TOPIC PROPOSAL

Studying the mapping class group of a surface through its actions

R. Jimenez Rolland
discussed with Professor B. Farb

November, 2008

Los grupos como los hombres son conocidos por sus acciones.[1]
G. Moreno

Our main purpose is to introduce the study of the mapping class group. We try to "know the group by its actions." Following mainly [FM08], we describe some sets on which the group acts and how this translates into properties of the group itself.[2]

1 The mapping class group

We denote by $S^n_{g,b}$ an orientable surface of genus g with b boundary components and n punctures (or marked points). The *mapping class group* of a surface $S = S^n_{g,b}$ is the group Γ_S ($\Gamma^n_{g,b}$) of isotopy classes of orientation-preserving self-homeomorphisms of S. We require the self-homeomorphism to be the identity on the boundary components and fixes the set of punctures. When including orientation-reversing self-homeomorphisms of S we use the notation $\Gamma^{\pm}_S$. The mapping class groups of the annulus A and the torus $T = S_1$ are $\Gamma_A \cong \mathbb{Z}$ and $\Gamma_T \cong SL_2(\mathbb{Z})$.

We define the *pure mapping class group* $P\Gamma_S$ by asking that the punctures remain pointwise fixed. If S is a surface with n punctures then $P\Gamma_S$ is related to Γ_S by the short exact sequence $1 \to P\Gamma_S \to \Gamma_S \to \mathrm{Sym}_n \to 1$.

For a surface S with $\chi(S) < 0$, let $S' = S \backslash \{p\}$. The *Birman exact sequence* is

$$1 \longrightarrow \pi_1(S) \xrightarrow{\text{Push}} P\Gamma_{S'} \xrightarrow{F} P\Gamma_S \to 1.$$

Here F is the map induced by the inclusion $S' \hookrightarrow S$. For an element $\gamma \in \pi_1(S)$ we consider the isotopy defined by "pushing" the base point along γ. Then $\mathrm{Push}(\gamma)$ is defined to be the homeomorphism at the end of the isotopy.

In this text we will not distinguish between "isotopic" and "homotopic", and between "diffeomorphic" and "homeomorphic" with respect to surfaces and curves. This is a classical fact for dimension 2 (see [FM08]).

2 The "outer action" on $\pi_1(S)$

There isn't a well-defined action of $\Gamma^{\pm}_S$ on $\pi_1(S)$ but only an "outer action": $\Gamma^{\pm}_S \to \mathrm{Out}(\pi_1(S))$, given by $[\phi] \mapsto [\phi_*]$. Here $\mathrm{Out}(\pi_1(S))$ is the group of outer automorphisms of $\pi_1(S)$, i.e., the quotient $\mathrm{Aut}(\pi_1(S))/\mathrm{Inn}(\pi_1(S))$.

The Dehn-Nielsen-Baer Theorem

This "outer action" gives an algebraic characterization of the mapping class group of closed surfaces.

Theorem 2.1 (Dehn-Nielsen-Baer). *If $S = S_g$ is a closed surface with genus $g > 0$, then the group homomorphism $\Gamma^{\pm}_S \to \mathrm{Out}(\pi_1(S))$ is an isomorphism.*

[1] Groups as men are known by their actions.
[2] See also [Iva02] for a survey about mapping class groups.

Proof. (*sketch*) Since S is closed, it is a $K(\pi_1(S), 1)$, and any based map is determined up to homotopy by its action on the fundamental group. Therefore the map is injective. Take $[\Phi] \in \mathrm{Out}(\pi_1(S))$, where Φ is a representative automorphism. Using again the fact that S is a $K(\pi_1(S), 1)$, we can get a continuous map $\phi : S \to S$ that realizes Φ. Moreover, since $\pi_i(S) = 0$ for $i > 1$ we can apply Whitehead's Theorem to conclude that ϕ is a homotopy equivalence. Surjectivity follows from the next result for closed hyperbolic[3] surfaces: if S is a closed surface with $\chi(S) < 0$, any homotopy equivalence of S is homotopic to a homeomorphism. The idea is that we can get a pants decomposition[4] of the surface and reduce to the case of getting a homeomorphism for each pair of pants, which can be done by an application of the Alexander trick (see [FM08]). ∎

3 The action on the complex of curves

The *complex of curves* $\mathcal{C}(S)$ is the flag complex with 1-skeleton defined as follows. The vertices are isotopy classes of essential (non null-isotopic) simple closed curves in S. A vertex is connected to another vertex by an edge if they have disjoint representatives. We denote the full subcomplex with vertices corresponding to non-separating curves by $\mathcal{N}(S)$. It turns out that this simplicial complex is connected when $3g-3+n+b \geq 2$, and the mapping class group Γ_S acts transitively on its vertices.

