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El teorema de estabilidad de Harer-Ivanov

Tesis que presenta

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Harer-Ivanov's Stability Theorem

Dissertation submitted by

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A Ramón, Monique, Marc, Daniel, Edith:

Los amo.

A Csar:

Sabes lo que yo ignoro

y me dices las cosas que no me digo.

Me aprendo en ti más que en mí misma.

Te amo.

Contents

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Introduction

I think mathematics is a vast territory. The outskirts of mathematics are the outskirts of mathematical civilization. There are certain subjects that people learn about and gather together. Then there is a sort of inevitable development in those fields. You get to the point where a certain theorem is bound to be proved, independent of any particular individual, because it is just in the path of development.

WILLIAM THURSTON.

Given a compact surface it is natural to look at its group of (self-)diffeomorphisms. Indeed, it is of particular interest to consider the group of isotopy classes of such diffeomorphisms: the *mapping class group*. This group, also called *Teichmüller modular group* or simply *modular group*, is fundamental in the theory of Teichmüller spaces and in algebraic geometry. It is also interesting in low-dimensional topology.

Let denote by $S^s_{g,b}$ a compact orientable surface of genus g with b boundary components and s punctures, and by $\Gamma^s_{g,b}$ its mapping class group. More precisely, $\Gamma^s_{g,b}$ (or Γ_S) is the group of isotopy classes of orientation-preserving diffeomorphisms of $S^s_{g,b}$. The diffeomorphisms and isotopies are supposed to be fixed at punctures and to be the identity when we restrict to the boundary of $S^s_{g,b}$. Depending on which conditions are imposed to the diffeomorphisms (for example fixed point-wise or set-wise at the punctures, at the boundary components, etc.), there are variants of this group that are also called mapping class groups. Some of these variants are introduced in Appendix A.

As an example, when the surface is the torus $S^0_{1,0}$, the group Γ_1 is the classical modular group $\mathrm{SL}_2(\mathbb{Z})$ in the theory of elliptic curves. The group $\mathrm{SL}_2(\mathbb{Z})$ is equal to the symplectic group $\mathrm{Sp}_2(\mathbb{Z})$ and has as higher dimensional generalizations both, the special linear groups $\mathrm{SL}_k(\mathbb{Z})$, and the symplectic groups $\mathrm{Sp}_{2k}(\mathbb{Z})$. These are typical examples of a sort of groups called *arithmetic groups.*[1] Alternatively the mapping class group may also be considered as a generalization of the group $\mathrm{SL}_2(\mathbb{Z})$. It is of interest to determine what relation may exist between both generalizations. Ivanov proved that the mapping class group is not arithmetic (see [Iva02]). However, a deep analogy between the mapping class group and arithmetic groups has been identified in the last twenty years and much of the research about mapping class group has been guided by this principle. In particular, the theorem that we are studying in this thesis, the stability theorem, fits nicely as a remarkable instance of this analogy.

In this field, an interesting question about the groups $\mathrm{SL}_k(\mathbb{Z})$ and $\mathrm{Sp}_{2k}(\mathbb{Z})$ is to establish how their homology groups are related with the parameter k. It is remarkable that a phenomenon of stabilization occurs: *The rational homology of* $\mathrm{SL}_k(\mathbb{Z})$ *and* $\mathrm{Sp}_{2k}(\mathbb{Z})$ *stabilizes as k increases.* That means that for k large enough with respect to n, the n-th homology group doesn't depend on k. Following the analogy with arithmetic groups, the natural question is whether the homology of the

[1] For definition and references about arithmetic groups see [Iva02].

mapping class group stabilizes as the genus of the surface goes to infinity. This question becomes of more interest when we look at the relation of the mapping class group with the moduli space of Riemman surfaces.

Let g and s be fixed. For the case $b = 0$, let denote the *Teichmüller space* by $\mathcal{T}_g^s$. It is the space of some equivalence classes of marked Riemman surfaces of genus g with s punctures. The mapping class group Γ_g^s acts properly discontinuously on $\mathcal{T}_g^s$ and the quotient space $\mathcal{M}_g^s = \mathcal{T}_g^s/\Gamma_g^s$ is called the *moduli space of Riemann surfaces*. Its points are the conformal equivalence classes of Riemman surfaces. The remarkable fact is that there is an isomorphism

$$H_n(\Gamma_g^s; \mathbb{Q}) \cong H_n(\mathcal{M}_g^s; \mathbb{Q}),$$

since the space $\mathcal{T}_g^s$ is diffeomorphic to an Euclidean space and the group Γ_g^s is virtually torsion free. More details of this relation between Teichmüller and moduli spaces with mapping class groups are presented in Appendix A.

Hence, answering the stability question for mapping class groups will also give information about the homology groups (and cohomology groups since coefficients are rational) of the moduli space of Riemann surfaces.

This moduli space falls in the intersection of hyperbolic geometry, complex analysis and algebraic geometry. Looking it as a quotient by the action of a group adds a topological and algebraic perspective. An exposition of how topological and geometrical tools have been used for studying this moduli space can be found in the Harer's work [Har88].

The only cases for which these groups have been explicitly computed are when $k = 1, 2$ (see [Har85] and references therein):

$$H_1(\Gamma_{g,b}^s; \mathbb{Z}) \cong 0, \ g \geq 3,$$

$$H_2(\Gamma_{g,b}^s; \mathbb{Z}) \cong \mathbb{Z}^{s+1}, \ g \geq 5.$$

Moreover, knowing the homology groups, or more precisely, the cohomology groups of the mapping class group of a surface is also useful in the problem of classification of surface bundles. A *surface bundle* or S_g*-bundle* is a differentiable bundle with fiber S_g. The structure group of an S_g-bundle is $\mathit{Diff}^+ S_g$, the group of orientation-preserving diffeomorphisms of S_g. For genus 0 and 1 the classification problem is already solved (see [Mor01] for references). In general, a common method for determining whether two S_g-bundles are isomorphic is the use of characteristic classes. These are classes in $H^*(B\mathit{Diff}^+ S_g, A)$, the cohomology ring of the classifying space of $\mathit{Diff}^+ S_g$, which are natural with respect to pull-backs.

For genus ≥ 2 it turns out that

$$B\mathit{Diff}^+ S_g \simeq B\Gamma_g = K(\Gamma_g, 1).$$

Hence, for all $k \geq 0$,

$$H^k(B\mathit{Diff}^+ S_g, A) \cong H^k(\Gamma_g, A),$$

and the characteristic classes of surface bundles are cohomology classes of the mapping class group. The existence of such characteristic classes was proved by Morita (see [Mor01]).

The first result about stability of homology groups for mapping class groups was obtained in 1985 by Harer. In his seminal work [Har85] Harer proved that the homology group $H_n(\Gamma_{g,b}^s; \mathbb{Z})$ is independent of g and b for $g \geq 3n+1$ and that $H_n(\Gamma^s_{g,b};\mathbb{Q})$ is independent of g, b and s for $g \geq 3n$. This implies that, with rational coefficients, the n-th homology group of the moduli of Riemman surfaces doesn't depend on g if $g \geq 3n$.

The restriction on g is called domain of stability. The best domain of stability known is due to Ivanov:

Theorem 0.1. *Let R be a connected subsurface of a connected surface S and let g_R denote the genus of R. The map*

$$i_* : H_n(\Gamma_R;\mathbb{Z}) \to H_n(\Gamma_S;\mathbb{Z}),$$

induced by the inclusion $i : R \hookrightarrow S$, is an epimorphism if

$$g_R \geq 2n+1,$$

and is an isomorphism if

$$g_R \geq 2n+2.$$

If S is a not closed surface, then the map i_* is an epimorphism if

$$g_R \geq 2n,$$

and is an isomorphism if

$$g_R \geq 2n+1.$$

The part of Theorem 0.1 where S is a closed surface is proved in [Iva93], and for S with non-empty boundary the proof is mostly contained in [Iva87], but refers strongly to some results of Harer in [Har85].

The objective of this thesis is to present a complete proof of Theorem 0.1 in the case when the surface S is not closed. This thesis is expository and although it contains no new results, explicit references are included.

From Theorem 0.1 it can be established the stability for closed surfaces: the homology group $H_n(\Gamma_g)$ doesn't depend on S_g provided S_g is closed and g is sufficiently large with respect to n. More precisely,

Corolary 0.2. *If S is a closed surface of genus $g_S \geq 2n+2$, then $H_n(\Gamma_S;\mathbb{Z})$ doesn't depend on S.*

Proof. First notice that if S and S' are closed surfaces of different genus, there is no natural morphism between their mapping class groups Γ_S and $\Gamma_{S'}$.

Consider R and R' diffeomorphic subsurfaces of S and S', respectively, of genus $g = \min\{g_S, g_{S'}\}$. If $g \geq 2n+2$, from Theorem 0.1 it follows that the inclusions $i : R \hookrightarrow S$ and $i : R' \hookrightarrow S'$ induce isomorphisms $i_* : H_n(\Gamma_R;\mathbb{Z}) \to H_n(\Gamma_S;\mathbb{Z})$ and $i'_* : H_n(\Gamma_{R'};\mathbb{Z}) \to H_n(\Gamma_{S'};\mathbb{Z})$. Since $R \simeq S$, the corollary follows. ∎

Since $H_*(\_,\mathbb{Q}) \cong H_*(\_,\mathbb{Z}) \otimes \mathbb{Q}$, Theorem 0.2 applies for coefficients in $\mathbb{Q}$. Hence, it also implies an improvement of the domain of stability for the homology of the moduli space: $H_n(\mathcal{M}_g;\mathbb{Q})$ doesn't depend on g for $g \geq 2n+2$. Indeed, for rational coefficients the domain of stability has been further improved.

Theorem 0.3. *Let R be a connected subsurface of a connected surface S with non-empty boundary. The map*

$$i_* : H_n(\Gamma_R; \mathbb{Q}) \to H_n(\Gamma_S; \mathbb{Q}),$$

induced by the inclusion $i : R \hookrightarrow S$, is an isomorphism if

$$g_R \geq 3n/2,$$

and, for n odd, is an epimorphism if

$$g_R \geq 3n/2 - 2.$$

The proof of this result is due to Harer (see [Iva02]). He also proved that this domain of stability cannot be improved. For integer coefficients this is still an open question. In [Iva93], Ivanov considers a version of the stability theorem for *twisted* coefficients, *i.e.* for coefficients in a non-trivial module.

The aim of this thesis is to present a proof of the stability theorem for mapping class groups. Specifically, Ivanov's improved domain of stability in Theorem 0.1 is proved for the case when the surface S has non-empty boundary.

Basically Ivanonv's ideas from [Iva87] are developed and some techniques that Harer introduced in [Har85]. The proof is presented in three parts: the construction of some complexes of curves on which Γ_S acts; the proof of their high degree of connectedness, and the stability proof by the association of a spectral sequence to each action. Ivanov's proof follows mainly the scheme of Harer's proof. The main difference between their proofs is the specific variants of the complex of curves that they consider and the methods that they follow for proving the high degree of connectedness of these complexes.

It will be showed that surface S in Theorem 0.1 may be obtained successively attaching pairs of pants to a surface R. This allows to restrict attention to the special case when S is obtained by attaching a single pair of pants. When the pair of pants is attached by one boundary component, we will be in the case of "stability by holes" since the number of boundary components is being increased (it corresponds to the inclusion $i : S_{g,b} \to S_{g,b+1}$, with $b \geq 1$). On the other hand, if the pair of pants is attached by two boundary components, the number of handles increases and this case will be referred as "stability by handles" (we have $i : S_{g,b} \to S_{g+1,b-1}$, with $b \geq 2$).

The proof is done using induction on n, the dimension of the homology group, and restricting attention to these two cases. The techniques of Ivanov are combined with those of Harer obtaining a self-contained proof.

The main idea is to consider complexes of curves on which the group Γ_S acts. A *complex of curves* is a collection of curves on the surface with a simplicial structure. Those considered are finite dimensional and highly connected. The first chapter of the thesis deals with the construction of the complexes. For "stability by holes", a complex $B_0(S)$ is defined in such way that if σ_0 is a vertex of $B_0(S)$ and $St(\sigma_0)$ denotes its stabilizer under the action of Γ_S, then the inclusion $St(\sigma_0) \hookrightarrow \Gamma_S$ corresponds to the map $\Gamma_{g,b} \to \Gamma_{g+1,b-1}$ induced by $i : S_{g,b} \to S_{g+1,b-1}$. Analogously, the complex $H(S)$ is defined for "stability by handles" and, for a vertex σ_0 in $H(S)$ the inclusion $St(\sigma_0) \hookrightarrow \Gamma_S$ is the map induced by $i : S_{g,b} \to S_{g,b+1}$.

The special property of these complexes is their high-connectedness which is proved in Chapter 2. The proofs are based on considering more general complexes containing the ones of interest. A more general definition allows the use of induction and some topological and geometrical techniques for proving high-connectedness. The corresponding results for $B_0(S)$ and $H(S)$ are then obtained by combinatorial arguments. In Chapter 1, some auxiliary complexes for this proof are constructed.

The complex $B_0(S)$ is studied following the methods presented by Ivanon in [Iva87]. He considers a more general complex $WB(S)$ and prove its high-connectedness transferring the problem into the context of spaces of functions. As a consequence, using combinatorial arguments firstly introduced by Harer in [Har85], the degree of connectedness is obtained for the subcomplexes $B(S)$ and $B_u(S)$. The result for $B_0(S)$ is then an easy consequence. In the case of $H(S)$, Harer's proof is sketched. It starts with the contractibility of auxiliary complex $AZ(S)$. This fact is proved by Harer using train tracks theory, but we outline a shorter proof due to Hatcher (see [Hat91]). Again Harer's combinatorial arguments imply the high-connectedness of the complexes $BZ(\Delta)$ and $BX(\Delta)$. $H(S)$ is a particular case of the last one.

This is the more technical part of the proof. A further improvement will be to get a unified argument, combining ideas from Harer, Ivanov and Hatcher, for proving in a simple way the high-connectedness for both complexes $B_0(S)$ and $H(S)$.

Chapter 3 is, finally, dedicated to the proof of the stability theorem. The action of Γ_S in highly acyclic complexes will allow to apply an analogous procedure to equivariant homology. For "stabilization by handles", a spectral sequence, which converges to zero, is associated to the action of Γ_S on $B_0(S)$. The degree of connectedness of $B_0(S)$ is enough for expressing the first term of the spectral sequence in terms of stabilizers of simplices in an appropriate range. It turns out that the morphism of interest for the Stability Theorem $i_* : H_n(\Gamma_{g,b}) \to H_n(\Gamma_{g,b+1})$ appears as a differential of this first term. An inductive argument allows to prove injectivity and surjectivity of the morphism. The case of "stabilization by holes" is proved in a similar way using the action of Γ_S on $H(S)$.

At the end of this thesis two appendices are included. Appendix A refers to the relation of mapping class group with the Teichmüller space and the moduli space of Riemann surfaces. Appendix B presents some tools of homology of groups and equivariant homology of groups useful for the construction of the spectral sequence in Chapter 3.

Finally I would like to add a word about one of the principal consequences of the Stability Theorem: the proof of *Mumford's conjecture*. The Stability Theorem opened up the possibility of considering an infinite genus and of defining the *stable cohomology ring* $H^*(\mathcal{M}_\infty;\mathbb{Q})$ by setting $H^n(\mathcal{M}_\infty;\mathbb{Q}) = H^n(\mathcal{M}_g;\mathbb{Q})$ for g sufficiently large. The stability property allows to define $H^*(\mathcal{M}_\infty;\mathbb{Q})$, although there is no actual moduli space $\mathcal{M}_\infty$. David Mumford conjectured that the stable cohomology ring $H^*(\mathcal{M}_\infty;\mathbb{Q})$ (also called *rational cohomology ring of Riemann surfaces*) is a polynomial algebra generated by certain cohomology classes κ_i of dimension $2i$ (these are the characteristic classes of surface bundles that we have mentioned before). As in the finite case, for calculating rational cohomology the stable moduli space of Riemann surfaces is replaced by $B\Gamma_\infty$, where Γ_∞ is the "mapping class group" of an oriented connected surface of "large" genus.

Morita and Miller proved, independently, that the polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, ...]$ maps injectivelly into $H^*(B\Gamma_g;\mathbb{Q})$ in the homological stability range of Harer (see [Mor01]). In [Til97] Tillmann established a relation of $B\Gamma_\infty$ with infinite loop spaces, having as a consequence a stable homotopy version of Mumford conjecture. Madsen and Weiss proved this version of the conjecture in [MW]. All these results rely in Harer's stability theorem and uses it as a fundamental piece. Following this direction, people are studying properties of the classes κ_i and some topological analogues that are called *stable Miller-Morita-Mumford classes* (see for example [GMT06]).

Not only the result is important, but the proof itself is of interest. It is unexpected that the proof of the stability theorem uses purely geometric and topological arguments and no proof using tools of algebraic geometry is known, in spite of the important role of this result in that field. "But, hidden in the proof of the homological stability theorem, there are vast landscapes of 2-dimensional topology and hyperbolic geometry, including Teichmüller theory and much of Thurston's work on surfaces".[2] Let's take a look at it.

[2]Quoting Michael S. Weiss on his review of [MT01].

Chapter 1

Complexes of curves

In this chapter we are concerned with the construction of some complexes of curves associated to a surface. These are simplicial complexes in the sense of [Spa66] that were first constructed by Harvey in [Har81]. Two properties of these complexes are the basis of the stability proof. First, they are geometric objects on which the mapping class group acts simplicially. Second, they have a high degree of connectedness.

In the following, we define the complexes needed in the stability proof and discuss some of their properties.

1.1 Preliminaries

This section outlines the principal results required for understanding the construction of complexes of curves on a surface. Most of the ideas presented here are based on [Thu97] and [Iva02]. Specially [Iva02] has a very comprehensive section of topology of surfaces needed for the study of mapping class groups.

Pants Decomposition

In dimension two, the classification of manifolds is a solved problem. In particular, compact, connected and orientable surfaces are determined, up to a diffeomorphism, by its genus and the number of boundary components. All the surfaces that we are considering here are of this type.

Let denote by $S_{g,b}$ a surface of genus g with b boundary components. A useful model of $S_{g,b}$ is the $4g-$gone with sides identified that defines $S_{g,0}$, with b discs removed. Thus the Euler characteristic relates these two parameters by

$$\chi(S_{g,b}) = 2 - 2g - b.$$

Most surfaces have negative Euler characteristic. The simplest of these surfaces is $S_{0,3}$, usually called a *pair of pants*. Topologically, it is a disc with two holes. The importance of this kind of surface is that they are the "building blocks" for all surfaces with negative Euler characteristic.

Theorem 1.1. *For every surface S such that $\chi(S) < 0$, there exists a collection of disjoint simple closed curves on S such that cutting S along their union C decomposes S into a collection of pairs of pants.*

Using the theorem of classification of surfaces, the collection of curves C can be exhibited. In [Iva02] is presented a proof of Theorem 1.1 using Morse theory.

Even though a pair of pants is very simple, sometimes it is useful to further decompose it. Cutting along the three non-trivial arcs in the Figure 1.1 (b), the result is a decomposition into two topological hexagons.

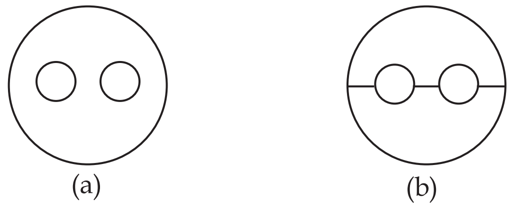


Figure 1.1: *(a) Representation of a pair of pants. (b) Decomposition of a pant in two hexagons.*

The basic objects: circles and arcs

The basic objects in the definition of complexes of curves are circles and arcs on the surface S.

Definition: A *circle* C is a simple closed curve on the surface. It is called *trivial* if it is contractible in S (bounds a disc in S) or homotopic to a boundary component B of S (bounds an annulus together with B). It is a *non-separating circle* if $S \backslash C$ is connected and it is *separating* otherwise.

Definition: An *arc* I is a compact interval embedded in the surface. We say that I is a *properly embedded arc on* S if $\partial I = I \cap \partial S$ and I is transverse to ∂S. An arc is *trivial* if it is homotopic, with fixed boundary, to an arc in a boundary component.

Let C be a circle on S. The surface S can be cut along this curve C, getting a new surface that we will denote by S_C. Gluing along the boundary components resulting from C, S can be recovered from S_C. S can be also cut along a proper arc. Notice that S_C is connected if and only if C is a non-separating curve.

Geometric structure

It will be useful to consider, if possible, a metric structure on the surface.

Definition: A Riemannian metric of constant curvature and geodesic boundary on a surface S is called a *metric or geometric structure on* S.

Notice that, scaling the metric, we can always consider curvatures $+1, 0, -1$. The relevant fact is that every surface admits a geometric structure:

- The sphere $S^2 = S_{0,0}$ has the metric induced by $\mathbb{R}^3$, while the disc $D = S_{0,1}$ can be considered as a hemisphere of the standard sphere.
- The annulus $A = S_{0,2}$ takes a metric product on $S^1 \times [0, 1]$, and the torus $T = S_{1,0}$ inherits the Euclidean metric of its universal cover $\mathbb{R}^2$.
- Surfaces with negative Euler characteristic can be endowed by a geometric structure of negative curvature (hyperbolic).

Theorem 1.2. *If $\chi(S) < 0$, then S admits a hyperbolic structure.*

Proof. This proof takes advantage of the "building blocks" of a negative Euler characteristic surface. We begin endowing a pair of pants with a hyperbolic metric with all boundary components having the same length. Then, by Theorem 1.1, every surface with negative Euler characteristic can be decomposed into pairs of pants with such a geometric structure. Taking isometries as gluing maps, the surface of interest can be reconstructed with a hyperbolic structure induced on it.

As we mentioned, a pair of pants P can be decomposed in two topological hexagons. They can be realized as isometric regular hexagons in the hyperbolic plane $\mathbb{H}$ with angles equal to $\pi/2$. Gluing again by the "cutting curves", we recover the pair of pants with an hyperbolic structure on it and boundary components with twice the length of a side of the hexagon. ■

Another proof is presented in [Iva02]. Basically, the idea is to represent a closed surface by a $4g$-gon and to consider it as a regular polygon on the hyperbolic plane. For surfaces with boundary, an hyperbolic metric is inherited by considering the double of the surface which becomes a closed surface.

In fact, a negative Euler characteristic surface doesn't admit just one hyperbolic structure, but many. A problem that appears immediately is the classification of "equivalent" geometric structures on a surface. This is one of the main problems studied by Teichmüller theory (see Appendix A for the complex approach).

On the other hand, giving a geometric structure to the surface allows to define the complexes of curves in terms of it, instead of in terms of isotopy classes, because of the following result (see [Iva02] for references).

Theorem 1.3. *Let be S a surface with a geometric structure. Then*

(i) Every non-trivial circle on S is isotopic to a geodesic circle on S. If S is endowed with a hyperbolic metric, then such geodesic circle is unique.

(ii) If two non-trivial circles do not intersect, then any two geodesic circles isotopic to them are either disjoint or equal.

1.2 Definition of the complexes of curves

Here certain complexes of curves are defined. For us, the most important of these are the complexes $B_0(S)$ and $H(S)$, since the proof of the stabilization theorem is based on their high- connectedness. We also define other complexes involved in the proof that $B_0(S)$ and $H(S)$ have this property. The results concerning the high connectedness of the complexes will be proved in Chapter 2.

The complex of curves $C(S)$

First we define a basic complex of curves on a surface S that is considered in [Iva87]. Even if this complex is not directly involved in the stability proof that we will present, all the other complexes of curves are defined following it as a model. It is denoted by $C(S)$ and is defined as follows.

Definition: $C(S)$ is the simplicial complex with vertices the set of isotopy classes of non-trivial circles. A set n distinct vertices $\{\gamma_0, \dots \gamma_n\}$ is a n-simplex of $C(S)$ if there exist representatives $C_0, \dots C_n$ of $\gamma_0, \dots \gamma_n$ that are pairwise disjoint.

Remark 1.1: For any non-trivial circle C, representative of a vertex in $C(S)$, its isotopy class will be denoted by $\langle C \rangle$. A set of vertices $\{\langle C_0 \rangle \cdots \langle C_n \rangle\}$ defining an n-simplex will be denoted by $\langle C_0 \cdots C_n \rangle$. In all the other complex of curves, the same notation will be considered.

Equipping the surface S with a geometric structure, by Theorem 1.3, the simplicial complex $C(S)$ can also be defined as follows. The vertices of $C(S)$ are the set of geodesic circles in $S\backslash\partial S$ and the simplices are the set of pairwise non-intersecting geodesics.

Notice the following properties of this complex of curves:

- For positive Euler characteristic surfaces, the annulus and a pair of pants, all the circles are trivial, then the complex $C(S)$ is empty.
- In the torus $S_{1,0}$, each torus knot of type (m,n) is a circle which represents a distinct isotopy class for each positive integer pair $\{m,n\}$ such that $(m,n)=1$. Then it is clear that $C(S_{1,0})$ has an infinite number of vertices. If $S_{1,0}$ is cut along any circle the result is an annulus. Then there are no 2-simplices in $C(S_{1,0})$ and it is zero dimensional.
- Removing a disc from the torus we get $S_{1,1}$, where isotopic curves to torus knots (avoiding the removed disc) represent also an infinite set of vertices of $C(S_{1,1})$. For genus $g\geq 1$, we can always include $S_{1,1}$ in $S_{g,b}$. Hence $C(S_{g,b})$ is also an infinite simplicial complex.
- When $C(S)$ has positive dimension it is always locally infinite.

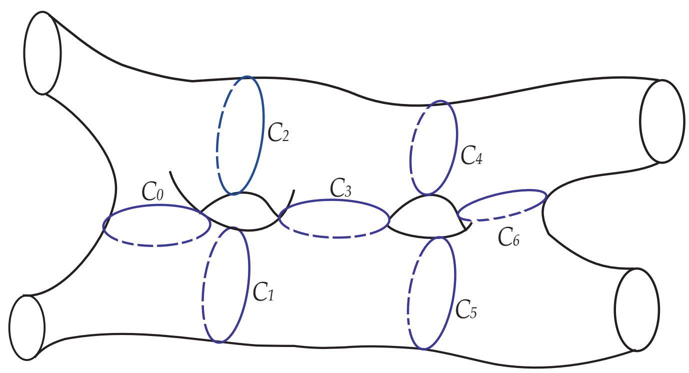


Figure 1.2: *For the surface $S=S_{2,4}$, the simplex $\langle C_0,\cdots,C_6\rangle$ of $C(S)$ is of maximal dimension.*

However this complex of curves is always finite dimensional.

Proposition 1.4. *dim* $C(S_{g,b})=3g-4+b$.

