so $V_n$ is an $S_n$-representation. The data of an FI-module $V$ is determined by the sequence of $S_n$-representations $\{V_n\}_n$, along with $S_n$-equivariant maps $\iota_n \colon V_n \to V_{n+1}$ induced by the inclusion $[n] \hookrightarrow [n+1]$. Figure 21 gives a schematic.

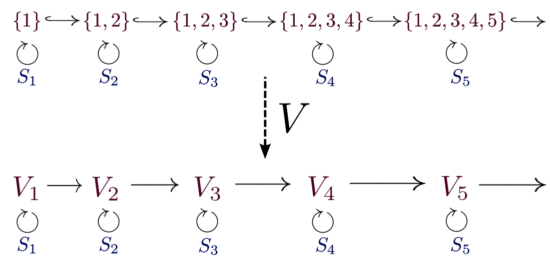



**Figure 21:** An FI-module $V$

We refer to (the morphisms of) the category FI *acting on* an FI-module $V$ in the same sense that a ring $R$ acts on an $R$-module.

We encourage the reader to verify that the following sequences of $S_n$-representations form FI-modules.

- $V_n = \mathbb{Q}$ the trivial $S_n$–representations, $\iota_n$ the identity maps.
- $V_n = \mathbb{Q}^n$, $S_n$ permutes the standard basis, $\iota_n \colon \mathbb{Q}^n \cong (\mathbb{Q}^n \times \{0\}) \hookrightarrow \mathbb{Q}^{n+1}$.
- $V_n = \mathbb{Q}[x_1, \ldots, x_n]$ the polynomial algebra with $S_n$ permuting the variables, $\iota_n$ the inclusion.

Applying any endofunctor of $R$-modules to an FI-module will produce another FI-module, so we can construct more examples by (say) taking tensor products or exterior powers of any of the above.

We leave it as an exercise to the reader to verify that the following sequences of $S_n$-representations do **not** form an FI-module. A hint to this exercise: first verify that if $\sigma \in S_n$ fixes the letters $\{1, 2, \ldots m\}$, then $\sigma$ must act trivially on the image of $V_m$ in $V_n$ under the map induced by the inclusion $[m] \subseteq [n]$.

- $V_n = \mathbb{Q}$ the alternating representation, i.e. $\sigma \cdot v = (-1)^{sgn(\sigma)} v$ for $v \in \mathbb{Q}$, $\iota_n$ the identity map.
- $V_n = \mathbb{Q}[S_n]$ the regular representation, $\iota_n$ induced by the inclusion $S_n \subseteq S_{n+1}$.

Importantly for present purposes, the (co)homology groups of ordered configuration spaces form FI-modules in many cases. If $M$ is any space, there is a contravariant action of FI on its ordered configuration spaces by continuous maps. If we view a point in $F_n(M)$ as an embedding $\rho\colon [n] \to M$, then an FI morphism $f\colon [m] \to [n]$ acts by precomposition,

$$\begin{aligned} f^*\colon F_n(M) &\longrightarrow F_m(M) \\ \rho &\longmapsto \rho \circ f. \end{aligned}$$

See Figure 22.

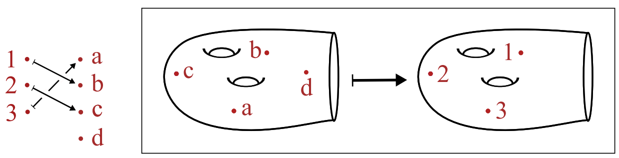



**Figure 22:** An FI morphism and its contravariant action on the configuration spaces $\{F_n(M)\}_n$

Composing this FI action with the (contravariant) cohomology functor gives a **covariant** action of FI on the cohomology groups $\{H^k(F_n(M))\}_n$.

To obtain a covariant action of FI on $\{F_n(M)\}_n$, we need additional assumptions on the space $M$. Let $M$ be the interior of a compact manifold of dimension at least 2 with nonempty boundary. Consider an FI morphism $f\colon [m] \to [n]$ and a configuration in $F_m(M)$. We relabel particles by their image under $f$, and apply the stabilization map of Section 2.4 to introduce any particles not in $f([m])$ in a neighbourhood of a distinguished boundary component. See Figure 23.

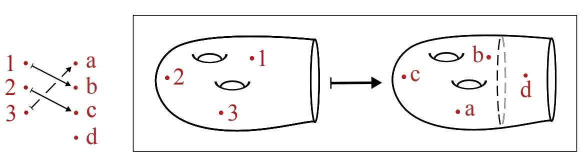



**Figure 23:** An FI morphism and its covariant action on the configuration spaces $\{F_n(M)\}_n$

This action of FI is only functorial up to homotopy, but this suffices to induce a well-defined FI-module structure on the sequence of homology groups $\{H_k(F_n(M))\}_n$.