The mapping class group is finitely generated

A *Dehn twist* T_c about a simple closd curve c on S is the homeomorphism of S defined as follows. Let N be a tubular neighborhood of c in S, parametrized by the annulus $\{z = (r+1)e^{2\pi i\theta} : 0 \leq r \leq 1, 0 \leq \theta \leq 1\}$. We define T_c as $(1+r)e^{2\pi i\theta} \mapsto (1+r)e^{2\pi i(\theta+r)}$ on N and the identity on $S\backslash N$. We now show how the action of Γ_S on $\mathcal{N}(S)$ implies that Γ_S is generated by finitely many Dehn twists.

Theorem 3.1 (Dehn). *Let S a surface of genus $g > 0$. The group $P\Gamma_S$ is finitely generated by Dehn twists about nonseparating curves. In particular, this is also true for Γ_S if S is a closed surface.*

Proof. (*sketch*) The proof is done by induction on the genus and the number of punctures.

First we prove that $P\Gamma_S$ is generated by Dehn twists. Consider a mapping class $f \in P\Gamma_S$ and a non-separating curve c. The connectedness of $\mathcal{N}(S)$ and some properties of Dehn twists imply that there is a product of Dehn twists taking $f(c)$ to c. Thus, after post-composing with this product we may assume that f fixes the vertex c of $\mathcal{N}(S)$. Let S_c be the surface resulting from cutting S along c and consider f as an element of Γ_{S_c}. By induction f is a product of Dehn twists about non-separating curves as an element of $P\Gamma_{S_c}$ and thus, up to a power of T_c, as an element of $P\Gamma_S$.

It remains to show the group is finitely generated. The group $P\Gamma_S$ acts transitively on the vertices of $\mathcal{N}(S)$. Fix a vertex c_0 and for each $P\Gamma_S$-orbit of edges choose one representative that is adjacent to c_0. Let D be the subcomplex of $\mathcal{N}(S)$ spanned by these edges. By definition, $\mathcal{N}(S) = \bigcup_{g\in\Gamma_S} gD$. The subcomplex D has finitely many vertices $c_0, c_1, \cdots, c_n$ and there exist $g_i \in \Gamma_S$ such that $g_i(c_0) = c_i$. Therefore if $\mathrm{Stab}_\Gamma(c_i)$ is the stabilizer of c_i, then $\mathrm{Stab}_\Gamma(c_i) = g_i \,\mathrm{Stab}_\Gamma(c_0) g_i^{-1}$. The connectedness of $\mathcal{N}(S)$ implies that a set of generators for Γ_S is given by $\{g \in \Gamma_S : gD \cap D \neq \varnothing\} = \bigcup_{i=0}^n g_i[\mathrm{Stab}_\Gamma(c_0)]g_i^{-1}$. The proof reduces then to an induction argument using the Birman exact sequence to show that $\mathrm{Stab}_\Gamma(c_0)$ is finitely generated. ∎

There are specific sets of Dehn twists that generate Γ_S, such as the *Lickorish* and the *Humphries generators*. The latter ones can be used to prove that the center of Γ_g is trivial for $g \geq 3$. Moreover, Theorem 3.1 together with the "lantern relation" and the fact that all Dehn twists about non-separating curves in S are conjugate in S imply (see [FM08]):

Theorem 3.2. *If $S = S_g$ with $g \geq 3$, then $H_1(\Gamma_S; \mathbb{Z}) = 0$.*

4 The action on the arc complex

Consider now a surface $S = S_{g,b}$ with $g \geq 1$ and $b \geq 1$ and a finite subset V of ∂S. We define an *arc* in (S, V) to be an embedding $\alpha : [0, 1] \to S$ such that $\alpha(0), \alpha(1) \in V$ and $\alpha^{-1}(\partial S) = \{0, 1\}$. We call α *essential* if it is not isotopic to an arc in ∂S.