Proof. Consider a surface $S=S_{g,b}$ with $\chi(S)=2-2g-b<0$. Take a simplex $\{\gamma_0,\dots\gamma_n\}$ in $C(S)$ and cut the surface S along pairwise disjoint representatives $C_0,\dots,C_n$ of the vertices of the simplex. Consider each piece after cutting. Note that if a piece is not a pair of pants, there is a non-trivial circle there. It is not isotopic to $C_0,\dots,C_n$, then it can represent a new vertex for having a $n+1$-simplex. Repeating this procedure for the new simplex, we will eventually get a maximal simplex that would result in a pants decomposition of the surface. As each pair of pants has Euler characteristic -1 and χ is additive, S decomposes in $|\chi(S)|$ pairs of pants. Hence $C(S)$ is finite dimensional. Reconstructing the surface S from the pairs of pants we get the exact dimension: the gluing of the pieces is done by the curves C_i along the ones that the surface was cut. There are $b+2g-2$ pieces, that are glued along exactly $b+2g-3$ curves for getting a connected piece. A new gluing is needed for each handle of S. Hence the maximum number of non-trivial circles that defines a simplex is $3g-3+b$. ■

Remark 1.2:

- The complex $C(S)$ corresponds to the auxiliary complex $Z(S)$ that Harer uses for proving the high-connectednes of the complex $X(S)$ ([Har85]).
- It is also the complex that Harvey defines in [Har81].
- The $3g-3+b$ curves that define a maximal simplex of the complex $C(S)$ can be used for defining the Fenchel-Nielsen coordinates of the Teichmüller space $\mathcal{T}_{g,b}$. In fact $\mathcal{T}_{g,b}$is diffeomorphic to $\mathbb{R}^{6g-6+3b}$: $(3g-3+b)+b$ parameters for the length for each curve of the simplex and the b boundary components of S, and $3g-3+b$ parameters for the twist for pasting along the curves (see Appendix A and [Jos97]).

Let $e(S) = -\chi(S)$ and, for any abstract simplicial complex X considered, denote its geometric realization by $|X|$. In [Iva87] the degree of connectedness of $|C(S)|$ is described:

Theorem 1.5.

(i) If S is a closed surface, then $|C(S)|$ is $(e(S)-1)$-connected.

(ii) If S has one boundary component, then $|C(S)|$ is $(e(S)-2)$-connected.

(iii) If S has ≥ 2 boundary components, then $|C(S)|$ is $(e(S)-3)$-connected.

This Theorem corresponds to Theorem 1.2 in [Har85].

The complexes for the stability proof

The hypothesis of the Stability Theorem considers connected surfaces R and S such that S can be obtained from R by successively attaching pairs of pants to one or two components of the boundary. Thus it will be sufficient to prove the result in the case that S is obtained from R by attaching a pair of pants to one boundary component, and the case when S is the result of attaching a pair of pants to two boundary components of R. The complexes $B_0(S)$ and $H(S)$ that we are going to define correspond to these two operation of attaching pairs of pants.

The complex $B_0(S)$: "Stabilization by handles"

Assume $\partial S \neq \phi$. Consider a single component bS of ∂S with a fixed orientation-reversing involution whose action we denote by $z \mapsto \overline{z}$. Choose two points b_0 and b_1 on it of the form z and $\overline{z}$.

Definition: The vertices of the simplicial complex $B_0(S)$ are defined as the isotopy classes $\langle I \rangle$ of arcs I on S joining b_0 with b_1. The isotopies are considered fixed at the ends and we consider non-trivial arcs properly embedded on S.

To define the simplices of $B_0(S)$ we first introduce the complex $B_0'(S)$ with the same vertices as $B_0(S)$. A set of vertices $\{\gamma_0, \ldots, \gamma_n\}$ is an n-simplex of $B_0'(S)$ if there are representatives $I_0, \ldots, I_n$ with pairwise disjoint interiors and such that $S \backslash (I_0 \cup \ldots \cup I_n)$ is connected. Fix an orientation in S and consider the one induced in bS. We can order the intervals in two ways: counterclockwise near b_0 and clockwise near b_1. The simplices of $B_0(S)$ are those of $B_0'(S)$ on which both orders coincide. Thus, $B_0(S)$ is a kind of ordered complex.

Definition: A simplicial complex of dimension n is called *spherical* if it is homotopy equivalent to a bouquet of n-dimensional spheres.

In Chapter 2 it will be proved that

Theorem 1.6. *The space $|B_0(S)|$ is spherical of dimension $g-1$.*

Let γ be a vertex of complex $B_0(S)$ and let I be an arc that represents γ. Let R be the surface resulting from cutting S along I, i.e. $R = S_I$. Then R has genus $g-1$ and $b+1$ boundary components.

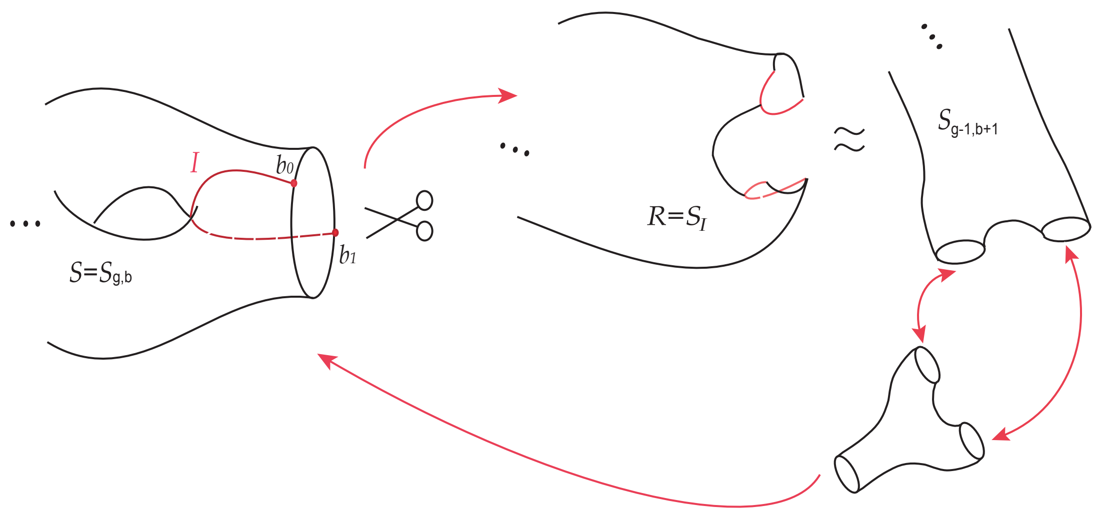


Figure 1.3: *"Stabilization by handles".*

We can recover the surface S from R by attaching a pair of pants to two boundary components of R (see Figure 1.3). Thus we are increasing by one the number of handles. In this case we speak of "stabilization by handles". The complex $B_0(S)$ will be the fundamental object for studying this case. The complex $B_0(S)$ plays a similar role than the complex $A(X)$ in Harer's proof.

The complex $H(S)$: "Stabilization by holes"

Now consider two fixed points b_0 and b_1 of ∂S lying in two different components of the boundary.

Definition: The vertices of the simplicial complex $H(S)$ are isotopy classes of arcs I joining b_0 and b_1. As set of vertices $\{\gamma_0, \ldots, \gamma_n\}$ is an n-simplex of $H(S)$ if $\gamma_1, \ldots, \gamma_n$ have representatives $I_0, \ldots, I_n$ such that the interior are pairwise disjoint and $S \backslash (I_0 \cup \ldots \cup I_n)$ is connected.

Theorem 1.7. *The dimension of $H(S)$ is $2g$*

Proof. The proof will be by induction on g. Consider first $S = S_{0,b}$. Let I_0 be an arc joining b_0 with b_1. Then S_{I_0} is a disc with $b-2$ holes and b_0 and b_1 lie in the boundary of the disc. Thus any other arc joining b_0 and b_1 will disconnect the surface. Then dim $H(S) = 0$. Take now $S = S_{1,b}$. Let I_0 be an arc joining b_0 and b_1. Then S_{I_0} is a surface $S_{1,b-1}$ with b_0 and b_1 lying in the same component of ∂S_{I_0}. Consider another non trivial arc I_1 in S_{I_0}. Cutting the surface along it we will have an annulus with b_0 and b_1 in different boundary components. This is the case that we considered before. Only one more arc I_2 can be considered. Then $H(S)$ has a maximum simplex of dimension 1.

Assume that $S_{g,b}$ has a maximal simplex of dimension $2g$, for all $b \geq 2$. Consider $S = S_{g+1,b}$ and cut it by an arc I_{2g+2}. We get a surface $S_{g+1,b-1}$. Cutting along a new arc I_{2g+1} we obtain a surface $S_{g,b-1}$ that has a maximum $2g$-simplex $\{\langle I_0\rangle, \dots, \langle I_{2g}\rangle\}$. All arcs considered can be chosen pairwise disjoint. ■

The high degree of connectedness of this complex will be essential in the stability proof.

Theorem 1.8. *The space $|H(S)|$ is spherical of dimension $2g$.*

Let I be a representative arc of a vertex γ of $H(S)$. Again take $R = S_I$. Now R will have the same genus as S and one boundary component less.

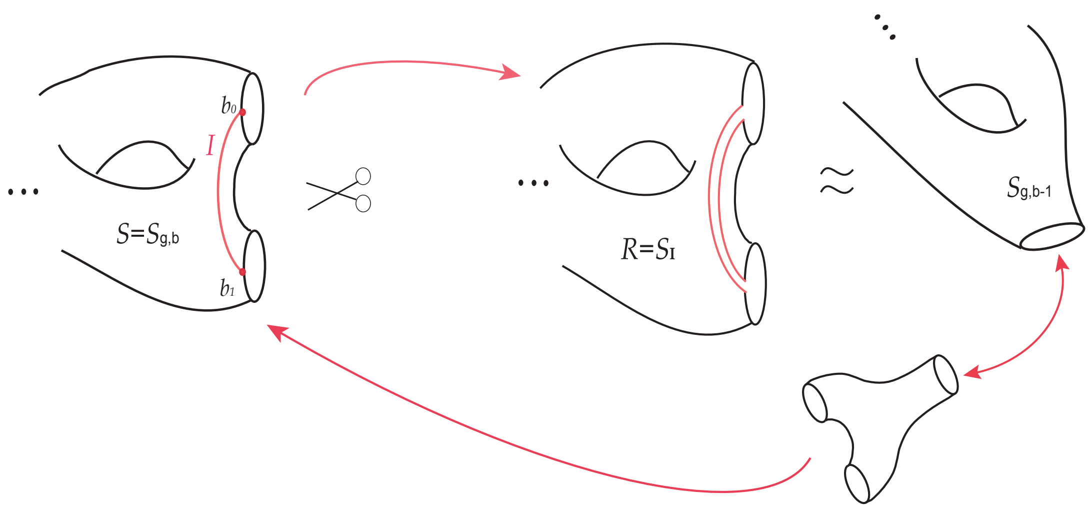


Figure 1.4: *"Stabilization by holes".*

The reverse operation for recovering S from R will be attaching a pair of pants to one boundary component (Figure 1.4). Thus we are increasing by one the boundary components of the surface ("holes"). This is the case of "stabilization by holes". The high connectedness of complex $|H(S)|$ is the clue for the stability proof in this case.

Auxiliary Complexes for $B_0(S)$

In Chapter 2 we present a proof of the high-connectedness of complex $B_0(S)$ proposed by Ivanov in [Iva87]. It depends on the high-connectedness of some auxiliary complexes that we define in the following. They are written as $WB(S)$, $B(S)$ and $B_u(S)$.

The complex of curves WB(S)

In order to define this complex we consider that S is connected and has a non-empty closed submanifold that we denote by bS. Often $bS \subset \partial S$. Assume that in each component of bS there is a fixed orientation-reversing involution whose action we denote by $z \mapsto \overline{z}$.

Definition: *WB(S)* is the simplicial complex with two kind of vertices:

- The white vertices are isotopy classes of non-trivial circles, i.e. the vertices of $C(S)$.
- The blue vertices are isotopic classes of arcs I on S, that we denote by $\langle I\rangle$. The isotopies of arcs are assumed to be connected at the ends and the arcs I satisfies: (i) I is properly embedded on S, and (ii) ∂I lies in one component of bS and has the form $\{z, \overline{z}\}$.

A set of vertices $\{\gamma_0, \dots \gamma_n\}$ is a n-simplex of *WB(S)* if there exist pairwise disjoint representatives.

Remark 1.3: It is clear that $C(S)$ is a subcomplex of $WB(S)$. In fact, it is a full sumbcomplex, i.e. if a set of vertices of $C(S)$ is a simplex in $WB(S)$, then it is also a simplex in $C(S)$.

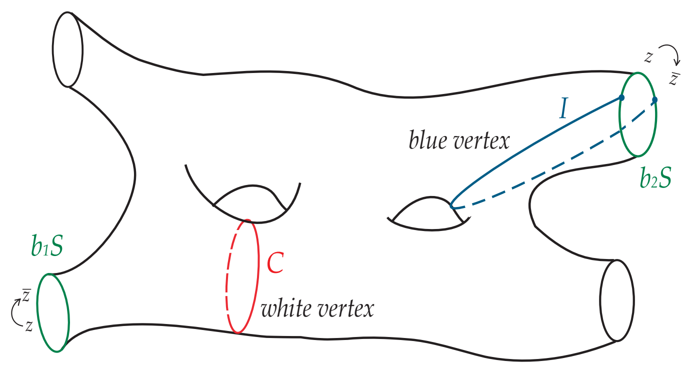


Figure 1.5: *Blue and white vertices of the complex of curves $WB(S)$.*

We will see in Chapter 2 that the geometric realization $|WB(S)|$ is $(e(S)-2)$-connected.

The complexes of curves $B(S)$ and $B_u(S)$

Definition: The complex $B(S)$ is the full subcomplex of $WB(S)$ whose vertices are the blue vertices of $WB(S)$.

Definition: The complex $B_u(S)$ has the same set of vertices of $B(S)$, but the simplices are defined differently. A set of blue vertices $\{\gamma_0, \dots \gamma_n\}$ is a n-simplex of $B_u(S)$ if there exist pairwise disjoint representatives $I_0, \dots, I_n$ that are pairwise non-intersecting and such that $S \backslash (I_0 \cup \dots \cup I_n)$ is connected[1].

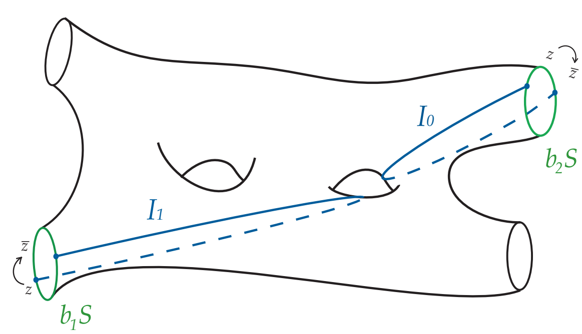


Figure 1.6: *The curves I_0 and I_1 define a 1-simplex in $B(S)$, but not in $B_u(S)$ because they disconnect the surface.*

As for $C(S)$, we can consider the definition of these complexes in terms of geodesics equipping the surface S with a geometric structure. The white vertices are geodesic circles in $S \backslash \partial S$ and the

[1] It corresponds to the notion of cut system in [Har85]

blue vertices are geodesic intervals satisfying conditions (i) and (ii). The simplices are the sets of pairwise non-intersecting geodesics.

Let S be a surface of genus g. It will be of interest knowing the dimension of the complex $B_u(S)$.

Proposition 1.9. *The dimension of $B_u(S)$ is $g-1$.*

Proof. Recall that representatives of vertices of $B_u(S)$ are non-trivial arcs I with $\partial I = \{z, \overline{z}\}$ in just one component of bS and such that don't disconnect the surface. Thus an arc that cut a handle of the surface defines a vertex. Arcs cutting different handles are non-isotopic and we can choose disjoint representatives. Moreover, for different handles the arcs don't disconnect the surface, while more than one arc cutting the same handle do. Hence in relation with arcs cutting handles, we get at most a $(g-1)$-simplex. Cutting the surface along disjoint representatives of the vertices of this simplex, we get a connected surface of genus 0. No new arcs with the above conditions can be considered. Thus it is a maximal simplex. ∎

Apropos the homotopy type of these complexes it can be proved that $|B(S)|$ is $(e(S)-2)$-connected and the space $|B_u(S)|$ is $(g-2)$-connected. Moreover $|B_u(S)|$ is spherical of dimension $g-1$.

Notice that the complex $B'_0(S)$ is a special case of complex $B_u(S)$, with bS a single component of ∂S. The degree of connectedness of complex $B_0(S)$ will be obtained by noticing that $B_0(S)$ is subcomplex of dimension $g-1$ of $B'_0(S)$ and all maximal simplices are of dimension $g-1$. The general case allows an inductive proof for the degree of connectedness of $|B(S)|$, using the high-connectedness of complex $|WB(S)|$.

Auxiliary Complexes for $H(S)$

The degree of connectedness of complex $H(S)$ is obtained, in Chapter 2, by following the proof of Harer in [Har85] and a simplification due to Hatcher ([Hat91]). Again some auxiliary complexes are defined: $AZ(\Delta)$, $BZ(\Delta, \Delta^0)$ and $BX(\Delta, \Delta^0)$.

The complex of curves AZ(Δ)

Let $\Delta = \{p_1, \dots, p_q\}$ a non-empty collection of points in ∂S, not necessarily in distinct components. We denote the order of Δ as q. Let ∂' be the union of the components in ∂S which contains points of Δ and let b' be the number of components of ∂'.

Definition: An *arc based in* Δ is

(a) the isotopy class, relative to Δ, of a C^∞ path imbedded in S connecting two points in Δ and meeting ∂S only at its endpoints, or

(b) a C^∞ loop imbedded in S based at a point of Δ and meeting ∂S only at this point.

Following Harer we define the simplicial complex $AZ(\Delta)$ as follows:

Definition: A vertex of the simplicial complex $AZ(\Delta)$ is an arc α based in Δ, non-isotopic (rel Δ) to a point or into an arc of $(\partial S - \Delta) \cup \{\text{end points of } \alpha\}$. A k-simplex is a set of vertices $\{\gamma_0, \cdots, \gamma_k\}$ with non-isotopic (rel Δ) representatives $\alpha_0, \cdots, \alpha_k$ such that $\alpha_i \cap \alpha_j \subset \Delta$, for all $i \neq j$.

The main characteristic of this complex is again its degree of connectedness. It is stated in Theorem 1.5 of [Har85], where section 2 is devoted to its proof using train tracks theory. However,

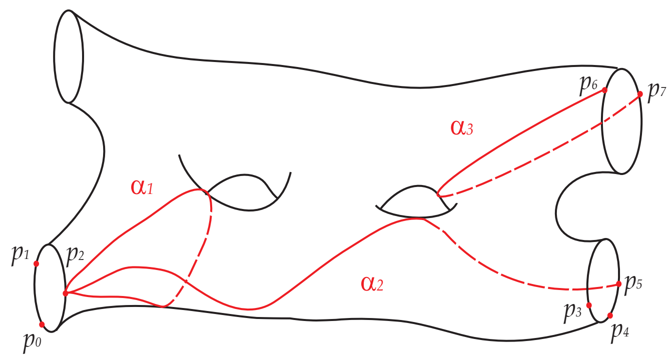


Figure 1.7: *Some arcs based on* $\Delta = \{p_0, p_1, \cdots, p_7\}$.

Hatcher, in [Hat91], considers a more general case of complexes, which includes $AZ(\Delta)$, and proves they were contractible using more elementary machinery.

The complexes of curves $BZ(\Delta, \Delta^0)$ *and* $BX(\Delta, \Delta^0)$

Consider now a non empty proper subset Δ^0 of Δ.

Definition: We define the simplicial complex $BX(\Delta, \Delta^0)$ as the subcomplex of $AZ(\Delta)$ consisting of those simplices $\langle \beta_0, \dots, \beta_k \rangle$, where each β_i connects a point of Δ^0 to one of $\Delta^1 = \Delta - \Delta^0$. Moreover, we ask that $S - \{\beta_0, \cdots, \beta_k\}$ is connected.

If $\Delta = \{p_1, p_2\}$, $\Delta^0 = \{p_1\}$ and p_1 and p_2 lie on distinct components of ∂S, we have the complex denoted by Harer as BX. Notice that it corresponds to the complex $H(S)$, previously defined (Ivanov's notation).

If $\Delta^1 \neq \phi$ and $\Delta^0 \neq \phi$, then we can take in $BX(\Delta, \Delta^0)$ a maximum simplex of dimension $2g - 2 + b'$. Hence, dim $BX(\Delta, \Delta^0) = 2g - 2 + b'$. Theorem 1.4 in [Har85] establishes that $|BX(\Delta, \Delta_0)|$ is spherical of dimension $2g - 2 + b'$. In particular, $|H(S)|$ is a wedge of spheres of dimension $2g$. The proof is based on the connectedness of a more general complex.

Definition: The auxiliary complex $BZ(\Delta, \Delta_0)$ is the same as $BX(\Delta, \Delta_0)$, but allowing the families of arcs to separate S.

The high degree of connectedness of this complex is established in Theorem 1.6 in [Har85] as a consequence of Theorem 1.5 and it is used for proving the corresponding result for complex $|BX(\Delta, \Delta_0)|$.

Chapter 2

The degree of connectedness of the complexes

A special feature of the complexes of curves is their high-connectedness. The aim of this chapter is proving the high degree of connectedness for the complexes of the stability proof: $|B_0(S)|$ and $|H(S)|$. Following a construction from Ivanov, that uses ideas going back to Cerf, we determine the degree of connectedness of $|B_0(S)|$. On the other hand, we outline Harer's combinatorial proof for complex $|H(S)|$. It is based on the contractibility of complex $|AZ(\Delta)|$. For this last one, a topological proof of Hatcher is sketched, instead of Harer's proof using Thurston's theory of train tracks. In both approaches, it is considered a larger complex having the complex of interest as a particular case. With enough generality a proof by induction becomes feasible.

2.1 The complex of curves $|B_0(S)|$

T get the degree of connectedness of $|B_0(S)|$, we will first prove the high-connectedness of complex $|WB(S)|$, and of complexes $|B_u(S)|$ and $|B_0(S)|$ as a consequence. Ivanov's proof is based on the application of families of functions to the study of diffeomorphisms (Cerf's theory [Cer70]). Here we reproduce this proof and determine the degree of connectedness of the complex of curves $|WB(S)|$. The main idea is having a two-direction assignment between families of functions and simplicial mappings of a particular finite complex to the complex of curves. Thus, the study of the high-connectedness for the complex of curves is changed to a problem of connectedness in the space of functions (which is contractible) and then, the conclusion is returned to the original context. Ivanov called this assignment "the fundamental construction". Later, following Harer's combinatorial methods, the degree of connectedness of complexes $|B_u(S)|$ and $|B_0(S)|$ is gotten.

The fundamental construction

Let bS be as in the definition of complex $WB(S)$ and let $b_1S, \ldots, b_nS$ be its components. Consider a collar T_k in S of b_kS, such that the T_k's are pairwise disjoint. Identify T_k with the annulus $A_k = \{z \in \mathbb{C} : 1/2 \leq |z - 3k| \leq 1\}$ in such a way that b_kS correspond to $\{z \in \mathbb{C} : |z - 3k| = 1\}$ (see Figure 2.1). Notice that these annulus are pairwise disjoint in $\mathbb{C}$. Let the involution on each b_kS correspond to the complex conjugation.

We are going to consider non-negative smooth functions $f : S \to \mathbb{R}$, such that $f(z) = \mathrm{Re}\, z$ for all $z \in T_1 \cup \ldots \cup T_n$ and $f(\partial S \backslash bS) = 0$, where zero is a regular value of f. Take the space of such functions with the C^∞-topology and denote it by $\mathcal{E}$.

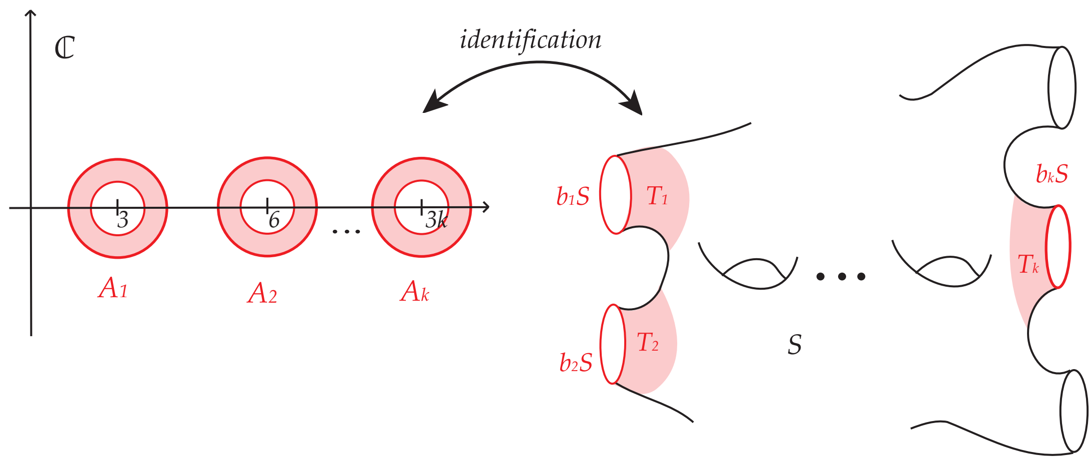


Figure 2.1: *Identification of the collars $T_k \subset S$ with the annulus $A_k \subset \mathbb{C}$.*

Definition: Let $a \in \mathbb{R}$. A component C of $f^{-1}(a)$ that does not contain a critical point of f is called *non-singular*.

Notice that for $a \in \mathbb{R}$, $f^{-1}(a) \subset S$ is a compact subspace of S. If C is a non-singular component of $f^{-1}(a)$ then C is a submanifold of codimension 1. Thus C is a circle or an arc. The identification of the neighborhoods T_k and the definition of f on it as $f : z \mapsto Rez$, guarantees that, if the arc I is a component of $f^{-1}(a)$, with $a =$Re z, for some $z \in A_k$, then I is properly embedded and ∂I lies on of $b_k S$ with the form $\{z, \overline{z}\}$.

Definition: A function f is called *non-degenerate* if there is a non-trivial circle or arc among the non-singular components at its level sets.

For the construction we are going to assume the following statement from [Iva87]:

(A) Let C be a non-singular component of the level set $f^{-1}(a)$ of a function f. If g is C^∞-close to f, then there is a non-singular component of $g^{-1}(a)$ isotopic to C.

I. From functions to circles

We begin with a family of non-degenerate functions and we assign to it a certain simplicial complex C together with a simplicial mapping $C \to WB(S)$.

Let $\{f_t : S \to \mathbb{R}\}_{t \in P}$ be a family of non-degenerate functions of $\mathcal{E}$. P is a parameter space such that nearness between points in P corresponds to nearness between the functions associated. Then for all $t \in P$, there exist $a_t \in \mathbb{R}$ and C_t, non-singular component of $f_t^{-1}(a_t)$, such that C_t is a non-trivial component. By the assumption (A), there is a small neighborhood U_t of $t \in P$ such that for all $u \in U_t$ there exists a non-singular component C_{t_u} of $f_u^{-1}(a_t)$ isotopic to C_t.

Definition: Let $\{U_t\}_{t \in V}$ a covering of the space P. The *nerve of the cover* is the simplicial complex whose vertices are the elements of V and we consider that a set $\{t_0 \dots t_n\} \subset V$ forms a n-simplex if $U_{t_0} \cap \dots \cap U_{t_n} \neq \phi$.

Consider the open covering $\{U_t\}_{t\in P}$ of P. Take a subcover $\{U_t\}_{t\in V}$, finite if possible, and let C be its nerve.

Definition: The map defined on vertices by

$$\begin{array}{rccc} \psi: & C & \longrightarrow & WB(S) \\ & t & \mapsto & \langle C_t \rangle \end{array}$$

is called a *realization of the family* $\{f_t\}_{t\in P}$.

