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EXAMPLES OF REPRESENTATION STABILITY PHENOMENA

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Para César y Adrián

Los grupos como los hombres son conocidos por sus acciones.

G. Moreno

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ABSTRACT

In this thesis we consider some sequences of groups or spaces $\{X_n\}_{n\in\mathbb{N}}$ and are interested in how the degree i cohomology $H^i(X_n;\mathbb{Q})$ changes as the parameter n increases. Our main examples are:

- the cohomology of the moduli space of n-pointed curves of genus g,
- the cohomology of the pure mapping class group of surfaces and some manifolds of higher dimension, and
- the cohomology of classifying spaces of some diffeomorphism groups.

We do not expect that our sequences satisfy *(co)homological stability*, meaning that the (co)homology groups are isomorphic for large n, because of existence of nontrivial symmetries.

We prove, in Chapter 3, that in each case we have a sequence of S_n-representations which is uniformly representation stable in the sense of Church–Farb. This condition puts strong constraints on the rate of growth of the representations and the patterns of irreducible S_n-representations occurring. In particular this result applied to the trivial S_n-representation implies rational "puncture homological stability" for the mapping class group Mod_g^n.

In Chapter 4 we apply the theory of finitely generated FI-modules developed by Church, Ellenberg and Farb to our examples. We introduce the notion of FI[G]-module and use it to strengthen and give new context to results on representation stability discussed in Chapter 3. With this approach we conclude that for each sequence of representations the characters are polynomial and find bounds on their degree. As a consequence we obtain that the Betti numbers of these spaces and groups are polynomial. Finally, rational homological stability of certain wreath products is obtained.

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CHAPTER 1

INTRODUCTION

Given a sequence of either groups or spaces $\{X_n\}_{n\in\mathbb{N}}$ there has been much interest in how the (co)homology changes as the parameter n gets large. A possible phenomenon is *homological stability*: as the parameter n is large enough, the homology groups are isomorphic. In the last fifty years several examples have exhibit this behavior. For example, it is known to hold for the symmetric groups S_n (Nakaoka, 1961 [39]); the braid groups B_n (Arnold' 1968 [3]); the mapping class groups $\mathrm{Mod}_{n,r}$ (Harer 1985 [23]) and Mod_g^n (see [47]); the outer automorphisms group of the free group $Out(F_n)$ (Hatcher–Vogtmann, 2004 [28]); the configuration spaces of unordered points $B_n(M)$ (by McDuff 1975 [38] and Segal 1979 [41] for open manifolds); linear and arithmetic groups (Vogtmann 1979 [45], Charney 1979 [8], Maazen, 1979 [35], van der Kallen, 1980 [44]). We refer the interested reader to [14], [46], [16] and the references therein.

However, this behavior generally does not occur when nontrivial symmetries are present. This is the case of the pure braid group P_n, where each cohomology group $H^i(P_n;\mathbb{Q})$ comes with an action of the symmetric group S_n. Arnold's computations in [2] imply that $\dim_{\mathbb{Q}} H^i(P_n;\mathbb{Q})$ grows to infinity with n, for any $i \geq 1$. In this thesis we are interested in some particular examples, such as the pure braid group P_n, where the phenomenon of homological stability fails. We focus on some sequences of groups or spaces $\{X_n\}$ where each cohomology group $H^i(X_n;\mathbb{Q})$ is equipped with an action of the symmetric group S_n. Our goal is to describe how the (co)homology groups change as the parameter n increases.

1.1 Examples of interest

We begin by describing the sequences of spaces and groups $\{X_n\}$ that we are interested in: the pure mapping class group $\mathrm{PMod}^n(M)$; the moduli space $\mathcal{M}_{g,n}$ of n-pointed curves of genus g, and the classifying spaces of some diffeomorphisms groups $B\,\mathrm{PDiff}^n(M)$.

1.1.1 Pure mapping class groups

Given M a connected, smooth manifold, consider an "ordered configuration" $\mathfrak{p} = (p_1, \dots, p_n)$ of n distinct points in the interior of M.

We denote by $\mathrm{Diff}^{\mathfrak{p}}(M)$ the subgroup in $\mathrm{Diff}(M \text{ rel } \partial M)$ of diffeomorphisms that leave invariant the set $\{p_1, \dots, p_n\}$. On the other hand, $\mathrm{PDiff}^{\mathfrak{p}}(M)$ is the subgroup that consists of the diffeomorphims that fix each point in $\{p_1, \dots, p_n\}$.

We refer to $p_1, \dots, p_n$ as the "punctures" or the "marked points".

The *mapping class group* is the group

$$\mathrm{Mod}^n(M) := \pi_0\big(\mathrm{Diff}^{\mathfrak{p}}(M)\big).$$

Similarly the *pure mapping class group* is

$$\mathrm{PMod}^n(M) := \pi_0(\mathrm{PDiff}^{\mathfrak{p}}(M)).$$

Notice that if we have two distinct configurations $\mathfrak{p} = (p_1, \dots, p_n)$ and $\mathfrak{q} = (q_1, \dots, q_n)$ in the interior of M, since M is connected, then $\mathrm{Diff}^{\mathfrak{p}}(M) \approx \mathrm{Diff}^{\mathfrak{q}}(M)$ and $\mathrm{PDiff}^{\mathfrak{p}}(M) \approx \mathrm{PDiff}^{\mathfrak{q}}(M)$. We refer to them by $\mathrm{Diff}^n(M)$ and $\mathrm{PDiff}^n(M)$, respectively.

Pure mapping class groups of surfaces. Our discussion and techniques below were originally motivated by the example of pure mapping class groups of surfaces. The understanding of their cohomology has relevance in both geometric group theory and algebraic geometry.

Let $\Sigma_{g,r}$ be a compact orientable surface of genus $g \geq 0$ with $r \geq 0$ boundary components. If $p_1, \dots, p_n$ are distinct points in the interior of $\Sigma_{g,r}$, then we write $\Sigma^n_{g,r}$ to denote the surface $\Sigma_{g,r} - \{p_1, \dots, p_n\}$. If $r = 0$ or $n = 0$, we omit it from the notation.

The *mapping class group* $\mathrm{Mod}^n_{g,r}$ is the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma^n_{g,r}$ that restrict to the identity on the boundary components (i.e. $\mathrm{Mod}^n_{g,r} = \mathrm{Mod}^n(\Sigma_{g,r})$). The *pure mapping class group* $\mathrm{PMod}^n_{g,r}$ is defined analogously by asking that the punctures remain fixed pointwise (i.e. $\mathrm{PMod}^n_{g,r} = \mathrm{PMod}^n(\Sigma_{g,r})$).

The group $\mathrm{PMod}_{0,1}^n$ is the *pure braid group* P_n and $\mathrm{Mod}_{0,1}^n$ is the *braid group* B_n.

Genus and puncture homological stability. One basic question is to understand how, for a fixed $i \geq 0$, the cohomology groups $H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Q})$ change as we vary the parameters g, r and n, in particular when the parameters are very large with respect to i.

It is known that the groups $\mathrm{PMod}_{g,r}^n$ and $\mathrm{Mod}_{g,r}^n$ satisfy "genus homological stability":

For fixed $i, n \geq 0$ the homology groups $H_i(\mathrm{PMod}_{g,r}^n; \mathbb{Z})$ and $H_i(\mathrm{Mod}_{g,r}^n; \mathbb{Z})$ do not depend on the parameters g and r, for $g \gg i$.

This was first proved in the 1980's by Harer [23] and the stable ranges have been improved since then by the work of several people (see Wahl's survey [47]).

An additional stabilization map can be defined by increasing the number of punctures. We refer to "puncture homological stability" when homological stability holds with respect to the parameter n. In the case of surfaces with non-empty boundary, we can consider a map $\Sigma_{g,r}^n \to \Sigma_{g,r}^{n+1}$ by gluing a punctured cylinder to one of the boundary components of $\Sigma_{g,r}^n$. This map gives a homomorphism

$$\mu_n \colon \mathrm{Mod}_{g,r}^n \to \mathrm{Mod}_{g,r}^{n+1}.$$

In [29, Proposition 1.5], Hatcher and Wahl proved that the map μ_n induces an isomorphism in $H_i(-; \mathbb{Z})$ if $n \geq 2i + 1$ (for fixed $g \geq 0$ and $r > 0$). Puncture stability for closed surfaces follows, as it is known that

$$H_i(\mathrm{Mod}_{g,1}^n; \mathbb{Z}) \approx H_i(\mathrm{Mod}_g^n; \mathbb{Z}) \text{ for } g \geq \frac{3}{2} i$$

(see [47, Theorem 1.2]). Handbury proved this puncture homological stability for non-orientable surfaces in [21] with techniques that can also be applied to the orientable case. When the surface is a punctured disk, this is Arnold's classical stability theorem for the cohomology of braid groups B_n [3]. Together, puncture and genus stability imply that the homology of the mapping class group of an orientable surface stabilizes with respect to connected sum with any surface.

The question of puncture homological stability can also be asked for mapping class groups of manifolds of higher dimensions. This was proved, with integral coefficients, by Hatcher–Wahl in [29, Proposition 1.5] for the mapping class group $\mathrm{Mod}^n(M)$ of connected manifolds M with boundary of dimension $d \geq 2$.

On the other hand, for the pure mapping class groups, attaching a punctured cylinder to $\Sigma_{g,r}^n$ also induces homomorphisms

$$\mu_n\colon \mathrm{PMod}_{g,r}^n \to \mathrm{PMod}_{g,r}^{n+1},$$

when $r > 0$. Hence we can ask whether $\mathrm{PMod}_{g,r}^n$ satisfies or not puncture homological stability.

The homology groups of $\mathrm{PMod}_{g,r}^n$ are largely unknown, apart from some low dimensional cases such as:

$$H_1(\mathrm{PMod}_{g,r}^n; \mathbb{Z}) = 0 \text{ for } g \geq 3$$

(see [18, Theorem 5.2] for a proof). Furthermore,

$$H_2(\mathrm{PMod}_{g,r}^n; \mathbb{Z}) \approx H_2(\mathrm{Mod}_{g,r+n}; \mathbb{Z}) \oplus \mathbb{Z}^n \text{ for } g \geq 3$$

(this is [34, Corollary 4.5], but the original computation for $g \geq 5$ is due to Harer [22]).

We notice that the rank of $H_2(\mathrm{PMod}_{g,r}^n; \mathbb{Z})$ blows up as $n \to +\infty$. Moreover, Arnold's computations in [2] imply that $\dim_{\mathbb{Q}} H^k(P_n; \mathbb{Q})$ grows to infinity with n, for any $k \geq 1$. Hence, the pure braid groups $P_n \approx \mathrm{PMod}_{0,1}^n$ fail in each dimension $i \geq 1$ to satisfy homological stability (see discussion in [12, Section 4]). This suggests to us the failure of puncture homological stability in the general case. In this thesis we consider the sequences

$\{\mathrm{PMod}^n_{g,r}\}_{n\in\mathbb{N}}$ and $\{\mathrm{PMod}^n(M)\}$ and address the problem of describing how their cohomology groups change as the parameter n increases.

For large g, Bödigheimer and Tillmann results in [6], combined with Madsen-Weiss' results in [37], give explicit calculations, although we do not discuss them in this thesis.

1.1.2 Moduli space of curves

A space of interest that lies at the juncture of complex analysis, geometric topology, algebraic topology and algebraic geometry is the moduli space $\mathcal{M}_{g,n}$. It is the space of n-pointed Riemann surfaces of genus g up to biholomorphism. The elements in $\mathcal{M}_{g,n}$ are equivalent classes, up to biholomorphism, of pairs $(X, \mathfrak{p})$, where X is a Riemann surface of genus g and $\mathfrak{p} = (p_1, \ldots, p_n)$ is an "ordered configuration of n points" in X. Moreover, $\mathcal{M}_{g,n}$ is also the space of metrics of constant curvature in Σ^n_g up to isometry. Furthermore, $\mathcal{M}_{g,n}$ is the space of n-pointed non-singular projective curves of genus g up to isomorphism.

Understanding its topology is a fundamental question, in particular, its cohomology ring. It is known to be an orbifold of dimension $3g - 3 + n$ and moreover, a quasiprojective non-compact algebraic variety. Nevertheless, the cohomology groups of $\mathcal{M}_{g,n}$ are largely unknown, besides some low dimensional and low genus cases. One question to ask is how, for a fixed $i \geq 0$, the cohomology groups $H^i(\mathcal{M}_{g,n}; \mathbb{Q})$ change as the parameters g and n vary.

The homology groups of the pure mapping class group PMod^n_g are of interest due to their relation with the topology of the moduli space $\mathcal{M}_{g,n}$. The space $\mathcal{M}_{g,n}$ is a rational model for the classifying space $B\,\mathrm{PMod}^n_g$ for $g \geq 2$. Hence

$$H^*(\mathcal{M}_{g,n}; \mathbb{Q}) \approx H^*(\mathrm{PMod}^n_g; \mathbb{Q}). \tag{1.1}$$

We refer the reader to [18], [20], [32] and [25] for more about the relation between $\mathcal{M}_{g,n}$ and PMod^n_g. Therefore Harer's stability theorem answers the previous question with respect to the parameter g. For the parameter n, there is a *forgetful morphism* between moduli spaces $f_n : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ that induces a map that relates the corresponding cohomology groups.

Again, by equation (1.1), we could answer the question of how $H^i(\mathcal{M}_{g,n};\mathbb{Q})$ change as n varies, by understanding what happens for the rational cohomology of the pure mapping class group PMod_g^n, our previous example.

A related space of interest is $\overline{\mathcal{M}}_{g,n}$, the Deligne–Mumford compactification of the moduli space $\mathcal{M}_{g,n}$.

$\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable n-pointed curves of arithmetic genus g. A *stable n-pointed curve* is a connected n-pointed curve C, with ordinary double points as worst singularities, that satisfy the following stability condition: every genus 0 component must have at least three special points and every genus 1 component must have at least one special point. By special point we refer to either a double point or a smooth marked point. This stability condition is equivalent to requiring that the curve has only finitely many automorphisms. Informally, the arithmetic genus of a curve with nodes is the genus of its smoothing.

It turns out that $\overline{\mathcal{M}}_{g,n}$ is a normal projective variety of dimension $3g-3+n$ and it is a compactification of $\mathcal{M}_{g,n}$. Few explicit computations of its cohomology are known and all have an algebro-geometric flavor. For an introduction to moduli space and its compactification we refer the reader to [43].

1.1.3 Classifying spaces for diffeomorphism groups

Consider the classifying space $B\,\mathrm{PDiff}^n(M)$ of the group of diffeomorphisms $\mathrm{PDiff}^n(M)$, where M is a smooth, compact and connected manifold of dimension $d \geq 3$. Notice that the forgetful homomorphisms $\mathrm{PDiff}^{n+1}(M) \to \mathrm{PDiff}^n(M)$ induce corresponding map between classifying spaces

$$f_n : B\,\mathrm{PDiff}^{n+1}(M) \to B\,\mathrm{PDiff}^n(M).$$

We are interested on the sequence $\{H^i(B\,\mathrm{PDiff}^n(M);\mathbb{Q})\}_{n\in\mathbb{N}}$ for any $i \geq 0$, and we want to understand how the cohomology $H^i(B\,\mathrm{PDiff}^n(M);\mathbb{Q})$ changes as the parameter n gets large.

1.1.4 Configuration spaces

Finally we mention the example of configuration spaces because of its relation with pure mapping class groups and $B\,\mathrm{PDiff}^n(M)$ (see Sections 3.4.1, 3.5.1 and 3.6). It turns out to be a key ingredient for our computations below.

We denote by $\mathrm{Conf}_n(M)$ the *configuration space of n distinct ordered points* in the interior of any topological space M:

$$\mathrm{Conf}_n(M) = \big\{(p_1, \ldots, p_n) \in M^n : p_i \neq p_j\big\}.$$

The symmetric group S_n acts freely on $\mathrm{Conf}_n(M)$ by permuting the coordinates. The The action of the symmetric group S_n on $M^n = \underbrace{M \times \cdots \times M}_{n}$ restricts to a free action on $C_n(M)$, and the quotient quotient $B_n(M) = \mathrm{Conf}_n(M)/S_n$ is the *configuration space of n distinct unordered points* in the interior of M.

We will usually take the n-tuple $\mathfrak{p} = (p_1, \ldots, p_n) \in \mathrm{Conf}_n(M)$ used in the definition of $\mathrm{PDiff}^n(M)$ as the base point of $\pi_1(\mathrm{Conf}_n(M))$.

We can also think of $\mathrm{Conf}_n(M)$ as the space of embeddings $\mathrm{Emb}([n], M)$ of the finite set $[n] = \{1, \ldots, n\}$ into M.

Homological stability for unordered configuration spaces. The first conguration spaces to be studied were $\mathrm{Conf}_n(\mathbb{R}^2)$ and $B_n(\mathbb{R}^2)$. These are aspherical spaces whose fundamental groups are the *pure braid group* P_n and the *braid group* B_n respectively. There are inclusions $B_n(\mathbb{R}^2) \to B_{n+1}(\mathbb{R}^2)$ and Arnold ([3]) proved that $B_n(\mathbb{R}^2)$ satisfies integral homological stability with respect to these maps. Cohen's 1972 thesis ([13]) computes the homology groups for $B_n(\mathbb{R}^q)$ for all $q \geq 2$ and for all $n \geq 0$. Homological stability in these cases is an immediate consequence of the computations. More generally, if M is the interior of a compact manifold with boundary, maps $B_n(M) \to B_{n+1}(M)$ can be defined by "pushing the additional point in from infinity". For such manifolds M, McDuff ([38, Theorem 1.2]) proved that $B_n(M)$ satises integral homological stability and extended the result to arbitrary open manifolds. The same result was proved by Segal [41, Appendix to 5] by an approach closer to Arnol'd's.

Nevertheless, the corresponding theorem is false for closed manifolds where the stabilization map $B_n(M) \to B_{n+1}(M)$ does not exist. The simplest example where homological stability fails is the 2-sphere, where (see e.g. [5, Theorem 1.11]): $H_1(B_n(S^2);\mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}$.

Instead we could consider the ordered conguration spaces $\mathrm{Conf}_n(M)$. In this case we have a natural map $\mathrm{Conf}_{n+1}(M) \to \mathrm{Conf}_n(M)$ given by "forgetting the last point", which induces maps in cohomology $H^i(\mathrm{Conf}_n(M)) \to H^i(\mathrm{Conf}_{n+1}(M))$ for any manifold M, whether open or closed. The action of S_n on $\mathrm{Conf}_n(M)$ induces an action on the cohomology groups and gives $H^i(\mathrm{Conf}_n(M);\mathbb{Q})$ the structure of an S_n-representation. It turns out that to understand the rational cohomology of $B_n(M)$, it suffices to understand the cohomology of $\mathrm{Conf}_n(M)$ together with the action of S_n on it. In [9], Church describes how, for any $i \geq 0$, the sequence of representations $\{H^i(\mathrm{Conf}_n(M);\mathbb{Q})\}$ changes as n gets large. From his main result he concludes rational homological stability for $B(M)$ when M is any connected orientable manifold, open or closed ([9, Corollary 3]). In [10], Church–Ellenberg–Farb give the corresponding description for the cohomology of $\mathrm{Conf}_n(M)$ with coefficients in any field of characteristic zero and $\mathbb{Z}$. In [11], Church–Ellenberg–Farb–Nagpal address the case with coefficients in any field k and prove that $\dim_k \left(H^i(\mathrm{Conf}_n(M);k)\right)$ is a polynomial in n for sufficiently large n ([11, Theorem 1.8]). In Section **??** we recall some of those results and compute specific ranges for the configuration space of surfaces.

1.2 Representation Stability

Church and Farb suggested that in many examples where homological stability fails there is a form of stability relative to the inherent symmetries of the spaces or groups. This is the notion of *representation stability* introduced in [12].

Roughly speaking, a sequence of rational S_n-representations $\{V_n\}$ is said to be *uniformly representation stable* if the decomposition of $V_n = \bigoplus_\lambda c_{\lambda,n} V(\lambda)$ into irreducible representations can be described independently of n for sufficiently large n: the multiplicities $c_{\lambda,n}$ are eventually independent of n, for each λ. This definition has proved to be the right one in many cases. For instance, it explains the behavior of the cohomology of the pure braid groups P_n (due to Farb–Church in [12]) and of the configuration space $\mathrm{Conf}_n(M)$ of n distinct ordered points on any connected oriented manifold M (see work by Church in [9]). We introduce the precise definition in Section 2.1.

In this thesis we prove that uniform representation stability holds for the rational cohomology of the pure mapping class groups of surfaces $\mathrm{PMod}^n_{g,r}$, the moduli space $\mathcal{M}_{g,n}$ of n-pointed curves of genus g, the pure mapping class groups $\mathrm{PMod}^n(M)$ and the classifying spaces $B\,\mathrm{PDiff}^n(M)$ for some manifolds M (see Theorem 1.3.1 below for precise result). Therefore the growth of the rational cohomology groups of these examples is explained precisely by the symmetries of the spaces or groups. We do this in Chapter 3 below which corresponds to the content of the paper [30].

As a consequence, classical homological stability for the sequences $\{X_n/S_n\}$ with coefficients in any finite dimensional rational S_n-representation is obtained. In particular, when taking the trivial representation, we recovered rational homological stability for $\mathrm{Mod}^n_{g,r}$, previously obtained integrally by Hatcher–Wahl ([29]) with completely different techniques. See 3.1.3 in Chapter 3 for the precise result.

After proving representation stability, several problems remain open for future research. For instance, for each of the examples above:

a. Determine how large n needs to be for the degree i cohomology to stabilize.

b. Compute the explicit decomposition into irreducible S_n-representations.

c. Understand cohomology classes geometrically and the meaning of the S_n-action.

d. Investigate stability behavior of cohomology groups with other coefficients besides $\mathbb{Q}$.

The answers to the previous problems are known when $g \geq 4$ for the second cohomology of PMod^n_g, for instance $H^2(\mathrm{PMod}^n_g;\mathbb{Z}) \approx \mathbb{Z}^{n+1}$, when $n \geq 3$ (see discussion in Section 3.2).

Moreover, the decomposition into irreducible representations is completely understood:

$$H^2(\mathrm{PMod}_g^n; \mathbb{Q}) \approx (\text{standard } S_n\text{-rep}) \oplus (\text{trivial } S_n\text{-rep}) \oplus (\text{trivial } S_n\text{-rep}).$$

In Chapter 3 we provide an answer for the first problem above. For each example we obtain a stable range N, which is quadratic in i, such that for $n \geq N$ the degree i cohomology stabilizes in the sense of representation stability. We improve this stable range to be linear in i in Chapter 4.