Modules over the category FI behave in many ways like modules over a ring (technically, they are an abelian category). We define a map of FI-modules $V \to W$ to be a natural transformation, that is, a sequence of maps $V_n \to W_n$ that commute with the FI morphisms. The kernels and images of these maps themselves form FI-modules, and we can define operations like tensor products and direct sums in a natural way. This structure allows us to import many of the standard tools from commutative and homological algebra to the study of FI-modules.

Church, Ellenberg, and Farb [CEF15] showed the answer to Question 5.1 is that the sequences in question are FI-modules that are finitely generated.

**Definition 5.3.** Let $V$ be an FI-module. A subset $S \subseteq \bigsqcup_{n \geq 0} V_n$ *generates* $V$ if the images of $S$ under the FI morphisms span $V_n$ for all $n \geq 0$. Equivalently, the smallest FI-submodule of $V$ containing $S$ is $V$ itself. The FI-module $V$ is *finitely generated* in *degree* $\leq d$ if there is a finite subset of elements $S \subseteq \bigsqcup_{n \leq d} V_n$ that generates $V$.

For example, consider the FI-module $V$ over a ring $R$ such that $V_n = R[x_1, \ldots, x_n]_{(d)}$ is the submodule of homogeneous degree-$d$ polynomials in $n$ variables, $S_n$ acts by permuting the variables, and $\iota_n\colon V_n \to V_{n+1}$ is the inclusion map. We encourage the reader to verify that $V$ is finitely generated in degree $\leq d$. Figure 24 shows a finite generating set when $d = 2$.

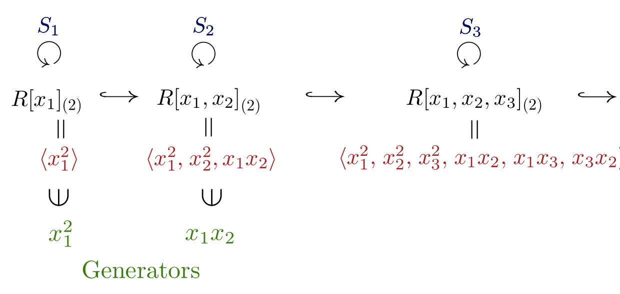



**Figure 24:** A finite generating set for the FI-module $R[x_1, \ldots x_n]_{(2)}$

Another example: from our description of the groups $\{H_1(F_n(\mathbb{C}); \mathbb{Q})\}_n$ in Figure 19, we see that this $\mathsf{FI}$-module is generated by the single element $\alpha_{1,2} \in H_1(F_2(\mathbb{C});\mathbb{Q})$ shown in Figure 25. Arnold's

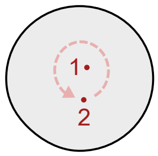



**Figure 25:** The homology class $\alpha_{1,2} \in H_1(F_2(\mathbb{C}))$ generates the FI-module $\{H_1(F_n(\mathbb{C});\mathbb{Q})\}_n$

description of the homology groups of $F_n(\mathbb{C})$ [Arn69] makes it straightforward to verify finite generation of $\{H_k(F_n(\mathbb{C});\mathbb{Q})\}_n$ in every degree $k$.

Church, Ellenberg and Farb [CEF15], and independently Snowden [Sno13] proved that $\mathsf{FI}$-modules over $\mathbb{Q}$ satisfy a *Noetherian* property: submodules of finitely generated modules are themselves always finitely generated. Using this result, Church–Ellenberg–Farb proved that, if $V$ is a finitely generated $\mathsf{FI}$-module, then the sequence $\{V_n\}_n$ of $S_n$–representations stabilizes in several senses.

**Theorem 5.4** (Church–Ellenberg–Farb [CEF15])**.** *Let $V$ be an $\mathsf{FI}$-module over $\mathbb{Q}$, finitely generated in degree $\leq d$. The following hold.*

- **Finite generation.** *For $n \geq d$,*

$$S_{n+1} \cdot \iota_n(V_n) \quad \textit{spans} \quad V_{n+1}.$$

- **Polynomial growth.** *There is a polynomial in $n$ of degree $\leq d$ that agrees with the dimension $\dim_{\mathbb{Q}}(V_n)$ for all $n$ sufficiently large.*

- **Multiplicity stability.** *For all $n \geq 2d$ the decomposition of $V_n$ into irreducible constituents stabilizes (in the sense illustrated in Figure 20).*

- **Character polynomials.** *The character of $V_n$ is independent of $n$ for all $n \geq 2d$.*

The characters of $V$ are in fact eventually equal to a *character polynomial*, independent of $n$; see [CEF15, Section 3.3].