Proposition 2.1. *The realization ψ is a simplicial map.*

Proof. Let Z be a simplex of C, then $Z \subset V$ and $\bigcap_{t\in Z} U_t \neq \phi$. Take $u \in \bigcap_{t\in Z} U_t$, then $\forall\, t \in Z$, C_{t_u} is isotopic to C_t. Thus $\{\langle C_{t_u}\rangle : t \in Z\} = \{\langle C_t\rangle : t \in Z\} = \psi(Z)$. Moreover, the C_{t_u} are components of the level sets of a unique function f_u. Thus two of them are disjoint or coincide. Hence $\psi(Z) = \{\langle C_{t_u}\rangle : t \in_Z\}$ is a simplex of $WB(S)$. ■

Now consider $Q \subset P$ and $\psi_Q : C_Q \to WB(S)$ a realization of the subfamily $\{f_t\}_{t\in Q}$. It will be desirable to have a realization $\psi_P : C_P \to WB(S)$ of $\{f_t\}_{t\in P}$ that extends the previous one: $C_Q \subset C_P$ and ψ_P is an extension of ψ_Q. We can construct it as follows. Let $\{U_t\}_{t\in V}$ be the cover of Q whose nerve is C_Q and C_t the components of $f^{-1}(a_t)$ that define ψ_Q. Every U_t has the form $U'_t \cap Q$, where U'_t is an open set of P. Choose U'_t in a way such that for all $u \in U'_t$, $f^{-1}(a_t)$ has a non-singular component isotopic to C_t. Finally C_P is constructed using any cover containing $\{U'_t\}_{t\in V}$ and the components C_t for defining the realization ψ_P that extends ψ_Q.

II. From circles to functions

Now we deal with the reverse situation. For a finite complex C and any simplicial mapping

$$\psi : C \to WB(S),$$

we construct a family of functions whose realization corresponds to this mapping.

Take the geometric realization of C as parameter space, i.e. $P = |C|$.

Definition: Let Z be a simplicial complex. The *closed star of a simplex* Σ *of* Z is the union of all closed simplices Σ' of Z such that Σ is a face of Σ'.

Let denote by St_v the closed star of a vertex v in the first barycentric subdivision of C. $\{St_v\}_{v\in V}$ is a closed cover of P, where V is a set of vertices whose nerve coincide with C. For all $v \in V$ consider sufficiently small neighborhoods U_v of St_v in such a way that the nerve of $\{U_v\}_{v\in V}$ also coincides with C. We use this cover for constructing the family $\{f_t : S \to \mathbb{R}\}_{t\in|C|}$.

Consider the definition of $WB(S)$ in terms of geodesics. Assume that $\psi(v)$ is the geodesic C_v. Since C is a finite complex, we can choose a closed neighborhood N_v for each component C_v so that
$N_v \cap N_w = \phi$ when $C_v \cap C_w = \phi$,
$N_v \cap \partial S = \phi$ if C_v is a circle and,
$N_v \cap (\partial S \backslash b_k S) = \phi$, if C_v is an arc with $\partial C_v \subset b_k S$.

For each component $b_k S$ of bS choose a collar T_k such that $T_k \cap T_l = \phi$ for all $k \neq l$, and $T_k \cap N_v = \phi$ for all $v \in V$ such that C_v is a circle. For each k consider, as before, an identification of T_k with the annulus A_k and the map $h_k : z \mapsto \text{Re } z$. Notice that this identification may be done in a way that for each arc C_v with $\partial C_v \subset b_k S$, $C_v \cap T_k$ corresponds, for some $z \in A_k$, to $\{x = \text{Re } z\} \cap A_k \subset \mathbb{C}$ as in Figure 2.3.

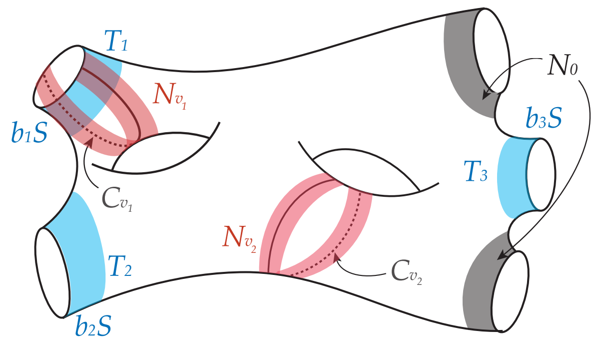


Figure 2.2: *Neighborhoods considered for the construction of the family* $\{f_t\}_{t\in V}$.

For each $v \in V$ such that C_v is a circle, choose a smooth positive function $g_v : N_v \to \mathbb{R}$ with some $a_v \in (3n+1, \infty)$ a regular value so that $g_v^{-1}(a_v) = C_v$ (identify, for example, the neighborhood N_v with some annulus in $\mathbb{C}^+$ disjoint to the annuli A_k, and take the projection into the real part). Make the election in a way such that $a_v \neq a_w$ if $C_v \cap C_w = \phi$. When C_v is an arc whose boundary lies in $b_k S$, choose a positive smooth function $g_v : N_v \to \mathbb{R}$ so that $g_v|_{N_v \cap T_k} = h_k|_{N_v \cap T_k}$ and with $a_v = h_k(C_v \cap T_k) \in \mathbb{R}$ a regular value of g_v. Finally, choose a closed neighborhood N_0 of $\partial S \backslash bS$ and a non-negative smooth function $g_0 : N_0 \to \mathbb{R}$, with zero as a regular value, such that N_0 doesn't intersect any of the neighborhoods considered before.

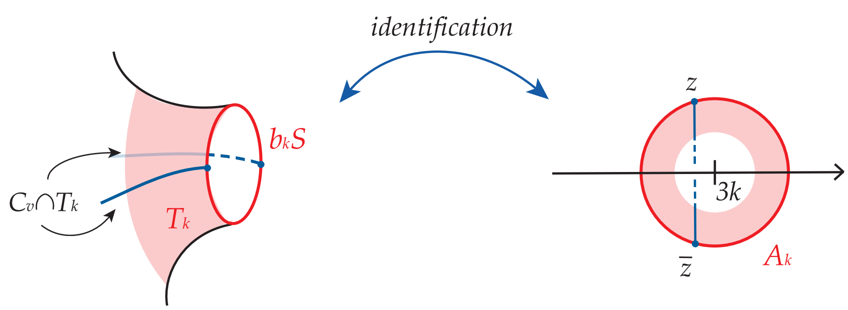


Figure 2.3: *Choose an identification between* T_k *and* A_k *in a way that* $C_v \cap T_k \approx \{x = Re\ \} \cap A_k$.

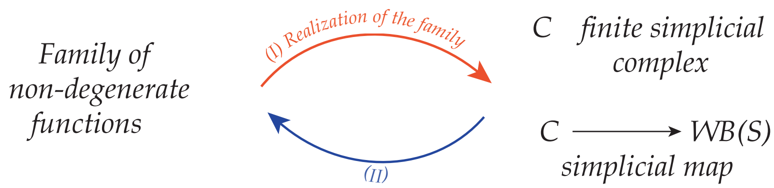
Consider the closed set $K = \bigcup_{v\in V}(St_v \times N_v) \cup \bigcup_{k=1}^{n}(|C| \times T_k) \cup (|C| \times N_0) \subset |C| \times S$ and define the map $G : K \to \mathbb{R}$ as

$$G(t,x) = \begin{cases} g_0(x), & \text{if } (t,x) \in |C| \times N_0 \\ g_v(x), & \text{if } (t,x) \in St_v \times N_v \\ h_k(x), & \text{if } (t,x) \in |C| \times T_k \end{cases}$$

Note that it is well defined. If $St_v \cap St_w \neq \phi$, then $\{v, w\}$ is a simplex in C and $C_v \cap C_w = \phi$, and so $N_v \cap N_w = \phi$. Moreover, if $(t, x) \in (t, x) \in (St_v \times N_v) \cap (|C| \times T_k)$, then $x \in T_k \cap N_v$, where g_v coincides with h_k.

The family that we are looking for results from extending G to some function $F : |C| \times S \to \mathbb{R}$ in such a way that all functions $f_t : x \mapsto F(x, t)$, $t \in |C|$ are smooth.

Because of the construction, the realization of such family coincides with the original mapping ψ as we wanted. We have constructed the two-direction association:



Now we can use the fact that the space of functions $\mathcal{E}$ is contractible to conclude something about the connectedness of $|WB(S)|$.

Theorem 2.2. *Let $m \in \mathbb{N}$ such that any family of functions $\{f_t : S \to \mathbb{R}\}_{t \in P}$, with $\dim P \leq m$, can be approximated by a family of non-degenerate functions. Then $|WB(S)|$ is $(m-1)$-connected.*

Proof. Let $n \leq m-1$ and $g : S^n \to |WB(S)|$ any continuous mapping. We shall prove that g is homotopic to the constant map. Take a triangulation of S^n and identify it with the geometric realization of some simplicial complex C, i.e. $|C| \simeq S^n$. Consider a triangulation sufficiently fine for g be homotopic to the geometric realization of some simplicial mapping $h : C \to WB(S)$. From the fundamental construction (II), h is the realization of some family $\{f_t : S \to \mathbb{R}\}_{t \in |C|}$. Since the space of functions is contractible, this family can be extended to a family $\{f_t\}_{t \in D^{n+1}}$, where D^{n+1} is a ball with boundary $|C| \simeq S^n$. Thus $\{f_t\}_{t \in D^{n+1}}$ is a family with $\dim D^{n+1} = n+1 \leq m$ and by hypothesis can be approximated by a family of non-degenerate functions $\{f'_t\}_{t \in D^{n+1}}$. For a good approximation h is the realization of $\{f'_t\}_{t \in |C|}$.

As we already saw in the part (I) of Ivanov's fundamental construction, we can construct a realization $\hat{h} : D \to WB(S)$ of $\{f'_t\}_{t \in D^{n+1}}$, with $|D| = D^{n+1}$, extending the realization h of the subfamily $\{f'_t\}_{t \in |C|}$. Hence $|\hat{h}|$ extends $|h| \simeq g$ and g is nullhomotopic. ■

From Theorem 2.2, the degree of connectedness of $|WB(S)|$ will be specified if, for a given surface S, we determine an integer m in such a way that any d-parameter family of functions, with $d \leq m$, can be approximated by a non-degenerate family. We establish this integer in the following.

Approximation by non-degenerate functions

Finite order critical points

Definition: Let $0 \in \mathbb{R}^n$ be a critical point of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Consider the quotient of $C^\infty(x_1 \ldots, x_n)$, the ring of germs of smooth functions of $x_1 \ldots, x_n$ at 0, by the ideal generated by $\frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f$, that is

$$Q = \frac{C^\infty(x_1 \ldots, x_n)}{\langle \frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f \rangle}.$$

Q is called the *local ring of the smooth function f at the critical point* 0. The *order*[1] μ of f at the critical point 0 is defined by $\mu = dim_{\mathbb{R}} Q$.

Example. The smooth function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ has order $\mu = 2$ at the critical point $x = 0$.

[1] Multiplicity in [Arn73]

Note that we can define μ even if 0 is not a critical point. In this case $\mu = 0$. We can say that the order of non-critical point is zero.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The next result follows from some propositions proposed in [Arn73].

Proposition 2.3. *If* 0 *is a finite order critical point of* f*, then there exists a sufficiently large* k *such that the Taylor polynomial, up to terms of degree* k*, is differentiably equivalent to* f*. That means that there is a change of coordinates such that* f *can be represented locally, near the critical point, as a polynomial.*

Remark 2.1: It follows that finite order critical points are isolated.

The next result is also established in [Arn73]:

Proposition 2.4. *Any finite-parameter family of functions can be approximated arbitrary well by a family of functions having only critical points of finite order.*

So we restrict our attention to functions with critical points of finite order. Note that these functions are "locally simple":

- Linear in a non-critical point.
- Polynomial in neighborhoods of finite order critical points.

Hence, level sets of this kind of functions have the same local structure as the level set of polynomials. Since we are interested in smooth functions in two variables, the level sets correspond to algebraic curves.

Branch number of critical points

Viewing level sets as algebraic curves we can define an algebraic invariant for finite order critical points.

Definition: A *branch* passing through a point $x \in \mathbb{R}^2$ is the image of a real analytic topological embedding $b : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that $b(0) = x$. The images of the half intervals $[0, \epsilon)$ and $(-\epsilon, 0]$ are called *half-branches.*

Take $x \in \mathbb{R}^2$ a non-isolated point of an algebraic curve $V \subset \mathbb{R}^2$. There exists a neighborhood of x, where V is the union of a finite number of branches intersecting only in x.

Example. (Nodal cubic) Let $V = \{(x, y) \in \mathbb{R}^2 : y^2 = x^3 + x^2\}$. This curve has a non injective parametrization $\phi : \mathbb{R} \to \mathbb{R}^2$ given by $\phi(t) = (t^2 - 1, t^3 - t)$. The point $(0, 0) \in V$ is not isolated and in a neighborhood of $(0, 0)$, V is the union of the branches

$$b_1 : (-\tfrac{1}{2}, \tfrac{1}{2}) \xrightarrow{+1} (\tfrac{1}{2}, \tfrac{3}{2}) \xrightarrow{\phi} \mathbb{R}^2$$

$$b_2 : (-\tfrac{1}{2}, \tfrac{1}{2}) \xrightarrow{-1} (-\tfrac{3}{2}, \tfrac{1}{2}) \xrightarrow{\phi} \mathbb{R}^2$$

In any other point of V, there is only one branch.

Since our level sets are algebraic curves, we can consider the branches through a critical point.

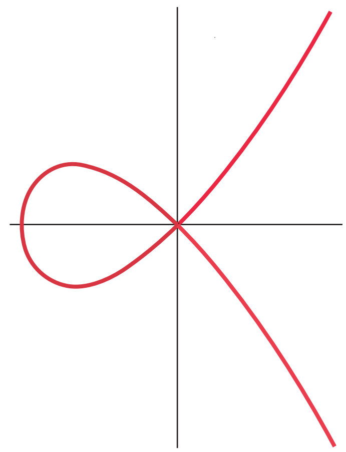


Figure 2.4: *Nodal cubic.*

Definition: The *branch number of a critical point* is the number of branches of a level set passing through this critical point in the neighborhood of this point.

The next result shows how the branch number of a critical point contains information about the local expression of the function around this point. That is why we called it an invariant.

Definition: Let $C^\infty(\mathbb{R}^2, \mathbb{R})$ be the vector space of smooth functions $f : \mathbb{R}^2 \to \mathbb{R}$. Let k be a non-negative integer, and let p be a point of $\mathbb{R}^2$. Declare that two functions f and g in $C^\infty(\mathbb{R}^2, \mathbb{R})$ are equivalent to order k if f and g have the same value at p, and all of their partial derivatives agree at p up to (and including) their k-th order derivatives. The *k-th order jet space of $C^\infty(\mathbb{R}^2, \mathbb{R})$ at p* is the set of such equivalence classes, and is denoted by $J^k_p(\mathbb{R}^2, \mathbb{R})$. The equivalence class of a function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ in $J^k_p(\mathbb{R}^2, \mathbb{R})$ is called the *k-th order jet of f at p*.

Recall that the *jet* takes a differentiable function f and produces a polynomial, the truncated Taylor polynomial of f, at each point p of its domain.

Lemma 2.5. *Let 0 be a critical point of finite order of a smooth function $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, with branch number $\geq n$. Then the $(n-1)-$jet of f at 0 is equal to 0, that is, $\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(0) = 0$ for $i + j \leq n - 1$.*

Proof. Since the order doesn't depend on the choice of coordinates, we can make any change of coordinates in $(\mathbb{R}^2, 0)$. Assume that f is a polynomial. By another change of coordinates we can ensure that the open right half-plane contains at least half of the half-branches of $f^{-1}(0)$ in a neighborhood of 0. Let the image of $b_i : [0, \epsilon) \to \mathbb{R}^2$ be the i-th half-branch. In a neighborhood of zero say $b_i(t) = a_{i1}t + a_{i2}t^2 + \ldots$ Denote by a_i the first non zero coefficient in the series above. A linear change of coordinates ensures that at least half of the vectors a_i, and then at least half ($\geq n$) of the half-branches, lie in the open right half plane.

Let m be the degree of f and $f(x, y) = \sum_{i+j \leq m} a_{ij} x^i y^j$ in a suitable neighborhood of 0. We will consider another change of coordinates such that f has a local expression with non zero coefficients for y^m at zero. Consider a change of the form $(x, y) \mapsto (x + \gamma y, y)$. Then we have $\sum_{i+j \leq m} a_{ij}(x + \gamma y)^i y^j$ and the coefficient of y^m at $x = 0$ is $\sum_{i+j=m} a_{ij}\gamma^i$. Choose a γ that guarantees that

$\sum_{i+j=m} a_{ij}\gamma^i \neq 0$. Take for example

$$\gamma = \frac{|a_{ij}|}{2}\left(\frac{1}{\sum_{\substack{k+l=m\\ k>i}} |a_{kl}| + 1}\right) \wedge \frac{1}{2},$$

where $i = min\{k : k = 0, \ldots, m, a_{kl} \neq 0, k + l = m\}$ and $i + j = m$. Note that a smaller γ is also useful. In fact, we want a γ sufficiently small for the composition preserves the number of half-branches lying in the right half plane. Thus we can assume that $a_{0m} \neq 0$.

Now regard $f(x, y)$ as a polynomial in y depending on the parameter x

$$f_x(y) = p_m(x)y^m + \ldots p_0(x),$$

where $p_j(x) = \sum_{i\leq m-j} a_{ij}x^i$.

Since $a_{0m} \neq 0$, then $p_m(0) \neq 0$ and $p_m(x) \neq 0\,\forall x$ in the neighborhood. Hence $f_x(y)$ has degree m for small x and has m complex roots, up to multiplicity. We can find m real-analytic functions $r_1 \ldots r_m : [0, \epsilon) \to \mathbb{C}$ where $\epsilon > 0$, such that the roots of $f_x(y)$ for $x < \epsilon$ are $r_1(x), \ldots r_m(x)$, that is

$$f_x(y) = c_x(y - r_1(x)) \ldots (y - r_m(x)),$$

where $c_x = p_m(x) \neq 0$. Note that for a point (x_0, y_0) in a half-branch, $y_0 \in \mathbb{R}$ is a real root of $f_{x_0}(y)$. We have guaranteed that the open right half-plane contains $\geq n$ half-branches of $f^{-1}(0)$. Thus the number of real roots for small $x > 0$ is at least n.

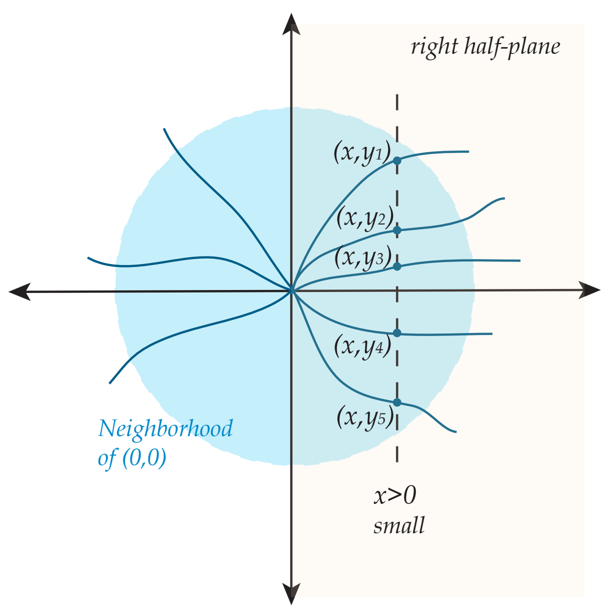


Figure 2.5: *Neighborhood of the critical point* $(0, 0)$*. We have at least the* n *real roots* $y_1, \ldots, y_n$ *of* $f_x(y)$*.*

Let $r_1, \ldots, r_n : [0, \epsilon) \to \mathbb{C}$ be such real roots that corresponds to n half branches in the right half-plane. Since each of these originates at 0, then we have $r_i(0) = 0$, $1 \leq i \leq n$. Thus f has a local expression

$$f(x, y) = q(x, y)(y - r_1(x)) \ldots (y - r_n(x)) \tag{2.1}$$

with $q, r_1, \ldots, r_n$ analytic functions. To complete the proof it remains to differentiate expression (2.1). Let α be the multiindex (i,j), where $i+j < n-1$. By the generalized Leibniz rule:

$$\begin{aligned}\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y) &= \frac{\partial^{i+j}}{\partial x^i \partial y^j}\left(q(x,y)(y-r_1(x))\ldots(y-r_n(x))\right) \\ &= \sum_{\beta_1+\ldots\beta_{n+1}=\alpha} \left(\begin{array}{c}\alpha \\ \beta_1 \ldots \beta_n\end{array}\right) \partial^{\beta_{n+1}}(q(x,y)) \prod_{i=1}^{n} \partial^{\beta_i}(y-r_i(x)).\end{aligned}$$

But $i+j \leq n-1$, hence at least two multiindices β_k are equal to $(0,0)$, and necessarily one corresponds to a factor of the form $(y-r_i(x))$. Then every term on the sums contains one of such factors and thus vanishes at at 0, i.e. $\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(0) = 0$ for $i+j \leq n-1$. ■

Branch number and non-degenerate approximating families

In the following, we will use the branch number for establishing conditions on certain approximating families. This will be useful to get the approximation by a non-degenerate family that we are looking for.

Let S be a surface and P a compact smooth manifold. For $n \in \mathbb{N}$, let denote $C_n = \frac{n(n+1)}{2} - 3$.

Lemma 2.6. *Any family of functions $\{f_t : S \to \mathbb{R}\}_{t \in P}$ can be approximated by a family $\{g_t : S \to \mathbb{R}\}_{t \in P}$ such that the set $P_n = \{t \in P :\ g_t \text{ has a critical point with } \geq n \text{ branches}\}$ has codimension at least C_n.*

Proof. Because of proposition 2.4 we restrict our attention to families with critical point of finite order. Let denote by $\mathcal{G}$ the space of germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. Consider the sets

$$\mathcal{F}'_n = \{f \in \mathcal{G} :\ f \text{ has a critical point of finite order with } \geq n \text{ branches}\}$$

and

$$\mathcal{F}_n = \{f \in \mathcal{G} :\ \frac{\partial^{i+j}}{\partial x^i \partial y^j}\Big|_0 f(x,y) = 0,\ 0 \leq i+j \leq n-1\}.$$

By lemma 2.5, $\mathcal{F}'_n \subset \mathcal{F}_n$. The functions $f \in \mathcal{F}'_n$ satisfies the $\sum_{k=0}^{n-1}(k+1) = \frac{n(n+1)}{2} - 1$ linearly independent conditions

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j}\Big|_0 f(x,y) = 0,\ 0 \leq i+j \leq n-1.$$

Then the codimension of $\mathcal{F}'_n$ is larger that $\frac{n(n+1)}{2} - 1$.

Since we are dealing with a local condition, we can assume that $S \subset \mathbb{R}^2$. The family $\{f_t\}_{t \in P}$ determines a mapping $\Phi : S \times P \to \mathcal{G}$ given by

$$\Phi(x,t) = f_t \circ \psi_x - f_t(x) : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0),$$

where $\psi_x : \mathbb{R}^2 \to \mathbb{R}^2$ is the translation by $x \in S \subset \mathbb{R}^2$. Consider the subspace $\mathcal{F}_n$ of $\mathcal{G}$. By moving the family a little the corresponding map Φ is transversal to $\mathcal{F}_n$. In fact, we can move the family, say $\{g_t\}_{t \in P}$, in such a way that Φ is transversal to $\mathcal{F}_n$ for any n. Hence the codimension of $\Phi^{-1}(\mathcal{F}_n)$ in $S \times P$ is equal to codim $(\mathcal{F}_n) \geq \frac{n(n+1)}{2} - 1$. Then, the set

$$\Phi^{-1}(\mathcal{F}'_n) = \{(x,t) \in S \times P :\ \text{ is a critical point of } f_t \text{ with branch number} \geq n\} \subset \Phi^{-1}(\mathcal{F}_n)$$

has codimension larger or equal to $\frac{n(n+1)}{2} - 1$ in $S \times P$. Under the projection in P there are two conditions less and the set $\pi \circ \Phi^{-1}(\mathcal{F}'_n) = P_n$ has codimension $\geq C_n$. ■

Let S and P be as in the previous theorem.

Definition: The *graph of a family* $\{f_t\}_{t\in P}$ [2] is defined as the set

$$G = \{(t,a) \in P \times \mathbb{R} : \ a \text{ is a critical value of } f_t\}.$$

Lemma 2.7. *Any family of functions $\{f_t : S \to \mathbb{R}\}_{t\in P}$can be approximated by a family $\{g_t : S \to \mathbb{R}\}_{t\in P}$ such that the set*

$$P_{n_1,\dots,n_k} = \{t \in P : \ \text{a level set } g_t^{-1}(a) \text{ contains } k \text{ critical points with branch numbers} \geq n_1,\dots,n_k\}$$

has codimension at least $\sum_{i=1}^{k} C_{n_i} + k - 1$.

Proof. Consider the graph G of $\{f_t\}_{t\in P}$. For each i take the subset

$$G_{n_i} = \{(t,a) \in P \times \mathbb{R} : a = f_t(x) \text{ for some critical point } x \text{ of } f_t \text{ with branch number} \geq n_i\}.$$

It is a submanifold of the graph. By lemma 2.6 the projection into P has codimension at least C_{n_i}. Since critical values are isolated the projection into P is locally one to one. Hence the submanifold G_{n_i} has codimension $\geq C_{n_i} + 1$ in $P \times \mathbb{R}$. If it has k critical points on a level set $f_t^{-1}(a)$ for some t and a, with branch numbers $\geq n_1,\dots,n_k$ respectively, then the corresponding G_{n_i} intersects in the point (t,a). Moving the family a little we get an approximating family $\{g_t\}_{t\in P}$ for which the corresponding intersection is transversal for any finite set of natural numbers. Then this intersection has codimension at least $\sum_{i=1}^{k}(C_{n_i} + 1)$ in $P \times \mathbb{R}$. By projecting into P it follows that $P_{n_1,\dots,n_k}$ has codimension $\geq \sum_{i=1}^{k}(C_{n_i} + 1) - 1$. ∎

Now we will use the previous lemmas to find an integer m, for a given surface S, such that any d-parameter family, with $d \leq m$, can be approximated by a non-degenerate family. Let S be a surface with a non-empty closed submanifold $bS \subset \partial S$.

Theorem 2.8. . *If $d \leq e(S)-1$, then any d-parameter family of functions in $\mathcal{E}$ can be approximated by a family of non-degenerate functions.*

Proof. Take an arbitrary d-parameter family $\{f_t\}_{t\in P}$ of functions in $\mathcal{E}$. Since dim $P = d$, then every non-empty $U \subset P$ has codimension $\leq d$.

By lemma 2.7 we can approximate the family $\{f_t\}_{t\in P}$ by a family $\{g_t\}_{t\in P}$ such that for any function g from it, the following condition holds:

(C) If in some level set $g^{-1}(a)$ there are k critical points with branch numbers $\geq n_1,\dots,n_k$, respectively, then $k - 1 + \sum_{i=1}^{k} C_{n_i} \leq d$.

This follows from the fact that if the set $P_{n_1,\dots,n_k}$ is non-empty, it has codimension at least $k - 1 + \sum_{i=1}^{k} C_{n_i}$. Thus $k - 1 + \sum_{i=1}^{k} C_{n_i} \leq \text{codim } P_{n_1,\dots,n_k} \leq d$.

For having the theorem, we will prove that any function satisfying condition (C) is non-degenerate when d satisfies the hypothesis.

Let g any function in $\mathcal{E}$ satisfying condition (C). Think g as a height function.

If all the components of the level sets intersecting b_iS are non-singular, then S is a disc with boundary b_iS. This case can be excluded from consideration.