1.3 The theory of FI-modules

The notion of an FI-module was introduced by Church, Ellenberg and Farb ([10]) in order to encode the information of a sequence of S_n-representations in a single object and convert the condition of representation stability into a finite generation property. An *FI-module* over a ring R is a functor from the category of finite sets and injections (**FI**) to R-modules. Their theory implied new theorems about configuration spaces, the diagonal coinvariant algebra on r sets of n variables and the space of polynomials on rank varieties of $n \times n$ matrices. We discuss the notion of FI-module and finite generation in Sections 2.2 and 2.4.

Some sequences of spaces or groups $\{X_n\}$ can be encoded by considering a functor X from $\mathbf{FI}^{op}$ to the category of topological spaces or groups. Then their cohomology can be understood as the FI-module $H^i(X; R)$ over R, obtained by composing X with the cohomology functor $H^i(\_\_; R)$. This is the case of the cohomology of our examples of interest (see Section 2.3): the pure mapping class groups of surfaces $\mathrm{PMod}_{g,r}^n$, the pure mapping class groups $\mathrm{PMod}^n(M)$ and the classifying spaces $B\,\mathrm{PDiff}^n(M)$ for some manifolds M, and the configuration space of ordered point in a surface $\mathrm{Conf}_n(\Sigma)$. In Chapter 4 we prove that each of such sequences has the structure of a *finitely generated FI-module* over any field k. Then we can use the theory developed by Church–Ellenberg–Farb and further results by Church–Ellenberg–Farb–Nagpal to obtain the following statement:

Theorem 1.3.1. *For $i \geq 0$, and each of the sequences $\{X_n\}$ in Table 1.1, there is an integer $N \geq 0$ such that for $n \geq N$ the following holds:*

i) *The decomposition of $H^i(X_n; \mathbb{Q})$ into irreducible S_n-representations stabilizes in the sense of uniform representation stability with stable range $n \geq N$.*

ii) *The sequence of quotients $\{X_n/S_n\}$ satisfies rational homological stability for $n \geq N$.*

iii) *The character χ_n of $H^i(X_n; \mathbb{Q})$ is of the form:*

$$\chi_n = Q_i(Z_1, Z_2, \ldots, Z_r),$$

where deg(χ_n) $= r > 0$ only depends on i and $Q_i \in \mathbb{Q}[Z_1, Z_2, \ldots]$ is a unique polynomial in the class functions

$$Z_l(\sigma) := \#\text{ cycles of length } l \text{ in } \sigma, \qquad \text{for any } \sigma \in S_n.$$

iv) *The length of the representation $\ell\big(H^i(X_n; \mathbb{Q})\big)$ is bounded above independently of n.*

The specific bounds for each example are presented in Table 1.1.

Moreover, if k is any field, for each X_n in Table 1.1 (except for $\mathcal{M}_{g,n}$) there exists an integer-valued polynomial $P(T) \in \mathbb{Q}[T]$ so that for all sufficiently large n,

$$\dim_k \big(H^i(X_n; k)\big) = P(n).$$

For $X_n = \mathcal{M}_{g,n}$ this is true when $k = \mathbb{Q}$.

Table 1.1: Main conclusions for the examples of interest in Theorem 1.3.1.

$\mathbf{X}_n$	**Hypotheses**	**N**	$\ell\big(H^i(X_n;\mathbb{Q})\big) \leq$	$\mathbf{deg}(\chi_n) \leq$
$\mathrm{Conf}_n(\Sigma)$ (Thm 4.3.3)	Σ a surface	$4i$	$2i+1$	$2i$
$\mathcal{M}_{g,n}$ (Thm 4.5.1)	$g \geq 2$	$6i$	$2i+1$	$2i$
$\mathrm{PMod}^n_{g,r}$ (Thm 4.5.3)	$2g+r>2$ and $r>0$	$4i$	$2i+1$	$2i$
$\mathrm{PMod}^n(M)$ (Thm 4.5.2)	M is a smooth connected manifold; $\dim M \geq 3$; $\pi_1(M)$ and $\mathrm{Mod}(M)$ are of type FP_∞, and either $\pi_1(\mathrm{Diff}(M))=0$ or $\pi_1(M)$ has trivial center	$3i$	$i+1$	i
$B\,\mathrm{PDiff}^n(M)$ (Thm 4.5.6)	M is a smooth, compact and connected manifold; $\dim M \geq 3$, and $B\,\mathrm{Diff}(M \mathrm{rel} \partial M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension	$3i$	$i+1$	i

Theorem 1.3.1 gives a unified understanding of the cohomology groups of such examples as sequences in n and puts strong constraints on their rate of growth, the patterns of irreducible S_n-representations occurring and the form of their characters. With the theory of FI-modules we also obtain homological stability results for some wreath products (see Theorem 4.1.5). Moreover, when considering surfaces with boundary ($r > 0$), we can conclude that the rank of $H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Z})$ and the number of cyclic summands in its p-torsion part are polynomials in n of degree at most $2i$ (Theorem 4.1.1).

The basic idea to prove Theorem 1.3.1 is to use a spectral sequence of FI-modules converging to the graded FI-module of interest. Closure properties of finite generation under subquotients and extensions allow us to reduce the argument to proving finite generation of the FI-modules in the E_2-page to get our conclusion. This approach was previously used successfully in [10] to obtain finite generation for configuration spaces. In Chapter 4, we develop the details to apply this to spectral sequences arising from "FI-fibrations" over a fixed space and "FI-group extensions" of a given group. In the case of pure mapping class groups our argument is made possible by the existence of a Birman exact sequence.

For a given group G, we introduce in Section4.4 the notion of a FI[G]-*module*: it is a functor V from the category **FI** to the category of G-modules over $\mathbb{Q}$. This definition incorporates the action of a group G on our sequences of S_n-representations and allows us to take V as twisted coefficients for cohomology. Therefore we can use finite generation of an FI[G]-module V to obtain finite generation and specific bounds for the new FI-modules $H^p(X; V)$ (Theorem 4.4.1). With this result we prove finite generation of the FI-modules in the E_2-page for each of our examples.

It remains open the computation of the specific Betti numbers and the character polynomials for each example. As an illustration, the character polynomial for the second cohomology of $\mathrm{PMod}_{0,1}^n$ (the pure braid group P_n) is given by

$$\chi_n = 2\binom{X_1}{3} + 3\binom{X_1}{4} + \binom{X_1}{2}X_2 - \binom{X_2}{2} - X_3 - X_4,$$

where $X_i(\sigma)$ is the number of i-cycles in $\sigma \in S_n$, for any $n \in \mathbb{N}$. For the second cohomology of $\mathrm{PMod}_{g,r}^n$ (when $g \geq 4$) the character polynomial is

$$\chi_n = X_1 + 1$$

and $b_2(\mathrm{PMod}_{g,r}^n) = n + 1$. In both cases the polynomial χ_n is independent of n. Obtaining the ring structure of the cohomology is a more ambitious goal for future research.

CHAPTER 2

PRELIMINARIES

In this chapter we go over the notions and consequences from the theory of representation stability and the theory of FI-modules that are the main tools for our computations in Chapters 3 and 4. The content presented here was mainly introduced in [12], [10] and [9]. We emphasize the results that are relevant for the discussion below and rearrange them in a convenient way for the sake of this thesis. Proofs can be found in [9, Section 2] and [10, Sections 1 & 2]. We finish the chapter by recalling some useful facts about group extensions and cohomology of groups.

2.1 Representation Stability and Monotonicity

The notion of representation stability for different families of groups was first defined in [12]. This language has proved to be useful since it allows generalization of stability theorems in topology and algebra. We recall this notion for the case of S_n-representations.

Definition 2.1. *A sequence $\{V_n\}_{n=1}^{\infty}$ of finite dimensional rational S_n-representations with linear maps $\phi_n\colon V_n \to V_{n+1}$ is said to be uniformly representation stable with stable range $n \geq N$ if the following conditions are satisfied for all $n \geq N$:*

0. ***Consistent Sequence.*** *The maps $\phi_n\colon V_n \to V_{n+1}$ are equivariant with respect to the natural inclusion $S_n \hookrightarrow S_{n+1}$.*

I. ***Injectivity.*** *The maps $\phi_n\colon V_n \to V_{n+1}$ are injective.*

II. ***Surjectivity.*** *The S_{n+1}-span of $\phi_n(V_n)$ equals V_{n+1}.*

III. ***Uniformly Multiplicity Stable with range $n \geq N$.*** *The decomposition of V_n into irreducible representations*

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)$$

can be eventually described independently of n; more precisely, when $n \geq N$, the multiplicities $c_{\lambda,n}$ are independent of n, for each λ.

Notation (Representations of S_n in characteristic zero): An *S_n-representation* over a field of characteristic zero k is a k-vector space equipped with a linear S_n-action. The irreducible representations of S_n over k are classified by partitions λ of n. By a partition of n we mean $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l > 0)$ where $l \in \mathbb{Z}$ and $\lambda_1 + \cdots + \lambda_l = n$. We will write $|\lambda| = n$. The corresponding irreducible S_n-representation will be denoted by V_λ. Every V_λ is defined over $\mathbb{Q}$ and any S_n-representation decomposes over $\mathbb{Q}$ into a direct sum of irreducibles ([19] is a standard reference). The decomposition of an S_n-representation over any such field k does not depend on k.

If λ is any partition of m, i.e. $|\lambda| = m$, then for any $n \geq |\lambda| + \lambda_1$ the *padded partition* $\lambda[n]$ of n is given by $\lambda[n] = (n - |\lambda|, \lambda_1, \cdots, \lambda_l)$. Keeping the notation from [12] we set $V(\lambda)_n = V_{\lambda[n]}$ for any $n \geq |\lambda| + \lambda_1$. Every irreducible S_n-representation is of the form $V(\lambda)_n$ for a unique partition λ.

We define the *length* of an irreducible representation of S_n to be the number of parts in the corresponding partition of n. The trivial representation has length 1, and the alternating representation has length n. We define the *length* $\ell(V)$ of a finite dimensional representation V of S_n to be the maximum of the lengths of the irreducible constituents. Notice that $\ell(V_\lambda) \leq |\lambda|$.

The notion of monotonicity introduced in [9] will be key in our arguments in Chapter 1.

Definition 2.2. *A consistent sequence $\{V_n\}_{n=1}^{\infty}$ of S_n-representations with injective maps $\phi_n \colon V_n \hookrightarrow V_{n+1}$ is monotone for $n \geq N$ if for each subspace $W < V_n$ isomorphic to $V(\lambda)_n^{\oplus k}$, the S_{n+1}-span of $\phi_n(W)$ contains $V(\lambda)_{n+1}^{\oplus k}$ as a subrepresentation for $n \geq N$.*

Now we point out the properties of monotone sequences that are useful for our purpose. These results are proven in [9, Sections 2.1 and 2.2].

Proposition 2.1.1. *Given $\{W_n\} < \{V_n\}$, if the sequence $\{V_n\}$ is monotone then so is $\{W_n\}$. If $\{V_n\}$ and $\{W_n\}$ are monotone and uniformly representation stable with stable range $n \geq N$, then $\{V_n/W_n\}$ is monotone and representation stable for $n \geq N$. Conversely, if $\{W_n\}$ and $\{V_n/W_n\}$ are monotone and uniformly representation stable with stable range $n \geq N$, then $\{V_n\}$ is monotone and uniformly representation stable for $n \geq N$.*

Proposition 2.1.2. *Let $\{V_n\}$ and $\{W_n\}$ be monotone sequences for $n \geq N$, and assume that $\{V_n\}$ is uniformly representation stable for $n \geq N$. Then for any consistent sequence of maps $f_n\colon V_n \to W_n$ that makes the following diagram commutative*

$$\begin{tikzcd} V_n \arrow[r, "f_n"] \arrow[d, "\phi_n"'] & W_n \arrow[d, "\psi_n"'] \\ V_{n+1} \arrow[r, "f_{n+1}"] & W_{n+1}, \end{tikzcd}$$

the sequences $\{kerf_n\}$ and $\{imf_n\}$ are monotone and uniformly representation stable for $n \geq N$.

The previous propositions apply also to $V(\lambda)_n$ for a single partition λ. In particular to the case of the trivial representation $V(0)_n$.

Proposition 2.1.3. *For a fixed partition λ, assuming monotonicity just for $V(\lambda)_n^{\otimes k}$, Propositions 2.1.1 and 2.1.2 hold if we replace "uniform representation stability" by "the multiplicity of $V(\lambda)_n$ is stable".*

2.2 FI and FI#-modules

Let **FI** be the category whose objects are natural numbers **n** and the morphisms **m**$\to$**n** are injections from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Similarly we denote by **FI#** the category whose objects are natural numbers **n** and the morphisms **m** $\mapsto$ **n** are triples (A, B, ψ), where $A \subset \{1, \ldots m\}$, $B \subset \{1, \ldots n\}$ and $\psi : A \to B$ is a bijection.

Remark. The category **FI#** can be also defined by adding in the base point 0 and taking based injections and based maps $\mathbf{m} \mapsto \mathbf{n}$ with at most one preimage of each non-basepoint. The categories **FI** and **FI#** are also called Λ and Π and have been used in algebraic topology since 1970's.

Definition 2.3. *An FI-module over a commutative ring k is a functor V from the category* **FI** *to the category* $\mathbf{Mod}_k$ *of modules over k. An FI#-module over k is a functor V from* **FI#** *to* $\mathbf{Mod}_k$*. We denote $V(\mathbf{n})$ by V_n and $V(f)$ by f_*, for any $f \in Hom_{FI}(\mathbf{m}, \mathbf{n})$. In the same manner a functor $V = \bigoplus V^i$ from* **FI** *to the category of graded modules over k is called a graded FI-module over k. In particular, each V^i is an FI-module over k.*

The category of FI-modules over k, $\mathbf{FI\text{-}Mod}_k$, is an abelian category. The concepts of kernel, cokernel, sub-FI-module, quotient, injection and surjection are defined "pointwise".

Remarks:

- For us k will be either a field or $\mathbb{Z}$. Most of the examples that we consider are finite dimensional vector spaces over $\mathbb{Q}$, unless otherwise specified. Therefore, we use the notation **FI-Mod** for the category of FI-modules over $\mathbb{Q}$. In that case, an FI-module V provides each V_n with the structure of a rational S_n-representation. The functor V allows us to encode the information of some sequences of rational S_n-representations in a single object (see Section 2.6 below).

- In general, the notation adopted below corresponds to the one for the case when k is a field. See Notational conventions in [10, Section 2.1].

One of the main advantages of the category $\mathbf{FI\text{-}Mod}_k$ is that we can define analogous concepts of the basic definitions coming from module theory.

Definition 2.4. *An FI-module V over k is said to be finitely generated in degree $\leq m$ if there exist $v_1, \ldots, v_s$, with each $v_i \in V_{n_i}$ and $n_i \leq m$, such that V is the minimal sub-FI-module of V containing $v_1, \ldots, v_s$. We write $V = span(v_1, \ldots, v_s)$. An FI#-module over k*

is finitely generated in degree $\leq m$ if the underlying FI-module is finitely generated in degree $\leq m$. A graded FI-module V over k is said to be of finite type if each FI-module V^i is finitely generated.

Finitely generated FI-modules have strong closure properties that allow our arguments below. In particular, extensions and quotients of finitely generated FI-modules are still finitely generated ([10, Proposition 2.17]). Furthermore they satisfy a "Noetherian property" in the following sense:

Theorem 2.2.1 (Theorem 1.1 in [11])**.** *If V is a finitely generated FI-module over a Noetherian ring R, and W is a sub-FI-module of V, then W is finitely generated.*

It was first proved for FI-modules over a field of characteristic zero in [10, Theorem 2.60].

2.3 Using the FI-module notation

Our objective in Chapter 4 is to study some examples of sequences of cohomology groups that have an underlying structure of an FI-module as defined by Church–Ellenberg–Farb in [10] and derive as many consequences as we can from this approach.

Recall from Section 2.2, that **FI** is the category of finite sets and injections. First we will consider a functor X from $\mathbf{FI}^{op}$ to the category **Top** of topological spaces or to the category **Gp** of groups. In the first case we call X a *co-FI-space*, in the second case X is a *co-FI-group*. Given such a functor X, for any $i \geq 0$, we will be interested in the *FI-module* $H^i(X; R)$ over a Noetherian ring R that we obtain by composing X with the cohomology functor $H^i(\_\_; R)$. We also consider the graded FI-module $H^*(X; R)$ over R.

We will focus in the FI-modules that arise from the following examples of co-FI-spaces and co-FI-groups.

(1) The co-FI-space $\mathrm{Conf}_\bullet(M)$. It is given by $\mathbf{n} \mapsto \mathrm{Conf}_n(M)$ and for a given inclusion $f : [m] \hookrightarrow [n]$ in $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, \mathbf{n})$ the corresponding restriction $f^* : \mathrm{Conf}_n(M) \to \mathrm{Conf}_m(M)$ is given by precomposition.

(2) The co-FI-space $\mathcal{M}_{g,\bullet}$. This functor is given by $\mathbf{n} \mapsto \mathcal{M}_{g,n}$ and such that assigns to $f \in \mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, \mathbf{n})$ the morphism $f^* : \mathcal{M}_{g,n} \to \mathcal{M}_{g,m}$ defined by $f^*\big([(X;\mathfrak{p})]\big) = [(X;\mathfrak{p}\circ f)]$.

(3) The co-FI-group $\mathrm{PMod}^\bullet(M)$. It is given by $\mathbf{n} \mapsto \mathrm{PMod}^n(M)$. Any inclusion $f : [m] \hookrightarrow [n]$ induces a restriction map $\mathrm{PDiff}^{\mathfrak{p}}(M) \to \mathrm{PDiff}^{\mathfrak{p}\circ f}(M)$, which gives us the morphism $f^* : \mathrm{PMod}^n(M) \to \mathrm{PMod}^m(M)$.

When $M = \Sigma_{g,r}$ we denote $\mathrm{PMod}^\bullet(\Sigma_{g,r})$ by $\mathrm{PMod}^\bullet_{g,r}$.

(4) The co-FI-space $B\,\mathrm{Diff}^\bullet(M)$. This is the functor $\mathbf{n} \mapsto B\,\mathrm{PDiff}^n(M)$. The morphisms are defined in a similar manner as for $\mathrm{PMod}^\bullet(M)$. If M is orientable, we can restrict to orientation-preserving diffeomorphisms.

Our main purpose in Chapter 4 is to prove finite generation of the corresponding FI-modules $H^i(X;k)$ over a field k.

2.3.1 A non-finitely generated FI-module

Consider the functor that assigns to $\mathbf{n}$ the space $\overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$. This is the co-FI-space $\mathcal{M}_{g,\bullet}$. Morphisms are defined by extending, in a careful way, the corresponding morphisms from $\mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$, so that the natural inclusions $\mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$ give us a map of co-FI-spaces $\mathcal{M}_{g,\bullet} \to \overline{\mathcal{M}}_{g,\bullet}$. See for example [33, Chapter 1] for a definition of $\overline{\mathcal{M}}_{g,n}$ and its corresponding forgetful maps. The co-FI-space $\overline{\mathcal{M}}_{g,\bullet}$ is an example of an FI-module that is not finitely generated. For instance $H^2(\overline{\mathcal{M}}_{g,\bullet})$ is not finitely generated for any $g > 2$ since computations of Arbarello and Cornalba in [1] show that for that case

$$\dim_{\mathbb{Q}}\left(H^2(\overline{\mathcal{M}}_{g,n};\mathbb{Q})\right) = 2^{n-1}(g+1),$$

thus violating Theorem 2.7.1 below.

2.4 Finitely generated FI#-modules

In some of our examples above, the FI-module V over k has actually the extra structure of an FI#-module over k. In that case more is true: if V is finitely generated in degree $\leq m$, then any sub-FI#-module is finitely generated in degree $\leq m$ (follows from [10, Corollaries 2.25 & 2.26]). Moreover, FI#-modules satisfy the following condition.

Theorem 2.4.1 (Corollary 2.27 in [10])**.** *If V is an FI#-module and k is a field of arbitrary characteristic, the following are equivalent:*

(a) V is finitely generated

(b) $\dim_k V_n = O(n^d)$ for some d

(c) $\dim_k V_n = P(n)$ for some polynomial $P \in \mathbb{Q}[t]$ and all $n \geq 0$

If k is an arbitrary commutative ring, then V is finitely generated if and only if V_n is generated by $O(n^d)$ elements for some d.

Notation (The FI-modules $M(W)$): Let $m \in \mathbb{N}$ and consider a fixed S_m-representation W over a field k or $k = \mathbb{Z}$. The FI-module $M(W)$ is defined as follows:

$$M(W)_n := \begin{cases} 0, & \text{if } n < m \\ \operatorname{Ind}_{S_m \times S_{n-m}}^{S_n} W \boxtimes k, & \text{if } n \geq m. \end{cases}$$

In particular, when $W = k[S_m]$ we will denote the FI-module $M(W)$ by $M(m)$. For a given partition $|\lambda| = m$, we use $M(\lambda)$ to denote the FI-module $M(V_\lambda)$. These FI-modules were introduced in [10, Section 2.1]. By definition, they are finitely generated in degree m. Moreover they have the structure of an FI#-module and have surjectivity degree at most m (actually $M(\lambda)$ has surjectivity degree λ_1).

The FI#-modules $M(W)$ are "building blocks" for general FI#-modules.

Theorem 2.4.2. *If V finitely generated FI#-module in degree $\leq d$, then V is of the form*

$$V = \bigoplus_{i=0}^{d} M(W_i)$$

where W_i is a representation (possibly zero) of S_i.

This follows from Theorem 2.24 and Corollary 2.26 in [10].

2.5 FI-modules over fields of characteristic zero

The proof of the "Noetherian property" does not give an upper bound on the degree in which a given subobject is generated. To deal with this for FI-modules over fields of characteristic zero the notion of *weight of an FI-module* was introduced in [10, Section 2.5].

In the remaining subsections, k is always a field of characteristic zero, unless otherwise indicated.