The answer of Question 1.2 for the family $\{F_n(M)\}_n$ is then given by the following result.

**Theorem 5.5** (Church [Chu12]; Church–Ellenberg–Farb [CEF15]; Miller–Wilson [MW19])**.** *Let $M$ be the interior of a compact connected smooth manifold of dimension at least 2 with nonempty boundary. In each degree $k$ the homology and cohomology of ordered configuration spaces $\{F_n(M)\}_n$ of $M$ are finitely generated $\mathsf{FI}$-modules. In particular, the rational (co)homology groups stabilize in the sense of Theorem 5.4.*

Heuristically, Theorem 5.5 states that the homology of $F_n(M)$ is spanned by classes of the form shown in Figure 26.

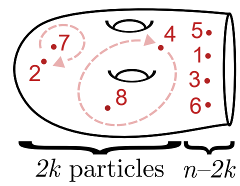



**Figure 26:** A homology class in the image of $H_k(F_{2k}(M);\mathbb{Z})$.

From the $S_n$-covering relationship (Figure 5) it follows that $\dim H^k(C_n(M);\mathbb{Q})$ is equal to the multiplicity of the trivial representation in $H^k(F_n(M);\mathbb{Q})$. Hence Theorem 5.5 implies classical cohomological stability with $\mathbb{Q}$-coefficients for unordered configuration spaces $\{C_n(M)\}_n$. Church [Chu12] used representation stability techniques to prove rational (co)homological stability results for the unordered configuration spaces $\{C_n(M)\}_n$ even in the case that $M$ is a closed manifold, so the isomorphisms are not necessarily induced by natural stabilization maps. See also Randal-Williams [RW13].

## 5.3 Other instances of representation stability

The definition of a finitely generated $\mathsf{FI}$-module makes sense for representations over the integers or other coefficients, even in situations where the representations are not semisimple and multiplicity stability is not well-defined. Moreover, this approach readily generalizes to analogous categories that encode actions by families of groups other than the symmetric groups. Some examples that have been studied are the classical Weyl groups, certain wreath products, various linear groups, and products or decorated variants of $\mathsf{FI}$. The term "representation stability" now refers to algebraic finiteness results (like finite generation or presentation degree) for a module over one of these categories. For further reading on representation stability, see the introductory notes and articles [Far14, Wil18, Sno19, Sam20].

The (co)homology of several families of groups and moduli spaces exhibit representation stability.

**Generalized ordered configuration spaces and pure braid groups.** There is a large and growing body of work on representation stability for the homology of configuration spaces: improving stable ranges, studying configuration spaces of broader classes of topological spaces, or studying alternate stabilization maps. See for example [EWG15, Cas16, Pet17, HR17, Lüt17, KM18, CMNR18, Bah18, MW19, Ram20, Alp20].

Other families generalizing the pure braid groups also have representation stable cohomology groups, including the pure virtual braid groups, the pure flat braid groups, the pure cactus groups, and the group of pure string motions [Wil12, Lee13, JRMD18].

**Pure mapping class groups and moduli spaces of surfaces with marked points**. Given a set of $n$ labelled marked points in a surface $\Sigma$, the *mapping class group* $\mathrm{Mod}^n(\Sigma)$ is the group of isotopy classes of (orientation-preserving if $\Sigma$ is orientable) diffeomorphisms of $\Sigma$ that fix $\partial\Sigma$ and stabilize the set of marked points. The *pure mapping class group* $\mathrm{PMod}^n(\Sigma)$ is the subgroup that fixes the marked points pointwise. These groups also generalize the braid groups since $\mathrm{Mod}^n(\mathbb{D}^2) \cong \mathbf{B}_n$ and $\mathrm{PMod}^n(\mathbb{D}^2) \cong \mathbf{P}_n$. There is a short exact sequence

$$1 \to \mathrm{PMod}^n(\Sigma) \to \mathrm{Mod}^n(\Sigma) \to S_n \to 1$$

that defines an action of $S_n$ on the (co)homology of $\mathrm{PMod}^n(\Sigma)$. Hatcher and Wahl [HW10] proved that the sequence $\{\mathrm{Mod}^n(\Sigma)\}_n$ satisfies homological stability. Jiménez Rolland [JR11, JR15, JR19] proved that the groups $H^k(\mathrm{PMod}^n(\Sigma); \mathbb{Z})$ assemble to a finitely generated $\mathsf{FI}$-module.