Take L a singular component of a level set $g^{-1}(a)$ intersecting some component b_iS. Assume that L contains exactly k critical points with branch numbers $n_1,\dots,n_k$, respectively. Then L is homeomorphic to a graphic with k vertices from which emanate $2n_1,\dots,2n_k$ edges, respectively,

[2] In the sense of [Cer70]

and with two more vertices from which only an edge emanate (from each one), corresponding to the points z and $\overline{z}$ in $L \cap b_i S$. Since the graphic is connected and compact, the total number of edges is $(2n_1 + \ldots + 2n_k + 2)/2 = n_1 + \ldots + n_k + 1$. Thus $\chi(L) = (k+2) - (\sum_{i=1}^{k} n_i + 1) = k + 1/\sum_{i=1}^{k} n_i$.

Take an $\epsilon > 0$ such that a is the unique critical value in the interval $[a-\epsilon, a+\epsilon]$. Since L is connected, it is contained in exactly one component L_ϵ of the set $g^{-1}([a-\epsilon, a+\epsilon])$. L_ϵ is a surface with boundary and is contractible into L. Hence, $\chi(L_\epsilon) = \chi(L)$. Note that the boundary ∂L_ϵ is contained in non-singular components of g. Thus, if a component of ∂L_ϵ is non-trivial, the function g is non-degenerate.

Assume that every component of ∂L_ϵ is trivial. That means that each component of ∂L_ϵ bounds (itself or together with an arc in $b_i S$) a disc, or, together with a component of ∂S, an annulus. Consider two cases:

(I) L is contained in one of these discs or annuli. Let h be a function that coincides with g outside of the disc (or annulus) and in has only one critical point (or in case of annulus there is no critical point). Since in the disc (annulus) all components of h are trivial, if h has a level set with a non-trivial component, then so does g (outside of the disc or annulus where $g = h$). But h satisfies condition (C) (because it is almost g) and has less critical points than g. Hence, by induction, it must have a non-trivial component.

(II) L is not contained in one of these discs or annuli. Then, the discs and annuli bounded by the components of ∂L_ϵ with value of g larger than $g(a)$ (less that $g(a)$) must themselves lie 'above' ('below') L. Hence S is the union of L_ϵ and these discs and annuli, and $\chi(S) = \chi(L_\epsilon) + m$, where m is the number of discs. Thus,

$$\chi(S) \geq \chi(L_\epsilon) = \chi(L) = 1 + k - \sum_{i=1}^{k} n_i,$$

i.e.

$$e(S) \leq \sum_{i=1}^{k} n_i - k - 1.$$

Assume that $n_i \geq 2$ and reduce k according with the number of those $n_i = 1$. Then

$$C_{n_i} = \frac{n_i(n_i+1)}{2} - 3 \geq (n_i + 1) - 3.$$

By hypothesis and condition (C) we have

$$e(S) - 1 \geq d > k - 2 + \sum_{i=1}^{k} C_{n_i} \geq \sum_{i=1}^{k} (n_i - 2) + k - 2 \geq e(S) - 1,$$

which is a contradiction. Then, one component of ∂L_ϵ must be non-trivial.

Thus, the result is proved. ■

From Theorems 2.2 and 2.8, we can finally establish Theorem 2.9 about the degree of connectedness of the geometric realization of the complex of curves $WB(S)$.

Theorem 2.9. *The space $|WB(S)|$ is $(e(S)-2)$-connected.*

The degree of connectedness of the complex $|B_u(S)|$

Here we will see how the high-connectedness of complex $|WB(S)|$ will imply the high-connectedness of the subcomplex $|B_u(S)|$.

In [Iva87] it is stated the degree of connectedness of complex $|B(S)|$ as a consequence of Theorem 2.9.

Theorem 2.10. *The space $|B(S)|$ is $(e(S)-2)$-connected.*

Denote by $\hat{S}$ the surface resulting from pasting discs on all components of $\partial S \backslash bS$. Ivanov reduces the proof of Theorem 2.10 to the case when $S = \hat{S}$, by the following lemma.

Lemma 2.11. *Let D be a disc in $S \backslash \partial S$ and let $T = S \backslash int\ D$. Consider $bT = bS$. If $|B(S)|$ is m-connected, then $|B(T)|$ is $(m+1)$-connected.*

The proof of Theorem 2.10 is done by induction on $e(S)$, considering only the case where $bS = \partial S$. The details of the proofs of Theorem 2.10 and 2.11 will be considered in a further work. Assuming Theorem2.10, we can get the result of high-connectedness for complex $B_u(S)$. Let S be a surface of genus g.

Theorem 2.12. *The space $|B_u(S)|$ is $(g-2)$-connected. Moreover $|B_u(S)|$ is spherical of dimension $g-1$.*

The proof of Theorem 2.12 is fully analogous to Harer's proof of Theorem 1.1 in [Har85] and depends on the high-connectedness of complex $|B(S)|$. In what follows we reproduce this proof.

Let $S = S_{g,b}$. If $g = 1$, dim $B_u(S) = g - 1 = 0$ and $|B_u(S)|$ is a wedge of 0-spheres.

The proof is done by induction. Fix $g > 1$ and $b \geq 1$. We want to prove that $|B_u(S)|$ is $(g-2)$-connected. For take $m < g-1$ and C a piece-linear manifold homeomorphic to S^m. Take $f : C \to |B_u(S)|$ any simplicial map. We shall prove that f is homotopic to the constant map.

Recall that $|B_u(S)|$ is a subcomplex of $|B(S)|$. We are going to use the high-connectedness of complex $|B(S)|$ for extending the map f form C to $B \approx D^{m+1}$. Since $g > 1$ and $b \geq 1$, then

$$e(S) - 2 = (g-1) + g - b - 3 \geq g - 1$$

and, by Theorem 2.10, $|B(S)|$ is at least $(g-1)$-connected. Hence we can extend simplicially f to $\hat{f} : B \to |B(S)|$, but we don't guarantee that the image lies in $|B_u(S)|$. There may be some simplices of B whose image under $\hat{f}$ doesn't lie completely on $|B_u(S)|$. We have to redefine $\hat{f}$ in such a way that we don't have these simplices.

Definition: σ is a *purely separating simplex of B* if $\hat{f}(\sigma) = \langle I_0, \ldots, I_t \rangle$, where each I_i separates $S \backslash \{I_0, \ldots, \hat{I}_i, \ldots, I_t\}$. The *complexity* $c(\hat{f})$ is the largest k for which some k-simplex of B is purely separating.

Take a "maximally problematic simplex": let σ be a purely separating k-simplex of B with $k = c(\hat{f})$ and let $\hat{f}(\sigma) = \langle I_0, \ldots, I_t \rangle$, $t \leq k \leq g-1$.

Let $S_1, \ldots, S_r$ be the connected components of the surface obtained by splitting S along $\{I_i\}$. Let bS_i be the subset of bS lying in S_i. Consider the join

$$S_\sigma = |B_u(S_1)| * \ldots * |B_u(S_r)|.$$

S_σ may be identified with a subcomplex of $|B_u(S)|$:

(i) Each simplex in $|B_u(S_i)|$ is a simplex in $|B_u(S)|$.

(ii) Simplices in $|B_u(S_i)|$ can be joined with simplices in $|B_u(S_j)|$: the vertices in $|B_u(S_i)|$ don't separate S_i and are disjoint of S_j, then we can join them with the vertices in $|B_u(S_j)|$ and we don't separate the surface S. Then $|B_u(S_i)| * |B_u(S_j)|$ is a subcomplex of $|B_u(S)|$.

We have split S along $t+1$ arcs, then we can get at most $t+2$ connected components, i.e. $r \leq t+2$. For each S_i let denote by g_i the genus and by b_i the number of boundary components. Notice that an arc I_i in S can have two effects: I_i separates a new component S_j of S, or I_i cuts a handle of the surface and doesn't disconnect the surface. We have $t+1$ curves in S from which $r-1$ correspond to the ones that separate S in a new component. Then there are $(t+1)-(r-1)$ handles that were broken, after splitting, that is:

$$\sum_{i=1}^{r} g_i = g - (t+2-r) = g + (r-t-2).$$

We have two cases:

(a) If $r < t+2$, then each S_i has genus $g_i < g$.

(b) If $r = t+2$, then $(\forall i, g_i < g)$ or $(g_i = g$ for some i and $g_j = 0, \forall i \neq j)$. But this last case is not possible because a non-trivial arc must cut a handle, so there is at least one handle less.

In both cases (a) and (b) we can apply induction hypothesis and get for all i:

$$|B_u(S_i)| \simeq Wedge\ of\ (g_i - 1) - spheres.$$

Lemma 2.13. *The subcomplex S_σ is $(g+r-t-4)$-connected.*

Proof. We shall prove that S_σ is a wedge of $(g+r-t-3)$-spheres. First note that for $n, m, l \in \mathbb{N}$ we have

$$\begin{aligned}(S^m \vee S^n) * S^l &\simeq S\left((S^m \vee S^n) \wedge S^l\right) = S\left((S^m \wedge S^l) \vee (S^n \wedge S^l)\right) \\ &= S\left((S^m \wedge S^l)\right) \vee S\left((S^n \wedge S^l)\right) = S(S^{m+l}) \vee S(S^{n+l}) \\ &\simeq S^{m+l+1} \vee S^{n+l+1}\end{aligned}$$

By induction we get that

$$S_\sigma = B_u(S_1) * \ldots * B_u(S_r) \simeq *_{i=1}^{r}\left(\bigvee_{k=1}^{n_i} S_k^{g_i-1}\right) \simeq \bigvee_l S_l^{(\sum g_i - 1)} = \bigvee_l S_l^{g+r-t-3},$$

as we wanted. ■

The high-connectedness of S_σ will allow us to redefine $\hat{f}$ near the purely separating simplex.

Let Σ de a simplex of a simplicial complex Z. We will denote here the closed star of Σ as $St_Z(\Sigma)$.

Definition: The *link of* Σ *in* Z, $L_Z(\Sigma)$, is the union of all simplices Σ_0 of Z such that $\Sigma_0 \leq \Sigma'$ for some Σ' in $St_Z(\Sigma)$, but Σ_0 doesn't intersect Σ.

Proposition 2.14. $\hat{f}(L_B(\sigma)) \subseteq S_\sigma$

Proof. We cut S along the curves I_i, hence the vertices that they represent in the complex $|B_u(S)|$ are not included on the subcomplex S_σ. It follows that $\hat{f}(\sigma) \cap S_\sigma = \phi$.

Take a simplex Σ' in $L_B(\Sigma)$. $\Sigma' \leq \Sigma$ for some $\Sigma \subseteq St_B(\Sigma)$ and is such that $\Sigma' \cap \sigma = \phi$. Since $\sigma \leq \Sigma$, $\hat{f}(\Sigma)$ is a simplex with vertices $I_0, \dots, I_t$ (the ones of $\hat{f}(\sigma)$), vertices $B_0, \dots, B_l$ in S_σ and other vertices $I_{t+1}, \dots I_s$ that separates the surface.

Under $\hat{f}$, $\hat{f}(\Sigma')$ must be a subcomplex of $\hat{f}(\Sigma)$ that doesn't intersect any separating curve. If some $I_i \subseteq \hat{f}(\Sigma')$, there is a vertex v in Σ' such that $\hat{f}(v) = I_i$. Then $\sigma * v$ is a purely separating simplex of dimension $k+1$. But this is a contradiction because $c(\hat{f}) = k$.

Hence, $\hat{f}(\Sigma')$ must lie in a join of a subset of vertices in $\{B_0, \dots, B_l\}$, that is, it must lie in S_σ.
■

$L_B(\sigma)$ is a sphere of dimension $(m-k)$. Notice that since σ is purely separating, σ cannot lie in $\partial B \approx S^m$.

Now, $m < g-1$ and S_σ has a degree of connectedness $g + r - t - 4 \geq (g-2-k) + (r-2) \geq g - 2 - k \geq m - k$, ($t \leq k$ and $r \geq 2$), then we can extend $\hat{f} \mid_{L_B(\sigma)}: L_B(\sigma) \to S_\sigma \subset |B_u(S)|$ to $\hat{f}': D^{m-k+1} \to S_\sigma$, where $\partial D^{m-k+1} \approx L_B(\sigma)$.

Identifying $(S_B(\sigma), LB(\sigma))$ with $(D^{m-k+1} * \partial\sigma, \partial D^{m-k+1})$ we can redefine $\hat{f}$ on $S_B(\sigma)$ as $\hat{f}' * \hat{f} \mid_{\partial\sigma}$. This eliminates σ from B without introducing a new purely separating simplex. Continuing in this way, we reduce the complexity of $\hat{f}$ and eventually eliminates all purely separating simplices from B. Since any simplex of B which is not mapped into $|B_u(S)|$ must have a non-empty purely separating face, we have some $g: B \to |B_u(S)|$ homotopic to $\hat{f}$ relative to f. Then we have gotten that f is homotopic to a nullhomotopic map $g \mid_{S^m}: S^m \to |B_u(S)|$. Hence $\pi_m(|B_u(S)|) = 0$, $\forall m < g-1$, i.e. $|B_u(S)|$ is $(g-2)$-connected.

Since $B_u(S)$ is $(g-2)$-connected is homotopic to a CW-complex with one cell in dimension 0 and all the other cells in dimension larger than $g-2$. But dim $B_u(S) = g-1$, then $B_u(S)$ cannot have cells in dimension larger than$g-1$. Hence $B_u(s)$ is homotopic to a wedge of cells of dimension $g-1$, and Theorem 2.12 is done.

For the case when bS is just a component of the boundary of S, Theorem 2.12 establishes that the complex $B_0'(S)$ is spherical of dimension $g-1$. Since $B_0(S)$ is a subcomplex of $B_0'(S)$ of the same dimension and all the maximal simplices are of that dimension, we can state Theorem 1.6 about the degree of connectedness of $B_0(S)$.

2.2 The complex of curves $|H(S)|$

Following Harer's proof, the degree of connectedness of complex $|H(S)|$ is gotten from the knowledge of the high-connectedness of some auxiliary complexes. First we observe complex $AZ(\Delta)$.

Theorem 2.15. *If S is not a disk or an annulus with Δ contained in one of its boundary components, then $|AZ(\Delta)|$ is contractible. If S is a disk ($b = 1$) or an annulus with Δ contained in one of its boundary ($b = 2$), then $|AZ(\Delta)|$ is homeomorphic to a sphere of dimension $q + 2b - 6$.*

In [Har85] this is Theorem 1.5, the principal technical result of the article, where Section 2 is entirely dedicated to prove it using machinery of Thurston's theory of projective lamination spaces. Fortunately for us, Hatcher gives in [Hat91] a shorter topological proof of this fact considering a larger complex. In the next, we sketch Hatcher's proof.

To define this larger complex we must define a new type of curves on the surface. Let S be a compact, connected surface and $V = \{v_1, \cdots, v_n\}$ be a finite subset of S.

Definition: An *essential arc in* (S,V) is an embedded arc $\alpha \subset S$ meeting $\partial S \cup V$ only in the endpoints, which lie in V and may coincide, with the condition that if S_α (the surface resulting from cutting S along α) has two components, neither component is a disk meeting V only in the endpoints of α. In other words, α is non isotopic to an arc in $\partial S - V$.

An *essential circle in* (S,V) is an embedded circle in $S - (\partial S \cup V)$ non-isotopic to a boundary component of S or to a point in S (including points in V).

A *curve system* is a set $\{\alpha_0 \cdots, \alpha_k\}$ of essential arcs and circles with α_i pairwise disjoint non-isotopic curves, except perhaps for the endpoints.

Definition: The complex of curves $\mathcal{C} = \mathcal{C}(S,V)$ is the simplicial complex having as k-simplices the isotopy classes (rel V) of curve systems $\{\alpha_0 \cdots, \alpha_k\}$.

The full subcomplex of $\mathcal{C}$ whose vertices are isotopy classes of essential arcs will be called $\mathcal{A} = \mathcal{A}(S,V)$.

The main result in [Hat91] is

Theorem 2.16. .

(a) The complex $\mathcal{A}(S,V)$ is contractible, except for S a disk with $V \subset \partial S$ or an annulus with V in one component of ∂S

(b) If V contains at least one point in the interior of S, then $\mathcal{C}(S,V)$ is contractible.

Remark 2.2:

1. In the exceptional cases in (a), $\mathcal{A}(S,V)$ is a sphere of dimension $q + 2b - 6$, where q is the order of V (see [Har86] for a proof).

2. When $V \subset \partial S$, $\mathcal{A}(S,V)$ is the complex $AZ(\Delta)$ with $\Delta = V$, defined by Harer, and thus Theorem 2.15 is a particular case of Theorem 2.16 (a).

Proof. (Sketch of proof) We outline the proof of part (a). The details and the proof of part (b) can be found in [Hat91].

The idea is to fix an essential arc β in (S,V) and construct a deformation retraction on $\mathcal{A}$ onto the closed star of the vertex $\langle\beta\rangle$, which is contractible. This star is the union of simplices $\langle\alpha_0, \cdots, \alpha_k\rangle$ in $\mathcal{A}$ with $\alpha_0 \cup \cdots \cup \alpha_k$ disjoint from β with at most one α_i isotopic to β.

Consider any simplex $\langle\alpha_0, \cdots, \alpha_k\rangle$ in $\mathcal{A}$ in such a way that $\alpha_0 \cup \cdots \cup \alpha_k$ intersects β minimally.

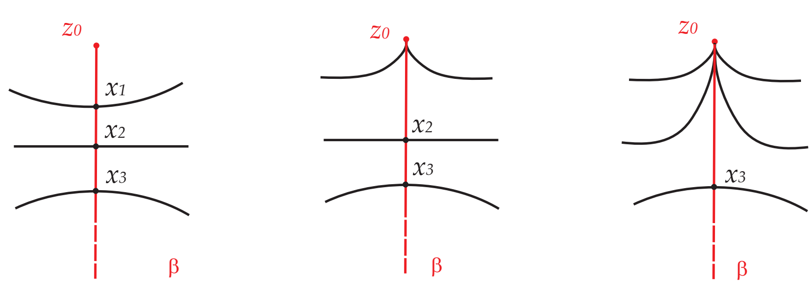


Figure 2.6: *Flow that moves a simplex in $\mathcal{A}$ to a simplex in the star of $\langle\beta\rangle$.*

The deformation retraction is a flow that gradually pushes the crossings of the curve system $\alpha_0 \cup \cdots \cup \alpha_k$ with β over to one endpoint, say z_0, of β (Figure 2.6).

More precisely, let $x_1, \cdots, x_m$ be the points of intersection of β with $c_1 = \alpha_0 \cup \cdots \cup \alpha_k$. Sliding the intersection point x_1 to the endpoint z_0, and discarding parallel and ∂-parallel components, we get a new curve system c_2 (Figure 2.7). The process is repeated for sliding x_2 to z_0 to convert c_2 into c_3, and successively, until all the x_i's have been eliminated. The resulting curve system is disjoint from β and is a simplex in the closed star of $\langle\beta\rangle$. Hatcher's construction guarantees that these slides are a continuous flow on all of $\mathcal{A}$.

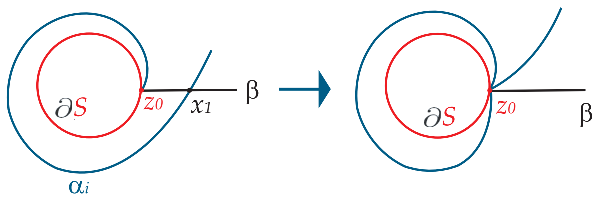


Figure 2.7: *The flow takes the crossing x_1 into z_0. The resulting curve is ∂-parallel, then it is discarded.*

To define the flow, he takes an arbitrary point P of the simplex $\langle\alpha_0, \cdots, \alpha_k\rangle$ expressed in barycentric coordinates t_i, with $\sum t_i = 1$, $P = \sum t_i \alpha_i$, and gives to it a geometrical interpretation. P is a weighted sum represented by replacing each α_i by a family of nearby parallel curves with total thickness t_i. Consider, in the crossings with β, all the families as a single family with total thickness θ. The point P is moved by the flow as follows. For $t \in [0, 1]$, it is taken to the point P_t obtained from P by cutting the band a thickness $t\theta$, staring at z_0 in the way crossing β, and then redirecting the two ends of the resulting cut band to the endpoint z_0.(Figure 2.8). Discard all ∂-parallel arcs (Figure 2.7).

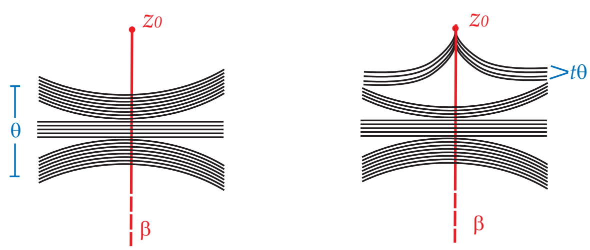


Figure 2.8: *Geometric representations for points P and P_t.*

This argument works for the case where V has at most one point in each component of ∂S, since not all of P_t is discarded. In the general case, Hatcher's argument is similar to that of Harer in the proof of Theorem 1.5.

Consider a component of ∂S with at least two adjacent points of V, say v_1 and v_2. Let v_0 and v_3 be the points preceding v_1 and following v_2, respectively (not necessarily distinct). Take δ the ∂-parallel arc in S joining v_2 with v_0 and γ the one joining v_1 with v_3. Let $\mathcal{A}(\gamma)$ be the subcomplex of $\mathcal{A}$ consisting of simplices $\langle\alpha_0, \cdots, \alpha_k\rangle$ with no α_i isotopic to γ. Repeating the above argument for this complex, using δ in the place of β, it can be shown that $\mathcal{A}(\gamma)$ deformation retract into $\mathcal{A}(v_1)$, the simplices not having an endpoint in v_1. Since $\mathcal{A}(v_1)$ is the star of $\langle\delta\rangle$, it is contractible. Thus,

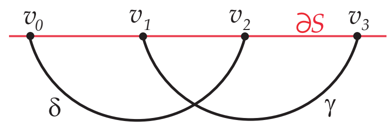


Figure 2.9: *Arcs δ and γ considered in the case when one boundary component has more than one point of V.*

$\mathcal{A}$ is the union of two contractible sets, $\mathcal{A}(\gamma)$ and the star of $\langle\gamma\rangle$, and they intersect in the link of $\langle\gamma\rangle$, which can be identified with $\mathcal{A}(S, V - \{v_2\})$ (non-empty except for the exceptional cases). By induction, this last is also contractible. ■

From Theorem 2.15, Harer got the degree of connectedness of the complex $|BZ(\Delta, \Delta^0)|$. Some extra definitions are needed for establishing the corresponding result of connectedness.

Definition: An edge of $\partial' - \Delta$ with both ends in Δ^0 or both in Δ^1 is called *pure*. If it has one end in Δ^0 and one in Δ^1 is called *impure*. A component of ∂' is called pure if it contains only pure edges.

Remark 2.3: Impure edges come in pairs. The endpoints of an impure edge, after a sequence of pure edges, become the endpoints of another impure edge. Hence we can write

$$q = 2l + u,$$

where $2l$ is the number of impure edges and u is the number of pure ones.

Thus, we have

Theorem 2.17. *The complex $|BZ(\Delta, \Delta^0)|$ is spherical of dimension $4g + b + b' + l + u - 5$. If ∂' has only components which are pure, $|BZ(\Delta, \Delta^0)|$ is contractible.*

This is Theorem 1.6 in [Har85] and the proof is done by induction on triples (g, b, q), where $q = |\Delta|$, ordered lexicographically.

The case when all components of ∂' are pure follows as consequence of Theorem 2.15 by an analogous argument of Theorem 1.1 in [Har85], considering $BZ(\Delta, \Delta^0)$ as a subcomplex of the contractible complex $AZ(S)$. The induction basis corresponds to the case when S is an annulus and Δ consists of a point from each boundary component, with Δ^0 one of such points. Thus $BZ(\Delta, \Delta^0) = AZ(\Delta)$ and is contractible. For the general case, he noticed that splitting the surface along the curves representing a simplex in $AZ(S)$, induction can be applied to the complexes $BZ(\Delta_i, \Delta_i^0)$ for each component of the splitting. Then the proof follows as in the one of Theorem 1.1 of [Har85].

When a component of ∂' has at least an impure edge, three cases are considered depending on if the next two edges are pure or impure. Details appear in the proof of Theorem 1.6 in [Har85] and are of combinatorial nature.

Finally we have

Theorem 2.18. *$BX(\Delta, \Delta^0)$ is spherical of dimension $2g - 2 + r'$.*

Again the proof is due to Harer (Theorem 1.4 in [Har85]) and is a consequence of Theorem 2.17 following essentially the same argument as for Theorem 1.1.

Proof. (Sketch of proof) Take $m < 2g - 2 + r'$. When $BX(\Delta, \Delta^0) \neq \phi$, i.e. for $g \geq 1$ and for $g = 0$ and $l + m \geq 2$, with $b \geq 1$ in both cases, the inequality

$$2g - 2 + r' \leq 4g + r + r' + l + u - 5,$$

holds. Then, from Theorem 2.17, any simplicial map $f : S^m \to BX(\Delta, \Delta^0)$ can be extended to $\hat{f} : B \approx D^{m+1} \to BZ(\Delta, \Delta^0)$.

For deforming $\hat{f}$ (relative f) into $BX(\Delta, \Delta^0)$ we look again in maximal purely separating simplexes. They are removed in a similar way. We notice that any component obtained by splitting S along a family of arcs $\langle \beta_0, \cdots, \beta_t \rangle$ of $BZ(\Delta, \Delta^0)$, inherits points from both Δ^0 and Δ^1. Then the corresponding complexes $BX(\Delta_i, \Delta_i^0)$ are non-empty and induction can be applied. It turns out that the complex $BX_\sigma = BX(\Delta_1, \Delta_1^0) * \cdots * BX(\Delta_n, \Delta_n^0)$ is spherical of dimension $\sum(2g_i + b_i' - 2) + n - 1 = 2g + b' + n - t - 4$, which is enough for redefining the map $\hat{f}$ near the purely separating simplex as in Theorem 1.1 of [Har85]. ■

In particular, it follows that the complex of the "stability by holes" $|H(S)|$ (complex $|BX|$ in [Har85]) is spherical of dimension $2g$ (Theorem 1.8).

Chapter 3

The Stability Theorem

In this chapter we are finally concerned with the phenomenon of stabilization of homology for mapping class groups of orientable surfaces. The main theorem asserts that the homology stabilizes as genus increase. Following Ivanov we will prove that for a surface $S_{g,b}$, the group $H_n(\Gamma_{g,b})$ depends only on n when $g \geq 2n+1$ and $b \geq 1$. The proof is based on equivariant homology theory applied to the highly connected complexes $B_0(S)$ and $H(S)$ that we have defined in previous chapters.

In the last twenty years an important analogy between mapping class groups and arithmetic groups has been identified. The stability phenomenon forms part of this identification. For mapping class groups, the Stabilization Theorem asserts that the homology group of a given dimension of $\Gamma_{g,b}$ does not depend on $S_{g,b}$ as soon as the genus g is sufficiently large. The exact restriction on the genus is called the *domain of stability*.

The first results about this are due to Harer. He got, in [Har85], a domain of stability of $g \geq 3n+1$. Since that, the domain of stability has been improved and some different versions of the stability theorem have been proved following essentially the same idea. Here we presents Ivanonv's proof that appeared in [Iva87], where he obtained a domain of stability of $g \geq 2n+1$ and $b \geq 1$.