Definition 2.5. *Let V be an FI-module over k. We say that V has weight $\leq d$ if for every $n \geq 0$ and every irreducible constituent $V(\lambda)_n$ we have $|\lambda| \leq d$.*

Notice that the weight of an FI-module is closed under subquotients and extensions. The subgroup of S_n that permutes $\{a+1, \ldots, n\}$ and acts trivially on $\{1, 2, \ldots, a\}$ is denoted by S_{n-a}. The coinvariant quotient $(V_n)_{S_{n-a}}$ is the S_a-module $V_n \otimes_{k[S_{n-a}]} k$, i.e. the largest quotient of V_n on which S_{n-a} acts trivially.

The following provides a notion of stabilization and range of stabilization for an FI-module (this is just a rephrasing of [10, Definitions 2.34 & 2.35]).

Definition 2.6. *Let V be an FI-module over k. If for every $a \geq 0$ and $n \geq N + a$ the map of coinvariants*

$$(V_n)_{S_{n-a}} \to (V_{n+1})_{S_{n-a}} \tag{2.1}$$

induced by the standard inclusion $I_n : \{1, \ldots n\} \hookrightarrow \{1, \ldots, n, n+1\}$, is an injection of S_a-modules, we say that V has injectivity degree $\leq N$. If the map (2.1) is surjective, we say *that V has surjectivity degree $\leq N$. The FI-module V has stability type (M, N) if it has injectivity degree M and surjectivity degree N. The stability degree of V is given by at most* $\max(M, N)$. *When V is an $FI\#$-module, the identity on V_n factors through I_n, hence the injectivity degree is always* 0.

The next two key properties that we will be using in our arguments below correspond to [10, Propositions 2.45 & 2.46].

Proposition 2.5.1. *Let k be a field of characteristic zero.*

(a) Let $U \xrightarrow{f} V \xrightarrow{g} W$ be a complex of FI-modules. If U has surjectivity degree A, V has stability type (B, C), and W has injectivity degree D; then the cohomology group $\ker g / \operatorname{im} f$ *has stability type* $\big(\max(A, B), \max(C, D)\big)$.

(b) Let V be an FI-module with a filtration by FI-modules $0 = F_k V \subseteq \ldots \subseteq F_1 V \subseteq F_0 V = V$. If for every $0 \leq i \leq k-1$ the quotient $F_i V / F_{i+1} V$ has stability type (A, B), then so does the FI-module V.

2.6 Finitely generated FI-modules and representation stability

As mentioned before, an FI-module over a field k containing $\mathbb{Q}$ encodes the information of a sequence of S_n-representations. More is true: if V is an FI-module and I_n the standard inclusion, then $\{V_n, V(I_n)\}$ is a consistent sequence of S_n-representations in the sense of Definition 2.1. We are interested in the behavior of consistent sequences arising from FI-modules.

The original motivation to introduce the notion of an FI-module was to give a new approach to representation stability that would allow to simplify many of the arguments. The precise correspondence between finitely generated FI-modules and uniformly representation stable sequences is contained in the following theorem that was proved in [10, Section 2.6].

Theorem 2.6.1. *Let V be an FI-module of weight $\leq d$ with stability degree $\leq N$. Then V_n is uniformly representation stable with stable range $n \geq N+d$. Conversely, if V_n is uniformly representation stable with stable range $n \geq M$, then V_n is a monotone sequence in the sense of [9] and the FI-module V is finitely generated in degree $\leq M$.*

2.7 Polynomiality of characters

For each $k \geq 1$ we let $Z_k : \coprod_{n\geq 1} S_n \to \mathbb{N}$ be the class function defined by

$$Z_k(\sigma) = \text{ number of } k\text{-cycles in the cycle decomposition of } \sigma.$$

Polynomials with rational coefficients in the variables Z_k are called *character polynomials*. It turns out that characters of finitely generated FI-modules have a uniform description in terms of the class functions Z_k.

Theorem 2.7.1 (Theorem 2.67 in [10]). *Let k be a field of characteristic zero. Let V be a finitely generated FI-module with stability degree $\leq N$ and weight $\leq d$. Then there is a unique character polynomial Q_V in the i-cycle counting functions Z_k, such that the character*

$$\chi_{V_n} = Q_V(Z_1, Z_2, \ldots, Z_d) \quad \textit{for all } n \geq N + d.$$

If V is an FI#-module such equality holds for any $n \geq 0$.

The degree of the polynomial Q_V is at most d (each Z_k has degree k).

Remarks:

- As a consequence we have that for $\sigma \in S_n$, the value of $\chi_{V_n}(\sigma)$ only depends on cycles of length $\leq d$ ("short cycles").

- In particular, that implies that for all $n \geq N + d$ the dimension $\dim_{\mathbb{Q}}\left(V_n\right) = \chi_{V_n}(\mathrm{id})$ is a polynomial in n of degree $\leq d$.

2.8 On the cohomology of group extensions

A *group extension* of a group Q by a group H is a short exact sequence of groups

$$1 \to H \to G \to Q \to 1. \tag{2.2}$$

Given a G-module M, the conjugation action $(h, m) \mapsto (ghg^{-1}, g\cdot m)$ of G on (H, M) induces an action of $G/H \cong Q$ on $H^*(H; M)$ as follows. Let $F \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and consider the diagonal action of G in the cochain complex $\mathcal{H}om(F, M)$ given by $f \mapsto [x \mapsto g \cdot f(g^{-1} \cdot x)]$, for $f \in \mathcal{H}om(F, M)$ and $g \in G$. This action restricts to the subcomplex $\mathcal{H}om_H(F, M)$ where H acts trivially by definition, hence we get an induced action of $Q \cong G/H$ on $\mathcal{H}om_H(F, M)$. But the cohomology of this complex is $H^*(H; M)$, giving the desired action of Q on $H^*(H; M)$.

The *cohomology Hochschild-Serre spectral sequence* for the group extension (2.2) is a first quadrant spectral sequence converging to $H^*(G; M)$ whose E_2 page is of the form

$$E_2^{p,q} = H^p(Q; H^q(H; M)).$$

Furthermore, from the construction of the Hochschild-Serre spectral sequence it can be shown that this spectral sequence is natural in the following sense. Assume we have group extensions (I) and (II) and group homomorphisms f_H and f_G making the following diagram commute

$$\begin{tikzcd}
1 \arrow[r] & H_1 \arrow[r] \arrow[d, "f_H"'] & G_1 \arrow[r] \arrow[d, "f_G"'] & Q \arrow[r] \arrow[d, equal, "\mathrm{id}"'] & 1 & \text{(I)} \\
1 \arrow[r] & H_2 \arrow[r] & G_2 \arrow[r] & Q \arrow[r] & 1 & \text{(II)}
\end{tikzcd}$$

Then the induced map

$$f_H^*\colon H^*(H_2; \mathbb{Q}) \to H^*(H_1; \mathbb{Q})$$

is Q-equivariant. Moreover, if $'E_*$ and $''E_*$ denote the Hochschild-Serre spectral sequences corresponding to the extensions (I) and (II), we have

1) Induced maps $(f_H)^*_r\colon {}''E_r^{p,q} \to {}'E_r^{p,q}$ that commute with the differentials.

2) The map $(f_G)^*\colon H^*(G_2;\mathbb{Q}) \to H^*(G_1;\mathbb{Q})$ preserves the natural filtrations of $H^*(G_1;\mathbb{Q})$ and $H^*(G_2;\mathbb{Q})$ inducing a map on the succesive quotients of the filtrations which is the map

$$(f_H)^*_\infty\colon {}''E_\infty^{p,q} \to {}'E_\infty^{p,q}.$$

3) The map $(f_H)^*_2\colon {}''E_2^{p,q} \to {}'E_2^{p,q}$ is the one induced by the group homomorphisms $\mathrm{id}\colon Q \to Q$ and $f_H\colon H_1 \to H_2$.

For an explicit description of the Hochschild-Serre spectral sequence we refer the reader to [7] and [36] (where it is called the Lyndon spectral sequence).

2.9 Finiteness Conditions

Finally we want to recall some finiteness conditions for groups that will be needed in our results below. We refer the reader to [7, Chapter VIII] for a more detailed expositions of these notions.

To compute $H^*(G,M)$ we can choose an arbitrary projective resolution $P=(P_i)_{i\geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Similarly, using the topological point of view, we can compute $H^*(G,M)$ in terms of an arbitrary $K(G,1)$-complex Y. Then it is reasonable to take the "smaller" complex Y possible. In the same way we can look for the "smaller" projective resolution: in terms of its length or such that each P_i is finitely generated.

The *cohomological dimension* of a group G, denoted by cd G, is defined as the projective dimension $\operatorname{projdim}_{\mathbb{Z}G}\mathbb{Z}$. An R-module M has projective dimension $\operatorname{projdim}_R M \leq n$ if and only if M admits a projective resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$ of length n. In other words

$$\text{cd } G = \inf\{n : \mathbb{Z} \text{ admits a projective resolution of length } n\}.$$

A module M over a ring R is said to be of *type* FP_n $(n \geq 0)$ if there is a partial projective resolution $P_n \to \cdots \to P_0 \to M \to 0$ of finite type, meaning that each P_i is a finitely generated projective module. The condition FP_0 corresponds to finite generation, FP_1 is finite presentation and the conditions FP_2, $FP_3, \ldots$ are successive strengthening of finite presentation. When M is type FP_n for all integers $n \geq 0$ we say that M is of type FP_∞. Moreover, we have the following characterization:

Proposition 2.9.1. *The following conditions on a module M are equivalent:*

(i) *M is of type FP_∞*

(ii) *M admits a free resolution of finite type*

(iii) *M admits a projective resolution of finite type.*

We are interested in the case when the ring R is a group ring $\mathbb{Z}G$ and $M = \mathbb{Z}$. We say that a group G is of type FP_n $(0 \leq n \leq \infty)$ if $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}G$-module.

The FP_n conditions behave nicely with respect to subgroups of finite index:

Proposition 2.9.2. *Let $H \leq G$ be a subgroup of finite index. Then G is of type FP_n if and only if H is of type FP_n.*

One of our main hypothesis in our theorems below is that G is of type FP_∞. The reason is the following proposition.

Proposition 2.9.3. *Let G be a group of type FP_n and M be a G-module which is finitely generated as an abelian group. Then $H_i(G, M)$ and $H^i(G, M)$ are finitely generated as abelian groups for $i \leq n$.*

CHAPTER 3

REPRESENTATION STABILITY

In this chapter we prove that, for any $i \geq 0$, uniform representation stability holds for the degree i rational cohomology of the pure mapping class groups of surfaces $\mathrm{PMod}_{g,r}^n$, the moduli space $\mathcal{M}_{g,n}$ of n-pointed curves of genus g, the pure mapping class groups $\mathrm{PMod}^n(M)$ and the classifying spaces $B\,\mathrm{PDiff}^n(M)$ for some manifolds M. This is the content of [30].

3.1 Introduction

Our motivating example for this chapter is the pure mapping class group of surfaces. Specifically, we want to compare $H^i(\mathrm{PMod}_{g,r}^n;\mathbb{Q})$ as the number of punctures n varies. The natural inclusion $\Sigma_{g,r}^{n+1} \hookrightarrow \Sigma_{g,r}^n$ induces the *forgetful map*

$$f_n\colon\ \mathrm{PMod}_{g,r}^{n+1} \to \mathrm{PMod}_{g,r}^n\,.$$

Notice that f_n is a left inverse for the map μ_n above, when $r > 0$, but can be defined even for surfaces without boundary. This map allows us to relate the corresponding cohomology groups:

$$f_n^*\colon H^*(\mathrm{PMod}_{g,r}^n;\mathbb{Q}) \to H^*(\mathrm{PMod}_{g,r}^{n+1};\mathbb{Q}).$$

Observe that f_n^* is also induced by the *forgetful morphism* between moduli spaces

$$\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}.$$

The key idea is to consider the natural action of the symmetric group S_n on $\mathcal{M}_{g,n}$ given by permuting the n labeled marked points. Thus we can regard $H^i(\mathcal{M}_{g,n};\mathbb{Q})$ as rational S_n-representations and compare them through the maps f_n^i. Moreover, we notice that the map f_n^i is equivariant with respect to the standard inclusion $S_n \hookrightarrow S_{n+1}$. In Section 3.2

below we explicitly compute the S_n-representation $H^2(\mathrm{PMod}_g^n;\mathbb{Q})$ and its decomposition into irreducibles.

3.1.1 Main results

Instead of asking if f_n^i is an isomorphism or not (puncture cohomological stability), we consider the question of whether the cohomology groups of the pure mapping class group satisfy representation stability. In [12, Theorem 4.2] Church-Farb prove that the sequence $\{H^i(P_n;\mathbb{Q}), f_n^i\}_{n=1}^{\infty}$ is representation stable. Our main result in this chapter shows that this is also the case for the pure mapping class group.

Theorem 3.1.1. *For any $i \geq 0$ and $g \geq 2$ the sequence $\{H^i(\mathrm{PMod}_g^n;\mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range*

$$n \geq min\ \{4i + 2(4g-6)(4g-5), 2i^2 + 6i\}.$$

Our arguments work for hyperbolic non-closed surfaces (Theorem 3.4.9). Hence Harer's homological stability and our main theorem imply that $H^i(\mathrm{PMod}_{g,r}^n;\mathbb{Q})$, as representation of S_n, is independent of g, r and n, provided n and g are large enough.

By (1.1), Theorem 3.1.1 can be restated as follows.

Corollary 3.1.2 (Representation stability for the cohomology of the moduli space $\mathcal{M}_{g,n}$)**.** *For any $i \geq 0$ and $g \geq 2$ the sequence of cohomology groups $\{H^i(\mathcal{M}_{g,n};\mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range $n \geq min\ \{4i+2(4g-6)(4g-5), 2i^2+6i\}$.*

Remark. In [6, Theorem 1.1] Bödigheimer and Tillmann proved that

$$B(\mathrm{PMod}_{\infty,r}^n)^+ \simeq B\,\mathrm{Mod}_{\infty}^+ \times (\mathbb{C}P^{\infty})^n.$$

Together with Harer's homological stability theorem this implies that, in dimensions $* \leq g/2$,

$$H^*(\mathrm{PMod}_{g,r}^n;\mathbb{Q}) \cong H^*(\mathrm{PMod}_{g,r};\mathbb{Q})\otimes\big(H^*(\mathbb{C}P^\infty;\mathbb{Q})\big)^{\otimes n} \cong H^*(\mathrm{PMod}_{g,r};\mathbb{Q})\otimes\mathbb{Q}[x_1,\ldots,x_n],$$

where each x_i has degree 2. The action of the symmetric group S_n on the left hand side corresponds to permuting the n factors $\mathbb{C}P^\infty$. In other words, it is given by the action of S_n on the polynomial ring in n variables by permutation of the variables x_i. On the other hand, Church and Farb proved in [12, Section 7] that representation stability holds for the S_n-action on the polynomial ring in n variables. Hence Bödigheimer and Tillmann result implies that for $i \leq g/2$ representation stability holds for $\{H^i(\mathrm{PMod}_{g,r}^n;\mathbb{Q})\}_{n=1}^\infty$. Notice that this only holds for large g with respect to i. In contrast, our Theorem 3.1.1 and Theorem 3.4.9 give uniform representation stability and monotonicy for arbitrary $g \geq 0$ such that $2g+r+s>2$ and large n.

Theorem 3.1.1 implies cohomological stability for Mod_g^n with twisted rational coefficients (see Section 3.4.3). For any partition λ, we denote the corresponding irreducible S_n-representation by $V(\lambda)_n$, as we have explained in Section 2.1 above. A transfer argument gives the proof of the following corollary of Theorem 3.1.1.

Corollary 3.1.3. *For any partition λ, the sequence $\{H^i\big(\mathrm{Mod}_g^n;V(\lambda)_n\big)\}_{n=1}^\infty$ of twisted cohomology groups satisfies classical cohomological stability: for fixed $i \geq 0$ and $g \geq 2$, there is an isomorphism*

$$H^i\big(\mathrm{Mod}_g^n;V(\lambda)_n\big) \approx H^i\big(\mathrm{Mod}_g^{n+1};V(\lambda)_{n+1}\big),$$

if $n \geq min\ \{4i+2(4g-6)(4g-5), 2i^2+6i\}$.

In [29, Proposition 1.5], Hatcher-Wahl obtained integral puncture homological stability for the mapping class group of surfaces with non-empty boundary and established a stable range linear in i. Plugging in the trivial representation $V(0)_n$ into Corollary 3.1.3, we recover rational puncture homological stability for Mod_g^n. The stable range that we obtain either depends on the genus g of the surface or is quadratic in i (see Corollary 3.4.8). Nonetheless, our approach by representation stability is completely different from the classical techniques used in the proofs of homological stability. Furthermore, we believe that our proof gives yet another example of how the notion of representation stability can give meaningful answers where classical stability fails.

In Section 3.5.2 we include a proof, for any group G, of representation stability for the sequence $\{H^i(G^n;\mathbb{Q})\}_{n=1}^{\infty}$. This is Proposition 3.5.5 below. We show how to use this result and the ideas developed in this chapter to establish the analogue of Theorem 3.1.1 and Corollary 3.1.3 for the pure mapping class groups of some connected manifolds of higher dimension.

Theorem 3.1.4. *Let M be a smooth connected manifold of dimension $d \geq 3$ such that $\pi_1(M)$ is of type FP_∞ (e.g. M compact). Suppose that $\pi_1(M)$ has trivial center or that $\mathrm{Diff}(M)$ is simply connected. If $\mathrm{Mod}(M)$ is a group of type FP_∞, then for any $i \geq 0$ the sequence of cohomology groups $\{H^i\big(\mathrm{PMod}^n(M);\mathbb{Q}\big)\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range $n \geq 2i^2+4i$.*

Corollary 3.1.5. *Let M be as in Theorem 3.1.4. For any partition λ, the sequence of twisted cohomology groups $\{H^i\big(\mathrm{Mod}^n(M);V(\lambda)_n\big)\}_{n=1}^{\infty}$ satisfies classical homological stability: for fixed $i \geq 0$, there is an isomorphism*

$$H^i\big(\mathrm{Mod}^n(M);V(\lambda)_n\big) \approx H^i\big(\mathrm{Mod}^{n+1}(M);V(\lambda)_{n+1}\big) \text{ if } n \geq 2i^2+4i.$$

Hatcher-Wahl proved integral puncture homological stability for mapping class group of connected manifolds with boundary of dimension $d \geq 2$ in [29, Proposition 1.5]. Our Corollary 3.1.5, applied to the trivial representation, gives rational puncture homological stability for $\mathrm{Mod}^n(M)$ for manifolds M that satisfy the hypothesis of Theorem 3.1.4, even if the manifold has empty boundary.

Ezra Getzler and Oscar Randal-Williams pointed out to me that the same ideas also give representation stability for the rational cohomology groups of the classifying space $B\,\mathrm{PDiff}^n(M)$ of the group $\mathrm{PDiff}^n(M)$ defined above.

Theorem 3.1.6. *Let M be a smooth, compact and connected manifold of dimension $d \geq 3$ such that $B\,\mathrm{Diff}(M \text{ rel } \partial M)$ has the homotopy type of CW-complex with finitely many cells in each dimension. Then for any $i \geq 0$ the sequence $\{H^i\big(B\,\mathrm{PDiff}^n(M);\mathbb{Q}\big)\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range $n \geq 2i^2 + 4i$.*

The details are described at the end of this chapter in Section 3.6.

3.1.2 Outline of the proof of Theorem 3.1.1

The proof of Theorem 3.1.1 is presented in Section 3.4 and relies on the existence of the Birman exact sequence which realizes $\pi_1(\mathrm{Conf}_n(\Sigma_g))$ as a subgroup of PMod_g^n. Here $\mathrm{Conf}_n(\Sigma_{g,r})$ denotes the configuration space of n distinct ordered points in the interior of $\Sigma_{g,r}$. Then for each n we can consider the associated Hochschild-Serre spectral sequence $E_*(n)$, which allows us to relate $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$ with $H^*\big(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q}\big)$. Following ideas of Church in [9], we use an inductive argument to show that the terms in each page of the spectral sequence are uniformly representation stable and thus we conclude the result in Theorem 3.1.1 from the E_∞-page.

The notion of *monotonicity* for a sequence of S_n-representations introduced in [9] is key in our inductive argument on the pages of the spectral sequence. The base of the induction is monotonicity and representation stability for the terms in the E_2-page of the Hochschild-Serre spectral sequence. In order to prove this, we introduce, in Section 3.3 below, the notion of a consistent sequence of rational S_n-representations *compatible with G-actions* and prove the following general result which we hope will be useful in future computations.

Theorem 3.1.7 (Representation stability with changing coefficients)**.** *Let G be a group of type FP_∞. Consider a consistent sequence $\{V_n, \phi_n\}_{n=1}^{\infty}$ of finite dimensional rational representations of S_n compatible with G-actions. If the sequence $\{V_n, \phi_n\}_{n=1}^{\infty}$ is monotone* *and uniformly representation stable with stable range $n \geq N$, then for any integer $p \geq 0$, the sequence $\{H^p(G; V_n), \phi_n^*\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with the same stable range.*

Monotonicity and uniform representation stability for the E_2-page follow from Theorem 3.1.7, as a consequence of the following result by Church [9, Theorem 1].

Theorem 3.1.8 (Church)**.** *For any connected orientable manifold M of finite type and any $q \geq 0$, the cohomology groups $\{H^q(\mathrm{Conf}_n(M); \mathbb{Q})\}$ of the ordered configuration space $\mathrm{Conf}_n(M)$ are monotone and uniformly representation stable, with stable range $n \geq 2q$ if $\dim M \geq 3$ and stable range $n \geq 4q$ if $\dim M = 2$.*

3.2 The second cohomology $H^2(\mathrm{PMod}_g^n; \mathbb{Q})$

First we understand the consistent sequence of S_n-representations $\{H^2(\mathrm{PMod}_g^n; \mathbb{Q}), f_n^2\}$ to give an explicit discussion of the phenomenon of representation stability.

The second cohomology group is given by:

$$H^2(\mathcal{M}_{g,n}; \mathbb{Q}) \approx H^2(\mathrm{PMod}_g^n; \mathbb{Q}) \approx H^2(\mathrm{Mod}_{g,n}; \mathbb{Q}) \oplus \mathbb{Q}^n, \text{ for } g \geq 3. \tag{3.1}$$

We want to compare $H^2(\mathrm{PMod}_g^n; \mathbb{Q})$ through the forgetful maps

$$f_n^2 \colon H^2(\mathrm{PMod}_g^n; \mathbb{Q}) \to H^2(\mathrm{PMod}_g^{n+1}; \mathbb{Q}).$$

We already know that f_n^2 is never an isomorphism (failure of homological stability). Instead, we consider $H^2(\mathrm{PMod}_g^n; \mathbb{Q})$ as an S_n-representation and we investigate how those representations depend on the parameter n.