For $g \geq 2$ the *moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points* is a rational model of the classifying space $\mathrm{BPMod}^n(\Sigma_g)$, and the symmetric group $S_n$ acts on $\mathcal{M}_{g,n}$ by permuting the $n$ marked points. Hence, the sequence $\{H^k(\mathcal{M}_{g,n}; \mathbb{Q})\}_n$ of $S_n$-representations stabilizes in the sense of Theorem 5.4.

In contrast, for fixed genus $g$ the cohomology groups $H^k(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ of the *Deligne-Mumford compactification of* $\mathcal{M}_{g,n}$ can grow exponentially in $n$. Thus these sequence cannot be finitely generated as $\mathsf{FI}$-modules. Tosteson [Tos21] proved, however, that the sequences $\{H^k(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})\}_n$ are subquotients of finitely generated $\mathsf{FS}^{op}$-modules, where $\mathsf{FS}^{op}$ is the opposite category of the category of finite sets and surjective maps. From this he deduced constraints on the growth rate and on the irreducible $S_n$-representations that occur.

**Flag varieties.** Let $\mathbf{G}_n^{\mathcal{W}}$ be a semisimple complex Lie group of type $A_{n-1}$, $B_n$, $C_n$, or $D_n$, with Weyl group $\mathcal{W}_n$ and $\mathbf{B}_n^{\mathcal{W}}$ a Borel subgroup. The space $\mathbf{G}_n^{\mathcal{W}}/\mathbf{B}_n^{\mathcal{W}}$ is called a *generalized flag variety.* Representation stability of these cohomology groups (as $S_n$- or $\mathcal{W}_n$-representations) has been studied by Church–Ellenberg–Farb [CEF15], Wilson [Wil14], and others.

**Complements of arrangements.** The cohomology of hyperplane complements associated to certain reflection groups $\mathcal{W}_n$ (and their toric and elliptic analogues) stabilizes as a sequence of $\mathcal{W}_n$-representations by the work of Wilson [Wil15] and Bibby [Bib18]. Representation stability holds for the cohomology of more general linear subspace arrangements with a wider class of groups actions; see Gadish [Gad17].

**Congruence subgroups.** Let $K$ be a commutative ring and $I \subseteq K$ a proper two-sided ideal. The *level $I$ congruence subgroups* $\mathrm{GL}_n(K, I)$ of $\mathrm{GL}_n(K)$ are defined to be the kernel of the "reduction modulo $I$" map $\mathrm{GL}_n(K) \to \mathrm{GL}_n(K/I)$. Representation stability of the sequence of homology groups $\{H_k(\mathrm{GL}_n(K, I); \mathbb{Z})\}_n$ (as $S_n$ or $\mathrm{GL}_n(K/I)$-representations) has been studied by Gan– Li [GL19], Putman [Put15], Putman–Sam [PS17], Church–

Ellenberg–Farb–Napgal [CEFN14], Miller–Patzt–Wilson [MPW19], Miller–Nagpal–Patzt [MNP20] and others.

## 6 Current research directions

Work continues on proving (co)homological stability for new families or new coefficients systems, improving stable ranges, and computing the stable and unstable (co)homology for families known to stabilize.

Recently Galatius, Kupers and Randal-Williams [GKRW18a, GKRW18b, GKRW19] identified and proved a new kind of stabilization result, which they describe by the slogan "the failure of homological stability is itself stable". They defined homological-degree-shifting stabilization maps and use them to prove *secondary homological stability* for the homology of mapping class groups and general linear groups. Himes [Him21] studied secondary stability for unordered configuration spaces. Miller–Patzt–Petersen [MPP21] studied stability with polynomial coefficient systems. Miller–Wilson [MW19], Bibby–Gadish [BG20], Ho [Ho20], and Wawrykow [Waw20] studied representation-theoretic analogues of secondary stability for ordered configuration spaces.

For a more in-depth introduction to homological stability and these current research directions, we recommend Kupers' minicourse notes [Kup21] and references therein.

## 7 Acknowledgments

We thank Omar Antolín Camarena, Jeremy Miller, and Nicholas Wawrykow for useful feedback on a draft of this article. We thank our referees for their extensive comments. The authors are grateful to Benson Farb and Tom Church for introducing them to the field of representation stability, and to many of the ideas in this survey. Rita Jiménez Rolland is grateful for the financial support by the CONACYT grant Ciencia Frontera CF-2019/217392. Jennifer Wilson is grateful for the support of NSF grant DMS-1906123.

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