More precisely, the result is stated as follows. Let R and S be connected surfaces with non-empty boundary such that R lies in S, i.e. each component of ∂R either coincides with a component of ∂S or lies in the interior of S. Assume also, that every component of $S \backslash R$ contains a component of ∂S. Then, there is a natural mapping $i : \Gamma_R \to \Gamma_S$, induced by the inclusion $R \subset S$, defined as follows:

If $f : R \to R$ is a diffeormorphism that fixes ∂R, then $i(f) : S \to S$ is the unique extension of f by the identity on $S \backslash R$.[1] *Note that i is well defined in Γ_R, since isotopic functions f and g on R extend to isotopic functions on S.*

The map i induces naturally a homomorphism in homology: $i_* : H_n(\Gamma_R) \to H_n(\Gamma_S)$. Now we can establish Ivanonv's version of the Stability Theorem:

Theorem 3.1 (Stability Theorem). *Let R and S be as before and let denote by g_R the genus of R. Then the natural homomorphism $i_* : H_n(\Gamma_R) \to H_n(\Gamma_S)$ is surjective when $g_R \geq 2n$, and injective when $g_R \geq 2n+1$.*

[1] We can always consider a function in the isotopy class of f such that the extension by identity is a diffeomorphism of S.

Remark 3.1:

- The homology is taken with coefficients in $\mathbb{Z}$.
- In the following we will denote the genus of a surface R by g_R.

Let $P = Cl(S \backslash R)$. Since R lies in S, $R \cap P = \partial R \cap P$ is the union of the boundary components of R lying in the interior of S. Notice that these components are also boundary components of P. Thus, P is a compact surface with boundary equal to $(P \cap R) \cup (\partial S \backslash R)$. By hypothesis, each connected component P_k of P has at least two boundary components: one from ∂S and one for attaching it to R. We will not consider the case where a component of P is an annulus. Therefore $\chi(P_k) < 0$ and from Theorem 1.1, each component P_k can be decomposed in pairs of pants. Indeed, S can be obtained from R by successively attaching pairs of pants to one or two boundary components.

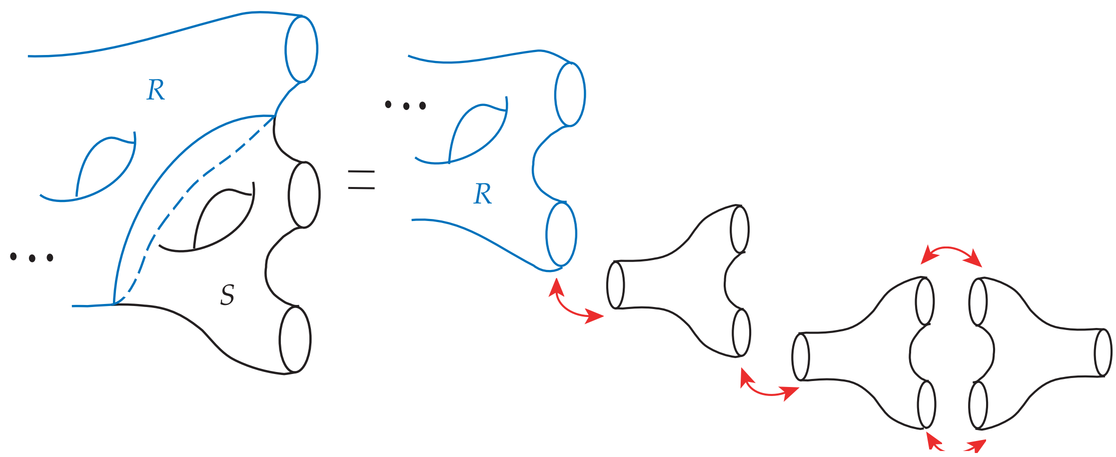


Figure 3.1: *Successive attaching of pairs of pants for getting S from R.*

Hence, it is enough to prove the theorem when P is a pair of pants. We have two cases:

- If $P \cap R$ is a single circle, then S has the same genus as R and one more boundary component ("hole"). We are considering "stability by holes" and it is the case why the complex $H(S)$ was defined for. The mapping $i : \Gamma_R \to \Gamma_S$ corresponds to the one induced by the inclusion that Harer, in [Har85], denotes by $\Phi : S_{g,b} \to S_{g,b+1}$, with $b \geq 1$.
- If $P \cap R$ consists of two circles, then $g_S = g_R + 1$, i.e. S has one more "handle", and the number of boundary components of S is one less than that of R. This case is referred as "stabilization by handles". The mapping i is the one induced by the inclusion that Harer denotes by $\Psi : S_{g,b} \to S_{g+1,b-1}$, with $b \geq 2$. The complex $B_0(S)$ is the essential object for this part of the proof.

The proof of the Stabilization Theorem 3.1 will be done by induction on n.

Notice that the inclusion $i : \Gamma_R \to \Gamma_S$ induces a chain morphism between the standard chain complexes $\hat{\imath} : C_*(\Gamma_R) \to C_*(\Gamma_S)$. In low dimensions we have

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ C_1(\Gamma_R) \arrow[r, "\hat{\imath}"] \arrow[d, "0"] & C_1(\Gamma_S) \arrow[d, "0"] \\ \mathbb{Z} \arrow[r, "id"] & \mathbb{Z}, \end{tikzcd}$$

where $\hat{\imath} : 1 \otimes [x] \mapsto 1 \otimes [i(x)]$ (see Appendix B). Thus, $H_0(\Gamma_R) = H_0(\Gamma_S) = \mathbb{Z}$ and $i_* : H_0(\Gamma_R) \to H_0(\Gamma_S)$ is the identity morphism. Hence, the case $n = 0$ of the stability theorem holds.

Even more, the first homology groups are known explicitly (see [Har85] for references). The results consider surfaces of genus g, with b boundary components and s punctures. For $n = 1$,

$$H_1(\Gamma^s_{g,b}, \mathbb{Z}) \cong 0, \text{ for } g \geq 3,$$

and $H_1(\Gamma^s_{2,b}, \mathbb{Z})$ is cyclic generated by the class of Dehn twist on any non-separating simple closed curve in S.

Thus, if R is a subsurface of S with $g_S \geq g_R \geq 3$, then $H_1(\Gamma_R) = H_1(\Gamma_S) = 0$ and i_* is an isomorphism. If $g_S \geq g_R = 2$, then $H_1(\Gamma_R) \cong \mathbb{Z}$, and under i_* the generator goes to zero ($H_1(\Gamma_S) = 0$) or to the generator of $H_1(\Gamma_S) \cong \mathbb{Z}$. Hence, i_* is surjective.

And for $n = 2$, it is known (see [Har85]) that

$$H_2(\Gamma^s_{g,b}, \mathbb{Z}) \cong \mathbb{Z}^{s+1}, \text{ for } g \geq 5,$$

and the generators lie on a subsurface of genus 3 with s punctures and one boundary component. Since i_* is induced by the inclusion, the generators are mapped into the generators and we have an isomorphism.

The inductive step will follow from the action of Γ_S on the complexes of curves $B_0(S)$ and $H(S)$.

3.1 The main tool: spectral sequences of a group action

In this section we outline the procedure that will be followed in the stability proof. It will be applied later to the high-connected complexes that we have constructed.

We have a discrete group, say G, and we are concerned with getting some homological information about G. A powerful tool for doing it is the equivariant homology theory. Having an interesting acyclic space X on which the group G acts, the homology groups of G correspond to the equivariant homology groups of (G, X) and the machinery of spectral sequences can be used for computing them. Now we describe an analogous construction useful for us. For more details refer to Appendix B, where the main properties of the equivariant homology groups are presented.

Let X be a d-connected simplicial complex on where the group G acts simplicially. Let $C_*(X)$ be the oriented chain complex of X. Note that $C_*(X)$ is a chain complex of G-modules.

Since X is acyclic for $q \leq d$, then we can consider the augmented chain complex and have the next exact sequence:

$$0 \to Z_{d+1}(X) \xrightarrow{\iota} C_{d+1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_0} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $Z_{d+1}(X)$ is the kernel of $\partial_d : C_{d+1}(X) \to C_d(X)$. To simplify the notation, take

$$L_i = \begin{cases} \mathbb{Z}, & i = 0 \\ C_{i-1}(X), & i = 1, \cdots, d+2 \\ Z_{d+1}, & i = d+3 \\ 0, & i > d+3 \end{cases}$$

Thus we have an acyclic complex $L = (L_i)_{i \geq 0}$:

$$0 \to L_{d+3} \to \cdots \to L_1 \to L_0 \to 0$$

We are going to consider the homology of the group G with coefficients in the chain complex L. Take a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$\cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

Then $H_*(G, L) = H_*(F \otimes_G L)$, where $F \otimes_G L$ is the total complex associated to the double complex formed by the products $F_p \otimes_{\mathbb{Z}G} L_q$. We can associate to this double complex two spectral sequences depending on whether we filter by columns or by rows (see Appendix B).

Since F_p is a free G-module, $F_p \otimes_G \_\_$ is an exact functor. Then, $H_q(F_p \otimes_G L_*) = 0$, because L_* is an acyclic complex. Thus, from filtering by columns, we have a spectral sequence with zero E^1-term.

Now consider the second spectral sequence obtained by filtering by rows. It must also converge to zero. The E^1-term and the differential d^1 of this spectral sequence is described in Appendix B. Indeed, E^1 can be expressed in terms of stabilizers of simplices. For $p \leq d+2$ we have that:

$$E^1_{pq} = H_q(F_* \otimes_G L_p) = H_q(F_* \otimes_G C_{p-1}(X)) = H_q(G, C_{p-1}(X)) \cong \bigoplus_{\sigma \in \Sigma_{p-1}} H_q(St(\sigma))$$

Where Σ_{p-1} denotes a set of representatives of the G-orbits of the $(p-1)$-simplices. For the second spectral sequence we will assume that:

(B) If $g \cdot \sigma = \sigma$ for some $g \in G$ and some simplex σ, then $g|_\sigma = id$. Thus, g preserves the orientation of σ and $St(\sigma)$ fixes σ point-wise. Moreover, for the augmented term of the chain complex, we introduce formally a (-1)-simplex σ_{-1} and declare that $St(\sigma_{-1}) = G$.

For $p \leq d+1$, the differential $d^1 : E^1_{p+1,q} \to E^1_{p,q}$, restricted to the summand $H_q(St(\sigma_p))$, with $\sigma_p \in \Sigma_p$, is the alternating sum of homomorphisms corresponding to the faces of σ_p. For a face $\tau \in X_{p-1}$ of σ_p, let $\sigma_{p-1} \in \Sigma_{p-1}$ be the representative of the G-orbit of τ in X_{p-1} and let $h_\tau \in G$ such that $\sigma_{p-1} = h_\tau \tau$. Then, the homomorphism that corresponds to the face τ of σ_p is

$$(int(h_\tau), f)_* : H_q(St(\sigma_p), \mathbb{Z}) \to H_q(St(\sigma_{p-1}), \mathbb{Z}),$$

induced by

$$\begin{array}{rccc} int(h_\tau): & St(\sigma_p) & \longrightarrow & St(\sigma_{p-1}) \\ & g & \mapsto & h_\tau g h_\tau^{-1} \end{array}$$

and

$$\begin{array}{cccccc} f: & \mathbb{Z} \cong \mathbb{Z}_{\sigma_p} & \longrightarrow & \mathbb{Z}_{\sigma_\tau} & \longrightarrow & \mathbb{Z} \cong \mathbb{Z}_{\sigma_{p-1}} \\ & z & \mapsto & \partial_{p-1}(z) & \mapsto & h_\tau(\partial(z)) \end{array},$$

where $\partial_{p-1} : C_p(X) \to C_{p-1}(X)$ is restricted to $\mathbb{Z}_p$. In the following we will denote $(int(h_\tau), f)_*$ simply by $int(h_\tau)_*$.

3.2 "Stabilization by handles"

In this section we use the spectral sequence described above for proving the Stability Theorem in the case of stabilization by handles.

Assume that Theorem 3.1 holds for $q \leq m$. Thus

$$i_* : H_q(\Gamma_R) \to H_q(\Gamma_S),$$

is surjective for $g_R \geq 2q$ and is injective for $g_R \geq 2q+1$.

We shall prove here that the morphism of "stabilization by handles"(Figure 3.2)

$$i_* = \Psi_* : H_{m+1}(\Gamma_{g,b}) \to H_{m+1}(\Gamma_{g+1,b-1}),$$

is surjective for $g_R = g \geq 2(m+1)$ and injective for $g_R = g \geq 2(m+1)+1$.

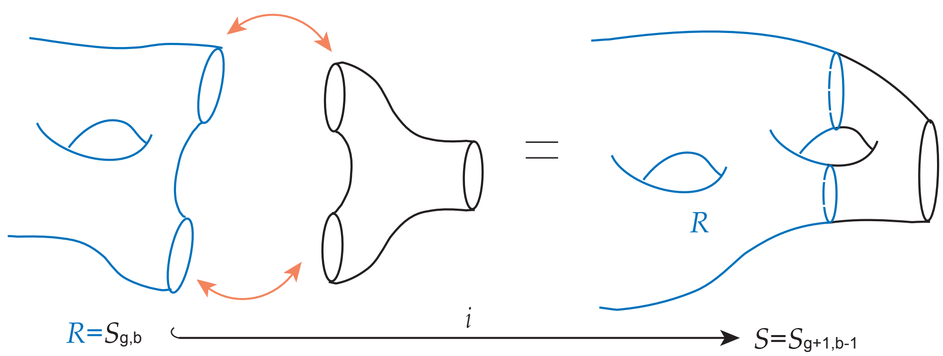


Figure 3.2: *The map of "Stabilization by handles".*

The action of Γ_S on $B_0(S)$

Consider the action of Γ_S on the complex of curves $B_0(S)$ defined as follows.

Let γ_0 be a vertex of $B_0(S)$ and let the arc I joining b_0 with b_1 be a representative of γ_0. Take $f : S \to S$ a diffeomorphism that fixes the boundary of S. Then $f(I)$ is also an arc joining b_0 with b_1. Since I is non-trivial, then $f(I)$ is non-trivial. Thus, on vertices, the action is defined by

$$f \cdot \langle I \rangle = \langle f(I) \rangle.$$

We can extend the action simplicially letting

$$f \cdot \{\gamma_0, \cdots, \gamma_n\} = \{\langle f(I_0) \rangle \cdots, \langle f(I_p) \rangle\},$$

where the arcs $I_0, \cdots, I_p$ are pairwise disjoint representatives of the vertices $\gamma_0, \cdots, \gamma_p$, respectively, of an p-simplex of $B_0(S)$. Since the elements of Γ_S are diffeomorphism classes, $f(I_0), \cdots, f(I_{p-1})$ are pairwise disjoint non-isotopic arcs. Moreover, $S\backslash f(I_0)\cup\cdots\cup f(I_p)$ is connected because $S\backslash I_0\cup\cdots\cup I_p$ is. Thus, the action of Γ_S on $B_0(S)$ sends p-simplices on p-simplices.

Moreover, the action is transitive.

Proposition 3.2. *The action of Γ_S is transitive on the set of p-simplices of $B_0(S)$.*

Proof. We will proceed by induction on the size of the simplices of $B_0(S)$. Let γ_0 and γ_1 be two vertices of $B_0(S)$ represented by arcs I_0 and I_1, respectively. Consider the surface S_{I_0} after cutting S along I_0 and identify naturally the points of S with the points of S_{I_0}. Notice that I_0 is contained in a boundary of S_{I_0}. In the same way identify the points of S with the points of S_{I_1}. Since cutting S along an arc of this kind results on a surface of genus one less and one more boundary component, by the classification of surfaces there is a diffeomorphism f from S_{I_0} to S_{I_1}. We can choose it in a way that the boundary component containing I_0 is mapped in the one containing I_1, and I_1 is the image of I_0. Thus, under the corresponding identification, we get a diffeomorphism of S sending I_0 on I_1.

Now, let $p \geq 1$. Let σ_p and τ_p be two p-simplices of $B_0(S)$. By induction hypothesis, assume that there exists an element $f \in \Gamma_S$ such that sends the first p arcs of σ_p on the first p arcs of τ_p. Let suppose that they are equal. If we split S along these arcs, we get a surface S' with two remaining non-separating arcs with the same endpoints. Applying the basis of induction we get the result. ■

Recall that the arcs defining a simplex of $B_0(S)$ are ordered. We are considering diffeomorphisms that fixe the boundary of S, then they preserve the order of arcs ending at a boundary component of S. Thus, if

$$f\cdot\{\langle I_0\rangle,\cdots,\langle I_n\rangle\}=\{\langle f(I_0)\rangle,\cdots,\langle f(I_n)\rangle\}=\{\langle I_0\rangle,\cdots,\langle I_n\rangle\},$$

then, since the vertices are ordered, we have $\langle f(I_j)\rangle = \langle I_j\rangle$ and f fixes all the faces of $\{\langle I_0\rangle\ \cdots,\langle I_n\rangle\}$. Hence, condition (B) holds for this action.

Stabilizers of the simplices of $B_0(S)$

Let $S = S_{g+1,b-1}$ and let $\sigma_0 = \langle I\rangle$ be some vertex of $B_0(S)$. Consider S_I, the surface resulting from cutting S along the arc I and denote it by R. Recall that $R \approx S_{g,b}$ and that S is gotten from R by attaching a pair of pants by two boundary components (see Figure 1.3 in Chapter 1). Thus we have $R \approx S_{g,b} \hookrightarrow S = S_{g+1,b-1}$, the case of stabilization by handles.

Note that f represents an element in Γ_R, it is a diffeomorphism $f : R \to R$ that fixes the boundary of R. Because of the cut, the boundary ∂R can be identified with $\partial S \cup I$ and $(R)^\circ$ with $(S\backslash I)^\circ$. Thus, f can be though as a diffeomorphism $f : S \to S$ such that fixes ∂S and I, i.e. f can be identified with an element in the stabilizer of σ_0, $St(\sigma_0) = \{[f] \in \Gamma_S :\ \langle f(I)\rangle = \langle I\rangle\}$.

Conversely, elements in $St(\sigma_0)$ are identified with elements in Γ_R. Thus, $\Gamma_{g,b} = \Gamma_R \cong St(\sigma_0)$ (Figure 3.3). Moreover, the inclusion of $St(\sigma_0)$ as a subgroup of $\Gamma_S = \Gamma_{g+1,b-1}$ can be identified with the inclusion $i : \Gamma_{g,b} \to \Gamma_{g+1,b-1}$.

Consider now the case of simplices of higher dimension. Let σ be some $(p-1)$-simplex ($p \geq 2$) with vertices $\langle I_0\rangle,\cdots,\langle I_{p-1}\rangle$, where the representatives $I_0,\cdots,I_{p-1}$ have pairwise disjoint interiors and $S\backslash(I_0\cup\cdots\cup I_{p-1})$ is connected. Let N be a regular neighborhood of the union $I_0\cup\cdots\cup I_{p-1}$ in S with boundary components that retracts on the arcs I_i. Take for example a cylindrical neighborhood N_i of each I_i, such that for $i \neq j$, N_i and N_j intersects only near the points b_0 and b_1 (see Figure 3.4). Let N be $N_0 \cup\cdots\cup N_{p-1}$ and we have the required neighborhood.

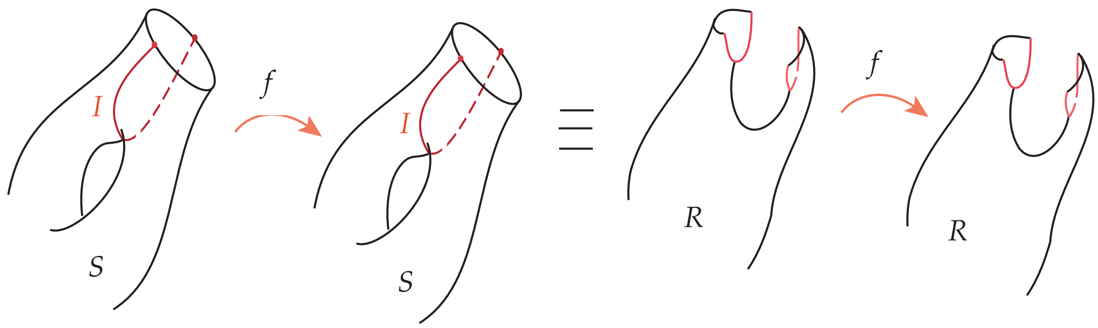


Figure 3.3: *Identification of $St(\sigma_0)$ with Γ_{S_I}. A diffeomorphism $f : S \to S$ that fixes ∂S and I can be identified with an $f : R \to R$ that fixes ∂R.*

Let R_σ be the complement in S of N [2]. Again, the stabilizer of σ may be identified with Γ_{R_σ} and the inclusion of the subgroup $St(\sigma)$ on Γ_S, with the inclusion $\Gamma_{R_\sigma} \hookrightarrow \Gamma_S$. Notice that for each arc I_j, the surface R_σ has one handle less than S, then $g_{R_\sigma} = g_S - p = (g+1) - p$.

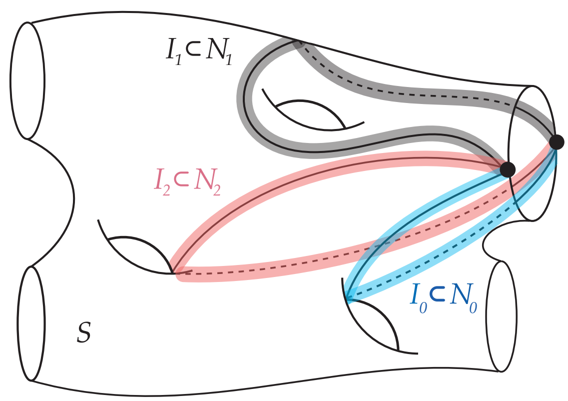


Figure 3.4: *Neighborhood $N = N_0 \cup N_1 \cup N_2$ of $I_0 \cup I_1 \cup I_2$.*

Now let $q \leq m$. Since $\Gamma_{R_\sigma} = St(\sigma)$, by induction hypothesis $i_* : H_q(St(\sigma)) \to H_q(\Gamma_S)$ is an isomorphism when $g_{R_\sigma} = g - p + 1 \geq 2q + 1$, and is onto if $g_{R_\sigma} = g - p + 1 \geq 2q$.

The spectral sequence associated

As before, consider $S = S_{g+1,b-1}$. By Theorem 1.6 the complex of curves $B_0(S)$ has degree of connectedness equal to $d = g_S - 2 = g - 1$. Take the spectral sequence arising from the action of Γ_S on $B_0(S)$. Following the general construction of Section 3.1, we have that, for $p \leq d + 2 = g + 1$, the E^1-term of the second spectral sequence has the form

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_{p-1}} H_q(St(\sigma)).$$

[2]Notice that we are taking the complement of N instead of cutting the surface along the arcs. With this we don't have the problem of duplicated boundaries.

Notice that if $(g-p+1) \geq 2q+1$, $p \geq 2$ and $q \leq m$, then $g+1 \geq 2q+1+p \geq p \geq 2$, and for $\sigma \in \Sigma_{p-1}$, we have $H_n(St(\sigma)) \cong H_n(\Gamma_S)$. Thus,

$$E^1_{pq} = \bigoplus H_q(\Gamma_S),$$

with one summand for each orbit of the $(p-1)$-simplices. Since the action of Γ_S on $B_0(S)$ is transitive, there is only one orbit and for $(g-p+1) \geq 2q+1$, $p \geq 2$ and $q \leq m$,

$$E^1_{pq} = H_q(\Gamma_S).$$

If $(g-p+1) \geq 2q$ and $q \leq m$, then we have a surjective function $i_* : H_q(St(\sigma)) \to H_q(\Gamma_S)$. In this range, the differential $d^1 : E^1_{p+1,q} \to E^1_{p,q}$ is an alternating sum of homomorphisms of the form $int(h)_* : H_q(St(\sigma_p)) \to H_q(St(\sigma_{p-1}))$, where $\sigma_p \in \Sigma_p$ and $\sigma_{p-1} \in \Sigma_{p-1}$, and the following diagram is commutative

$$\begin{tikzcd} int(h)_* : & H_q(St(\sigma_p)) \arrow[r] \arrow[d] & H_q(St(\sigma_{p-1})) \arrow[d] \\ int(h)_* : & H_q(\Gamma_S) \arrow[r] & H_q(\Gamma_S) \end{tikzcd}$$

Since the map $int(h)_* : H_q(\Gamma_S) \to H_q(\Gamma_S)$ is induced by an inner automorphism, it is the identity map.

Hence, for $g-p+1 \geq 2q+1$, $q \leq m$ and $p \geq 2$, we have

$$\begin{tikzcd} \cdots & E^1_{2,q} \arrow[l, "d^1"'] \arrow[d, "\cong"'] & \cdots \arrow[l] & E^1_{p,q} \arrow[l] \arrow[d, "\cong"] & E^1_{p+1,q} \arrow[l, "d^1"'] \arrow[d, "\cong \text{ or onto}"] \\ \cdots & H_q(\Gamma_S) \arrow[l, "id"] & \cdots \arrow[l] & H_q(\Gamma_S) \arrow[l] & H_q(\Gamma_S) \arrow[l, "id"] \end{tikzcd}$$

Proposition 3.3. *Let C_* and C'_* be chain complexes and $f : C'_* \to C_*$ be a chain map such that $f : C'_i \to C_i$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$. If $H_i(C_*) = 0$ for $i \leq n$, then $H_i(C'_*) = 0$ for $i \leq n$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd} C'_{i-1} \arrow[d, "f_{i-1}"', "\cong"] & C'_i \arrow[l, "\partial'_i"'] \arrow[d, "f_i"', "\cong"] & C'_{i+1} \arrow[l, "\partial'_{i+1}"'] \arrow[d, "f_{i+1}"', "\cong \text{ or onto}"] \\ C_{i-1} & C_i \arrow[l, "\partial_i"'] & C_{i+1} \arrow[l, "\partial_{i+1}"'] \end{tikzcd}$$

By hypothesis ker $\partial_i =$ im ∂_{i+1}. Let $x \in$ker ∂'_i, then $f \circ \partial'_i(x) = 0 = \partial_i \circ f(x)$. Thus $f(x) \in$ ker ∂_i and there exists some $y \in C_{i+1}$ such that $\partial_{i+1}(y) = f(x)$. Since f is at least onto for $i \leq n+1$, there is some $z \in C'_{i+1}$ such that $f(z) = y$ and $f \circ \partial'_{i+1}(z) = \partial_{i+1} \circ f(z) = \partial_{i+1}(y) = f(x)$. But $f_i : C'_i \to C_i$ is an isomorphism for $i \leq n$, then $x = \partial'_{i+1}(z) \in$ im ∂'_{i+1}. ■

We have seen that the complexes $(E^1_{*,q}, d^1)$ coincide with an acyclic complex in some range. From the previous proposition, $E^2_{p,q} = 0$ for $g-p+1 \geq 2q+1$, $q \leq m$ and $p \geq 2$.

Remember that we want to find ranges of injectivity and surjectivity for the homomorphism

$$i_* : H_{m+1}(\Gamma_R) \to H_{m+1}(\Gamma_S),$$

where $R = S_{g,b}$, $S = S_{g+1,b-1}$ and the map i_* is induced by $i : \Gamma_R \to \Gamma_S$. Recall that we have taken $St(\sigma_{-1})$ as Γ_S, and we have identified Γ_R with $St(\sigma_0)$ and the map i with the natural inclusion of $St(\sigma_0)$ as a subgroup of Γ_S. Moreover,

$$E^1_{1,m+1} = H_{m+1}(St(\sigma_0)),$$

$$E^1_{0,m+1} = H_{m+1}(St(\sigma_{-1})) = H_{m+1}(\Gamma_S),$$

and the differential $d^1 : E^1_{1,m+1} \to E^1_{0,m+1}$ is the homomorphism induced by the inclusion $St(\sigma_0) \hookrightarrow \Gamma_S$. Hence, proving that $d^1 : E^1_{1,m+1} \to E_{0,m+1}$ is a surjection for $g \geq 2(m+1)$ and an injection for $g \geq 2(m+1)+1$, we are done for this part of the proof.