When $g \geq 4$, $H^2(\mathrm{Mod}_{g,n}; \mathbb{Q}) \approx \mathbb{Q}$ [22] and the S_n-action on this summand is trivial. On the other hand, the summand $\mathbb{Q}^n$ is generated by classes $\tau_i \in H^2(\mathrm{PMod}_g^n; \mathbb{Q})$ $(i = 1, \ldots, n)$

corresponding to the central extensions $\mathrm{PMod}(X_i)$:

$$1 \to \mathbb{Z} \to \mathrm{PMod}(X_i) \to \mathrm{PMod}_g^n \to 1.$$

The right map above is induced from the inclusion $X_i := \Sigma_g - N_\epsilon(p_i) \hookrightarrow \Sigma_g^n$, where $N_\epsilon(p_i) = \{x \in \Sigma_g^n : d(x,p_i) < \epsilon\}$ for a small $\epsilon > 0$. Notice that $X_i \simeq \Sigma_{g,1}^{n-1}$. The kernel is generated by a Dehn twist around the boundary component, which is the simple loop $\partial N_\epsilon(p_i)$ around the puncture p_i in Σ_g^n. Observe that a permutation of the punctures induces a corresponding permutation of the surfaces $\{X_1, \ldots, X_n\}$, hence of the classes τ_i in $H^2(\mathrm{PMod}_g^n;\mathbb{Q})$.

We can also think of τ_i as the first Chern class of the line bundle $\mathbb{L}_i$ over $\mathcal{M}_{g,n}$ defined as follows: at a point in $\mathcal{M}_{g,n}$, i.e. a Riemann surface X with marked points $p_1, \ldots, p_n$, the fiber of $\mathbb{L}_i$ is the cotangent space to X at p_i. In fact, the τ-classes are the image of the ψ-classes under the surjective homomorphism $H^2(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \to H^2(\mathcal{M}_{g,n};\mathbb{Q})$, where $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ (see [20]). A permutation of the marked points induces the same permutation of the classes τ_i in $H^2(\mathcal{M}_{g,n};\mathbb{Q})$. Therefore, S_n acts on the summand $\mathbb{Q}^n$ in (3.1) by permuting the generators.

Thus, for $g \geq 4$ and $n \geq 3$, the decomposition of (3.1) into irreducibles is given by

$$H^2(\mathrm{PMod}_g^n;\mathbb{Q}) \approx V(0)_n \oplus V(0)_n \oplus V(1)_n,$$

where, following our notation from Section 2.1, $V(0)_n$ is the trivial S_n-representation and $V(1)_n$ is the standard S_n-representation. Notice that, even though the dimension of the vector space $H^2(\mathrm{PMod}_g^n;\mathbb{Q})$ blows up as n increases, the decomposition into irreducibles stabilizes. In terms of definition of representation stability stated in Section 2.1, we have shown that the sequence of S_n-representations $\{H^2(\mathrm{PMod}_g^n;\mathbb{Q})\}$ is *uniformly multiplicity stable* with stable range $n \geq 3$. This indicates to us that representation stability of the cohomology groups of PMod_g^n may be the phenomena to expect.

3.3 Representation stability of $H^*(G; V_n)$

We discuss here when representation stability for a sequence $\{V_n\}$ of G-modules will imply representation stability for the cohomology of a group G with coefficients V_n. This is Theorem 3.1.7 below and it is a key ingredient for the base of the induction in the proof of Theorem 3.1.1.

Definition 3.1. *Let G be a group. We will say that a sequence of rational vector spaces V_n with given maps $\phi_n \colon V_n \to V_{n+1}$ is consistent and compatible with G-actions if it satisfies the following:*

- ***Consistent Sequence.*** *Each V_n is a rational S_n-representation and the map $\phi_n \colon V_n \to V_{n+1}$ is equivariant with respect to the inclusion $S_n \hookrightarrow S_{n+1}$.*
- ***Compatible with G-actions.*** *Each V_n is a G-module and the maps $\phi_n \colon V_n \to V_{n+1}$ are G-maps. The G-action commutes with the S_n-action.*

Notice that for a sequence as in the previous definition we have that $\{H^p(G; V_n); \phi_n^*\}$ is a consistent sequence of rational S_n-representations for $p \geq 0$. Here

$$\phi_n^* \colon H^p(G; V_n) \to H^p(G; V_{n+1})$$

denotes the map induced by $\phi_n \colon V_n \to V_{n+1}$.

Proof of Theorem 3.1.7. Take $E \to \mathbb{Z}$ a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of finite type. This means that each E_p is a free G-module of finite rank, say $E_p \approx (\mathbb{Z}G)^{d_p}$ generated by $x_1, \ldots, x_{d_p}$.

There is an S_n-action on the chain complex $\mathcal{H}om(E, V_n)$ given by $\sigma \cdot h \colon x \mapsto \sigma \cdot h(x)$ for any $h \in \mathcal{H}om(E, V_n)$ and $\sigma \in S_n$. Since the S_n-action and the G-action on V_n commute, this action restricts to a well-defined S_n-action on $\mathcal{H}om_G(E, V_n)$ which makes each $\mathcal{H}om_G(E, V_n)^p := Hom_G(E_p, V_n)$ into a rational S_n-representation.

Observe that any G-homomorphism $h\colon E_p \to V_n$ is completely determined by the d_p-tuple $(h(x_1), \ldots, h(x_{d_p}))$. Then the assignment $h \mapsto (h(x_1), \ldots, h(x_{d_p}))$ gives us an isomorphism

$$\mathcal{H}om_G(E, V_n)^p \approx V_n^{\oplus d_p}$$

not just of rational vector spaces, but of S_n-representations. Notice that since V_n is finite dimensional, $\mathcal{H}om_G(E, V_n)^p$ also has finite dimension. Moreover, under this isomorphism the map

$$\phi_n^p := \mathcal{H}om_G(E, \phi_n)^p \colon \mathcal{H}om_G(E, V_n)^p \to \mathcal{H}om_G(E, V_{n+1})^p$$

is just $(\phi_n)^{\oplus d_p} \colon V_n^{\oplus d_p} \to V_{n+1}^{\oplus d_p}$. From Proposition 2.1.1, it follows that the sequence $\{\mathcal{H}om_G(E, V_n)^p; \phi_n^p\}$ is monotone and uniformly representation stable for $n \geq N$.

The differentials δ_p^n of the cochain complex $\mathcal{H}om_G(E, V_n)$ are a consistent sequence of maps, meaning that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}om_G(E, V_n)^p \arrow[r, "\phi_n^p"] \arrow[d, "\delta_p^n"'] & \mathcal{H}om_G(E, V_{n+1})^p \arrow[d, "\delta_p^{n+1}"'] \\
\mathcal{H}om_G(E, V_n)^{p+1} \arrow[r, "\phi_n^{p+1}"] & \mathcal{H}om_G(E, V_{n+1})^{p+1}
\end{tikzcd}$$

From Proposition 2.1.2 the subsequences $\{\ker\, \delta_p^n\}$ and $\{\mathrm{im}\, \delta_p^n\}$ are monotone and uniformly representation stable for $n \geq N$. Finally Proposition 2.1.1 gives the desired result for $H^p(G; V_n) := \ker\, \delta_p^n / \mathrm{im}\, \delta_{p+1}^n$. $\square$

Since $H^0(G; V_n)$ is equal to the G-invariants V_n^G, as a particular case of Theorem 3.1.7, we get the following.

Corollary 3.3.1. *The sequence of G-invariants $\{V_n^G, \phi_n\}$ is monotone and uniformly representation stable with the same stable range as $\{V_n, \phi_n\}$.*

3.4 Representation stability of $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$

In this section we prove our main result Theorem 3.1.1 and some consequences of it. We will focus on the sequence of pure mapping class groups PMod_g^n and its cohomology with rational coefficients. We consider the case $g \geq 2$.

3.4.1 The ingredients for the proof of the main theorem

Here we describe three of the four main ingredients needed in our proof of Theorem 3.1.1 in Section 3.4.2. The ingredient (iv) is Theorem 3.1.8 [9, Theorem 1].

(i) The Birman exact sequence. Our approach relies on the existence of a nice short exact sequence, introduced by Birman in 1969, that relates the pure mapping class group with the pure braid group of the surface: the *Birman exact sequence* $(Bir1_n)$.

The map in $(Bir1_n)$ that realizes $\pi_1(\mathrm{Conf}_n(\Sigma_g),\mathfrak{p})$ as a subgroup of PMod_g^n is the *point-pushing map* Push. For an element $\gamma \in \pi_1(\mathrm{Conf}_n(\Sigma_g),\mathfrak{p})$, consider the isotopy defined by "pushing" the n-tuple $(p_1,\cdots,p_n)$ along γ. Then $\mathrm{Push}(\gamma)$ is represented by the diffeomorphism at the end of the isotopy. The map f in $(Bir1_n)$ is a *forgetful morphism* induced by the inclusion $\Sigma_g^n \hookrightarrow \Sigma_g$.

Taking the quotient $(Bir1_n)$ by the S_n-action there, we obtain the Birman exact sequence $(Bir2_n)$. The relation between these two sequences is illustrated in the following diagram.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & & \\
1 \arrow[r] & \pi_1(\mathrm{Conf}_n(\Sigma_g)) \arrow[r, "\mathrm{Push}"] \arrow[d, "q"'] & \mathrm{PMod}_g^n \arrow[r, "f"] \arrow[d, "q"'] & \mathrm{Mod}_g \arrow[r] \arrow[d, equal, "id"'] & 1 & (Bir1_n) \\
1 \arrow[r] & \pi_1(B_n(\Sigma_g)) \arrow[r, "\mathrm{Push}"] \arrow[d] & \mathrm{Mod}_g^n \arrow[r, "f"] \arrow[d] & \mathrm{Mod}_g \arrow[r] & 1 & (Bir2_n) \\
 & S_n \arrow[r, equal, "id"] \arrow[d] & S_n \arrow[d] & & & \\
 & 1 & 1 & & &
\end{tikzcd} \tag{3.2}$$

The columns in this diagram relate the groups $\pi_1(\mathrm{Conf}_n(\Sigma_g))$ and PMod_g^n with $\pi_1(B_n(\Sigma_g))$ and Mod_g^n, respectively, in the same way as the pure braid group P_n is related to the braid group B_n by the short exact sequence

$$1 \to P_n \to B_n \to S_n \to 1.$$

Proofs of the exactness of the sequences in diagram (3.2) can be found in [5] and [18]. The exactness of $(Bir1_1)$ and $(Bir2_n)$ requires $g \geq 2$.

From the short exact sequence $(Bir1_n)$ we get a Mod_g-action on $H^*(\pi_1(\mathrm{Conf}_n(\Sigma_g)); \mathbb{Q})$. The second column in diagram (3.2) defines an S_n- action on $H^*(\mathrm{PMod}_g^n; \mathbb{Q})$ which restricts to the S_n-action on $H^*(\pi_1(\mathrm{Conf}_n(\Sigma_g)); \mathbb{Q})$ defined by the short exact sequence in the first column. The induced map

$$\mathrm{Push}^* \colon H^*(\mathrm{PMod}_g^n; \mathbb{Q}) \to H^*(\pi_1(\mathrm{Conf}_n(\Sigma_g)); \mathbb{Q})$$

is a S_n-map between rational S_n-representations. Moreover, from the commutativity of diagram (3.2) we have the following.

Proposition 3.4.1. *The actions of* S_n *and* Mod_g *on* $H^*(\pi_1(\mathrm{Conf}_n(\Sigma_g)); \mathbb{Q})$ *commute.*

(ii) The Hochschild-Serre spectral sequence. We denote the Hochschild-Serre spectral sequence associated to the short exact sequence $(Bir1_n)$ by $E_*(n)$, where the E_2-page is given by:

$$E_2^{p,q}(n) = H^p\big(\mathrm{Mod}_g; H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g)); \mathbb{Q})\big),$$

and the spectral sequence converges to $H^{p+q}(\mathrm{PMod}_g^n; \mathbb{Q})$. This spectral sequence gives a natural filtration of $H^i(\mathrm{PMod}_g^n; \mathbb{Q})$:

$$0 \leq F_i^i(n) \leq F_{i-1}^i(n) \leq \cdots \leq F_1^i(n) \leq F_0^i(n) = H^i(\mathrm{PMod}_g^n; \mathbb{Q}), \tag{3.3}$$

where the successive quotients are $F^i_p(n)/F^i_{p+1}(n) \cong E^{p,i-p}_\infty(n)$.

The following lemma is due to Harer ([24, Theorem 4.1]) and establishes that Mod_g satisfies the finiteness conditions that our argument requires.

Lemma 3.4.2. *For $2g+s+r>2$, the mapping class group $\mathrm{Mod}^s_{g,r}$ is a virtual duality group with virtual cohomological dimension $d(g,r,s)$, where $d(g,0,0)=4g-5$, $d(g,r,s)=4g+2r+s-4$, $g>0$ and $r+s>0$, and $d(0,r,s)=2r+s-3$. In particular, $\mathrm{Mod}^s_{g,r}$ is a group of type FP_∞, and for any rational $\mathrm{Mod}^s_{g,r}$-module M, we have $H^p(\mathrm{Mod}^s_{g,r};M)=0$ for $p>d(g,r,s)$.*

We now see that the terms of the spectral sequence $E_*(n)$ are finite dimensional S_n-representations.

Proposition 3.4.3. *For $2\le r\le\infty$, each $E^{p,q}_r(n)$ is a finite dimensional rational S_n-representation and the differentials*

$$d^{p,q}_r(n)\colon E^{p,q}_r(n)\to E^{p+r,q-r+1}_r(n)$$

are S_n-maps.

Proof. Let $\sigma\in S_n$ and take $\tilde\sigma\in\mathrm{Push}(\pi_1(B_n(\Sigma_g))<\mathrm{Mod}^n_g$ (see $(Bir2_n)$). Denote by $c(\tilde\sigma)$ the conjugation by $\tilde\sigma$. Diagram (3.2) then gives:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathrm{Conf}_n(\Sigma_g)) \arrow[r] \arrow[d, "c(\tilde\sigma)"'] & \mathrm{PMod}^n_g \arrow[r] \arrow[d, "c(\tilde\sigma)"'] & \mathrm{Mod}_g \arrow[r] \arrow[d, equal, "id"'] & 1 \\
1 \arrow[r] & \pi_1(\mathrm{Conf}_n(\Sigma_g)) \arrow[r] & \mathrm{PMod}^n_g \arrow[r] & \mathrm{Mod}_g \arrow[r] & 1
\end{tikzcd}$$

The induced maps $c(\tilde\sigma)^*_r\colon E^{p,q}_r(n)\to E^{p,q}_r(n)$ do not depend on the lift of $\sigma\in S_n$ and, by naturality of the Hochschild-Serre spectral sequence, they commute with the differentials. Hence we get an S_n-action on each $E^{p,q}_r(n)$ for $2\le r\le\infty$ that commutes with the differentials. Moreover, naturality also implies that the S_n-action on $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$ induces the corresponding S_n-action on $E_\infty^{p,q}(n)$.

By Lemma 3.4.2, the group Mod_g is of type FP_∞. Totaro showed in [42, Theorem 4] that the cohomology ring $H^*(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q})$ is generated by cohomology classes from the rings $H^*(\Sigma_g;\mathbb{Q})$ and $H^*(P_n;\mathbb{Q})$. In particular, his result implies that $H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space for $q \geq 0$. It follows that

$$E_2^{p,q}(n) = H^p\big(\,\mathrm{Mod}_g; H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q})\big)$$

is a finite dimensional $\mathbb{Q}$-vector space, and likewise for the subquotients $E_r^{p,q}(n)$. □

(iii) The forgetful map. For the pure braid group, there is a natural map $f_n\colon P_{n+1} \to P_n$ given by "forgetting" the last strand. Similarly, the inclusion $\Sigma_g^{n+1} \hookrightarrow \Sigma_g^n$ induces a homomorphism

$$f_n\colon\ \mathrm{PMod}_g^{n+1} \to \mathrm{PMod}_g^n$$

that we call the *forgetful map*. We can also think of this map as the one induced by "forgetting a marked point" in Σ_g^n. When restricted to the subgroup $\mathrm{Push}\big(\pi_1(\mathrm{Conf}_{n+1}(\Sigma_g))\big)$ it corresponds to the homomorphism in fundamental groups induced by the map $\mathrm{Conf}_{n+1}(\Sigma_g) \to \mathrm{Conf}_n(\Sigma_g)$ given by "forgetting the last coordinate". This gives rise to the commutative diagram (3.4) that relates the exact sequences $(Bir1_{n+1})$ and $(Bir1_n)$.

Diagram (3.4) and our remarks in Section 2.8 imply the following.

Proposition 3.4.4. *The induced maps* $f_n^*\colon H^*(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q}) \to H^*(\pi_1(\mathrm{Conf}_{n+1}(\Sigma_g);\mathbb{Q})$ *are* Mod_g*-maps.*

Moreover, diagram (3.4) and naturality of the Hochschild-Serre spectral sequence give us:

1) Induced maps $(f_n)_r^*\colon E_r^{p,q}(n) \to E_r^{p,q}(n+1)$ that commute with the differentials. This means that the differentials $d_r^{p,q}(n)$ are consistent maps in the sense of Proposition 2.1.2.

2) The map $(f_n)^*\colon H^*(\mathrm{PMod}_g^n;\mathbb{Q}) \to H^*(\mathrm{PMod}_g^{n+1};\mathbb{Q})$ preserves the filtrations (3.3) inducing a map on the succesive quotients $E_\infty^{p,q}(n)$ which is the map $(f_n)_\infty^*\colon E_\infty^{p,q}(n) \to E_\infty^{p,q}(n+1)$.

3) The map $(f_n)_2^*\colon E_2^{p,q}(n) \to E_2^{p,q}(n+1)$ is the one induced by the group homomorphisms $\mathrm{id}\colon \mathrm{Mod}_g \to \mathrm{Mod}_g$ and $f_n\colon \pi_1(\mathrm{Conf}_{n+1}(\Sigma_g)) \to \pi_1(\mathrm{Conf}_n(\Sigma_g))$.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & & \\
 & \pi_1(\Sigma_g^n) \arrow[r, equal, "id"] \arrow[d] & \pi_1(\Sigma_g^n) \arrow[d] & & & \\
1 \arrow[r] & \pi_1(\mathrm{Conf}_{n+1}(\Sigma_g)) \arrow[r] \arrow[d, "f_n"'] & \mathrm{PMod}_g^{n+1} \arrow[r] \arrow[d, "f_n"'] & \mathrm{Mod}_g \arrow[r] \arrow[d, equal, "id"'] & 1 & (Bir_{n+1}) \\
1 \arrow[r] & \pi_1(\mathrm{Conf}_n(\Sigma_g)) \arrow[r] \arrow[d] & \mathrm{PMod}_g^n \arrow[r] \arrow[d] & \mathrm{Mod}_g \arrow[r] & 1 & (Bir1_n) \\
 & 1 & 1 & & &
\end{tikzcd} \tag{3.4}$$

3.4.2 The proof of Theorem 3.1.1

In order to prove Theorem 3.1.1 we use an inductive argument on the pages of the spectral sequence described in Section 3.4.1 (ii). The following lemma gives us the base of the induction.

Lemma 3.4.5. *For each $p \geq 0$ and $q \geq 0$, the consistent sequence of rational representations of S_n*

$$\{E_2^{p,q}(n) = H^p\big(\mathrm{Mod}_g; H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q})\big)\}$$

is monotone and uniformly representation stable with stable range $n \geq 4q$.

Proof. Let $q \geq 0$. Since $\mathrm{Conf}_n(\Sigma_g)$ is aspherical, by Theorem 3.1.8 of Church we have that the consistent sequence of rational S_n-representations $\{H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q})\}$ with the forgetful maps

$$f_n\colon H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g));\mathbb{Q}) \to H^q(\pi_1(\mathrm{Conf}_{n+1}(\Sigma_g));\mathbb{Q})$$

is monotone and uniformly representation stable with stable range $n \geq 4q$. Moreover, Propositions 3.4.1 and 3.4.4 imply that the sequence is compatible with the Mod_g-action. The group Mod_g is FP_∞ (Lemma 3.4.2). Hence we can apply Theorem 3.1.7. □

From Lemma 4.4.3, we follow the same type of inductive argument from [9, Section 3] that Church uses in order to prove his main result [9, Theorem 1]. Here we get monotonicity and uniform representation stability for all the pages of the spectral sequence $E_*(n)$. We include the proofs here for completeness.

Lemma 3.4.6. *The sequence $\{E_r^{p,q}(n)\}$ is monotone and uniformly representation stable with stable range $n \geq 4q + 2(r-1)(r-2)$.*

Proof. The proof is done by induction on r where the base case $r = 2$ is given by Lemma 4.4.3. Assume that $\{E_r^{p,q}(n)\}$ is monotone and uniformly representation stable for $n \geq 4(q + \sum_{k=1}^{r-2} k)$.

As noted before, the differentials

$$d_r^{p,q}(n)\colon E_r^{p,q}(n) \to E_r^{p+r,q-r+1}(n)$$

are a consistent sequence of maps in the sense of Proposition 2.1.2. Then $\{\ker d_r^{p,q}(n)\}$ is monotone and uniformly representation stable for $n \geq 4(q+\sum_{k=1}^{r-2} k)$ and $\{\mathrm{im}\ d_r^{p-r,q+r-1}(n)\}$ is monotone and uniformly representation stable for $n \geq 4(q+(r-1)+\sum_{k=1}^{r-2} k)$. Therefore by Proposition 2.1.1 the next page in the spectral sequence $E_r^{p,q}(n) \cong \ker d_r^{p,q}(n)/\mathrm{im}\ d_r^{p-r,q+r-1}$ is monotone and uniformly representation stable for $n \geq 4(q + \sum_{k=1}^{r-1} k)$. □

Lemma 3.4.7. *For every $p, q \geq 0$ and every $n \geq 2$, we have $E_\infty^{p,q}(n) = E_R^{p,q}(n)$, where*

$$R = 4g - 4 = vcd(\mathrm{Mod}_g) + 1.$$

Proof. The Hochschild-Serre spectral sequence $E_*(n)$ is a first-quadrant spectral sequence. Moreover, from Lemma 3.4.2 it follows that for every $p > 4g - 5$

$$0 = H^p(\mathrm{Mod}_g; H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g)) = E_2^{p,q}(n) = E_r^{p,q}(n).$$

Therefore for $R = 4g - 4$, $q \geq 0$ and $0 \leq p \leq 4g - 5$, we have that $E_R^{p-R,q+R-1}(n) = 0$ since $p - R < 0$ and $E_R^{p+R,q-R+1}(n) = 0$ since $p + R > 4g - 5$. Then the differentials $d_R^{p,q}$ and $d_R^{p-R,q+R-1}$ are zero and hence

$$E_{R+1}^{p,q}(n) = \text{ ker } d_R^{p,q}/\text{im } d_R^{p-R,q+R-1} = E_R^{p,q}(n).$$

□

Having built up, we are now able to prove our main result Theorem 3.1.1: uniform representation stability of $\{H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Q})\}_{n=1}^\infty$.