3.2.1 Surjectivity

Let $g \geq 2(m+1)$. If $p+q \leq m+2$, $2 \leq p$ and $q \leq m$, then $g-p+1 \geq 2q+1$ which implies that $E^2_{p,q} = 0$. Take any $r \geq 2$, $p = r$ and $q = m+2-r$, then $E^2_{r,m+2-r} = 0$ and so $E^r_{r,m+2-r} = 0$. Thus, the differential of bi-degree $(-r, r-1)$, $d^r : E^r_{r,m+2-r} \to E^r_{0,m+1}$, is identically zero for $r \geq 2$.

$$\begin{tikzcd}
E^r_{0,m+1} & E^1_{1,m+1} \arrow[l, dashed, "d^1"'] \\
& E^r_{r,m+2-r} \arrow[ul, "d^r\equiv 0"]
\end{tikzcd}$$

Since $E^\infty = 0$, then $0 = E^\infty_{0,m+1} = \cdots = E^r_{0,m+1} = \cdots = E^2_{0,m+1} = E^1_{0,m+1}/\operatorname{im} d^1$. Then, necessarily, $\operatorname{im} d^1 = E^1_{0,m+1}$ and d^1 is surjective as we wanted.

3.2.2 Injectivity

Let now $g \geq 2(m+1)+1$. Then $E^2_{p,q} = 0$ for $p+q \leq m+3$, $2 \leq p$ and $q \leq m$. Hence $E^r_{r+1,m+2-r} = 0$ for any $r \geq 2$ and the differentials d^r acting on $E^r_{1,m+1}$ must be identically zero. Moreover, for $r \geq 2$, $E^r_{1-r,m+r} = 0$, and $d^r : E^r_{1,m+1} \to E^r_{1-r,m+r}$ is also zero. We shall show that $d^1 : E^1_{2,m+1} \to E^1_{1,m+1}$ is also identically zero.

$$\begin{tikzcd}
E_{1-r,m+r} & & & & \\
& E^1_{0,m+1} & E^r_{1,m+1} \arrow[l, dashed, "d^1"] \arrow[ull, "d^r\equiv 0"'] & E^1_{2,m+1} \arrow[l, dashed, "d^1\equiv 0"] & \\
& & & & E^r_{1+r,m+2-r} \arrow[ull, "d^r\equiv 0"]
\end{tikzcd}$$

Recall that $E^1_{2,m+1} = H_{m+1}(St(\sigma_1))$ and $E^1_{1,m+1} = H_{m+1}(St(\sigma_0))$, where σ_1 and σ_0 are representatives of the Γ_S-orbits of 1-simplices and 0-simplices, respectively. In this case the differential d^1 is the alternating sum of morphisms corresponding to the faces, say τ_1 and τ_2, of σ_1. Let h_1 and h_2 in Γ_S such that $h_1\tau_1 = \sigma_0$ and $h_2\tau_2 = \sigma_0$. Choose the representatives σ_1 and σ_0 in such a way that σ_0 is the face τ_1 (thus $h_1 = 1$) and h_2 is fixed in some subsurface $Q \subset S$ of genus $g-1$ disjoint to the curves defining simplex σ_1. Then, extending by identity, the elements of Γ_Q fixe σ_1 and Γ_Q can be thought as subgroup of $St(\sigma_1)$. Since $g - 1 \geq 2(m+1)$, assuming that the surjectivity part of the Stability Theorem[3] is true, we have that $H_{m+1}(\Gamma_Q) \to H_{m+1}(St(\sigma_1))$ is an epimorphism. Thus, we have the diagram

$$\begin{tikzcd}
H_{m+1}(St(\sigma_1)) \arrow[r, "int(h_i)_*"] & H_{m+1}(St(\sigma_0)) \\
H_{m+1}(\Gamma_Q) \arrow[u, "\text{onto}"] \arrow[ur, "int(h_i)_*"'] &
\end{tikzcd}$$

which is commutative for $i = 1, 2$. Since h_2 is fixed in Q, $int(h_2)|_{\Gamma_Q} = int(h_1)|_{\Gamma_Q}$ and the induced isomorphisms coincide. Hence, $d^1 = int(h_1)_* - int(h_2)_* = 0$.

Then, since $E^\infty = 0$, we have that $0 = E^\infty_{1,m+1} = \cdots = E^r_{1,m+1} = \cdots = E^2_{1,m+1} = \ker d^1$, and $d^1 : E^1_{1,m+1} \to E_{0,m+1}$ is an injection as we expected.

3.3 "Stabilization by holes"

Now we are concerned with the case of "stabilization by holes" (Figure 3.5). Again we proceed by induction assuming that Theorem 3.1 holds for $q \leq m$. In this section we will prove that the morphism

$$i_* = \Phi_* : H_{m+1}(\Gamma_{g,b}) \to H_{m+1}(\Gamma_{g,b+1}),$$

is surjective for $g \geq 2(m+1)$ and injective for $g \geq 2(m+1)+1$.

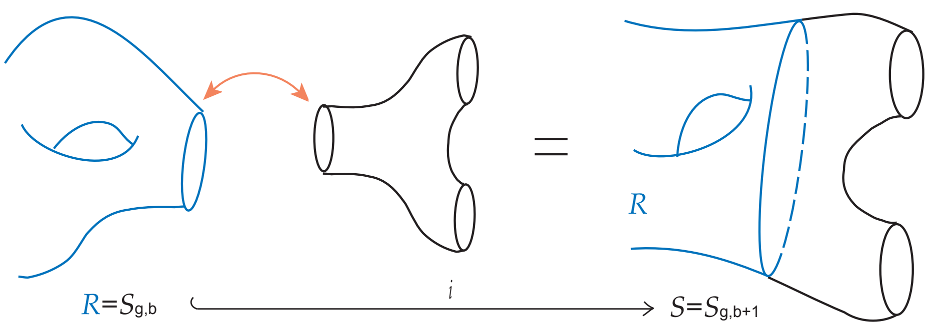


Figure 3.5: *The map of "Stabilization by holes".*

The action of Γ_S on $H(S)$

The group Γ_S acts naturally on complex $H(S)$ by

$$f \cdot \{\langle I_0\rangle \cdots, \langle I_p\rangle\} = \{\langle f(I_0)\rangle \cdots, \langle f(I_p)\rangle\},$$

for an f representing an element of Γ_S and $I_0, \cdots, I_p$ pairwise disjoint arcs joining b_0 and b_1, such that $\{\langle I_0\rangle \cdots, \langle I_p\rangle\}$ is an p-simplex of $H(S)$. Since $f : S \to S$ is a diffeomorphism (isotopy class) that fixes ∂S, then $\{\langle f(I_0)\rangle \cdots, \langle f(I_p)\rangle\}$ is a p-simplex of $H(S)$ and the action is well-defined.

[3]The surjectivity part for "stabilization by holes" is proved in the next section.

Recall that a p-simplex of $H(S)$ is determined by $(p+1)$ arcs from b_0 to b_1, on distinct components b_1S and b_2S of ∂S. Fix an orientation of b_1S and b_2S and label the arcs by $I_0, \cdots, I_p$ in order as they leave b_0. The order in which they come into ∂_2 gives a permutation of $(0, \cdots, p)$. Following Harer in [Har85], we label this p-simplex with this permutation.

Lemma 3.4. *Let σ_p and τ_p be two p-simplices of $H(S)$. Then, there is an $f \in \Gamma_S$ which takes σ_p to τ_p if and only if the orders of σ_p and τ_p coincide.*

Proof. Any $f \in \Gamma_S$ preserves orders. Suppose, conversely, that the orders of two p-simplices agree. Assume, inductively, that the first p-arcs are matched by some $f_0 \in \Gamma_S$ and therefore that they are equal. If we split S along these arcs, we get a connected surface S' with the two remaining non-separating arcs with the same endpoints (one from each original p-simplex). Any two such are equivalent in $\Gamma_{S'}$, by the classification of surfaces. ■

Remark 3.2:

- The previous lemma implies that the action of Γ_S on p-simplices of $H(S)$ is not transitive.
- If for some simplex σ of $H(S)$, there is an $f \in \Gamma_S$ such that $f \cdot \sigma = \sigma$, the order of the arcs defining σ must be preserved and then f fixes the faces of σ. Thus condition (B) holds.

It will be of interest to look at the factor complex $H(S)/\Gamma_S$. It is a finite complex, and since p-simplices with the same label are in the same Γ_S-orbit, there are $n_p \leq (p+1)!$ p-simplices. The useful thing of the complex $H(S)/\Gamma_S$ is that it is acyclic in some range.

Lemma 3.5. *$H_p(H(S)/\Gamma_S) = 0$, for $1 \leq p \leq g-1$.*

Proof. For $1 \leq p \leq g-1$, we can identify each p-simplex σ_p^i with its order $(i_0, \cdots, i_p)$ and label it. Then

$$\partial_p(i_0, \cdots, i_p) = \sum_{j=0}^{p} (-1)^j (\tau_j(i_0), \cdots, \hat{i_j}, \cdots, \tau_j(i_p)),$$

where τ_j is given by

$$\tau_j(n) = \begin{cases} n, & n < i_j, \\ n-1, & n > i_j. \end{cases}$$

Consider the map

$$\begin{array}{rccc} D_p : & C_p(H(S)/\Gamma_S) & \longrightarrow & C_{p+1}(H(S)/\Gamma_S) \\ & (i_0, \cdots, i_p) & \mapsto & (p+1, i_0, \cdots, i_p) \end{array}$$

Then,

$$\begin{array}{rcl} D_{p-1} \circ \partial_p(i_0, \cdots, i_p) & = & D_{p-1}\left(\sum_{j=0}^{p} (-1)^j (\tau_j(i_0), \cdots, \hat{i_j}, \cdots, \tau_j(i_p))\right) \\ & = & \sum_{j=0}^{p} (-1)^j (p+1, \tau_j(i_0), \cdots, \hat{i_j}, \cdots, \tau_j(i_p)), \end{array}$$

and,

$$\begin{array}{rcl} \partial_{p+1} \circ D_p(i_0, \cdots, i_p) & = & \partial_{p+1}(p+1, i_0, \cdots, i_p) \\ & = & \sum_{j=0}^{p} (-1)^{j+1} (p+1, \tau_{j+1}(i_0), \cdots, \hat{i_j}, \cdots, \tau_{j+1}(i_p)) + (i_0, \cdots, i_p). \end{array}$$

Hence, $D\partial + \partial D = id - 0$, and D is a chain homotopy (see [Spa66], chapter 4, sec. 2) in the range where $p < g$ from the chain map $id : C_*(H(S)/\Gamma_S) \to C_*(H(S)/\Gamma_S)$ and the zero chain map. Hence, $id_* = 0_*$ for $p < g$ and in this range $H_p(H(S)/\Gamma_S) = 0$. ■

The full value $n_p = (p+1)!$ occurs when $p \leq g$.

Stabilizers of the simplices of $H(S)$

Again we are interested in the isotropy groups of the simplices of $H(S)$. Let $S = S_{g,b-1}$ and $\sigma_0 = \langle I \rangle$ some vertex of $H(S)$. Cutting along the arc I, we get the surface S_I that we will denote by R. In this case the surface is homeomorphic to $S_{g,b}$ and S can be recovered from R by attaching a pair of pants by a boundary component (see Figure 1.4 in chapter 1). Notice that the inclusion $R \hookrightarrow S$ is the one of the “stabilization by holes”. Moreover, the elements of $St(\sigma_0)$ can be naturally identified elements of Γ_R (analogous to the case of complex $B_0(S)$), i.e. $St(\sigma_0) \cong \Gamma_R = \Gamma_{g,b}$.

For simplices of higher dimension we proceed as in the case of $B_0(S)$. Consider σ a $(p-1)$-simplex of $H(S)$ with $p \geq 2$). Let $I_0, \cdots, I_n$ be pairwise disjoint representatives of σ. Take a regular neighborhood N of $I_0 \cup \cdots \cup I_{p-1}$ in S with boundary components abutting on the arcs I_i. Let R_σ be the complement in S of N. The stabilizer $St(\sigma)$ can be identified with Γ_{R_σ} and the inclusion $St(\sigma) \subset \Gamma_S$ with the inclusion $\Gamma_{R_\sigma} \to \Gamma_S$.

Notice that $g_{R_\sigma} \geq g - p + 1$. When S is cut along a single arc, the resulting surface has the same genus of S. It remains cutting by the other $p-1$ arcs defining σ, which at most cut $p-1$ handles of the surface. Thus R_σ has, after cutting, at most $p-1$ handles less.

Now let $q \leq m$. By hypothesis of induction $i_* : H_q(St(\sigma)) \to H_q(\Gamma_S)$ is surjective if $g_{R_\sigma} \geq g - p + 1 \geq 2q$ and an isomorphism if $g_{R\sigma} \geq g - p + 1 \geq 2q + 1$.

The spectral sequence associated

Let $S = S_{g,b+1}$ and consider the complex of curves $H(S)$ where Γ_S acts. From Theorem 1.8 $H(S)$ is $d = 2g_S - 1 = 2g - 1$)-connected and we can construct the spectral sequence of Section 3.1. The E^1-term for $p \leq d + 2 = 2g + 1$ is then

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_{p-1}} H_q(St(\sigma)).$$

If $(g - p + 1) \geq 2q + 1$ and $q \leq m$, then $g + 1 \geq 2q + 1 + p \geq p$. Thus, $2g + 1 \geq p$ and $p - 1 \leq g$. By induction hypothesis, $H_n(St(\sigma)) \cong H_n(\Gamma_S)$ for $\sigma \in \Sigma_{p-1}$, with $(g - p + 1) \geq 2q + 1$ and $q \leq m$. In this range, the E^1-term has the form

$$E^1_{pq} = \bigoplus H_q(\Gamma_S),$$

with one summand for each element in Σ_{p-1}.

Recall that the differential $d^1 : E^1_{p+1,q} \to E^1_{p,q}$ restricted to $H_q(St(\sigma_p))$, for $\sigma_p \in \Sigma_p$, is the alternating sum of the mappings $int(h)_* : H_q(St(\sigma_p)) \to H_q(St(\sigma_{p-1}))$ corresponding to the faces $h\sigma_{p-1}$ of σ_p. In this range for p and q, these morphisms $int(h)_*$ make the following diagram commutative:

$$\begin{tikzcd} int(h)_* : H_q(St(\sigma_p)) \arrow[r] \arrow[d] & H_q(St(\sigma_{p-1})) \arrow[d] \\ int(h)_* : H_q(\Gamma_S) \arrow[r] & H_q(\Gamma_S) \end{tikzcd} .$$

Consider the chain complex $C_*(H(S)/\Gamma_S, H_q(\Gamma_S))$, where $C_p(H(S)/\Gamma_S, H_q(\Gamma_S)) \cong C_p(H(S)/\Gamma_S) \otimes H_q(\Gamma_S)$ and $C_p(H(S)/\Gamma_S)$ is the free group generated by the n_p p-simplices of the factor complex

$H(S)/\Gamma_S$. The differential of this complex is $\tilde{\partial} = \partial \otimes id_{H_q(\Gamma_S)}$. Notice that

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_{p-1}} H_q(St(\sigma)) \cong \underbrace{H_q(\Gamma_S) \oplus \cdots \oplus H_q(\Gamma_S)}_{n_{p-1}} \cong C_{p-1}(H(S)/\Gamma_S) \otimes H_q(\Gamma_S).$$

Moreover, there are morphisms such that make the following diagram commutative:

$$\begin{tikzcd}
E^1_{2,q} \arrow[d, "\cong"] & \cdots \arrow[l] & E^1_{p,q} \arrow[l] \arrow[d, "\cong"] & E^1_{p+1,q} \arrow[l, "d^1"'] \arrow[d, "\cong \text{ or onto}"] \\
C_1(H(S)/\Gamma_S, H_q(\Gamma_S)) & \cdots \arrow[l] & C_{p-1}(H(S)/\Gamma_S, H_q(\Gamma_S)) \arrow[l] & C_p(H(S)/\Gamma_S, H_q(\Gamma_S)) \arrow[l, "\tilde{\partial}"]
\end{tikzcd}$$

By Lemma B.11, the complex $C_*(H(S)/\Gamma_S, H_q(\Gamma_S))$ is acyclic for $2 \leq p \leq g$. Since for $g-p+1 \geq 2q+1$, $p \geq 2$ and $q \leq m$, $(E^1_{*,q}, d^1)$ coincides with $C_*(H(S)/\Gamma_S, H_q(\Gamma_S))$, it follows from Proposition 3.3 that $E^2_{p,q} = 0$ in this range.

Notice that $n_{-1} = 1$ and $n_0 = 1$, then $|\Sigma_{-1}| = |\Sigma_0| = 1$ and thus

$$E^1_{0,m+1} = H_{m+1}(St(\sigma_{-1})) \cong H_{m+1}(\Gamma_S)$$

and

$$E^1_{1,m+1} = H_{m+1}(St(\sigma_0)) \cong H_{m+1}(\Gamma_R).$$

Moreover, the differential $d^1 : E^1_{1,m+1} \to E^1_{0,m+1}$ corresponds to the morphism $i_* : H_{m+1}(\Gamma_R) \to H_{m+1}(\Gamma_S)$ whose range of surjectivity and injectivity we are interested in.

3.3.1 Surjectivity

For $g \geq 2(m+1)$, we proceed in a fully analogous way to the surjectivity proof for stabilization by handles.

3.3.2 Injectivity

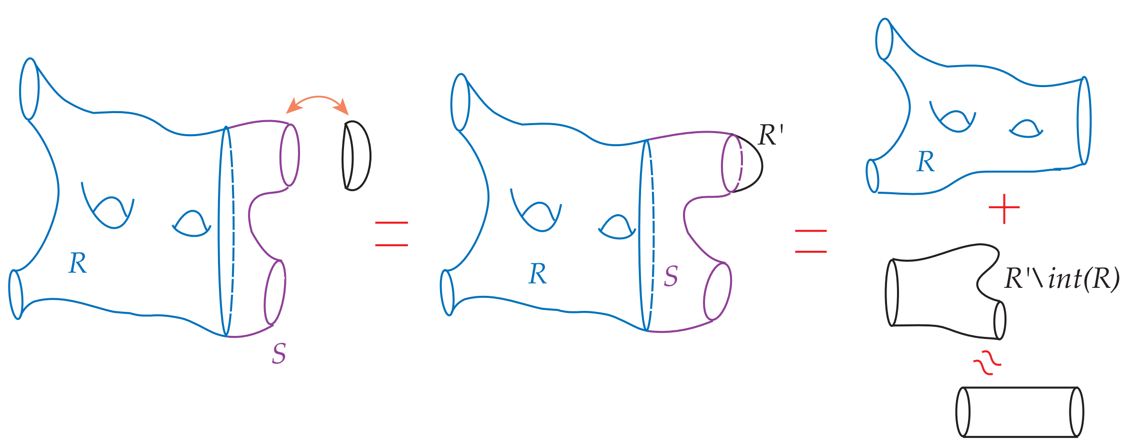


Figure 3.6: *Surface R' gotten from attaching a disc to a boundary component of surface S.*

Let $R \approx S_{g,b}$ a surface lying in $S \approx S_{g,b+1}$ as in the case of stabilization by holes. Without induction, we shall prove that $i_* : H_n(\Gamma_{g,b}) \to H_n(\Gamma_{g,b+1})$ is injective for all $n \in \mathbb{N}$. For attach a disc to one of the components of ∂S and get a new surface $R' \approx S_{g,b}$.

Note that $(R' \setminus \text{int}\ (R))$ is a cylinder (Figure 3.6) and then the composition map

$$\begin{array}{ccccc} \Gamma_R & \xrightarrow{i} & \Gamma_S & \longrightarrow & \Gamma_{R'} \\ (f : R \to R) & \mapsto & (f' : S \to S) & \mapsto & (f'' : R' \to R') \end{array},$$

where f' extends f by identity and so does f'' with respect to f', is an isomorphism. Hence the composition of the induced mappings $H_n(\Gamma_R) \xrightarrow{i_*} H_n(\Gamma_S) \to H_n(\Gamma_{R'})$ is an isomorphism and then i_* is a monomorphism as expected.

3.4 Stability Theorem for surfaces with boundary: an improvement

The Stability Theorem 3.1 can be generalized as follows:

Theorem 3.6. *Let R and S be connected surfaces with non-empty boundary such that R lies in S. The natural homomorphism $i_* : H_n(\Gamma_R) \to H_n(\Gamma_S)$ is surjective when $g_R \geq 2n$ and is injective when $g_S \geq 2n+1$.*

This version is due to Ivanov (see [Iva93]). Notice that Theorem 3.6 is the same as Theorem 3.1 removing the assumption: *every component of $S \setminus R$ contains a component of ∂S.* The importance of this assumption is that it allowed us to restrict the attention only to the cases of attaching a pair of pants to R to get S.

To prove Theorem 3.6, only assume that ∂S is not empty. Let Q be a subsurface of R, $Q \subset \text{int}(R)$, having the same genus as R and with only one boundary component. Because of that, $S \setminus Q$ and $R \setminus Q$ are connected and contain ∂S and ∂R, respectively (Figure 3.7).

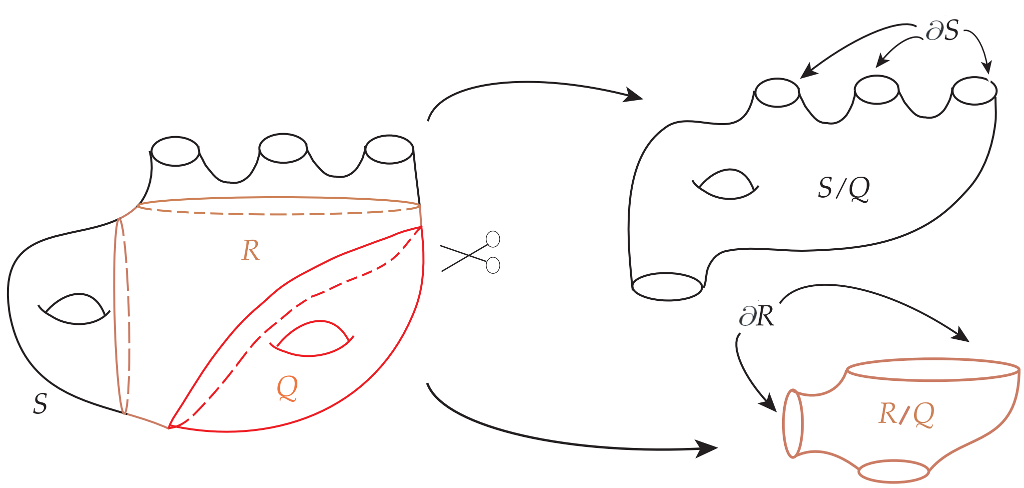


Figure 3.7: *The subsurface Q such that $S \setminus Q$ and $R \setminus Q$ are connected and contain ∂S and ∂R, respectively.*

Thus, Theorem 3.1 applies to both inclusions $Q \hookrightarrow R$ and $Q \hookrightarrow S$. From the following commutative diagram of induced maps

$$\begin{tikzcd} H_n(\Gamma_Q) \arrow[r] \arrow[d] & H_n(\Gamma_R) \arrow[dl] \\ H_n(\Gamma_S) & \end{tikzcd}$$

we have that $H_n(\Gamma_R) \to H_n(\Gamma_S)$ is surjective if $H_n(\Gamma_Q) \to H_n(\Gamma_S)$ is, and it is an isomorphism if both, $H_n(\Gamma_Q) \to H_n(\Gamma_S)$ and $H_n(\Gamma_Q) \to H_n(\Gamma_R)$, are isomorphisms. Since Q has the same genus as R, it follows that $H_n(\Gamma_R) \to H_n(\Gamma_S)$ is surjective (isomorphism) in the same range of the Stability Theorem, but without the additional assumption.

Appendix A

Teichmüller space, moduli space and mapping class group

A.1 The mapping class group: definition

Topologists use the name of *mapping class group* to refer to several related groups of isotopy classes of diffeomorphisms on a surface. In Teichmüller theory and algebraic geometry, *Teichmüller modular group* and *modular group* are also frequent names. In the following we describe some variants of this group.

Let $S = S^s_{g,b}$ be a compact orientable surface of genus g with b boundary components and s punctures. Let denote by $\text{Diff}^{\,+}S$ the group of orientation-preserving diffeomorphisms of S that fix the punctures and are the identity restricted to the boundary of S. Let $\text{Diff}^{\,+}_{\,0}S$ be the subgroup of all such diffeomorphisms isotopic to the identity on S.

Definition: The *mapping class group of* S, denoted by $\Gamma^s_{g,b} = \Gamma_S$ is the group of isotopy classes of orientation preserving diffeomorphism of S. Diffeomorphisms and isotopies are supposed to be the identity on the boundary of S and fix the punctures, i.e.

$$\Gamma^s_{g,b} = \Gamma_S = \text{Diff}^{\,+}S/\text{Diff}^{\,+}_{\,0}S.$$

Equivalently, $\Gamma^s_{g,b}$ is the group of path components of $\text{Diff}^{\,+}S$:

$$\Gamma^s_{g,b} = \pi_0(\text{Diff}^{\,+}S)$$

It is important that isotopies keep ∂S fixed point-wise for having a distinction between $\Gamma^s_{g,b}$ and $\Gamma^s_{g,0}$. This definition of mapping class group is the more appropriate in the context of the stability proof. The main advantage is that there exists a natural map $\Gamma_R \to \Gamma_S$ when R is a subsurface of S: a diffeomorphism of R fixed on ∂R can be extended by the identity to a diffeomorphism of S fixed on ∂S. Along the proof of the Stability Theorem, the case without punctures is considered and the group $\Gamma^s_{g,b}$ is just denoted by $\Gamma_{g,b}$.

When not only diffeomorphisms fixed at boundary and punctures are considered, the mapping class group is a larger group.

Definition: For a surface $S = S_{g,b}$ we denote by Mod_S the group of isotopy classes of orientation preserving diffeomorphism of S.

These groups are related by the next exact sequence:

$$1 \to \mathbb{Z}^b \to \Gamma_S \to Mod_S \to \mathcal{P}_b \to 1,$$

where $\mathbb{Z}^b$ is the group generated by the twists along the components of the boundary, and $\mathcal{P}_b$ is the group of permutations of the boundary components.

Some other variants are the following.

Definition: The *extended mapping class group* Mod_S is the group of isotopy classes of all preserving diffeomorphism of S.

Definition: The *pure mapping class group*[1] $PMod_S$ is the group of isotopy classes of orientation-preserving diffeomorphism of S preserving set-wise all the components of ∂S.

In the context of configuration spaces and braid groups it is useful to look at the mapping class groups for closed surfaces with punctures. Birman consider in [Bir75] two definitions: the case when the diffeomorphisms fix the punctures point-wise[2] (she calls it *pure mapping class group*) and the case when the punctures are fixed set-wise (called *full mapping class group*).

A.2 Metric and complex structures on surfaces

It is well known that there is a complete topological classification for two-dimensional compact and connected manifolds without boundary (closed surfaces). The Euler characteristic together with the property of being orientable or not can serve as a complete set of invariants.