Proof of Theorem 3.1.1. Each of the successive quotients of the natural filtration (3.3) of $H^i(\mathrm{PMod}_g^n; \mathbb{Q})$ give us a sequence

$$\{F_p^i(n)/F_{p+1}^i(n) \approx E_\infty^{p,i-p}(n)\}$$

which, by Lemmas 4.2.1 and 3.4.7, is monotone and uniformly representation stable with stable range $n \geq 4(i-p)+2(4g-6)(4g-5)$. This is the case, in particular, for $F_{i-1}^i(n)/F_i^i(n)$ and $F_i^i(n) \approx E_\infty^{i,0}(n)$. Then by Proposition 2.1.1 we have that $F_{i-1}^i(n)$ is monotone and uniformly representation stable. Reverse induction and Proposition 2.1.1 imply that the sequences $\{F_p^i(n)\}$ $(0 \leq p \leq i)$ are monotone and uniformly representation stable with the same stable range. In particular this is true for $F_0^i(n) = H^i(\mathrm{PMod}_g^n; \mathbb{Q})$.

Observe that

$$4(i-p) + 2(4g-6)(4g-5) + 4p \geq 4(i-p) + 2(4g-6)(4g-5)$$

for all $0 \leq p \leq i$, which give us the desired stable range.

Finally, we notice that for a fixed $i \geq 0$, the group $H^i(\mathrm{PMod}_g^n; \mathbb{Q})$ only depends on the terms $E_\infty^{p,i-p}(n) = E_{i+2}^{p,i-p}(n)$, $i \geq p \geq 0$. Hence from Lemma 4.2.1 we get a stable range that does not depend on the genus g. However, this stable range is quadratic on i: the sequence $\{H^i(\mathrm{PMod}_g^n; \mathbb{Q})\}$ is monotone and uniformly representation stable for $n \geq 4i + 2(i+1)(i) = (2i)(i+3)$. □

3.4.3 Rational homological stability of Mod_g^n

From the short exact sequence in the second column of diagram (1), we have that any rational S_n-representation can be regarded as a representation of Mod_g^n by composing with the projection $\mathrm{Mod}_g^n \to S_n$. As a consequence of Theorem 3.1.1 we get cohomological stability for Mod_g^n with twisted coefficients. This is Corollary 3.1.3 above.

Proof of Corollary 3.1.3. This is just the argument by Church-Farb in [12, Corollary 4.4]. The group PMod_g^n is a finite index subgroup of Mod_g^n and the coefficients $V(\lambda)_n$ are rational vector spaces, therefore the transfer map (see [7]) give us an isomorphism

$$H^i\big(\,\mathrm{Mod}_g^n; V(\lambda)_n\big) \approx H^i\big(\,\mathrm{PMod}_g^n; V(\lambda)_n\big)^{S_n}.$$

Moreover, $V(\lambda)_n$ is a trivial PMod_g^n-representation, since the action of Mod_g^n on $V(\lambda)_n$ factors through S_n. Hence, from the universal coefficient theorem, we have

$$H^i\big(\,\mathrm{PMod}_g^n; V(\lambda)_n)\big)^{S_n} \approx \Big(H^i(\mathrm{PMod}_g^n;\mathbb{Q})\otimes V(\lambda)_n\Big)^{S_n}. \tag{3.5}$$

For two partitions λ and μ of n the representation $V(\lambda)\otimes V(\mu)$ contains the trivial representation if and only if $\lambda=\mu$, in which case it has multiplicity 1 (see [19]) . Therefore the dimension of (3.5) is the multiplicity of $V(\lambda)_n$ in $H^i(\mathrm{PMod}_g^n;\mathbb{Q})$ which is constant for $n\geq 4i+2(4g-6)(4g-5)$ by Theorem 3.1.1. □

In particular, the multiplicity of the trivial representation in $H^i(\mathrm{PMod}_g^n;\mathbb{Q})$, which equals $H^i(\mathrm{Mod}_g^n;\mathbb{Q})$, is constant for $n\geq 4i+2(4g-6)(4g-5)$. In fact, the stable range in this case can be slightly improved.

Corollary 3.4.8. *For any $i\geq 0$ and a fixed $g\geq 2$, the sequence of mapping class groups $\{\mathrm{Mod}_g^n\}_{n=1}^{\infty}$ satisfies rational cohomological stability:*

$$H^i(\mathrm{Mod}_g^n;\mathbb{Q})\approx H^i(\mathrm{Mod}_g^{n+1};\mathbb{Q}),$$

if $n\geq max\ \{i+(2g-3)(4g-5), 2i^2+4i\}$.

Proof. For any n the S_n-invariants of the spectral sequence $(E_2^{p.q})^{S_n}$ form a spectral sequence that converges to $H^{p+q}(\mathrm{PMod}_g^n;\mathbb{Q})^{S_n}$. In fact, $(E_2^{p.q})^{S_n}$ is just the (p,q)-term of the E_2-page of the Hochschild-Serre spectral sequence of the group extension $(Bir2_n)$ converging to $H^{p+q}(\mathrm{Mod}_g^n;\mathbb{Q})$. In [9, Corollary 3] a better stable range than the one in Theorem 3.1.8 is obtained when restricted to the S_n-invariants: the dimension of $H_q(\mathrm{Conf}_n(\Sigma_g);\mathbb{Q})^{S_n}$ is constant for $n>q$. As a consequence the dimension of $(E_2^{p.q})^{S_n}$ is constant for $n\geq q$. Proposition 2.1.3 allows us to repeat the general argument for this spectral sequence of S_n-invariants in order to get the desired stable range. □

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3.4.4 Non-closed surfaces

Our main result is also true if we consider a non-closed surface $\Sigma_{g,r}^{s}$ of genus g, with r boundary components and s punctures with $2g+r+s>2$.

Let $p_1,\ldots,p_n$ be distinct points in the interior of $\Sigma_{g,r}^{s}$. We define the *mapping class group* $\mathrm{Mod}^n(\Sigma_{g,r}^{s})$ as the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_{g,r}^{s}$ that permute the distinguished points $p_1,\ldots,p_n$ and that restrict to the identity on the boundary components. The *pure mapping class group* $\mathrm{PMod}_{g,r}^{n}$ is defined analogously by asking that the distinguished points $p_1,\ldots,p_n$ remain fixed pointwise.

When $2g+r+s>2$ we have again a Birman exact sequence (see [18]):

$$1\to\pi_1(\mathrm{Conf}_n(\Sigma_g^{r+s}))\to\mathrm{PMod}^n(\Sigma_{g,r}^{s})\to\mathrm{Mod}_{g,r}^{s}\to 1.$$

In particular, this includes the three punctured sphere Σ_0^3 and the punctured torus Σ_1^1.

Using this short exact sequence and Theorem 3.1.8 we can use the previous arguments to get representation stability for the cohomology of $\mathrm{PMod}^n(\Sigma_{g,r}^{s})$, when $2g+s+r>2$.

Theorem 3.4.9. *For any $i\geq 0$ and $2g+s+r>2$ the sequence $\{H^i(\mathrm{PMod}^n(\Sigma_{g,r}^{s});\mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range*

$$n\geq min\ \{4i+2\big(d(g,r,s)\big)\big(d(g,r,s)-1\big),2i^2+6i\}.$$

Furthermore for any partition λ and any fixed $i\geq 0$ and $2g+s+r>2$, there is an isomorphism

$$H^i\big(\mathrm{Mod}^n(\Sigma_{g,r}^{s});V(\lambda)_n\big)\approx H^i\big(\mathrm{Mod}^{n+1}(\Sigma_{g,r}^{s});V(\lambda)_{n+1}\big),$$

if $n\geq min\ \{4i+2\big(d(g,r,s)\big)\big(d(g,r,s)-1\big),2i^2+6i\}$.

Here $d(g,r,s)$ denotes the virtual cohomological dimension of $\mathrm{Mod}_{g,r}^{s}$ as in Lemma 3.4.2.

In the case of trivial coefficients $V(0)_n = \mathbb{Q}$ we recover puncture stability for the rational cohomology groups of $\mathrm{Mod}^n(\Sigma_{g,r}^s)$ for $2g + s + r > 2$.

3.5 Pure mapping class groups of higher dimensional manifolds

We now explain how the key ideas from before can be applied to obtain representation stability for the cohomology of pure mapping class groups of higher dimensional manifolds.

3.5.1 Representation stability of $H^*(\mathrm{PMod}^n(M);\mathbb{Q})$

Let M be a connected, smooth manifold and consider the mapping class group $\mathrm{Mod}^n(M)$ and the pure mapping class group $\mathrm{PMod}^n(M)$ as defined in Section 1.1.1. We now show how, in some cases, the previous techniques and Proposition 3.5.5 from Section 3.5.2 can be used to prove representation stability for $\{H^i(\mathrm{PMod}^n(M);\mathbb{Q}), f_n^i\}$.

The inclusion

$$\big(M - \{p_1, \ldots, p_n, p_{n+1}\}\big) \hookrightarrow \big(M - \{p_1, \ldots, p_n\}\big)$$

induces the *forgetful homomorphism*

$$f_n \colon \ \mathrm{PMod}^{n+1}(M) \to \mathrm{PMod}^n(M).$$

Recall that one of the main ingredients needed in our proof of Theorem 3.1.1 is the existence of a Birman exact sequence that allows us to relate $\pi_1(\mathrm{Conf}_n(M), \mathfrak{p})$ with $\mathrm{PMod}^n(M)$. First we notice that, when the dimension of M is $d \geq 3$, the group $\pi_1(\mathrm{Conf}_n(M))$ can be completely understood in terms of $\pi_1(M)$.

Lemma 3.5.1. *Let M be a smooth connected manifold of dimension $d \geq 3$. Then for any $n \geq 1$ the inclusion map $\mathrm{Conf}_n(M) \hookrightarrow M^n$ induces an isomorphism $\pi_1(\mathrm{Conf}_n(M), \mathfrak{p}) \cong \pi_1(M^n, \mathfrak{p}) \cong \prod_{i=1}^n \pi_1(M, p_i)$.*

The case for closed manifolds is due to Birman ([4, Theorem 1]). As Allen Hatcher explained to me, there are many manifolds for which there is a Birman exact sequence.

Lemma 3.5.2 (Existence of a Birman Exact Sequence)**.** *Let M be a smooth connected manifold of dimension $d \geq 3$. If the fundamental group $\pi_1(M)$ has trivial center or* $\mathrm{Diff}(M)$ *is simply connected, then there exists a Birman exact sequence*

$$1 \longrightarrow \pi_1(\mathrm{Conf}_n(M)) \longrightarrow \mathrm{PMod}^n(M) \longrightarrow \mathrm{Mod}(M) \longrightarrow 1. \tag{3.6}$$

Proof. The evaluation map

$$ev\colon \mathrm{Diff}(M) \to \mathrm{Conf}_n(M),$$

given by $f \mapsto (f(p_1), \ldots, f(p_n))$ is a fibration with fiber $\mathrm{PDiff}^n(M)$. Consider the associated long exact sequence in homotopy groups

$$\cdots \longrightarrow \pi_1(\mathrm{Diff}(M)) \longrightarrow \pi_1(\mathrm{Conf}_n(M)) \xrightarrow{\delta} \pi_0(\mathrm{PDiff}^n(M)) \longrightarrow \pi_0(\mathrm{Diff}(M)) \longrightarrow 1.$$

If $\mathrm{Diff}(M)$ is simply connected, then the existence of the short exact sequence (4.2) follows. On the other hand, we may consider the map

$$\psi : \pi_0(\mathrm{PDiff}^n(M)) \to \mathrm{Aut}[\pi_1(\mathrm{Conf}_n(M))]$$

given by $[f] \mapsto [\gamma \mapsto f \circ \gamma]$.

The composition

$$\pi_1(\mathrm{Conf}_n(M)) \xrightarrow{\delta} \pi_0(\mathrm{PDiff}^n(M)) \xrightarrow{\psi} \mathrm{Aut}[\pi_1(\mathrm{Conf}_n(M))]$$

sends $\sigma \in \pi_1(\mathrm{Conf}_n(M))$ to the inner automorphism $c(\sigma)$ given by conjugation by σ. If the dimension $d \geq 3$ and $\pi_1(M)$ has trivial center, then so does $\pi_1(\mathrm{Conf}_n(M))$ by Lemma 3.5.1. In this case, the boundary map δ is injective and we get the desired Birman exact sequence

(4.2). $\square$

The E_2-page of the Hochschild-Serre spectral sequence associated to (4.2) is then

$$E_2^{p,q}(n) = H^p\big(\operatorname{Mod}(M); H^q(\pi_1(\operatorname{Conf}_n(M)); \mathbb{Q})\big).$$

By Lemma 3.5.1

$$H^q(\pi_1(\operatorname{Conf}_n(M)); \mathbb{Q})) = H^q(\pi_1(M)^n; \mathbb{Q}).$$

Moreover, by Proposition 3.5.5 below, if the group $\pi_1(M)$ is of type FP_∞, the consistent sequence $\{H^q(\pi_1(M)^n; \mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable, with stable range $n \geq 2q$. Hence when $\operatorname{Mod}(M)$ is also of type FP_∞ (e.g. M is compact), Theorem 3.1.7 and the same inductive argument on the succesive pages of spectral sequence yield the following:

Lemma 3.5.3. *For every $i \geq 0$ and every $n \geq 2$, the consistent sequence of rational S_n-representations*

$$\{E_2^{i-q,q}(n) = H^{i-q}\big(\operatorname{Mod}(M); H^q(\pi_1(\operatorname{Conf}_n(M)); \mathbb{Q})\big)\}_{n=1}^{\infty}$$

is monotone and uniformly representation stable with stable range $n \geq 2q$. Furthermore $E_\infty^{i-q,q}(n) = E_{i+2}^{i-q,q}(n)$, which is monotone and uniformly representation stable with stable range

$$n \geq 2q + 2(i+1)(i).$$

Observe that now we have all the ingredients needed in order to reproduce our arguments from Section 3.4.2 and prove Theorem 3.1.4 and Corollary 3.1.5.

3.5.2 Representation stability of $H^*(G^n;\mathbb{Q})$

Given a group G, we may consider the sequence of groups $\{G^n = \prod_{i=1}^n G\}$ with the corresponding S_n-action given by permuting the factors. The natural homomorphism $G^{n+1} \to G^n$ by forgetting the last coordinate is equivariant with respect to the inclusion $S_n \hookrightarrow S_{n+1}$. For a fixed $q \geq 0$ the induced maps

$$\phi_n \colon H^q(G^n;\mathbb{Q}) \to H^q(G^{n+1};\mathbb{Q})$$

give us a consistent sequence of S_n-representations. If G is of type FP_∞, we have finite dimensional representations. Monotonicity and uniform representation stability of this sequence are a particular case of [9, Proposition 3.1] (corresponding to the first row in the spectral sequence). Since this result gives us the inductive hypothesis for the proof of Theorem 3.1.4, we present here a complete proof for the reader's convenience.

For a fixed S_l-representation V and each $n \geq l$, we denote by $V_\alpha \boxtimes \mathbb{Q}$ the corresponding $(S_l \times S_{n-l})$-representation, where the factor S_{n-l} acts trivially. We can then consider the sequence of S_n-representation $\{\mathrm{Ind}_{S_l\times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q}\}$ with the natural inclusions

$$\iota_n \colon \mathrm{Ind}_{S_l\times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q} \hookrightarrow \mathrm{Ind}_{S_l\times S_{n+1-l}}^{S_{n+1}} V_\alpha \boxtimes \mathbb{Q}.$$

This sequence is monotone and uniform representation stable as proved in [9, Theorem 2.11]:

Lemma 3.5.4. *Let V be a finite dimensional S_l-representation, then the sequence of induced representations $\{Ind_{S_l\times S_{n-l}}^{S_n} V \boxtimes \mathbb{Q}\}_{n=1}^\infty$ is monotone and uniformly representation stable for $n \geq 2l$.*

This lemma and the Künneth formula give us the following result.

Proposition 3.5.5. *Let G be any group of type FP_∞ and $q \geq 0$. The consistent sequence of S_n-representations $\{H^q(G^n;\mathbb{Q}), \phi_n\}_{n=1}^\infty$ is monotone and uniformly representation stable for $n \geq 2q$.*

Proof. By the Künneth formula we have

$$H^q(G^n;\mathbb{Q}) \approx \bigoplus_{\mathfrak{a}} H^{\mathfrak{a}}(G^n)$$

where the sum is over all tuples $\mathfrak{a} = (a_1, \ldots, a_n)$ such that $a_j \geq 0$ and $\sum a_j = q$ and $H^{\mathfrak{a}}(G^n)$ denotes $H^{a_1}(G;\mathbb{Q}) \otimes \cdots \otimes H^{a_n}(G;\mathbb{Q})$.

Let $\overline{\mathfrak{a}} = \alpha$ where $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l)$ is a partition of q and the α_j are the positive values of $\mathfrak{a}$ arranged in decreasing order. We define $\mathrm{supp}(\mathfrak{a})$ as the subset of $\{1, 2, \ldots, n\}$ for which $a_i \neq 0$. Observe that the length of α is $l =| \mathrm{supp}(\mathfrak{a})| \leq q$. Therefore we have

$$H^q(G^n;\mathbb{Q}) = \bigoplus_{\alpha} H^{\alpha}(G^n)$$

where now the sum is over all partitions α of q of length $l \leq q$ and $H^{\alpha}(G^n) = \bigoplus_{\overline{\mathfrak{a}}=\alpha} H^{\mathfrak{a}}(G^n)$.

The natural S_n-action on G^n induces an S_n-action on $H^q(G^n;\mathbb{Q})$. More precisely, the group S_n acts on n-tuples $\mathfrak{a}$ by permuting the coordinates. This induces an action on $\bigoplus_{\overline{\mathfrak{a}}=\alpha} H^{\mathfrak{a}}(G^n)$ by permuting the summands accordingly (with a sign, since cohomology is graded commutative) . Hence, under this action, each $H^{\alpha}(G^n)$ is S_n-invariant. We now describe $H^{\alpha}(G^n)$ as an induced representation.

For a given α, take $\mathfrak{b} = (\alpha_1, \ldots, \alpha_l, 0, \cdots, 0)$. Observe that we can identify the S_n-translates of $H^{\mathfrak{b}}(G^n)$ with the cosets $S_n/\mathrm{Stab}(\mathfrak{b})$ by an orbit-stabilizer argument. Thus

$$H^{\alpha}(G^n) = \mathrm{Ind}_{\mathrm{Stab}(\mathfrak{b})}^{S_n} H^{\mathfrak{b}}(G^n).$$

Moreover, $S_{n-l} < \mathrm{Stab}(\mathfrak{b}) < S_l \times S_{n-l}$, where S_l permutes coordinates $\{1, \ldots, l\}$ and S_{n-l} permutes coordinates $\{l+1, \ldots, n\}$. Therefore $\mathrm{Stab}(\mathfrak{b}) = H \times S_{n-l}$, for some subgroup $H < S_l$.

Notice that

$$H^{\mathfrak{b}}(G^n) = H^{b_1}(G;\mathbb{Q}) \otimes \cdots \otimes H^{b_l}(G;\mathbb{Q}) \otimes \cdots \otimes H^0(G;\mathbb{Q}) \approx H^{b_1}(G;\mathbb{Q}) \otimes \cdots \otimes H^{b_l}(G;\mathbb{Q})$$

can be regarded as an H-representation.

Let $V_\alpha := \mathrm{Ind}_H^{S_l} H^{\mathfrak{b}}(G^n)$ and let $V_\alpha \boxtimes \mathbb{Q}$ denote the corresponding $(S_l \times S_{n-l})$-representation. Then

$$\begin{aligned} H^\alpha(G^n) = \mathrm{Ind}_{\mathrm{Stab}(\mathfrak{b})}^{S_n} H^{\mathfrak{b}}(G^n) &= \mathrm{Ind}_{H\times S_{n-l}}^{S_n} \Big(H^{\mathfrak{b}}(G^n) \boxtimes \mathbb{Q} \Big) \\ &= \mathrm{Ind}_{S_l \times S_{n-l}}^{S_n} \Big(\mathrm{Ind}_{H\times S_{n-l}}^{S_l \times S_{n-l}} \big(H^{\mathfrak{b}}(G^n) \boxtimes \mathbb{Q} \big) \Big) \\ &= \mathrm{Ind}_{S_l \times S_{n-l}}^{S_n} \Big(\big(\mathrm{Ind}_H^{S_l} H^{\mathfrak{b}}(G^n) \big) \boxtimes \mathbb{Q} \Big) \\ &= \mathrm{Ind}_{S_l \times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q} \ . \end{aligned}$$

Moreover, we notice that the forgetful map ϕ_n restricted to the summand $H^\alpha(G^n)$ corresponds to the inclusion

$$\mathrm{Ind}_{S_l \times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q} \hookrightarrow \mathrm{Ind}_{S_l \times S_{n+1-l}}^{S_{n+1}} V_\alpha \boxtimes \mathbb{Q}.$$

Therefore, by Lemma 3.5.4, the consistent sequence $\{H^\alpha(G^n)\}$ is monotone and uniformly representation stable with stable range $n \geq 2l$, where l is the length of α and $l \leq q$. The result for $\{H^q(G^n;\mathbb{Q}), \phi_n\}$ then follows from Proposition 2.1.1. $\square$

Finally we illustrate the notation in the previous proof with the concrete case of $G = \mathbb{Z}$.

By the Künneth formula we have

$$H^q(\mathbb{Z}^n;\mathbb{Q}) \approx \bigoplus_{\sum a_i = q} H^{a_1}(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^{a_n}(\mathbb{Z};\mathbb{Q}).$$

Following our previous notation we take the n-tuple $\mathfrak{b} = (1,\ldots,1,0,\ldots,0)$ with $|\mathrm{supp}(\mathfrak{b})| = q$

and $\alpha := \overline{\mathfrak{b}}$. Since $H^q(\mathbb{Z};\mathbb{Q}) = \mathbb{Q}$ for $q = 0, 1$ and zero otherwise, we have that

$$H^q(\mathbb{Z}^n;\mathbb{Q}) = \bigoplus_{\overline{\mathfrak{a}}=\alpha} H^{\mathfrak{a}}(\mathbb{Z}^n) = \mathrm{Ind}_{\mathrm{Stab}(\mathfrak{b})}^{S_n} H^{\mathfrak{b}}(\mathbb{Z}^n).$$

Notice that $\mathrm{Stab}(\mathfrak{b}) = S_q \times S_{n-q}$. The corresponding $(S_q \times S_{n-q})$-representation is

$$H^{\mathfrak{b}}(\mathbb{Z}^n) = H^1(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^1(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^0(\mathbb{Z};\mathbb{Q}) \approx V_\alpha \boxtimes \mathbb{Q}$$

where $V_\alpha := H^1(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^1(\mathbb{Z};\mathbb{Q}) \approx H^{\mathfrak{b}}(\mathbb{Z}^n)$ is regarded as an S_q-representation. Then, as an induced representation,

$$H^q(\mathbb{Z}^n;\mathbb{Q}) = \mathrm{Ind}_{S_q \times S_{n-q}}^{S_n} V_\alpha \boxtimes \mathbb{Q}.$$

Moreover, if $\mathbb{Q}^n$ denotes the permutation S_n-representation, then

$$\begin{aligned} \mathrm{Ind}_{S_q \times S_{n-q}}^{S_n} V_\alpha \boxtimes \mathbb{Q} = \bigwedge^q(\mathbb{Q}^n) &= \bigwedge^q\big(V(0)_n \oplus V(1)_n\big) \\ &= \Big(\bigwedge^q V(1)_n\Big) \oplus \Big(\bigwedge^{q-1} V(1)_n\Big) \\ &= V(\underbrace{1, \ldots, 1}_{q})_n \oplus V(\underbrace{1, \ldots, 1}_{q-1})_n \end{aligned}$$

Hence, we see explicitly how uniform multiplicity stability holds for this particular case.

3.6 Classifying spaces for diffeomorphism groups

In this last section we see how the same ideas also imply representation stability for the cohomology of classifying spaces for diffeomorphism groups.

Let M be a connected and compact smooth manifold of dimension $d \geq 3$. We denote by $\mathcal{E}(M, \mathbb{R}^\infty)$ the space of smooth embeddings $M \to \mathbb{R}^\infty$. It is a contractible space and

$\operatorname{Diff}(M \text{ rel } \partial M)$ acts freely by pre-composition. The quotient space

$$\mathcal{E}(M, \mathbb{R}^\infty)/\operatorname{Diff}(M \text{ rel } \partial M)$$

is the space of smooth submanifolds of $\mathbb{R}^\infty$ diffeomorphic to M and it is a classifying space $B\operatorname{Diff}(M \text{ rel } \partial M))$ for $\operatorname{Diff}(M \text{ rel } \partial M))$. Similarly we can consider the action of the subgroup $\operatorname{PDiff}^n(M)$ of $\operatorname{Diff}(M \text{ rel } \partial M)$ on $\mathcal{E}(M, \mathbb{R}^\infty)$. The quotient space is a classifying space $B\operatorname{PDiff}^n(M)$ for $\operatorname{PDiff}^n(M)$ and we have a fiber bundle

$$B\operatorname{PDiff}^n(M) \to B\operatorname{Diff}(M \text{ rel } \partial M) \tag{3.7}$$

where the fiber is given by $\operatorname{Diff}(M \text{ rel } \partial M)/\operatorname{PDiff}^n(M) \approx \operatorname{Conf}_n(M)$, the configuration space of n ordered points in M.

There is a Leray-Serre spectral sequence associated to the fiber bundle (4.3) that converges to the cohomology $H^*(B\operatorname{PDiff}^n(M); \mathbb{Q})$ with E_2-page given by

$$E_2^{p,q}(n) = H^p\big(B\operatorname{Diff}(M \text{ rel } \partial M); H^q(\operatorname{Conf}_n(M); \mathbb{Q})\big). \tag{3.8}$$

Here, we regard (3.8) as the pth cohomology group of $B\operatorname{Diff}(M \text{ rel } \partial M)$ with local coefficients in the G-module $H^q(\operatorname{Conf}_n(M); \mathbb{Q})$, where $G = \pi_1(B\operatorname{Diff}(M \text{ rel } \partial M))$ (see [26, Section 3.H]). Notice that the actions of S_n and G on $H^q(C_n(M); \mathbb{Q})$ commute. Therefore $\{H^q(\operatorname{Conf}_n(M); \mathbb{Q})\}_{n=1}^\infty$ is a consistent sequence compatible with G-actions. Moreover, by Theorem 3.1.8, it is monotone and uniformly representation stable, with stable range $n \geq 2q$. Monotonicity and uniform representation stability for the terms in the E_2-page will be a consequence of the following result, which is essentially Theorem 3.1.7 from before.

Theorem 3.6.1. *Let G be the fundamental group of a connected CW complex X with finitely many cells in each dimension. Consider a consistent sequence $\{V_n, \phi_n\}_{n=1}^\infty$ of finite dimensional rational representations of S_n compatible with G-actions. If the sequence*

$\{V_n, \phi_n\}_{n=1}^{\infty}$ *is monotone and uniformly representation stable with stable range* $n \geq N$*, then for any non-negative integer* p*, the sequence of cohomology groups with local coefficients* $\{H^p(X; V_n), \phi_n^*\}_{n=1}^{\infty}$ *is monotone and uniformly representation stable with the same stable range.*

Proof. Since $G = \pi_1(X)$, the universal cover $\tilde{X}$ of X has a G-equivariant cellular chain complex. Given that X has finitely many cells in each dimension, for each p the group $C_p(\tilde{X})$ is a free G-module of finite rank, where a preferred G-basis can be provided by selecting a p-cell in $\tilde{X}$ over each p-cell in X. Hence, the proof of Theorem 3.6.1 is the same as the one for Theorem 3.1.7, by replacing the notions of cohomology of groups by cohomology of a space with local coefficients. □

Hence when $B\operatorname{Diff}(M \text{ rel } \partial M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension, we can apply the inductive argument from Section 3.4.2 on the successive pages of the Leray-Serre spectral sequence from above and obtain the following result.

Lemma 3.6.2. *For every* $i \geq 0$ *and every* $n \geq 2$*, the consistent sequence of rational* S_n*-representations*

$$\{E_2^{i-q,q}(n) = H^{i-q}\big(B\operatorname{Diff}(M \ rel\ \partial M); H^q(\operatorname{Conf}_n(M); \mathbb{Q})\big)\}_{n=1}^{\infty}$$

is monotone and uniformly representation stable with stable range $n \geq 2q$*. Furthermore* $E_{\infty}^{i-q,q}(n) = E_{i+2}^{i-q,q}(n)$*, which is monotone and uniformly representation stable with stable range*

$$n \geq 2q + 2(i+1)(i).$$

As a consequence we get Theorem 3.1.6 for the cohomology of the classifying space of a group of diffeomorphisms.

If the manifold M is orientable, we can replace $\mathrm{Diff}(M, \text{ rel } \partial M)$ by the group of orientation preserving diffeomorphims $\mathrm{Diff}^+(M, \text{ rel } \partial M)$ in the above argument. In particular, Hatcher and McCullough proved in [27] that if M is an irreducible, compact connected orientable 3-manifold with nonempty boundary, then $B\,\mathrm{Diff}^+(M \text{ rel } \partial M)$ is a finite $\mathrm{K}(\pi, 1)$-space for the mapping class group $\mathrm{Mod}(M)$. Therefore, Theorem 3.1.6 is true for this type of manifold. Moreover, if M satisfies conditions (i)-(iv) in [27, Section 3], then $\pi_1(M)$ is centerless and we can apply Theorem 3.1.4 to get uniform representation stability for the cohomology of $\mathrm{PMod}^n(M)$.

CHAPTER 4

THE THEORY OF FI-MODULES

In this chapter we consider the cohomology of our examples in Section 2.3 and prove that each of such sequences has the structure of a *finitely generated FI-module* over any field k. The content of this chapter corresponds to [31].

4.1 Introduction

Let X be any of the co-FI-spaces or co-FI-groups from Section 2.3. A combination of the work by Church ([9]), Church–Ellenberg–Farb ([10]) and myself ([30]) implies that, under some hypotheses recalled below, for any $i \geq 0$ the FI-module $H^i(X; \mathbb{Q})$ is *finitely generated*.

In this chapter we develop a unified approach to proving finite generation for FI-modules that arise as in the examples above. In Section 4.2 we present a general spectral sequence argument that allows us to prove finite generation for our examples in Theorems 4.3.3, 4.5.1, 4.5.2 and 4.5.6. Furthermore, this approach applies to spectral sequences arising from "FI-fibrations" over a fixed space and "FI-group extensions" of a given group (see Section 4.4.2).

The basic idea is to use a spectral sequence of FI-modules converging to the graded FI-module of interest. We then use knowledge about finite generation of the FI-modules in the E_2-page and an inductive process together with closure properties of finite generation under subquotients and extensions to get our conclusion. The main difference between Theorems 4.3.3 and the other theorems are the type of spectral sequence that we use and the way that finite generation is proved for the E_2-page.

Given finite generation, the conclusions in Theorem 1.3.1 are consequences of [10, Proposition 2.58 and Theorem 2.67] and [11, Theorem 1.2].

4.1.1 Specific Bounds

When $k = \mathbb{Q}$, our direct proofs of finite generation below allow us to obtain linear bounds on i for the degrees of the character polynomials Q_i and the lengths of the representations. Moreover our new stable ranges for uniform representation stability are also linear in i, instead of the quadratic bounds in i that were obtained in Chapter 3. The precise bounds obtained in each case are summarized in Table 4.1.

Remarks:

- The explicit bounds for the first two examples in Table 4.1 are due to Church–Ellenberg–Farb. We recall them here as they are used as main ingredients in our proofs below. For each n, the cohomology ring $H^*(M^n;\mathbb{Q})$ is completely understood and $H^*(\mathrm{Conf}_n(M);\mathbb{Q})$ is described by Totaro in [42]. However, few explicit computations are known for the other examples in Table 4.1.

- From Theorem 1.3.1 (iii) it follows that for each $\sigma \in S_n$, the character $\chi_{V_n}(\sigma)$ only depends on "short cycles", more precisely on the cycles of σ of length $\leq \deg(\chi_n)$. Then in Table 4.1 we provide an upperbound for the lengths of the cycles.

- Our result for $\mathrm{Conf}_n(\Sigma)$ recovers the same stable range of $n \geq 4i$ obtained in [9, Theorem 1] for the case when Σ is a closed surface or has non-empty boundary. The statment about $\dim_k \left(H^i(\mathrm{Conf}_n(\Sigma);k)\right)$ being a polynomial in n for any field k is a particular case of [11, Theorem 1.8].

- Theorem 4.5.1 below implies that the consistent sequences of rational S_n-representations $\{H^i(\mathrm{PMod}^n_{g,r};\mathbb{Q})\}$ and $\{H^i(\mathcal{M}_{g,n};\mathbb{Q})\}$ satisfy uniform representation stability with stable range $n \geq 4i$ when $r > 0$ and $n \geq 6i$ in general. This improves the stable range

$$n \geq \min\{4i + 2(4g-6)(4g-5), 2i^2 + 6i\}$$

  obtained in Theorem 3.1.1.

Table 4.1: Specific bounds for the FI-modules of interest.

Group or Space X_n	**Stable Range**	$\ell(V_n) \leq$	**degree of** $\chi_{V_n} \leq$	**Stability Type**
M^n [10, Thm 4.1]	$n \geq 2i$	$i+1$	i	$(0, i)$
$\mathrm{Conf}_n(M)$ $\dim M = d \geq 3$ [10, Thm 4.2]	$n \geq 2i$	$i+1$	i	$(i+2-d, i)$ $(0,i)$ if $\partial M \neq \emptyset$
$\mathrm{Conf}_n(\Sigma)$ $\dim \Sigma = 2$ Thm 4.3.3	$n \geq 4i$ $n \geq 5i$	$2i+1$	$2i$	$(2i, 2i)$ if Σ closed $(0, 2i)$ if $\partial\Sigma \neq \emptyset$ $(3i-1, 2i)$ O/W
$\mathcal{M}_{g,n}$ with $g \geq 2$ Thm 4.5.1	$n \geq 6i$	$2i+1$	$2i$	$(4i, 2i)$
$\mathrm{PMod}^n_{g,r}$ $2g+r>2$ and $r>0$ Thm 4.5.1	$n \geq 4i$	$2i+1$	$2i$	$(0, 2i)$
$\mathrm{PMod}^n(M)$ $\dim M \geq 3$ Thm 4.5.2	$n \geq 3i$ $n \geq 2i$	$i+1$	i	$(2i, i)$ $(0,i)$ if $\partial M \neq \emptyset$
$B\,\mathrm{PDiff}^n(M)$ $\dim M \geq 3$ Thm 4.5.6	$n \geq 3i$	$i+1$	i	$(2i, i)$

- Theorems 4.5.2 and 4.5.6 below apply to irreducible, compact, orientable 3-manifolds M with nonempty boundary satisfying conditions (i)-(iv) in [27, Section 3].

4.1.2 Other results

FI[G]-modules. Let G be a group. In Section 4.4 we introduce the notion of an FI[G]-*module*: it is a functor V from the category **FI** to the category **G-Mod** of G-modules over R. This definition incorporates the action of a group G on our sequences of S_n-representations and allows us to take V as twisted coefficients for cohomology. For X, a path-connected space with fundamental group G, and $p \geq 0$, we are interested in the FI-module $H^p(X; V)$ over R given by $\mathbf{n} \mapsto H^p(X; V_n)$. Our major result in Section 4.4 is Theorem 4.4.1 which uses finite generation of an FI[G]-module V to obtain finite generation and, when $R = \mathbb{Q}$, specific bounds for the new FI-modules $H^p(X; V)$. This is our tool to prove the base of the induction in the spectral sequence argument for Theorems 4.5.1, 4.5.2 and 4.5.6.

Remark: It was pointed out to me by Ian Hambleton that FI[G]-modules can be understood in the framework of modules over EI-categories. An *EI-category* Γ is a small category in which each endomorphism is an isomorphism. An FI[G]-module corresponds to a left $R\Gamma$-module, where R is the group ring $\mathbb{Z}G$ and Γ is the EI-category **FI**. The theory of $R\Gamma$-modules and its homological algebra have been developed and applied in the context of transformation groups (see for example [15, Chapter I.11]).

Manifolds with boundary. If we assume that M is a manifold with non-empty boundary, the examples above of configuration spaces and pure mapping class groups have the extra structure of an FI#-module that allows us to conclude the following results from the arguments in Section 4.5.3.

Theorem 4.1.1. *Let $\Sigma = \Sigma_{g,r}$ be a connected compact oriented surface with non-empty boundary ($r > 0$). For any $i \geq 0$ and $n \geq 0$, each of the following invariants of $\mathrm{PMod}^n_{g,r}$ is given by a polynomial in n of degree at most $2i$:*

- *The i^{th} rational Betti number $b_i(\mathrm{PMod}^n_{g,r})$ and the i^{th} mod-p Betti number of $\mathrm{PMod}^n_{g,r}$.*

- *The rank of $H^i(\mathrm{PMod}^n_{g,r};\mathbb{Z})$ and the rank of the p-torsion part of $H^i(\mathrm{PMod}^n_{g,r}\,\mathbb{Z})$.*

Theorem 4.1.2. *Let M be a manifold with non-empty boundary that satisfies the hypothesis of Theorem 4.5.2. For $n \geq 0$ each of the following is given by a polynomial in n:*

- *$b_i(\mathrm{PMod}^n(M))$ and the i^{th} mod-p Betti number of $\mathrm{PMod}^n(M)$.*

- *The rank of $H^i(\mathrm{PMod}^n(M);\mathbb{Z})$ and the rank of the p-torsion part of $H^i(\mathrm{PMod}^n(M);\mathbb{Z})$.*

The polynomial is of degree at most i for rational Betti numbers and degree at most $2i$ in the other cases.

Closed Surfaces. For a fixed $n \geq 0$, we can relate the mapping class group of a closed surface with the one of a surface with non-empty boundary. Let

$$\delta_g : \mathrm{PMod}^n_{g,1} \to \mathrm{PMod}^n_g$$

be the group homomorphism induced by gluing a disk to the boundary component. The following result is part of the so called Harer's stability Theorem and was proved initially by Harer ([23]). A proof of it with the improved bounds that we use can be found in [47].

Theorem 4.1.3. *If $i \leq \frac{2}{3}g$, we have following isomorphism:*

$$H_i(\delta_g) : H_i(\mathrm{PMod}^n_{g,1};\mathbb{Z}) \to H_i(\mathrm{PMod}^n_g;\mathbb{Z}).$$

When the genus of the surface is large, by combining the previous result with Theorem 4.1.1 we obtain the following information in the case of closed surfaces.

Theorem 4.1.4. *If $g \geq \max\{2, \frac{3}{2}i\}$, then each of the following invariants of PMod^n_g is given by a polynomial in n for $n \geq 0$:*

- *the i-th rational Betti number $b_i(\mathrm{PMod}^n_g)$*

- *the rank of* $H^i(\mathrm{PMod}^n_g;\mathbb{Z})$
- *the rank of the* p*-torsion part of* $H^i(\mathrm{PMod}^n_g;\mathbb{Z})$

In each case the polynomial is of degree at most $2i$*.*

Cohomological stability of some wreath products. Let $\{K_n\}$ be a sequence of groups with surjections $K_n \twoheadrightarrow S_n$. Given a group G the wreath product $G \wr K_n$ is the semidirect product $G^n \rtimes K_n$, where K_n acts on G^n through the surjection $K_n \twoheadrightarrow S_n$.

Notation: The *surface pure braid group* is the group $\pi_1\big(\mathrm{Conf}_n(\Sigma_{g,r})\big)$ and will be denoted by $P_n(\Sigma_{g,r})$. The *surface braid group* is $\pi_1\big(\mathrm{Conf}_n(\Sigma_{g,r})/S_n\big)$ and we use $B_n(\Sigma_{g,r})$ to denote it. When $g = 0$ and $r = 1$, these are the pure braid group P_n and the braid group B_n, respectively. On the other hand, the *braid permutation group* Σ_n^+ is the group of string motions that preserve orientation of the circles (see [48, Section 8] for a precise definition).

In Section 4.6 we discuss how our previous results and the closure of finite generation of FI-modules under tensor products can be used to get information about homological stability of some wreath products.

Theorem 4.1.5. *Let* G *be any group of type* FP_∞ *and let* K_n *be one of the following groups:*

(i) The symmetric group S_n*,*

(ii) The surface braid group $B_n(\Sigma_{g,r})$*, with* $g, r \geq 0$*,*

(iii) The mapping class group $\mathrm{Mod}^n_{g,r}$*, with* $2g + r > 2$*,*

(iv) The mapping class group $\mathrm{Mod}^n(M)$*, where* M *is a smooth connected manifold of dimension* $d \geq 3$ *such that the hypotheses in Theorem 4.5.2 are satisfied,*

(v) The braid permutation group Σ_n^+*.*

Then the wreath product $G \wr K_n$ *satisfies rational homological stability.*

Remarks: In general we do not have explicit stable ranges. The following is known about stable ranges:

- For (i) we get the stable range $n \geq 2i$. Homological stability is known to hold integrally in this case for $n \geq 2i+1$ (see [29, Propositions 1.6]). Therefore our bound suggests that the possible failure of injectivity when $n = 2i$ should come from torsion.

- For the case (ii), Hatcher–Wahl have shown that if $r > 0$ the group $G \wr B_n(\Sigma_{g,r})$ satisfies integral homological stability when $n \geq 2i+1$ ([29, Propositions 1.7]). Rationally, the stable range has been improved to $n \geq 2i$ by Randall-Williams (see [40, Theorem A]).

4.1.3 Speculation on the existence of non-tautological classes in $\mathcal{M}_{g,n}$

The *tautological ring* of $\mathcal{M}_{g,n}$ is defined to be a subring $\mathcal{RH}^*(\mathcal{M}_{g,n})$ of the cohomology ring $H^*\big(\mathcal{M}_{g,n};\mathbb{Q}\big)$ generated by certain "geometric classes": the kappa-classes $\kappa_j \in H^{2j}\big(\mathcal{M}_{g,n};\mathbb{Q}\big)$, for $j \geq 0$, and the psi-classes $\psi_i \in H^2\big(\mathcal{M}_{g,n};\mathbb{Q}\big)$, for $1 \leq i \leq n$. In $\mathcal{RH}^*(\mathcal{M}_{g,n})$, the class κ_j has grading j and ψ_i has grading 1 (half the cohomological grading). We refer the reader to [17, Section 1] for precise definitions of the tautological rings of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

In [10, Section 5.1] it is proved that $\mathcal{RH}^*(\mathcal{M}_{g,\bullet})$ is a graded FI-module of finite type for $g \geq 2$. This follows from the fact that this graded FI-module is a quotient of the free commutative algebra

$$\mathbb{Q}\big[\{\kappa_j : j \geq 0\} \cup \{\psi_i : 1 \leq i \leq n\}\big],$$

where S_n acts trivially on the kappa-classes and permutes the psi-classes. From this description, we can see that for any $k \geq 0$ the weight of the FI-module $\mathcal{RH}^k(\mathcal{M}_{g,\bullet})$ is at most k. As a consequence we obtain an upper bound for the length of the representation $\ell\big(\mathcal{RH}^k(\mathcal{M}_{g,n})\big) \leq k+1$.