Theorem A.1 (Classification of surfaces). *Every closed surface S is homeomorphic to one and only one of the following surfaces:*

$$S^2\#\underbrace{T\#\ldots\#T}_{g},\ \text{for } g \geq 0 \qquad \text{(orientable)}$$

$$S^2\#\underbrace{\mathbb{R}P^2\#\ldots\#\mathbb{R}P^2}_{h},\ \text{for } h \geq 1 \qquad \text{(non-orientable)}$$

Here # denotes the connected sum of surfaces. Recall that the Euler characteristic of the surface determines the genus of the surface and viceversa. For orientable surfaces $\chi(F_g) = 2 - 2g$ and for non-orientable surfaces $\chi(F_h) = 2 - h$.

In addition to the topological information, we can endow a surface S with some extra structure.

Definition: A *metric (or geometric) structure on S* is a complete Riemannian metric of constant curvature[3].

We can give a metric structure to a surface S as follows. Suppose that we have an oriented polygonal region in dimension two with an even number of boundary edges. Give the edges the orientation induced by the region. Choosing a "gluing pattern", that is, pairing edges by orientation preserving or orientation-reversing homemorphisms, we get a surface (orientable or non-orientable). Except for positive Euler characteristic, all other surfaces can be constructed in this way and then inherit a Riemannian structure of constant curvature. Details of this construction can be found in [Thu97]. Thus,

[1] This name is used by Ivanov in [Iva02].

[2] It corresponds to $\Gamma^s_{g,0}$ previously defined.

[3] And geodesic boundary if $\partial S \neq \phi$. We are considering closed surface, unless the opposite is specified.

- If $\chi(S) > 0$, then $S \approx S^2$ $(g = 0)$ or $S \approx S^2/\{\pm 1\} = \mathbb{R}P^2$ $(h = 1)$ and S has a spherical (*elliptic*) structure.
- If $\chi(S) = 0$, then S is the torus $(g = 0)$ in the orientable case and the Klein bottle $(h = 1)$ in the non-orientable case. In both cases S can be obtained by gluing edges of a square so that all vertices are identified together. We can consider a tiling of the Euclidean plane by squares, and then identify the edges. Hence S inherits naturally an *Euclidean structure.*
- If $\chi(S) < 0$, then $g < 0$ or $h < 1$ and for constructing S we need a polygon with more than six edges. Hence the angle sum of the polygon is greater than 2π and we cannot have a tiling in Eucledian plane, but in the hyperbolic plane $\mathbb{H}^2$. So S inherits a hyperbolic structure form $\mathbb{H}^2$ (see Theorem 1.1.2 in Chapter 1).

In what follows we will only consider orientble surfaces. See also section of Preliminaries in Chapter 1 and references therein for orientable surfaces with boundary components.

On the other hand, a surface can be endowed with a complex structure.

Definition: Let S be a connected surface. Consider an open covering $\{U_\alpha\}$ of S, where each U_α is furnished with a topological mapping $z_\alpha : U_\alpha \to z_\alpha(U_\alpha)$, where $z_\alpha(U_\alpha) \subseteq \mathbb{C}$ is open and the transition mappings

$$z_{\alpha\beta} = z_\alpha \circ \left(z_\beta \mid_{z_\beta(U_alpha \cap U_\beta)} \right)^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic The collection $\{U_\alpha, z_\alpha\}$ is called a*coordinate atlas* or a *complex structure on S.*

A *Riemann surface* is a surface S together with a complex structure $\{U_\alpha, z_\alpha\}$ on it.

Remark A.1: Since a homeomorphism in $\mathbb{C}$ is holomorphic if and only if is conformal and preserves orientation, then the transition mappings of a Riemann surface preserve orientation and every Riemann surface is orientable.

Metric and complex structures are closely related. Since orientation preserving isometries of $\mathbb{H}^2$ and $\mathbb{E}^2$ are biholomorphic, then every hyperbolic and Euclidean orientable surface inherit the structure of a complex manifold. Moreover the stereographic projection from the unit sphere to $\mathbb{C}$ is a conformal map. Hence, a complex structure can be given to S^2, the *Riemann sphere.*

It follows that elliptic, Euclidean and hyperbolic structures on any orientable surface give rise to a complex structure. This observation has a converse: "every complex structure on a surface S corresponds to a hyperbolic, Euclidean or elliptic metric". This applies to any Riemann surface (even non-closed or with infinitely many holes) and it is called the *uniformization theorem*:

Theorem A.2. *Each Riemann surface S is conformally equivalent to either one of the following:*

(i) $\mathbb{C}$

(ii) $\mathbb{C} - \{0\}$

(iii) $S^2 (= \mathbb{C} \cup \{\infty\})$

(iv) $\mathbb{C}/\Lambda$, *where Λ is a lattice.*

(v) $\mathbb{H}^2/\Lambda$, *where Λ is a fix point free, properly discontinuous group of conformal automorphisms of $\mathbb{H}^2$.*

Remark A.2:

- It is clear that each Riemann surface naturally inherits a metric of constant curvature. We have curvature 0 for cases (i), (ii) and (iv), 1 for case (iii) and -1 for (v).
- Only in cases (iii), (iv) and (v) we can have compact surfaces. (v) may also be noncompact.

Then, every Riemann surface S is conformally equivalent to one which admits a metric structure, and this metric is uniquely determined, up to isometry, by the conformal equivalence class of S. Hence, we can study compact Riemman surfaces looking at its complex structure or at its metric structure. In any way we are interested in classifying Riemann surfaces. Now the question is what notion of equivalence between surfaces are we considering.

A.3 On the definition of Teichmüller space

Complex approach

When we are considering surfaces with a complex structure, the topological classification is not enough. The theory of Teichmüller space stems from the fact that topologically equivalent Riemann surfaces need not to be equivalent in a holomorphic fashion.

There are two notions of equivalence between Riemann surfaces that give rise to two spaces: *moduli space* and *Teichmüller space*. Informally, in Teichmüller space we consider what complex structure the surface is wearing and how it is worn. In moduli space, all surfaces wearing the same complex structure are equivalent.

The notion of "wearing" the same complex structure is given by the following definition:

Definition: Let S_1 and S_2 be compact Riemann surfaces of genus g. We say that S_1 and S_2 are *conformally equivalent* if there is a conformal homeomorphism $k : S_1 \to S_2$. Denote an equivalence class by $[S]$. The space of equivalence classes is called the *moduli space of Riemann surfaces* $\mathcal{M}_g$.

Remark A.3: The moduli space $\mathcal{M}$ is not generally a complex-analytic manifold. Thus is convenient to have a representation

$$\mathcal{M} = \mathcal{T}/\Gamma,$$

where $\mathcal{T}$ is a complex manifold and Γ is a discontinuous group of analytic self-mappings of $\mathcal{T}$.

Definition: Let S be a compact topological surface, S_1 a Riemann surface and $f : S \to S_1$ an orientation-preserving homemorphism . Denote by $[f]$ the homotopy class of f. An ordered triple $(S, [f], S_1)$ is called a *marked surface with respect to S.*

Two marked surfaces $(S, [f_1], S_1)$ and $(S, [f_2], S_2)$ are called equivalent if there exists a conformal mapping $h : S_1 \to S_2$ such that $f_2 \simeq h \circ f_1$, i.e. the following diagram is commutative, up to homotopy.

$$\begin{tikzcd} S \arrow[r, "f_1"] \arrow[dr, "f_2"'] & S_1 \arrow[d, "h"] \arrow[dl, phantom, "\simeq"] \\ & S_2 \end{tikzcd}$$

The *Teichmüller space* $\mathcal{T}_g$ is the set of equivalence classes of marked Riemann surfaces.

Remark A.4:

(i) Isotopy is much stronger than homotopy; however, for surfaces, it is known that they are equivalent (see [Mor01]). Moreover, the previous definition remains the same if we consider diffeomorphisms instead of homeomorphisms.

(ii) The purpose of $[f]$ on a marked surface is to restrict the admissible conformal mappings between one surface and another (how the surface "is wearing" the conformal structure).

(iii) By "forgetting" the marking $[f]$ we get a natural projection mapping that relate the two spaces:

$$\begin{array}{ccc} \mathcal{T}_g & \longrightarrow & \mathcal{M}_g \\ [(S,[f],S_1)] & \mapsto & [S_1] \end{array}$$

(iv) The Teichmüller space was defined by Teichmüller in the 1930's.

Let denote by Γ_g the mapping class group $\Gamma^0_{g,0}$. Γ_g acts on $\mathcal{T}_g$ as follows:

$$\begin{array}{ccc} \mathcal{T}_g \times \Gamma_g & \longrightarrow & \mathcal{T}_g \\ ([(S,[f],S_1)],g) & \mapsto & [(S,[g\circ f],S_1)] \end{array}.$$

Notice that the space of orbits "forgets" the marking and we have

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g.$$

Example. For genus $g=0$ the Teichmüller space and the moduli space consist each of a single point, corresponding to the Riemann sphere. For $g=1$ the associated complex dimension is 1, parametrizing the space of elliptic curves. In this case $\mathcal{T}_g = \mathbb{H}$, $\Gamma_g = EMG =$ elliptic modular group and

$$\mathcal{M}_g = \mathbb{H}/EMG \simeq \mathbb{C} - 0.$$

Metric approach

We can think about equivalent metrics instead of equivalent complex structures. Consider a surface S which admits a hyperbolic structure, and let $\mathcal{S}_S$ be the space of such structures.

Definition: The *moduli space of* S is

$$\mathcal{M}S = \mathcal{S}_S/\mathrm{Diff}\ S.$$

The Teichmüller space of S is

$$\mathcal{T}S = \mathcal{S}_S/\mathrm{Diff}\ _0 S.$$

Remark A.5:

- We can reformulate the previous definition as follows:

  A triple (S,f_1,S_1), where S is a topological surface, S_1 is a hyperbolic surface and $f_1 : S \to S_1$ is a diffeomorphism, is called a *marked surface*. Consider all surfaces marked by S. To get $\mathcal{T}S$, identify (S,f_1,S_1) and (S,f_2,S_2) if and only if $f_2\circ f_1$ is isotopic to an isometry. To get $\mathcal{M}S$ forget the marking and identify isometric surfaces S_1 and S_2.

- If Γ_S is the mapping class group defined before, it is clear again that

$$\mathcal{M}S = \mathcal{T}S/\Gamma_S.$$

Because of the uniformization theorem we know that, for genus larger than 1, the study of complex structures on a surface is also the study of its hyperbolic structures. Thus, $\mathcal{T}S = \mathcal{T}_S$ and $\mathcal{M}S = \mathcal{M}_S$.

Teichmüller space for surfaces with punctures

A distinguished point in a Riemann surface will be called a puncture on the surface. Take s distinct ordered points $p_1, \cdots, p_s$ on the surface $S = S_g$.

Consider the triples $(S_1, (q_1, \cdots, , q_s), [f_1])$, where S_1 is a Riemann surface, $q_1, \cdots, , q_s$ are distinct ordered points in S_1, and $f_1 : S \to S_1$ is a homeomorphism with $f_1(p_i) = q_i$ for each i. Let denote by $[f_1]$ the homotopy class of f_1 rel $\{p_i\}$. Again define an equivalence relation between these marked surfaces as follows: $(S_1, (q_1^1, \cdots, , q_s^1), [f_1]) \sim (S_2, (q_1^2, \cdots, , q_s^2), [f_2])$ if and only if there exists a conformal diffeomorphism $h : S_1 \to S_2$ with $h(q_i^1) = q_i^2$ for each i and such that $[h \circ f_1] = [f_2]$. The space of equivalence classes is the Teichmüller space and is denoted by $\mathcal{T}_g^s$.

The mapping class group Γ_g^s is the group of diffeomorphisms $f : S \to S$ such that $f(p_i) = p_i$ for all i, up to isotopy fixing each p_i. Γ_g^s acts on $\mathcal{T}_g^s$ as before and $\mathcal{M}_g^s = \mathcal{T}_g^s / \Gamma_g^s$ is the moduli space.

Fenchel-Nielsen Coordinates

Roughly speaking, the "moduli problem" consists in assigning a set of parameters to a point in $\mathcal{M}_S$ in such a way that any variation of these parameters correspond to a "nice" variation of the associated surfaces. In our case this is done by determining how the elements of $\mathcal{T}_S$ can be parametrized and then using at the relation between the moduli space of Riemann surfaces and the Teichmüller space. The parameters for the elements of $\mathcal{T}_S$ are the Fenchel-Nielsen coordinates.

Let $S = S_g$ be an oriented Riemann surface of genus g with $\chi(S_g) < 0$. Select a maximal collection of pairwise disjoint and non-isotopic simple closed curves on S_g and denote them by $C_0^0, \cdots, C_{3g-3}^0$. Notice that $\langle C_0^0, \cdots, C_{3g-3}^0 \rangle$ is a simplex of maximal dimension on complex $C(S)$ defined in Chapter 1.

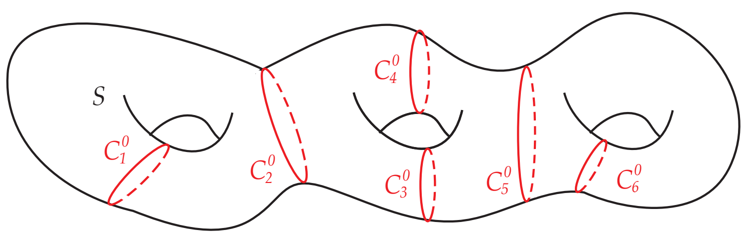


Figure A.1: Collection of curves $\{C_0^0, \cdots, C_{3g-3}^0\}$ on S_g for defining the Fenchel-Nielsen coordinates of Teichmüller space.

Let the triple $(S, [f], S_1)$ represents an element of $\mathcal{T}_g$ and let C_i be the closed geodesic on S_1 isotopic to $f(C_i^0)$. Again the curves $C_0, \cdots, C_{3g-3}$ are simple and mutually disjoint.

Cutting the surface S_1 along the curves $C_0, \cdots, C_{3g-3}$ we have a decomposition of the surface in $|\chi(S_g)| = 2g - 2$ pairs of pants[4]

[4]They are also called three-circle domains.

As we have mentioned in Theorem 1.1.2 of Chapter 1, we can endow each pair of pants with an hyperbolic metric. Indeed, the lengths $l_1, l_2, l_3 > 0$ of the boundaries of a pair of pants determines its conformal class.

We may rebuilt the marked Riemann surface by gluing hyperbolic pairs of pants together following the pattern determined by the curves C_i. The *Fenchel-Nielsen coordinates* are the free parameters for this construction. The pairs of pants have a total of $3(2g-2)$ boundary components that must be glued by pairs along $3g-3$ curves. There are two parameters for each curve. A first parameter corresponds to the length of the curve l_i and takes values in $\mathbb{R}^+$. The second parameter, the twist parameter, has to do with how the boundary curves are glued and takes values in $\mathbb{R}$ (see [Jos97] or [Thu97] for a specific description of this twist parameter).

Thus, we obtain a mapping

$$\begin{array}{cccc} F: & \mathcal{T}_g & \longrightarrow & (\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3} \\ & [(S,[f],S_1)] & \mapsto & (l_1, \cdots, l_{3g-3}, \tau_1, \cdots, \tau_{3g-3}) \end{array}$$

Definition: $(l_1, \cdots, l_{3g-3}, \tau_1, \cdots, l_{3g-3}) = F([(S,[f],S_1)])$ are called the *Fenchel-Nielsen coordinates of* $[(S,[f],S_1)]$.

Fenchel and Nielsen proved

Theorem A.3. *The map $F : \mathcal{T}_g \to (\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$is a diffeomorphism. Hence, $\mathcal{T}_g$ is diffeomorphic to $\mathbb{R}^{6g-6}$.*

A proof of this result can be found in [Thu97]. Analogous constructions can be done for the cases when the surface has b boundary components (Teichmüller space can be defined in a similar way, see [Jos97]) and when the surface has s punctures. The corresponding results are

Theorem A.4. *$\mathcal{T}_{g,b}$ is diffeomorphic to $\mathbb{R}^{6g-6+3b}$.*

Theorem A.5. *$\mathcal{T}_g^s$ is diffeomorphic to $\mathbb{R}^{6g-6+2s}$.*

In [Jos97] we can find some remarks about the non-orientable case. In all cases the Teichmüller space is diffeomorphic to an Euclidean space, thus it is contractible. Indeed, for closed surfaces, the Teichmüller even has a canonical complex analytic structure.

A.4 Moduli space and mapping class group.

We have already described the action of the mapping class group $\Gamma_{g,b}^s$ on $\mathcal{T}_{g,b}^s$. For closed surfaces with negative Euler characteristic, i.e. $b = 0$ and $2g + s \geq 3$, this action is properly discontinuous (see [Har85]).

Even if the action is not free, the stabilizers of the points are finite and the action is as close to being free as possible: any torsion free subgroup acts freely on $\mathcal{T}_g^s$. It is also known that Γ_g^s contains torsion free subgroups, even of finite index, i.e. it is virtually torsion free[5] (see [Har88] for an explicit subgroup).

Since $\mathcal{T}_g^s$ is contractible, Γ_g^s is virtually torsion free and the action is properly discontinuous, then $\mathcal{M}_g^s$ is "rationally a $K(\Gamma_g^s, 1)$", that is, there is an isomorphism for all $k \geq 0$

$$H_k(\mathcal{M}_g^s; \mathbb{Q}) = H_k(\Gamma_g^s; \mathbb{Q}).$$

[5] We say that a property for a group is satisfied *vitually* if some subgroup of finite index satisfies it.

The only cases for which these groups are explicitly are $k = 1, 2$ (see [Har85] for references):

$$H_1(\Gamma^s_{g,b}; \mathbb{Z}) \cong 0,\ g \geq 3,$$

$$H_2(\Gamma^s_{g,b}; \mathbb{Z}) \cong \mathbb{Z}^{s+1},\ g \geq 5.$$

Appendix B

Homology of groups

Here we present some definitions and results apropos the homology of groups that are needed in the stability proof. Most of the proofs are omitted and can be found in [Bro94] or in [AM04].

B.1 Some homological algebra

Group ring

Let G be a group, written multiplicatively.

Definition: Let $\mathbb{Z}G$ be the $\mathbb{Z}$-module generated by the elements of G.

$$\mathbb{Z}G = \left\{ \sum_{g \in G} a(g)g : \ a(g) \in \mathbb{Z}, \ a(g) = 0 \text{ for almost all } g \in G \right\},$$

with the multiplication in G extended uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}G$, is called the *integral group ring of* G.

Notice that $G \leq (\mathbb{Z}G)^*$ and it satisfies a *universal mapping property*:

> For any ring R and a group homomorphism $f : G \to R^*$, there exists a unique ring homomorphism $\hat{f} : \mathbb{Z}G \to R$ that extends f. So the next diagram is commutative

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[d, "f"'] & (\mathbb{Z}G)^* \arrow[r, hook, "\iota"] & \mathbb{Z}G \arrow[dl, dashed, "\hat{f}"] \\ R^* \arrow[r, hook, "\iota'"] & R & \end{tikzcd}$$

Thus, there is an "adjunction formula":

$$\mathrm{Hom}_{\mathrm{rings}}(\mathbb{Z}G, R) \cong \ \mathrm{Hom}_{\mathrm{groups}}(G, R^*).$$

Definition: A $\mathbb{Z}G$*-module* or G*-module* is an abelian group A together with a ring homomorphism $\mathbb{Z}G \to \mathrm{End}(A)$.

By the adjunction formula, a G-module is then an abelian group A with an action of G on A.

Example. $\mathbb{Z}$ is a G-module with the trivial G action $g \cdot z = z$, $(\forall\, g \in G, \ z \in \mathbb{Z})$.

Example: Cyclic groups. Let G be a cyclic group of order n and let t be a generator. Then a $\mathbb{Z}$-basis for $\mathbb{Z}G$ is t^i, $0 \leq i \leq n-1$ and $t^n = 1$. Thus,

$$\mathbb{Z}G \cong \mathbb{Z}[x]/\langle x^n - 1 \rangle.$$

If G is cyclic infinite, then

$$\mathbb{Z}G \cong \mathbb{Z}[x, x^{-1}],$$

the ring of Laurent polynomials.

Definition: A *G-complex* (*G-set*) is a CW-complex (set) X together with an action of G on X which permutes the cells. X is a *free* G-complex (G-set) if the action of G freely permutes the cells of X (is free on X).

Example. If X is a simplicial complex on which G acts simplicially, then X is a G-complex.

Given a G-set, we can construct a G-module as follows. Let X be a G-set. Consider $\mathbb{Z}X$ the free abelian group generated by X. Extending the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}G$, the group $\mathbb{Z}X$ becomes a G-module called the *permutation module.*

Example. For every subgroup H of G, we can consider the action of G on G/H by left translation, which gives rise to the permutation module $\mathbb{Z}[G/H]$.

Definition: Let X be a G-set and $x \in X$. The *stabilizer of x at G* or the *isotropy subgroup of G at x* is the subgroup $St(x) = G_x = \{g \in G : \ gx = x\}$.

Remark B.1:

- The disjoint union in the category of G-sets corresponds to the direct sum in the category of G-modules:
$$\mathbb{Z}\left[\coprod X_i\right] = \bigoplus \mathbb{Z}X_i.$$

- Let E be a set of representatives for the G-orbits on X. We can identify the orbit of $x \in E$, Gx with $G/St(x)$. Thus
$$\mathbb{Z}X = \mathbb{Z}\left[\coprod_{x \in E} Gx\right] \cong \bigoplus_{x \in E} \mathbb{Z}[G/St(x)].$$

It follows that:

Proposition B.1. *Let X be a free G-set and E as before. Then the permutation module $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.*

Resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$

Let R be a ring (associative, with identity) and M a left R-module.

Definition: A *resolution of M* is a long exact sequence of R-modules of the form

$$\ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

If each F_i is free, it is called a *free resolution.*

Remark B.2:

- For any G-module M, we can construct a free resolution as follows. Take a set S of $\mathbb{Z}G$-generators $m_1, m_2 \dots$ for M and consider the free G-module generated by S, $C_0 = \coprod(\mathbb{Z}G)_{m_i}$, and the natural projection

$$\begin{array}{cccc} \partial_0 : & C_0 & \longrightarrow & M \\ & \sum \theta_i 1_{m_i} & \mapsto & \sum \theta_i m_i. \end{array}$$

  Consider now ker ∂_0 and construct as before a surjective map

$$\partial_1 : C_1 \longrightarrow (\ker \partial_0) \subset C_0.$$

  Repeating this procedure we get a free resolution for M.

- Any two free resolutions of an R-module M are homotopy equivalent (see [Bro94] chap 1). The uniqueness is the algebraic analogue of Hurewicz's result about the uniqueness, up to homotopy, to an aspherical space given a fundamental group.

We are interested in free resolutions of $\mathbb{Z}$ as a G-module. They arise from free actions of G on contractible complexes.

Proposition B.2. *Let X be a contractible free G-complex. Then the augmented cellular chain*

$$\dots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \dots \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Proposition B.3. *If Y is a $K(G,1)$, then the augmented chain complex of the universal cover of Y is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Remark B.3: When the action of G on a contractible simplicial complex X is only free on vertices, but not on higher dimension simplices, we use the ordered chain complex $C'_*(X)$ instead of the oriented one (see [Spa66], chap 4), in which an ordered n-simplex is sequence $v_0, \dots, v_n$ of $n+1$ vertices of X which belongs to some simplex of X.

Example: The standard resolution. Consider the "simplex" X spanned by G: the vertices of X are the elements of G and every finite subset of G is a simplex of X. When G is finite, X is actually a simplex. The action of G on X is by left translation and X is contractible. The corresponding free resolution $F_* = C'_*(X)$ is called the *standard resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

F_* is described explicitly as follows. F_n is the $\mathbb{Z}$-module generated by the $(n+1)$-tuples $(g_0, \dots, g_n)$ of elements of G, and G acts on F_n by $g \cdot (g_0, \dots, g_n) = (g \cdot g_0, \dots, g \cdot g_n)$. The boundary operator is given by $\partial = \sum_{i=0}^n (-1)^i d_i$ with

$$d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n),$$

and the augmentation map $\epsilon : F_0 \to \mathbb{Z}$ is $\epsilon(g_0) = 1$.

Notice that each G-orbit in F_n has a representative of the form $(1, h_1, h_2 \dots, h_n)$. Usually the bar notation $[g_1|g_2|\dots|g_n]$ is used for representatives of the form $(1, g_1, g_1g_2, \dots, g_1g_2\dots g_n)$. Thus F_n is a G-module with basis $[g_1|g_2|\dots|g_n]$. For $n = 0$, $F_0 \cong \mathbb{Z}G$ and the unique basis element is denoted by $[\,]$. In terms of this $\mathbb{Z}G$-basis, the boundary $\mathbb{Z}G$-homomorphism is given by $\partial = \sum_{i=0}^n (-1)^i d_i$ with

$$d_i([g_1|\dots|g_n]) = \begin{cases} g_1[g_2|\dots|g_n], & i = 0 \\ [g_1|\dots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\dots|g_n], & 0 \leq i \leq n \\ [g_1|\dots|g_{n-1}], & i = n \end{cases}$$

F_* is often called the bar resolution. In low dimensions F_* takes the form

$$\dots F_2 \overset{\partial_2}{\to} F_1 \overset{\partial_1}{\to} \mathbb{Z}G \overset{\epsilon}{\to} \mathbb{Z} \to 0,$$

with $\partial_2[g|h] = g[h] - [gh] + [g]$, $\partial_1[g] = g[\,] - [\,] = g - 1$, and $\epsilon([\,]) = 1$.

Example: Finite cyclic groups. Let G be a finite cyclic group with generator t of order n. Consider S^1 as a CW-complex with n vertices and n cells of dimension 1. Then G acts freely as a group of rotations of S^1, and S^1 is a G-complex. The group $H_1(S^1) \cong \mathbb{Z}$ is generated by the cycle

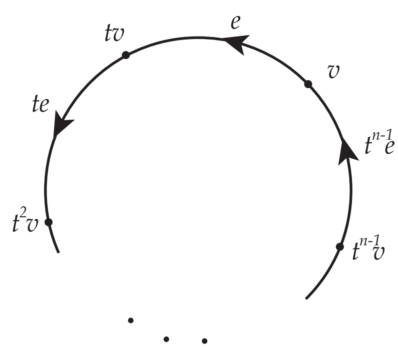


Figure B.1: *The group G acts on S^1 by rotation.*

$e + te + t^2e + \dots + t^{n-1}e = Ne$ (e as in the Figure B.1), where $N = 1 + t + \dots + t^{n-1} \in \mathbb{Z}G$. From the augmented cellular chain complex of S^1 we have the following exact sequence:

$$0 \to \mathbb{Z} \overset{\iota}{\to} C_1(S^1) \overset{t-1}{\to} C_0(S^1) \overset{\epsilon}{\to} \mathbb{Z} \to 0,$$

where $C_0(S^1) \cong C_1(S^1) \cong \mathbb{Z}G$ and $\iota(1) = N$. Pasting an infinite number of copies of this short sequence, we get the following free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$\dots \to \mathbb{Z}G \overset{t-1}{\to} \mathbb{Z}G \overset{N}{\to} \mathbb{Z}G \overset{t-1}{\to} \mathbb{Z}G \overset{\epsilon}{\to} \mathbb{Z} \to 0, \tag{B.1}$$

where $N = \iota \circ \epsilon$.

B.2 The homology of a group

Now we are interested in constructing some homological invariants of groups.

Let M be a right R-module and N a left one. Recall that the tensor product $M \otimes_R N$ is the quotient $M \otimes_{\mathbb{Z}} N = M \otimes N$ by introducing the relations $mr \otimes n = m \otimes rn$, ($m \in M$, $n \in N$, $r \in R$). Let G be a group and M a left G-module.

Definition: The *group of co-invariants of M*, denoted by M_G, is the quotient of M by the additive subgroup generated by the elements $gm - m$, where $g \in G$ and $m \in M$.