On the other hand, Faber and Pandharipande studied in [17] the S_n-action on $H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ and get an upper bound for the length of the irreducible representations occurring in the tautological ring $\mathcal{RH}^*(\overline{\mathcal{M}}_{g,n})$. Their interest is to exhibit, by other methods (counting, boundary geometry), several classes of Hodge type that cannot be tautological classes because the lengths of the corresponding S_n-representations are larger than their upper bound. In particular, they have established the existence of many non-tautological cohomology classes on $\overline{\mathcal{M}}_{2,21}$. They obtained that $\ell\big(\mathcal{RH}^k(\overline{\mathcal{M}}_{g,n})\big) \leq k+1$ ([17, Section 4]). Since $\mathcal{RH}^k(\overline{\mathcal{M}}_{g,n})$ surjects onto $\mathcal{RH}^k(\mathcal{M}_{g,n})$, that implies that $\ell\big(\mathcal{R}^k(\mathcal{M}_{g,n})\big) \leq k+1$, which is the same bound that we obtained directly with the FI-module approach. In contrast, their method involves studying representations induced from the boundary strata.

Finally we would like to point out that from Table 1 we have the upper bounds

$$\ell\big(H^{2k}(\mathcal{M}_{g,n};\mathbb{Q})\big) \leq 4k+1.$$

This, contrasted with $\ell\big(\mathcal{RH}^k(\mathcal{M}_{g,n})\big) \leq k+1$, suggests that there is room for the existence of non-tautological classes $\mathcal{M}_{g,n}$ and that an approach à la Faber and Pandharipande could demonstrate that some explicit classes are non-tautological. However, we have no indication that our bounds are sharp. As matter of fact, the only completely known case shows evidence of the contrary since $H^2\big(\mathcal{M}_{g,n};\mathbb{Q}\big) = \mathcal{RH}^1(\mathcal{M}_{g,n})$ has length 2.

4.2 A spectral sequence argument

In this section we present the general spectral sequence argument that will give us finite generation and specific bounds in Theorems 4.3.3 and 4.4.4. We basically apply the idea used in the proof of [10, Theorem 4.2] to a more general context.

Setting: Suppose that we have a first quadrant spectral sequence of FI-modules $E_*^{p,q}$ over $\mathbb{Q}$ converging to a graded FI-module $H^*(E)$ over $\mathbb{Q}$. Let α and β be two non-negative constants such that $2\alpha \leq \beta$. In what follows, we assume that for any $p,q \geq 0$ the FI-module $E_2^{p,q}$ is finitely generated with injectivity degree at most βq and surjectivity degree at most $\alpha p + \beta q$.

In our applications below, $E_*^{p,q}$ is either a Leray, Leray–Serre or Hochschild–Serre spectral sequence.

Lemma 4.2.1. *For any $p, q \geq 0$ and $r \geq 3$, the FI-module $E_r^{p,q}$ is finitely generated with injectivity degree at most $\alpha p+\beta q+(\beta-\alpha)r+(\alpha-2\beta)$ and surjectivity degree at most $\alpha p+\beta q$.*

Proof. Finite generation of an FI-module is closed under subquotients. To verify the stated stability type we proceed by induction on $r \geq 3$. The base of induction is the case $r = 3$. To compute $E_3^{p,q}$ we consider the complex of FI-modules

$$E_2^{p-2,q+1} \longrightarrow E_2^{p,q} \longrightarrow E_2^{p+2,q-1} ,$$

where the left map is the differential $d_2^{p-2,q+1}$ and the right map is $d_2^{p,q}$. By hypothesis the left hand side term in the previous complex has surjectivity degree at most $\alpha(p-2)+\beta(q+1) = \alpha p+\beta q+(\beta-2\alpha)$. The middle term has stability type at most $\big(\beta q, \alpha p+\beta q\big)$ and the right hand side term has injectivity degree at most $\beta(q-1)$. Hence, by applying [10, Proposition 2.45] to the complex of FI-modules above, we obtain that the quotient FI-module

$$E_3^{p,q} \approx \ker d_2^{p,q} / \operatorname{im} d_2^{p-2,q+1}$$

has injectivity degree at most

$$\max\big(\alpha p+\beta q+(\beta-2\alpha), \beta q\big) = \alpha p+\beta q+(\beta-2\alpha) = \alpha p+\beta q+(\beta-\alpha)(3)+(\alpha-2\beta)$$

since $2\alpha \leq \beta$, and surjectivity degree at most

$$\max\big(\alpha p+\beta q, \beta q-\beta\big) = \alpha p+\beta q$$

since $\alpha, \beta \geq 0$.

Now suppose that the statement is true for $E_r^{p,q}$. To compute $E_{r+1}^{p,q}$ we consider the complex of FI-modules

$$E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

where the left map is the differential $d_r^{p-r,q+r-1}$ and the right map $d_r^{p,q}$. By induction, the left hand side term in the previous complex has surjectivity degree at most $\alpha p + \beta q + (\beta - \alpha)(r+1) + (\alpha - 2\beta)$. The middle term has stability type at most

$$\big(\alpha p + \beta q + (\beta - \alpha)r + (\alpha - 2\beta), \alpha p + \beta q\big).$$

Finally the right hand side term has injectivity degree at most $\alpha p + \beta q + \alpha - \beta$. By applying again [10, Proposition 2.45] we get the desired stability type for the quotient

$$E_{r+1}^{p,q} \approx \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}.$$

□

For a given $i \geq 0$ and $0 \leq p \leq i$, we have that $E_\infty^{p,i-p} = E_{i+2}^{p,i-p}$. From Lemma 4.2.1 we get the immediate corollary.

Corollary 4.2.2. *The FI-module $E_\infty^{p,i-p}$ has injectivity degree at most*

$$\alpha p + \beta(i-p) + (\beta - \alpha)(i+2) + (\alpha - 2\beta) = (2\beta - \alpha)i + (\alpha - \beta)p - \alpha \leq (2\beta - \alpha)i - \alpha$$

and surjectivity degree at most

$$\alpha p + \beta(i-p) = \beta i + (\alpha - \beta)p \leq \beta i.$$

As assumed at the beginning of this section, the spectral sequence $E_*^{p,q}$ converges to a graded FI-module $H^*(E)$. From Lemma 4.2.1 and Corollary 4.2.2 we can conclude the following about the stability type of each FI-module $H^i(E)$.

Theorem 4.2.3. *Suppose that we have a first quadrant spectral sequence of FI-modules $E_*^{p,q}$ over a Noetherian ring R converging to a graded FI-module $H^*(E;R)$ over R. If the FI-module $E_2^{p,q}$ is finitely generated, then the FI-module $H^i(E;R)$ is finitely generated, for any $i \geq 0$.*

Furthermore, asume that $R = \mathbb{Q}$ and that for any $p, q \geq 0$ the FI-module $E_2^{p,q}$ has injectivity degree at most βq and surjectivity degree at most $\alpha p + \beta q$, where $\alpha, \beta \geq 0$ such that $2\alpha \leq \beta$. Then, the FI-module $H^i(E) = H^i(E;\mathbb{Q})$ is finitely generated with stability type at most $((2\beta - \alpha)i - \alpha, \beta i)$.

Proof. The first statement follows from the fact that finite generation of an FI-module over a Noetherian ring R is closed under subquotients ([11, Theorem 1.1]).

For each $i \geq 0$, there is a natural filtration of $H^i(E)$ by FI-modules

$$0 \subseteq F_i^i \subseteq F_{i-1}^i \subseteq \ldots \subseteq F_1^i \subseteq F_0^i = H^i(E), \tag{4.1}$$

where, for $0 \leq p \leq i$, the successive quotients $F_p^i/F_{p+1}^i \approx E_\infty^{p,i-p}$. The second statement for the case $k = \mathbb{Q}$ follows from combining the bounds in Lemma 4.2.2 with [10, Proposition 2.46], which states injectivity and surjectivity degrees for filtrations of FI-modules satisfying the conditions above. $\square$

4.2.1 Spectral sequences and FI#-modules

We conclude this section with an argument that allows us to take advantage of the extra structure of finitely generated FI#-modules to get information about the cases where k is a field of arbitrary characteristic or $\mathbb{Z}$. This follows essentially the proof of [10, Theorem 4.7].

Setting: Suppose that we have a first quadrant spectral sequence of FI-modules $E_*^{p,q}$ over k converging to a graded FI#-module $H^*(E;k)$ over k. Let α and β be two non-negative constants such that $\alpha \leq \beta$. Assume that for any $p, q \geq 0$ each term $E_2^{p,q}$ is an FI#-module which is finitely generated in degree $\leq \alpha p + \beta q$.

Theorem 4.2.4. *Let k be any field or $\mathbb{Z}$. For any $i \geq 0$ the FI#-module $H^i(E;k)$ is finitely generated in degree $\leq \beta i$.*

Proof. Suppose first that k is a field. We have that $E_2^{p,q}$ is an FI#-module which is finitely generated in degree $\leq \alpha p + \beta q$. [10, Corollary 2.27] allows to relate this upper bound on the degree of generation with the dimension of the k-vector space $E_2^{p,q}(n)$ and conclude that $\dim_k E_2^{p,q}(n) = O(n^{\alpha p + \beta q})$. Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ and k is a field, then $\dim_k E_\infty^{p,q}(n) = O(n^{\alpha p + \beta q})$. Finally for each $i \geq 0$, from the filtration (4.1) of $H^i(E;k)$, we have that $\dim_k H^i(E;k) = O(n^{\beta i})$ (since $\alpha p + \beta(i-p) \leq \beta i$ for any $0 \leq p \leq i$). Hence, by applying again [10, Corollary 2.27] we get the desired implication. The case when $k = \mathbb{Z}$ can be treated similarly because the rank of a $\mathbb{Z}$-module is non-increasing when passing to submodules. $\square$

4.3 Sequences of cohomology groups as FI-modules (part I)

In this section we revisit two examples of FI-modules that are key ingredients to understand our main examples in Section 4.5.

4.3.1 The FI-module $H^i(M^\bullet;k)$

Given M a topological space, consider the co-FI-space $M^\bullet$. It is the functor that assigns $\mathbf{n} \mapsto M^n := \underbrace{M \times \cdots \times M}_{n}$. Morphisms are defined as follows:

If $f \in \mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, \mathbf{n})$, then $f^* : M^n \to M^m$ is given by $f^*(x_1, \ldots, x_n) = (x_{f(1)}, \ldots, x_{f(m)})$. For each $i \geq 0$ we compose with the contravariant functor $H^i(\_\_;k)$ to get an FI-module over a field k.

Proposition 4.3.1. *Let M be a connected CW-complex with $\dim_k\big(H^i(M;k)\big) < \infty$ for any $i \geq 0$. Then $H^i(M^\bullet;k)$ is an FI#-module finitely generated over any field k. If $k = \mathbb{Q}$ then it has weight $\leq i$ and has stability type at most $(0,i)$.*

Proof. This is a consequence of the Künneth formula. As pointed out in [10, Section 4] the graded FI-module $H^*(M^\bullet;k)$ coincides, apart from signs, with the graded FI-module $H^*(M;k)^{\otimes\bullet}$ (see [10, Definition 2.71]) which is a finitely generated FI#-module since M is connected and $\dim_k\big(H^i(M;k)\big) < \infty$. When $k = \mathbb{Q}$, it can actually be shown that $H^i(M^\bullet;\mathbb{Q})$ is a direct sum of FI#-modules of the form $M(W_j)$, where W_j is some S_j-representation and each summand satisfies that $j \leq i$ (see for example [30, Proposition 6.5]). Then the weight and the stability type claimed in Proposition 4.3.1 follow. □

Similarly, if G is a group, we can consider the co-FI-group $G^\bullet$. If G is a group of type FP_∞ (see for example [7, Chapter VIII] for definition), then the CW-complex $M = K(G,1)$ satisfies the hypotheses in Proposition 4.3.1 and it follows that the FI#-module $H^i(G^\bullet;k) = H^i(M^\bullet;k)$ is finitely generated .

4.3.2 Cohomology of configuration spaces

Let k be any field and let M be a connected, oriented manifold of dimension $d \geq 2$ and assume that $\dim_k\big(H^*(M);k)\big) < \infty$. Since the inclusion $\mathrm{Conf}_n(M) \hookrightarrow M^n$ is S_n-equivariant, we get a corresponding map of co-FI-spaces $\mathrm{Conf}_\bullet(M) \to M^\bullet$. We recall here how a spectral sequence argument can be used to obtain finiteness conditions for the FI-modules $H^q\big(\mathrm{Conf}_\bullet(M);k\big)$.

Let us take, together for all n, the Leray spectral sequences of $\mathrm{Conf}_n(M) \hookrightarrow M^n$. The functoriality of the Leray spectral sequence implies that we have a spectral sequence of FI-modules

$$E_*^{p,q} = E_*^{p,q}(\mathrm{Conf}_\bullet(M) \to M^\bullet)$$

converging to the graded FI-module $H^*(\mathrm{Conf}_\bullet(M);k)$.

Using this spectral sequence, finite type of this graded FI-module has been proved over any field k in [11, Proposition 4.1]. For the case when $k = \mathbb{Q}$ and the dimension of M is $d \geq 3$, particular bounds for the stability degree have been obtained.

Theorem 4.3.2 ([10], Theorem 4.2)**.** *Let M be a connected, oriented manifold of dimension $d \geq 3$. For any $i \geq 0$, the FI-module $H^i(\mathrm{Conf}_\bullet(M); \mathbb{Q})$ has weight $\leq i$ and stability type at most $(i + 2 - d, i)$.*

We now focus in the case where Σ is a connected, oriented surface ($d = 2$). Following the approach in [10, Section 4] we get a better bound for the degree of the FI-module $H^i\big(\mathrm{Conf}_\bullet(\Sigma); \mathbb{Q}\big)$ and get the specific bounds for the stability type.

Theorem 4.3.3. *Let Σ be a connected, oriented manifold of dimension 2. For any $i \geq 0$, the FI-module $H^i(\mathrm{Conf}_\bullet(\Sigma); \mathbb{Q})$ is finitely generated of weight $\leq 2i$ and has stability type at most $(2i, 2i)$ when Σ is a closed surface, at most $(0, 2i)$ when $\partial\Sigma$ is nonempty, and at most $(3i - 1, 2i)$ otherwise.*

Proof. We have a spectral sequence of FI-modules

$$E_*^{p,q} = E_*^{p,q}(\mathrm{Conf}_\bullet(\Sigma) \to \Sigma^\bullet)$$

converging to the graded FI-module $H^*(\mathrm{Conf}_\bullet(\Sigma))$. For any $p, q \geq 0$ the FI-module $E_2^{p,q}$ is the direct sum of FI-modules of the form $M(W_k)$ where W_k is a certain S_k-representation. Moreover, each summand satisfies $k \leq p + 2q$ (see [9, Section 3.3.]). Hence, for every $p, q \geq 0$ we have that $E_2^{p,q}$ is finitely generated in degree $\leq p + 2q$ and has stability type at most $(0, p+2q)$. This is precisely the setting needed for our spectral sequence argument in Section 4.2 with constants $\alpha = 1$ and $\beta = 2$. Then for each $i \geq 0$ the FI-module $H^i(\mathrm{Conf}_\bullet(\Sigma))$ is finitely generated with stability type at most $(3i - 1, 2i)$.

In addition, Totaro proved in [42, Theorem 3] that if M is a smooth complex projective variety, then $E_\infty\big(\mathrm{Conf}_n(M) \hookrightarrow M^n\big) = E_3$. This is the case when Σ is a closed surface.

Therefore we can use Lemma 4.2.1 to improve the bounds for the stability type of $E_\infty^{p,i-p}$ to be at most $(2i, 2i)$, which gives the corresponding bounds stated before.

On the other hand, if $\partial\Sigma$ is nonempty, [10, Proposition 4.6] implies that $H^i(\mathrm{Conf}_\bullet(\Sigma))$ has an $FI\#$-module structure and the injectivity degree is 0.

Finally, observe that for any $0 \leq i$ and $0 \leq p \leq i$ the FI-module $E_\infty^{p,i-p}$ is a subquotient of the FI-module $E_2^{p,i-p}$ of weight $p + 2q$. It follows that $\mathrm{weight}(E_\infty^{p,i-p}) \leq 2i - p \leq 2i$, which implies that $H^i(\mathrm{Conf}_\bullet(\Sigma))$ has weight at most $2i$. $\square$

Remark: In [10, Section 2.6] finite generation of an FI-module is related with representation stability. In particular, Theorem 4.3.3 together with [10, Proposition 2.58] imply that the sequence $H^i(\mathrm{Conf}_n(\Sigma); \mathbb{Q})$ is uniformly representation stable and we recover the stable range of $n \geq 4i$ for the cases when Σ is closed or has non-empty boundary, which was first obtained in [9, Theorem 1].

It follows from [10, Proposition 4.6] that $H^i(\mathrm{Conf}_\bullet(M); R)$ is an FI#-module for any commutative ring R, when the manifolds M has non-empty boundary . The argument of Section 4.2.1 implies the following result.

Theorem 4.3.4 (Theorem 4.7 in [10])**.** *Let M be a connected, oriented manifold of dimension $d \geq 2$ which is the interior of a compact manifold with non-empty boundary. Let k be any field or $\mathbb{Z}$, then for each $i \geq 0$ the FI#-module $H^i(\mathrm{Conf}_\bullet(M); k)$ over k is finitely generated by $O(n^{2i})$ elements.*

4.4 FI[G]-modules

Here we introduce the notion on an FI[G]-module. Basically we want to incorporate the action of a group G on our sequences of S_n-representations. These types of FI-modules will allow us to construct new FI-modules by taking cohomology with twisted coefficients. We will see how in some situations we can use finite generation of the original FI[G]-module to get finite generation and specific bounds for the new FI-modules. In Section 4.4.2 we use this setting in spectral sequence arguments for cohomology of fibrations and groups extensions.

Definition 4.1. *Let R be any commutative ring and let G be a group. An FI$[G]$-module V over R is a functor from the category* **FI** *to the category* **G-Mod**$_R$ *of G-modules over R. We say that an FI$[G]$-module V is finitely generated if it is finitely generated as an FI-module. Similarly an* FI#[G]*-module V over R is a functor from the category* **FI#** *to the category* **G-Mod**$_R$.

FI$[G]$-modules and consistent sequences compatibles with G-actions: For an FI$[G]$-module V, for each $\sigma \in S_n$ the induced linear automorphim $\sigma_* : V_n \to V_n$ is a G-map. Hence the S_n-action and the G-action on V_n commute. If we denote by ϕ_n the map obtained by applying V the standard inclusion I_n (i.e. $\phi_n = V(I_n)$), we have that $\{V_n, \phi_n\}$ is a *consistent sequence of S_n-representations compatible with G-actions* as defined in Section 3.3.

4.4.1 Getting new FI-modules from FI$[G]$-modules

Let V be an FI$[G]$-module over R. Consider a path connected space X with fundamental group G. For each integer $p \geq 0$ we have a covariant functor $H^p(X; \_\_)$ from the category **G-Mod**$_R$ to the category **Mod**$_R$. Hence we have a new FI-module $H^p(X;V)$ over R where $H^p(X;V)_n := H^p(X;V_n)$, the pth cohomology of X with local coefficients in the G-module V_n (see [26, Section 3.H]). Moreover the functor $H^*(X;V)$ given by $H(X;V)_n := H^*(X;V_n)$ is a graded FI-module over R.

Theorem 4.4.1 (Cohomology with coefficients in a f.g. FI[G]-module)**.** *Let G be the fundamental group of a connected CW complex X with finitely many cells in each dimension. If V is a finitely generated FI$[G]$-module over a Noetherian ring R, then for every $p \geq 0$, the FI-module $H^p(X;V)$ is finitely generated over R.*

Moreover, if $R = \mathbb{Q}$ and V has weight $\leq m$ and stability degree N, then the FI-module $H^p(X;V)$ has weight $\leq m$ and stability degree N.

Proof. Given that $G = \pi_1(X)$, the universal cover $\tilde{X}$ of X has a G-equivariant cellular chain complex. Since X has finitely many cells in each dimension, for each $p \geq 0$ the group $C_p(\tilde{X})$ is a free G-module of finite rank, say $C_p(\tilde{X}) \approx (\mathbb{Z}G)^{d_p}$. A preferred G-basis $x_1, \ldots, x_{d_p}$ can be provided by selecting a p-cell in $\tilde{X}$ over each cell in X.

For each $p \geq 0$ and $n \in \mathbb{N}$ we have an isomorphism of G-modules $\mathcal{H}om_G(C_p(\tilde{X}), V_n) \approx V_n^{\oplus d_p}$, given by $h \mapsto \big(h(x_1), \ldots, h(x_{d_p})\big)$. Moreover, for any morphism $\phi : V_m \to V_n$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}om_G(C_p(\tilde{X}), V_m) \arrow[r, "\phi\circ_"] \arrow[d, "\approx"'] & \mathcal{H}om_G(C_p(\tilde{X}), V_n) \arrow[d, "\approx"] \\
V_m^{\oplus d_p} \arrow[r, "\phi^{\oplus d_p}"] & V_n^{\oplus d_p}
\end{tikzcd}$$

Therefore the FI[G]-module $C^p(X;V)$ given by $C^p(X;V)_n := \mathcal{H}om_G(C_p(\tilde{X}), V_n)$ is precisely the direct sum of FI[G]-modules $V^{\oplus d_p}$. Therefore, finite generation of the FI-module $H^p(X;V)$ follows since it is a subquotient of the finitely generated FI-module $C^p(X,V)$.

If $R = \mathbb{Q}$, since the weight of an FI-module does not increase when taking extensions, then we have that $V^{\oplus d_p}$ is finitely generated of weight $\leq m$. Moreover, $V^{\oplus d_p}$ has stability degree N because V has stability degree N. Furthermore, the FI-module $H^p(X;V)$ is obtained from the complex of FI-modules

$$C^{p-1}(X,V) \xrightarrow{\delta_{p-1}} C^p(X,V) \xrightarrow{\delta_p} C^{p+1}(X,V)$$

where we have that each FI-module has stability degree N and is finitely generated of degree $\leq m$. The weight of an FI-module is preserved under subquotients and from [10, Proposition 2.45] applied to the previous complex we get the desired stability degree. $\square$

Remark: For each integer $p \geq 0$ we have a covariant functor $H^p(G; \_\_)$ from $\mathbf{G\text{-}Mod}_R$ to $\mathbf{Mod}_R$ (see [7]). Hence, if V is an FI[G]-module, we have the FI-module $H^p(G;V)$ given by

$H^p(G;V)_n := H^p(G;V_n)$. If G is a group of type FP_∞, then the space $X = K(G,1)$ satisfies the hypotheses of Theorem 4.4.1 and the FI-module $H^p(X;V)$ is precisely $H^p(G;V)$.