Remark B.4:

- M_G is gotten from M by "dividing out" by the G action. M_G is the largest quotient of M on which G acts trivially.

- $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$. Consider the homomorphism $\overline{m} \mapsto 1 \otimes m$ and its inverse $x \otimes m \mapsto x\overline{m}$.

Thus we have the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} \_\_$ from the category of G-modules to the category of abelian groups, with the following properties:

1. It is right exact.
2. If F is a free G-module, then F_G is a free $\mathbb{Z}$-module.

The co-invariant functor arises naturally in the topological context.

Proposition B.4. *Let X be a free G-complex and Y the orbit complex X/G. Then $C_*(Y) \cong C_*(X)_G$.*

We are now ready for defining the homology of a group. Let G be a group and $\epsilon : F \to \mathbb{Z}$ a free resolution[1] of $\mathbb{Z}$ over $\mathbb{Z}G$, and consider the reduced resolution

$$... \to F_2 \to F_1 \to F_0 \to 0.$$

Definition: The *homology groups of G* are defined by $H_iG = H_i(F_G)$.

Example: Finite cyclic groups. Consider the resolution (B.1) and apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} \_\_$. We have the isomorphism $\phi : \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \to \mathbb{Z}$ given by $\phi(1 \otimes t^i) = (t^i) \cdot 1 = 1$. Moreover, $[1_{\mathbb{Z}} \otimes (t-1)](1 \otimes N) = 0$ and $[1_{\mathbb{Z}} \otimes N](1 \otimes 1) = \sum_{i=0}^{n-1} 1 \otimes t^i$. Then the following diagrams commute

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \arrow[r, "\phi"] \arrow[d, "1_{\mathbb{Z}}\otimes(t-1)"'] & \mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \arrow[r, "\phi"] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \arrow[r, "\phi"] \arrow[d, "1_{\mathbb{Z}}\otimes N"'] & \mathbb{Z} \arrow[d, "n"] \\
\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \arrow[r, "\phi"] & \mathbb{Z}
\end{tikzcd}$$

Hence we have the chain complex F_G:

$$\ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

and,

$$H_i(G) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}_n, & i \text{ odd}, \\ 0, & i > 0 \text{ even}. \end{cases}$$

Example. Let G be any group and take F the standard resolution. Denote by $C_*(G)$ the chain complex F_G.

$C_n(G) = \mathbb{Z} \otimes_{\mathbb{Z}G} F_n$ has a $\mathbb{Z}$-basis $[g_0, \ldots, g_n] = 1 \otimes (g_0, \ldots, g_n) = 1 \otimes g \cdot (g_0, \ldots, g_n)$ and the boundary operator $\partial : C_n(G) \to C_{n-1}(G)$ is given by $\partial = \sum_{i=0}^{n} (-1)^i d_i$ with

$$d_i([g_0, \ldots, g_n]) = [g_0, \ldots, \hat{g}_i, \ldots, g_n].$$

[1] A projective resolution is enough.

$C_*(G)$ written in this form is called the homogeneous chain complex of G. Using the bar notation we get the non-homogeneous description of $C_*(G)$. $C_n(G)$ has a $\mathbb{Z}$-basis with elements of the form $1 \otimes [g_1|\dots|g_n] = 1 \otimes [g \cdot g_1|\dots|g \cdot g_n]$, and the components of the boundary operator are

$$d_i(1 \otimes [g_1|\dots|g_n]) = \begin{cases} 1 \otimes [g_2|\dots|g_n], & i = 0 \\ 1 \otimes [g_1|\dots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\dots|g_n], & 0 \leq i \leq n \\ 1 \otimes [g_1|\dots|g_{n-1}], & i = n \end{cases}$$

In low dimensions $C_*(G)$ has the form

$$\dots C_2(G) \xrightarrow{\partial_2} C_1(G) \xrightarrow{\partial_1} \mathbb{Z},$$

with $\partial_2[g|h] = [h] - [gh] + [g]$ and $\partial_1[g] = [\,] - [\,] = 0$. Hence,

$$H_0(G) = H_0(C_0(G)) = \mathbb{Z}/\mathrm{im}\ \partial_1 = \mathbb{Z}.$$

Notice that if $f : G \to H$ is a group homomorphism, a chain map between the standard complexes $C_*(G)$ and $C_*(H)$ is induced: $\hat{f} : C_n(G) \to C_n(H)$ is given by $[g_1|\dots|g_n] \mapsto [f(g_1)|\dots|f(g_n)]$. Hence $H_*(G)$ is a functor of G.

An equivalent topological definition can be given. Let BG be a $K(G,1)$-complex with universal cover EG. Then, from Proposition B.4, $C_*(EG)_G \cong C_*(BG)$ and the homology of the group G corresponds to the homology of the space BG.

Proposition B.5. *If BG is a $K(G,1)$-complex, then $H_*(G) \cong H_*(BG)$.*

Homology with coefficients

Note that any left G-module M can be regarded as a right G-module setting $mg = g^{-1}m$ ($m \in M$, $g \in G$). Then, G acts "diagonally" on $M \otimes N$ by $g \cdot (m \otimes n) = gm \otimes gn$. We denote $(M \otimes N)/G$ by $M \otimes_G N$ and since $m \otimes n = gm \otimes gn = g \cdot (m \otimes n)$, we have

$$M \otimes_G N = (M \otimes N)_G.$$

Let F be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let M be a G-module.

Definition: The *homology of G with coefficients in M* is defined by $H_*(G, M) = H_*(F \otimes_G M)$.

Remark B.5: Taking $M = \mathbb{Z}$ we recover $H_*(G)$.

Example: Finite cyclic groups. Consider the resolution B.1. Since $\mathbb{Z}G \otimes_{\mathbb{Z}G} M \cong M$, then $H_*(G, M)$ is the homology of

$$\dots \xrightarrow{N} M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M.$$

Recall that $F \otimes_G \_\_$ is a covariant functor, then $H_*(G, \_)$ is a covariant functor of the coefficient module. If $f : M \to M'$ is a G-module homomorphism, we will denote by $H_*(G, f)$ the induced morphism from $H_*(G, M)$ to $H_*(G, M')$.

$H_*(\_, \_)$ as a functor of two variables

Let $\mathcal{C}^2$ be the category with objects the pairs (G, M), with G a group and M a G-module, and morphisms $(\alpha, f) : (G, M) \to (G', M')$, where $\alpha : G \to G'$ is a group homomorphism and $f : M \to M'$ is a map of abelian groups such that $f(gm) = \alpha(g)f(m)$, for all $g \in G$ and $m \in M$.

For a given (α, f), take F and F' projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$, respectively. Let $\tau : F \to F'$ a chain map compatible with α. Thus, we have the map $\tau \otimes f : F \otimes_G M \to F' \otimes_{G'} M'$ and it induces a well-defined map in homology $(\alpha, f)_*$:

$$\begin{tikzcd} H_*(G,M) \arrow[rr, "(\alpha,f)_*"] \arrow[dr, "H_*(G,f)"'] & & H_*(G',M') \\ & H_*(G,M') \arrow[ur, "\alpha_*"'] & \end{tikzcd}$$

Where α_* denotes the morphism $(\alpha, id_{M'})_* : H_*(G, M') \to H_*(G', M')$. Thus H_* becomes a covariant functor on $\mathcal{C}^2$.

Example. Let $H \leq G$ and M be a G-module. Let $g \in G$. Consider in $\mathcal{C}^2$ the objects (H, M) and (gHg^{-1}, M) and the morphism $c(g) : (H, M) \to (gHg^{-1}, M)$ given by $(h \mapsto ghg^{-1}, m \mapsto gm)$. Applying functor H_*, there is an induced morphism in homology groups

$$c(g)_* : H_*(H, M) \to H_*(gHg^{-1}, M).$$

Extensions of scalars and Shapiro's lemma

Let R and S be rings, $\alpha : R \to S$ be a ring homomorphism. Then S can be regarded as an R-module via α. This is a functor from S-modules to R-modules called *restriction of scalars*. Let M be a left R-module.

Definition: The left S-module $S \otimes_R M$ is said to be obtained from M by *extension of scalars from R to S.*

Definition: When we consider $H \leq G$ and the natural inclusion homomorphism $\mathbb{Z}H \to \mathbb{Z}G$, the extension of scalars is called *induction from H to G*. If M is an H-module, then we write

$$Ind_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M.$$

The G-modules of the form $Ind_H^G M$ are characterized as follows (see proofs in [Bro94]):

Proposition B.6.

$$Ind_H^G M = \bigoplus_{g \in G/H} gM.$$

Proposition B.7. *Let N be a G-module whose underlying abelian group is of the form $\bigoplus_{i \in I} M_i$. Suppose that there is a G action on I that corresponds to the permutations of the summands by the G action. Let E be a set of representatives for I/G. Then M_i is a G_i-module and there is a G-isomorphism $N \cong \bigoplus_{i \in E} Ind_{G_i}^G M_i$.*

Example. The permutation module $\mathbb{Z}[G/H]$ is a direct sum of copies od $\mathbb{Z}$, as many as cosets gH. The G-action permutes these copies transitively. Thus, by Proposition B.7, considering the trivial action of H on $\mathbb{Z}$, we have

$$\mathbb{Z}[G/H] \cong Ind_H^G \mathbb{Z} = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}.$$

Example. Let X be a G-complex. Then the G-action induces an action on the sets X_p of p-cells and the abelian groups $C_p(X)$ are the permutation modules $\mathbb{Z}X_p$. Thus, the cellular chain complex $C_* = (C_p(X))_{p\geq 0}$ is a G-chain complex.

As an abelian group, each $C_p(X)$ is a direct sum of copies of $\mathbb{Z}$, one for each p-cell of X and the summands are permuted by the G-action. Let Σ_p be a set of representatives of the G-orbits on X_p. Thus, by Proposition B.7 there is a G-isomorphism

$$C_p(X) \cong \bigoplus_{\sigma\in\Sigma_p} Ind^G_{St(\sigma)}\mathbb{Z}_\sigma = \bigoplus_{\sigma\in\Sigma_p} \mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} \mathbb{Z}_\sigma.$$

$\mathbb{Z}_\sigma$ is the orientation module, an infinite cyclic group whose generators correspond to the two orientations of σ. More precisely,

Definition: The *orientation module* $\mathbb{Z}_\sigma$ is the abelian group $\mathbb{Z}$ with the following $St(\sigma)$-action: an element $g \in St(\sigma)$ acts on $\mathbb{Z}_\sigma$ by the identity if g preserves the orientation of σ, and by multiplication by -1 if it reverses it.

Note that if G acts freely on X, then the above isomorphism corresponds to Proposition B.1.

Now consider the case of a G-module M. Let $M_\sigma = M \otimes \mathbb{Z}_\sigma$. M_σ is equal to M additively, but the $St(\sigma)$-action is twisted by the orientation character $St(\sigma) \to \{\pm 1\}$. Thus, as an abelian group,

$$C_p(X, M) = C_p(X) \otimes M = \left(\bigoplus_{\sigma\in X_p} \mathbb{Z}_\sigma\right)\otimes M = \bigoplus_{\sigma\in X_p} M_\sigma.$$

Again by Proposition B.7, we get a G-module decomposition

$$C_p(X, M) \cong \bigoplus_{\sigma\in\Sigma_p} Ind^G_{St(\sigma)} M_\sigma = \bigoplus_{\sigma\in\Sigma_p} \mathbb{Z}G \otimes_{\mathbb{Z}St(\sigma)} M_\sigma. \tag{B.2}$$

The next proposition will be useful for defining the transfer map and a further decomposition in terms of stabilizers of simplices.

Proposition B.8 (Shapiro's Lemma). *If $H \leq G$ and M is an H-module, then*

$$H_*(H, M) \cong H_*(G, Ind^G_H M).$$

Proof. Let F be a free (projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then F can be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. So $F \otimes_{\mathbb{Z}H} M \cong F \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong F \otimes_G (Ind^G_H M)$. ■

The transfer map

Let $H \leq G$ and M a G-module. We have just seen that the inclusion $i : H \hookrightarrow G$ induces a morphism $i_* = (i, id_M)_* : H_*(H, M) \to H_*(G, M)$. If $[G : H] < \infty$, a map going in the other direction can be defined. It is called the *transfer map.* Consider the following injection

$$\begin{array}{rccl} t: & M & \longrightarrow & Hom_{\mathbb{Z}H}(\mathbb{Z}G, M) \\ & m & \mapsto & (z \mapsto zm) \end{array}$$

Applying the functor $H_*(G, \_)$ to t we get

$$t_* = (id_G, t)_* : H_*(G, M) \to H_*(G, Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)).$$

In [Bro94] is proved the following proposition.

Proposition B.9. *If $[G:H] < \infty$, then $\mathbb{Z}G \otimes_{\mathbb{Z}H} M \cong Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$.*

Hence, by the previous proposition and Shapiro's lemma (B.8), we have

$$H_*(G, Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)) \cong H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong H_*(H, M),$$

and t_* is the transfer map that we are looking for

$$t_* : H_*(G, M) \to H_*(H, M).$$

Other ways of defining the transfer map are described in [Bro94]. Moreover the transfer map satisfies

Proposition B.10. *If $[G:H] < \infty$, then the composition*

$$H_*(G, M) \xrightarrow{t_*} H_*(H, M) \xrightarrow{i_*} H_*(G, M)$$

is multiplication by $[G:H]$.

Double complexes

Definition: A *double complex* is a bigraded module $C = (C_{pq})_{p,q\in\mathbb{Z}}$ with a "horizontal" differential ∂' of bidegree $(-1, 0)$ and a "vertical" differential ∂'' of bidegree $(0, -1)$ such that $\partial'\partial'' = \partial''\partial'$. Then the next diagram is commutative:

$$\begin{tikzcd} C_{p-1,q} \arrow[d, "\partial''"'] & C_{p,q} \arrow[l, "\partial'"'] \arrow[d, "\partial''"] \\ C_{p-1,q-1} & C_{p,q-1} \arrow[l, "\partial'"] \end{tikzcd}$$

A double complex C gives rise to an ordinary chain complex.

Definition: The *total complex TC* is defined by

$$(TC)_n = \bigoplus_{p+q=n} C_{pq},$$

with the differential defined as $D|_{C_{pq}} = \partial' + (-1)^p \partial''$.

Example. Tensor product of complexes. Consider two complexes C' and C''. Then we have a double complex C with $C_{pq} = C'_p \otimes C''_q$. The total complex TC is simply the usual tensor product $C' \otimes C''$ of chain complexes.

A total complex admits two filtrations and two associated spectral sequences converging to $H(TC)$.

1. *Filtering by columns.* We can filter TC by setting

$$F_p(TC)_n = \bigoplus_{i\leq p} C_{i,n-i}.$$

Thus we have a spectral sequence $\{{}^{I}\!E^r\}$ converging to $H_*(TC)$:

$$\begin{aligned} {}^{I}\!E^0_{pq} &= F_p(TC)_{p+q}/F_{p-1}(TC)_{p+q} = C_{pq} \\ {}^{I}\!E^1_{pq} &= H_q^{\partial''}(C_{p*}) \\ {}^{I}\!E^2_{pq} &= H_p^{\partial'}H_q^{\partial''}(C_{**}) \end{aligned}$$

With differentials $d^0 = \pm\partial''$ and $d^1 : {}^{I}\!E^1_{p,q} \to {}^{I}\!E^1_{p-1,q}$ induced by $\partial' : C_{p,*} \to C_{p-1,*}$.

2. *Filtering by rows.* In this case we filter TC by setting

$$F_p(TC)_n = \bigoplus_{j \le p} C_{n-j,j}.$$

Thus we have another spectral sequence $\{{}^{II}\!E^r\}$ converging to $H_*(TC)$:

$$\begin{aligned} {}^{II}\!E^0_{pq} &= F_p(TC)_{p+q}/F_{p-1}(TC)_{p+q} = C_{qp} \\ {}^{II}\!E^1_{pq} &= H_q^{\partial'}(C_{*p}) \\ {}^{II}\!E^2_{pq} &= H_p^{\partial''}H_q^{\partial'}(C_{**}) \end{aligned}$$

And the differential $d^1 : {}^{II}\!E^1_{p,q} \to {}^{II}\!E^1_{p-1,q}$ is equal, up to sign, to the map induced by $\partial'' : C_{p,*} \to C_{p-1,*}$.

Homology of a group with coefficients in a chain complex

We have already defined homology with coefficients in an G-module M. It is useful to generalize this by taking a non-negative chain complex $C = (C_n)_{n\ge 0}$ of coefficients. We set

$$H_*(G, C) = H_*(F \otimes_G C)^2,$$

where F is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Remark B.6: If C is a single module M concentrated in dimension 0, then $H_*(G, C)$ reduces to $H_*(G, M)$.

Note that $F \otimes_G C$ is the total complex of a double complex of abelian groups $(F_p \otimes_G C_q)$. Hence we have two spectral sequences converging to $H_*(G, C) = H_*(F \otimes_G C)$:

1. Filtering by columns we have

$$ {}^{I}\!E^1_{pq} = H_q(F_p \otimes_G C_*)) = F_p \otimes_G H_q(C),$$

since $F_p \otimes_G \_\_$ is an exact functor (being F_p free), and

$$ {}^{I}\!E^2_{pq} = H_p(F_* \otimes_G H_q(C)) = H_p(G, H_q(C)) \Rightarrow H_{p+q}(G, C).$$

2. Filtering by rows we get a spectral sequence with ${}^{II}\!E^1$-term

$$ {}^{II}\!E^1_{pq} = H_q(F_* \otimes_G C_p) = H_q(G, C_p) \Rightarrow H_{p+q}(G, C).$$

Both spectral sequences can be thought of as approximations to $H_*(G, C)$ in terms of ordinary homology groups $H_*(G, M)$.

[2] In the case of cohomology, the analogous construction is often called the G-hypercohomology of a G-cochain complex C^* (see for example [**?**])

B.3 Equivariant homology

Let X be a G-complex.

Definition: The homology groups of G with coefficients the cellular chain complex $C(X)$, that is $H_*(G, C(X))$, is denoted by $H_*^G(X)$ and called the *equivariant homology groups of* (G, X). The version with coefficients in a R-module M is $H_*^G(X, M) = H_*(G, C(X, M))$, where $C(X, M) = C(X) \otimes M$.

Remark B.7:

- $H_*^G(pt, M) = H_*(G, M)$.
- Since any G-complex X admits a (unique) map $X \to pt$, it induces a canonical map

$$H_*^G(X, M) \to H_*(G, M).$$

The spectral sequences for equivariant homology

We have again two spectral sequences I and II associated to the vertical and horizontal directions:

1. The first spectral sequence (filtering by columns) has the form:

$$^{I}E^1_{pq} = F_p \otimes_G H_q(X, M),$$

$$^{I}E^2_{pq} = H_p(F_* \otimes_G H_q(X, M)) = H_p(G, H_q(X, M)) \Rightarrow H^G_{p+q}(X, M).$$

This spectral sequence is functorial with respect to G-invariant morphisms, which imply the following proposition. See [Iva93] for a sketch of the proof.

Proposition B.11. *If X is acyclic, then the canonical map is an isomorphism, i.e. $H_*^G(X, M) \cong H_*(G, M)$. Moreover, if $H_q(X, M) = 0$ for $q \leq Q$ (if X is Q-connected, for example), then $H_q^G(X, M) \cong H_q(G, M)$ and $H^G_{Q+1}(X, M) \to H_{Q+1}(G, M)$ is an epimorphism.*

2. The second spectral sequence that we can consider takes the form:

$$^{II}E^1_{pq} = H_q(F_* \otimes_G C_p(X, M)) = H_q(G, C_p(X, M)) \Rightarrow H^G_{p+q}(X, M).$$

From the G-module decomposition of $C_p(X, M)$ in (B.2) and Shapiro's Lemma (B.8) we have that

$$\begin{aligned} H_q(G, C_p(X, M)) &= H_q\left(G, \bigoplus_{\sigma \in \Sigma_p} Ind^G_{St(\sigma)} M_\sigma\right) \\ &= \bigoplus_{\sigma \in \Sigma_p} H_q(G, Ind^G_{St(\sigma)} M_\sigma) \\ &\cong \bigoplus_{\sigma \in \Sigma_p} H_q(St(\sigma), M_\sigma). \end{aligned}$$

Then, the second spectral sequence has the form

$$^{II}E^1_{pq} = H_q(G, C_p(X, M)) \cong \bigoplus_{\sigma \in \Sigma_p} H_q(St(\sigma), M_\sigma) \Rightarrow H^G_{p+q}(X, M).$$

If X is acyclic (but the action of G is non necessarily free), by Proposition B.11 we have $H_*^G(X,M) \cong H_*(G,M)$. Then the homology groups of G can be computed, using the second spectral sequence, in terms of the homology groups of stabilizers of simplices, which sometimes are simpler than G:

$$^{II}E^1_{pq} \cong \bigoplus_{\sigma\in\Sigma_p} H_q(St(\sigma), M_\sigma) \Rightarrow H_{p+q}(G,M).$$

Hence, in the computation of the homology of a group G, it is helpful to find an adequate acyclic space X on which G acts and then, look at the homology of the stabilizers of simplices $St(\sigma)$ that may be simpler to analyze.

The differential d^1

The second spectral sequence above, corresponds to filtering the double complex $F \otimes_G C(X,M)$ by rows. Then the differential $d^1 : {}^{II}E^1_{p,q} \to {}^{II}E^1_{p-1,q}$ is equal, up to sign, to the map $H_*(G,\partial)$, induced by the boundary operator $\partial : C_p(X,M) \to C_{p-1}(X,M)$ of the chain complex $C(X,M)$. We are interested in a more explicit description of the differential d^1.

Recall that ∂ sends a p-simplex σ in the alternating sum of its faces, which are $p-1$-simplices. Since $C_p(X,M) = \bigoplus_{\sigma\in X_p} M_\sigma$ and $C_{p-1}(X,M) = \bigoplus_{\tau\in X_{p-1}} M_\tau$, we can consider, for each $\sigma \in X_p$ and $\tau \in X_{p-1}$, the (σ,τ)-component of ∂ and denote it by $\partial_{\sigma,\tau} : M_\sigma \to M_\tau$. Notice that $\partial_{\sigma,\tau} \neq 0$ only if τ is a face of σ. Then the set $\mathcal{F}_\sigma = \{\tau : \partial_{\sigma,\tau} \neq 0\}$ is finite. Moreover, it is $St(\sigma)$-invariant.

Let $G_{\sigma\tau} = St(\sigma) \cap St(\tau)$. Then $[St(\sigma) : G_{\sigma\tau}] < \infty$, for all $\tau \in \mathcal{F}_\sigma$ (since $gG_{\sigma\tau} = hG_{\sigma\tau}$ iff $g\tau = h\tau$, for g, $h \in G_\sigma$). Since $[St(\sigma) : G_{\sigma\tau}] < \infty$, there is a transfer map

$$t_{\sigma\tau} : H_*(St(\sigma), M_\sigma) \to H_*(G_{\sigma\tau}, M_\sigma).$$

Now we look at the map induced by ∂. Take the inclusion $\iota_{\sigma\tau} : G_{\sigma\tau} \to St(\tau)$ and the (σ,τ)-component $\partial_{\sigma\tau} : M_\sigma \to M_\tau$, which is a $G_{\sigma\tau}$-map. This is a compatible pair that induces a map in homology

$$(\iota_{\sigma\tau}, \partial_{\sigma\tau})_* : H_*(G_{\sigma\tau}, M_\sigma) \to H_*(St(\tau), M_\tau).$$

Finally we put all in terms of the representant τ_0 of the G-orbit of τ in X_{p-1}. Say $\tau_0 = h_\tau\tau$, for some $h_\tau \in G$. Consider the pair $int(h_\tau) : St(\tau) \to G_{St(\tau_0)}$ given by $g \mapsto h_\tau g h_\tau^{-1}$ and $f : M_\tau \to M_{\tau_0}$ by $m \mapsto h_\tau m$. There is an induced map in homology

$$(int(h_\tau), f)_* : H_*(St(\tau), M_\tau) \to H_*(St(\tau_0), M_{\tau_0}).$$

The composition of these three maps

$$H_*(St(\sigma), M_\sigma) \xrightarrow{t_{\sigma\tau}} H_*(G_{\sigma\tau}, M_\sigma) \xrightarrow{(\iota,\partial)_*} H_*(St(\tau), M_\tau) \xrightarrow{(int(h_\tau),f)_*} H_*(St(\tau_0), M_{\tau_0}),$$

is the differential d^1. More precisely, in [Bro94] is proved that, up to sign, the differential d^1 is the map

$$\phi : \bigoplus_{\sigma\in\Sigma_p} H_*(St(\sigma), M_\sigma) \to \bigoplus_{\tau_0\in\Sigma_{p-1}} H_*(St(\tau_0), M_{\tau_0}),$$

given by

$$\phi|_{H_*(St(\sigma),M_\sigma)} = \sum_{\tau\in\mathcal{F}'_\sigma} (int(h_\tau), f)_* \circ (\iota, \partial)_* \circ t_{\sigma\tau},$$

where $\mathcal{F}'_\sigma$ is a set of representatives of $\mathcal{F}_\sigma / St(\sigma)$.

Suppose that $St(\sigma)$ fixes σ point-wise for each simplex σ of X. Each simplex σ of X can be oriented in such a way that the G-action preserves orientations. Whit this orientation, for each σ there is an isomorphism $M_\sigma \cong M$ of $St(\sigma)$-modules and we can write $H_*(St(\sigma), M)$ instead of $H_*(St(\sigma), M_\sigma)$. Therefore, $\tau \in \mathcal{F}_\sigma$ is in the closure of σ and then it is fixed by $St(\sigma)$. Then $G_{\sigma\tau} = St(\sigma)$, $\mathcal{F}'_\sigma = \mathcal{F}_\sigma$ and the transfer map is the identity. Hence, an easier expression of d^1 is gotten in this case: $d^1|_{H_*(St(\sigma),M)}$ is the alternating sum of the homomorphisms corresponding to the faces τ of σ (the elements of $\mathcal{F}_\sigma$). These are the compositions

$$(int(h_\tau) \circ \iota, f \circ \partial)_* : H_*(St(\sigma), M) \to H_*(St(\tau_0), M)$$

induced by $int(h_\tau) : St(\sigma) \to St(\tau_0)$ given by $g \mapsto h_\tau g g_\tau^{-1}$ and by $f \circ \partial_{\sigma\tau} : M \cong M_\sigma \to M \cong M_{\tau_0}$ given by $m \mapsto h_\tau(\partial_{\sigma\tau}(m))$. We will denote it simply by $(int(h_\tau)_)*$.

The topological definition

The equivariant homology groups can also be defined in a topological way. Let EG be a contractible free G-complex.

Definition: The *Borel construction on a G-complex X* is the orbit space

$$X \times_G EG = (X \times EG)/G,$$

where G acts diagonally on $X \times EG$.

Remark B.8: For $X = *$, $X \times_G EG$ is the classifying space of G, BG.

The natural projection $\pi : X \times EG \to EG$ induces a map

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] & X \times_G EG \arrow[d, "\pi'"'] \\ & BG \end{tikzcd}$$

Since the action of G on EG is free, the vertical map π' is a fibration with fiber X. Thus, topologically, we have the following equivalent definition.

Definition: The *G-equivariant homology of X with coefficients in a G-module M* is

$$H_*^G(X; M) = H_*(X \times_G EG, M).$$

If X is a free G-complex, then the natural projection $X \times_G EG \to X/G$ is also a fibration with contractible fiber. In this case, $X \times_G EG \simeq X/G$.

List of Figures

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