The case of FI#[G]**-modules:** Next we state the equivalent result to Theorem 4.4.1 when we take coefficients in a finitely generated FI#[G]-module.

Theorem 4.4.2 (Cohomology with coefficients in a f.g. FI#[G]-module)**.** *Let k be any field or $\mathbb{Z}$ and suppose that G is the fundamental group of a connected CW complex X with finitely many cells in each dimension. If V is an* FI#[G]*-module over k finitely generated in degree $\leq m$, then for every $p \geq 0$, the FI#-module $H^p(X;V)$ is finitely generated in degree $\leq m$.*

Proof. Clearly $H^p(X;V)$ is a covariant functor from **FI#** to $\mathbf{Mod}_k$. First suppose that k is any field. Keeping the notation from the previous proof we have that

$$\dim_k C^p(X, V_n) = \dim_k V_n^{\oplus d_p} = O(n^m).$$

By hypothesis and [10, Corollary 2.27] it follows that $\dim_k V_n = O(n^m)$. Then dimension over k of the subquotient $H^p(X, V_n)$ is $O(n^m)$ and [10, Corollary 2.27] gives us the desired conclusion. A similar argument applies for $k = \mathbb{Z}$ considering rank instead of dimension. □

4.4.2 FI[G]*-modules and spectral sequences*

Let X be a connected CW complex with finitely many cells in each dimension and let $x \in X$ be a fixed base point. Suppose that the fundamental group $\pi_1(X, x)$ is G. Consider a functor from $\mathbf{FI}^{op}$ to the category $\mathbf{Fib}(X)$ of fibrations over X (a *co-FI-fibration over* X). Let

$$E_n \to X$$

be the fibration associated to $\mathbf{n}$, and H_n the fiber over the basepoint x. We denote by E the co-FI space of total spaces $\mathbf{n} \mapsto E_n$ and by H the co-FI space of fibers $\mathbf{n} \mapsto H_n$. We can think of $E \to X$ as a pointwise fibration over X with "fiber" H.

Let us take, together for all n, the Leray-Serre spectral sequences associated to each fibration $E_n \to X$. The functoriality of the Leray-Serre spectral sequence implies that we have a spectral sequence of FI-modules

$$E_*^{p,q} = E_*^{p,q}\big(E \to X\big)$$

converging to the graded FI-module $H^*(E)$.

The E_2-page of this spectral sequence is the FI[G]-module

$$E_2^{p,q} = H^p\big(X; H^q(H)\big)$$

Remark: Observe that for any $q \geq 0$ and $n \geq 1$, we get an action of the fundamental group G on $H^q(H_n;\mathbb{Q})$ from the n-th fibration, which gives to the FI-module $H^q(H;R)$ the structure of an FI[G]-module over R.

With this setting, we want to use Theorem 4.4.1 and the spectral argument given in Section 4.2 to determine finiteness conditions for the graded FI-module $H^*(E;R)$ given that we know that the FI[G]-module $H^q(H;R)$ is finitely generated over R and we have upper bounds for its degree and its stability degree when $R = \mathbb{Q}$.

The typical situation that we will have in the examples in Section 4.5 below is that the FI[G]-module $H^q(H;\mathbb{Q})$ is finitely generated of weight $\leq \beta q$ with stability degree $\leq \beta q$, for some positive constant β. Then Theorem 4.4.1 gives us the following information about the E_2-page.

Lemma 4.4.3. *Suppose that for any $q \geq 0$ the* FI[G]*-module $H^q(H;\mathbb{Q})$ is finitely generated of weight $\leq \beta q$ with stability degree $\leq \beta q$. Then, for any $p, q \geq 0$, the FI-module $E_2^{p,q} = H^p\big(X; H^q(H)\big)$ has weight $\leq \beta q$ and stability degree $\leq \beta q$.*

For a given $i \geq 0$ and $0 \leq p \leq i$, the FI-module $E_\infty^{p,i-p}$ is a subquotient of $E_2^{p,i-p}$. Since the weight of an FI-module cannot increase when taking subquotients, it follows that the

FI-module $E_\infty^{p,i-p}$ is finitely generated of weight $\leq \beta i$. Moreover, the spectral sequence gives a natural filtration of $H^i(E)$ by FI-modules

$$0 \subseteq F_i^i \subseteq F_{i-1}^i \subseteq \ldots \subseteq F_1^i \subseteq F_0^i = H^i(E),$$

where, for $0 \leq p \leq i$, the FI-module F_p^i is an extension of F_{p+1}^i by $E_\infty^{p,i-p}$ of weight $\leq \beta i$. Since, by definition, the weight of an FI-module is preserved under extensions, therefore $H^i(E)$ has weight at most βi.

Furthermore, we have precisely the setting described in Section 4.2 for constants $\alpha = 0$ and $\beta > 0$ and Theorem 4.2.3 takes the following form.

Theorem 4.4.4. *For any $i \geq 0$ the FI-module $H^i(E;\mathbb{Q})$ is finitely generated of weight at most βi and has stability type at most $(2\beta i, \beta i)$.*

The case of group extensions: Let G be a group of type FP_∞. Consider a functor from $\mathbf{FI}^{op}$ to the category of group extensions with quotient G and isomorphisms of such (a *co-FI-group extension of* G). Let

$$1 \to H_n \to E_n \to G \to 1$$

be the group extension associated to $\mathbf{n}$ and denote by E and H the corresponding co-FI groups $\mathbf{n} \mapsto E_n$ and by $\mathbf{n} \mapsto H_n$. For each group extension there is an associated fibration

$$K(E_n, 1) \to K(G, 1)$$

with fiber over a fixed base point $x \in K(G,1)$ an Eilenberg-Maclane space $K(H_n, 1)$. Observe that the space $K(G,1)$ has the homotopy type of a connected CW complex with finitely many cells in each dimension since G is of type FP_∞. Hence, this gives us a functor from $\mathbf{FI}^{op}$ to $\mathbf{Fib}(K(G,1))$ as in the setting of Section 4.4.2 and we obtain the conclusion of Theorem 4.4.4 about the FI-modules $H^i(E)$.

Remarks: The Leray-Serre spectral sequence associated to the fibration above corresponds to the Hochschild-Serre spectral sequence associated to the original group extension. Notice that we could have considered this spectral sequence in our previous discussion.

The Hochschild-Serre spectral sequence and FI#-modules: Assume that we have a functor from $\mathbf{FI\#}^{op}$ to the category of group extensions with quotient G, and not just from $\mathbf{FI}^{op}$ as before. By taking the Hochschild-Serre spectral sequence associated to each group extension with coefficients in any field k or $\mathbb{Z}$, we obtain with a first quadrant spectral sequence of **FI#**-modules converging to the graded **FI#**-modules $H^*(E;k)$. Furthermore, suppose that for any $q \geq 0$ the FI#-module $H^q(H;k)$ is finitely generated over k in degree $\leq \beta q$, for some $\beta > 0$. Then Theorem 4.4.2 implies that for any $p, q \geq 0$, the FI#-module $E_2^{p,q}$ is finitely generated in degree $\leq \beta q$. Since we have the setting from Section 4.2.1 with $\alpha = 0$ and $\beta > 0$, we can conclude that for any $i \geq 0$ the FI#-module $H^i(E, k)$ is finitely generated in degree $\leq \beta i$.

4.5 Sequences of cohomology groups as FI-modules (part II)

Let us apply the perspective described in Section 4.4 to understand other sequences of cohomology groups as finitely generated FI-modules. Most of these sequences were already considered in Chapter 3. We will see here how the FI-module approach allows us to obtain more information.

4.5.1 Moduli spaces $\mathcal{M}_{g,n}$ and pure mapping class groups of surfaces

Let $2g + r > 2$ and consider the functor from $\mathbf{FI}^{op}$ to the category of group extensions of $G = \mathrm{Mod}_{g,r}$ given as follows. The group extension associated to $\mathbf{n}$ is

$$1 \to \pi_1(\mathrm{Conf}_n(\Sigma_g^r)) \to \mathrm{PMod}^n(\Sigma_{g,r}) \to \mathrm{Mod}_{g,r} \to 1.$$

This is the *Birman exact sequence* introduced by Birman in [5]. A proof of the exactness can be found in [18]. To see that this association is indeed functorial we refer the reader to Section 3.4.1.

From [11, Proposition 4.1] we have that $H^q\big(\pi_1[\mathrm{Conf}_n(\Sigma_g^r)];k\big) = H^q\big(\,\mathrm{Conf}_n(\Sigma_g^r);k\big)$ is a finitely generated FI[G]-module over any field k. When $k=\mathbb{Q}$, it follows from Theorem 4.3.3 that it has weight $\leq 2q$ and stability degree $\leq 2q$. From our discussion in Section 4.4.2 with $\beta = 2$, we obtain the following statement.

Theorem 4.5.1. *Let k be a field. For any $i \geq 0$ and $2g+r>2$ the FI-module $H^i\big(\,\mathrm{PMod}^{\bullet}_{g,r};k\big)$ is finitely generated over k. If $k=\mathbb{Q}$, it has weight $\leq 2i$ and stability type at most $(4i,2i)$.*

4.5.2 Pure mapping class groups for higher dimensional manifolds

Let M be a smooth connected manifold of dimension $d \geq 3$ and suppose that the fundamental group $\pi_1(M)$ has trivial center or $\mathrm{Diff}(M)$ is simply connected. Moreover, we assume that $\mathrm{Mod}(M)$ is of type FP_∞.

Consider the functor from $\mathbf{FI}^{op}$ that associates to each $\mathbf{n}$ the group extensions of $\mathrm{Mod}(M)$

$$1 \longrightarrow \pi_1(\mathrm{Conf}_n(\mathring{M})) \longrightarrow \mathrm{PMod}^n(M) \longrightarrow \mathrm{Mod}(M) \longrightarrow 1, \tag{4.2}$$

where $\mathring{M}$ denotes the interior of M. For a proof of the existence of this Birman exact sequence see Lemma 3.5.2.

Let $\mathfrak{p} = (p_1,\dots,p_n) \in \mathrm{Conf}_n(\mathring{M})$ be a fixed base point. Since $d \geq 3$, then from [4, Theorem 1] it follows that the fundamental group

$$\pi_1(\mathrm{Conf}_n(\mathring{M}),\mathfrak{p}) \approx \pi_1(\mathrm{Conf}_n(M),\mathfrak{p}) \approx \pi_1(M^n,\mathfrak{p}) \approx \prod_{i=1}^{n} \pi_1(M,p_i).$$

Hence the FI-module $H^q\big(\pi_1(\mathrm{Conf}_\bullet(\mathring{M});k)\big)$ is precisely $H^q\big(\pi_1(M)^\bullet;k\big)$. If the group $\pi_1(M)$ is of type FP_∞, then from Proposition 4.3.1 we have that this FI[G]-module is finitely generated over k and has weight $\leq q$ with stability degree $\leq q$ when $k = \mathbb{Q}$. From our discussion in Section 4.4.2 with $\beta = 1$ we can conclude the following result.

Theorem 4.5.2. *Let M be a smooth connected manifold of dimension $d \geq 3$ such that $\pi_1(M)$ is of type FP_∞ (e.g. M compact). Suppose that $\pi_1(M)$ has trivial center or that $\mathrm{Diff}(M)$ is simply connected and assume that the group $\mathrm{Mod}(M)$ is of type FP_∞. Then for any field k and $i \geq 0$ the FI-module $H^i\big(\mathrm{PMod}^\bullet(M); k\big)$ is finitely generated over k and has weight $\leq i$ and stability type at most $(2i, i)$ when $k = \mathbb{Q}$.*

4.5.3 The case of manifolds with boundary

When the surface $\Sigma_{g,r}$ in Section 4.5.1 or the manifold M in Section 4.5.2 has nonempty boundary, the cohomology of the corresponding pure mapping class groups actually has an FI#-module structure.

Proposition 4.5.3. *Let k be a field or $\mathbb{Z}$ and $i \geq 0$. If M is a connected smooth manifold of dimension $d \geq 2$ with nonempty boundary, then the FI-module $H^i\big(\mathrm{PMod}^\bullet(M); k\big)$ has the structure of an FI#-module. In particular, $H^i\big(\mathrm{PMod}^\bullet(M)\big)$ has injectivity degree 0 (when $k = \mathbb{Q}$).*

Proof. We just prove that $\mathrm{PMod}^\bullet(M)$ has the structure of an FI#-group. Consider $(A, B, \psi) \in \mathrm{Hom}_{\mathrm{FI\#}}(\mathbf{m}, \mathbf{n})$ where $A \subset [m]$, $B \subset [n]$ and $\psi : A \to B$ is a bijection. In particular the orders $|A| = |B|$. The corresponding morphism

$$(A, B, \psi)_* : \mathrm{PMod}^m(M) \to \mathrm{PMod}^n(M)$$

is induced from the following composition:

$$\mathrm{PDiff}^m(M) \xrightarrow{|_A} \mathrm{PDiff}^{|A|}(M) \xrightarrow{\psi} \mathrm{PDiff}^{|B|}(M) \xrightarrow{\Upsilon \circ c_\varphi} \mathrm{PDiff}^n(M).$$

The map $|_A : \mathrm{PDiff}^{\mathfrak{p}} \to \mathrm{PDiff}^{\mathfrak{p}_A}(M)$ is given by restricting the configuration $\mathfrak{p} \in$

$\mathrm{Conf}_m(M)$ to the configuration $\mathfrak{p}_A : A \hookrightarrow M$ in $\mathrm{Conf}_{|A|}(M)$.

Abusing notation, $\psi : \mathrm{PDiff}^{|A|}(M) \to \mathrm{PDiff}^{|B|}(M)$ corresponds to the isomporphism $\mathrm{PDiff}^{\mathfrak{p}_A} \approx \mathrm{PDiff}^{\mathfrak{p}_A \circ \psi^{-1}}$ induced by taking $\mathfrak{p}_A : A \hookrightarrow M$ to the embedding $\mathfrak{p}_A \circ \psi^{-1} : B \hookrightarrow M$ in $\mathrm{Conf}_{|B|}(M)$, using the bijection $\psi : A \to B$.

Let R be a collar neighborhood of one component of ∂M and fix a diffeomorphism $\varphi : M \to M \backslash R$. Then, conjugation by φ gives us the identification $c_\varphi : \mathrm{PDiff}^{|B|}(M) \approx \mathrm{PDiff}^{|B|}(M \backslash R)$. Finally, we can extend any diffeomorphism $h \in \mathrm{Diff}\big((M \backslash R) \text{ rel } \partial(M \setminus R)\big)$ to a diffeomorphism $\Upsilon(h) \in \mathrm{Diff}(M \text{ rel } \partial M)$ by letting $\Upsilon(h) = h$ in $M \setminus R$ and $\Upsilon(h)$ be the identity in R. Therefore we obtain a group homomorphism

$$\Upsilon : \mathrm{PDiff}^{\mathfrak{q}}(M \setminus R) \to \mathrm{PDiff}^{\Psi_{B,[n]}(\mathfrak{q})}(M),$$

that takes any diffeomorphism h that fixes the configuration $\mathfrak{q} : B \hookrightarrow (M \setminus R)$ in $\mathrm{Conf}_{|B|}(M \setminus R)$ to a diffeomorphism $\Upsilon(h)$ of M that fixes the configuration $\Psi_{B,[n]}(\mathfrak{q}) : [n] \hookrightarrow M$ in $\mathrm{Conf}_n(M)$ as defined in [10, Proof of Proposition 4.1]. $\square$

When a Birman sequence exists (hypothesis of Theorems 4.5.1 and 4.5.2), we do have a co-FI#-group extension of $\mathrm{Mod}(M)$. Moreover, Theorem 4.3.4 states the finite generation of the cohomology of configuration spaces of manifolds with boundary. Hence, the argument at the end of Section 4.4 implies the following results.

Theorem 4.5.4. *Let k be any field or $\mathbb{Z}$. For any $i \geq 0$, $2g+r > 2$ and $r > 0$ the FI-module $H^i\big(\mathrm{PMod}^\bullet_{g,r}; k\big)$ has the structure of an FI#-module which is finitely generated in degree $\leq 2i$.*

Theorem 4.5.5. *Let k be any field or $\mathbb{Z}$. Let M be a smooth connected manifold of dimension $d \geq 3$ with non-empty boundary that satisfies the hypotheses of Theorem 4.5.2. Then, for any $i \geq 0$, the FI-module $H^i\big(\mathrm{PMod}^\bullet(M); k\big)$ has the structure of an FI#-module that is finitely generated in degree $\leq 2i$.*

From the classification of FI#-modules given in [10, Theorem 2.24] and the cases when k is either $\mathbb{Z}$ or the fields $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ in Theorems 4.5.4 and 4.5.5, we obtain Theorems 4.1.1 and 4.1.2, respectively.

4.5.4 Classifying spaces for some diffeomorphism groups

Let M be a connected and compact smooth manifold of dimension $d \geq 3$. We have a fiber bundle

$$B\,\mathrm{PDiff}^n(M) \to B\,\mathrm{Diff}(M \text{ rel } \partial M) \tag{4.3}$$

where the "fiber" is given by $\mathrm{Diff}(M \text{ rel } \partial M)/\,\mathrm{PDiff}^n(M) \approx \mathrm{Conf}_n(\mathring{M})$, the configuration space of n ordered points in $\mathring{M}$, the interior of M. This gives us a functor from $\mathbf{FI}^{op}$ to the category $\mathbf{Fib}\big(B\,\mathrm{Diff}(M \text{ rel } \partial M)\big)$. The hypotheses in the Theorem below give the setting needed to apply the arguments in Section 4.4.2 with $\beta = 1$ to get the desired conclusion.

Theorem 4.5.6. *Let M be a connected real manifold of dimension $d \geq 3$. Suppose that the classifying space $B\,\mathrm{Diff}(M$ rel $\partial M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension. Then, the FI-module $H^i(B\,\mathrm{PDiff}^\bullet M; k)$ is finitely generated over k, for any field k and $i \geq 0$, and has weight $\leq i$ and stability type at most $(2i, i)$ when $k = \mathbb{Q}$.*

4.6 Application to cohomology of some wreath products

Let G be a group of type FP_∞. The wreath product $G \wr S_n$ is the semidirect product $G^n \rtimes S_n$, where S_n acts on G^n by permuting the coordinates. Therefore there is a split short exact sequence

$$1 \to G^n \to G \wr S_n \to S_n \to 1$$

For any $i \geq 0$ and any partition λ, a transfer argument implies that the dimension of $H^i\big(G \wr S_n; V(\lambda)_n\big)$ is equal to the multiplicity of $V(\lambda)_n$ in $H^i\big(G^n; \mathbb{Q}\big)$. But, from Proposition

4.3.1 and [10, Proposition 2.58], this multiplicity is constant for $n \geq 2i$. Hence we obtain cohomological stability for the group $G \wr S_n$ with coefficients in any S_n-representation for any $n \geq 2i$.

More generally, let PK be a co-FI-group given by $\mathbf{n} \mapsto PK_n$. Assume that there is a sequence of groups K_n such that, for each n, we have the following short exact sequence:

$$1 \to PK_n \to K_n \to S_n \to 1$$

The wreath product $G \wr K_n$ is the semidirect product $G^n \rtimes K_n$, where K_n acts on G^n via the surjection $K_n \to S_n$. Therefore there is a split short exact sequence

$$1 \to G^n \times PK_n \to G \wr K_n \to S_n \to 1$$

On the other hand, for any $i \geq 0$, the naturality of the Künneth formula implies the following isomorphism of FI-modules:

$$H^i(G^\bullet \times PK) = \bigoplus_{p+q=i} H^p(G^\bullet) \otimes H^q(PK).$$

Suppose that the graded FI-module $H^*(PK)$ is known to be of finite type. In [10, Proposition 2.61] is proved that finite generation is closed under tensor products, therefore the FI-modules $H^p(G^\bullet) \otimes H^q(PK)$ are finitely generated for $p, q \geq 0$ such that $p + q = i$. Moreover,

$$\mathrm{weight}\big(H^p(G^\bullet) \otimes H^q(PK)\big) \leq \mathrm{weight}\big(H^p(G^\bullet)\big) + \mathrm{weight}\big(H^p(PK)\big).$$

It follows that the consistent sequence $H^i(G^n \times PK_n; \mathbb{Q})$ is monotone and uniformly representation stable (although we do not always get a specific stable range).

As before, the dimension of $H^i\big(G \wr K_n; V(\lambda)_n\big)$ is given by the multiplicity of $V(\lambda)_n$ in $H^i\big(G^n \times PK_n; \mathbb{Q}\big)$, which is eventually constant by uniform representation stability. Therefore we have that $H^i\big(G \wr K_n; V(\lambda)_n\big) \approx H^i\big(G \wr K_{n+1}; V(\lambda)_n\big)$ for any n sufficiently large. In particular, we obtain rational homological stability for the groups $G \wr K_n$.

Parts (ii), (iii) and (iv) of Theorem 4.1.5 follow from applying the above discussion to the short exact sequences:

$$1 \to P_n(\Sigma_{g,r}) \to B_n(\Sigma_{g,r}) \to S_n \to 1$$

$$1 \to \mathrm{PMod}_{g,r}^n \to \mathrm{Mod}_{g,r}^n \to S_n \to 1$$

$$1 \to \mathrm{PMod}^n(M) \to \mathrm{Mod}^n(M) \to S_n \to 1$$

To obtain Theorem 4.1.5 part (v), we consider the co-FI-groups $\mathrm{P}\Sigma_\bullet$ and $\Sigma_\bullet^+$, which are functors from $\mathbf{FI}^{op}$ to $\mathbf{Gp}$ given by $\mathbf{n} \mapsto \mathrm{P}\Sigma_n$, the *pure string motion group* (see definition in [48, Sections 1 & 2]) and $\mathbf{n} \mapsto \Sigma_n^+$, the braid permutation group, respectively. In [48, Theorem 6.4] Wilson proved that for any $k \geq 0$ the sequence $\big\{H^k(\mathrm{P}\Sigma_n; \mathbb{Q})\big\}$ satisfies uniform representation stability with stable range $n \geq 4k$. Therefore, [10, Theorem 1.14] implies that for any $k \geq 0$ the FI-module $H^k(\mathrm{P}\Sigma_\bullet)$ is finitely generated.

The co-FI-groups $\Sigma_\bullet^+$ and $\mathrm{P}\Sigma_\bullet$ are related by the following short exact sequence:

$$1 \to P\Sigma_n \to \Sigma_n^+ \to S_n \to 1,$$

which give us again the setting discussed above.

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