The *arc complex* $\mathcal{A}(S, V)$ is the flag complex with 1-skeleton given as follows: the vertices are isotopy classes (rel V) of essential arcs in (S, V); two vertices are connected by an edge if we can realize these isotopy

[3] See [BP92] and [Kat92] for definitions and properties.

[4] See Section 6.1.

classes disjointly. Hatcher proved in [Hat91] that the arc complex $\mathcal{A}(S,V)$ is contractible. The idea is to fix an essential arc β in (S,V) and construct a continuous flow from all of $\mathcal{A}(S,V)$ onto the closed star of β, which is itself contractible.

The mapping class group is finitely presented

The group $\Gamma_{g,b}$ acts on the vertices of $\mathcal{A}(S,V)$. We can extend this action to get a simplicial action of $\Gamma_{g,b}$ on $\mathcal{A}(S,V)$. The action of $\Gamma_{g,b}$ on such a contractible complex will give us:

Theorem 4.1. *The group $\Gamma_{g,b}$ with $g \geq 1$ and $b \geq 1$ is finitely presented.*

Proof. (*sketch*) Consider a model of $K(\Gamma,1)$ for $\Gamma := \Gamma_{g,b}$ with universal cover $E\Gamma$. Since the diagonal action of Γ on the contractible space $E\Gamma \times \mathcal{A}(S,V)$ is free, then $E\Gamma \times_\Gamma \mathcal{A}(S,V)$ is a $K(\Gamma,1)$. It turns out that $E\Gamma \times_\Gamma \mathcal{A}(S,V)$ is a complex of spaces whose underlying complex is $\mathcal{A}(S,V)/\Gamma$: the vertices are $K(\pi,1)$ models for each vertex stabilizer and the edges are $K(\pi,1)$ models for each edge stabilizer crossed with intervals.

For each vertex β, let S_β be the surface resulting by cutting along the arc β. Then, we can identify $\mathrm{Stab}_\Gamma(\beta)$ with $\Gamma_{S_\beta} = \Gamma_{g',b'}$ with $g' \leq g$ and $b' < b$, which by induction is finitely presented. Similary for each edge e of $\mathcal{A}(S,V)$, the stabilizer $\mathrm{Stab}_\Gamma(e)$ is finitely generated. Therefore, for each vertex and edge in our complex we may take the associated space to have finite 2-skeleton. We get then a homotopy equivalent complex of spaces with finite 2-skeleton which is a $K(\Gamma,1)$. ■

Furthermore Harer (see [Har85] and references in therein) used the action of Γ_S on the arc complex to prove:

Theorem 4.2 (Harer). *Let S be S_g or S_g^1 with $g \geq 4$; then $H_2(\Gamma_S;\mathbb{Z}) \cong \mathbb{Z}$.*

Another proof of Theorem 4.2 is due to Pitsch who applied Hopf's formula to Wajnryb's explicit presentation of Γ_S (see [FM08]).

5 The action on the first homology of the surface

The symplectic representation and the Torelli group

The action of Γ_S on the first homology of the surface gives us a natural linear representation of Γ_S in $\mathrm{GL}(2g;\mathbb{Z})$. Moreover, algebraic intersection number gives a symplectic form $\hat{\imath}$ on homology, so that $(H_1(S_g;\mathbb{Z}),\hat{\imath})$ is a symplectic $\mathbb{Z}$-module. Since the homeomorphisms don't change intersection numbers,Γ_S preserves the symplectic form $\hat{\imath}$ and we get a *symplectic representation*

$$\psi_{\mathrm{Sp}} : \Gamma_S \to \mathrm{Sp}(2g;\mathbb{Z}).$$

The kernel of ψ_{Sp} is called the *Torelli group* $\mathcal{I}_S$. It is known that ψ_{Sp} is surjective, so we get a short exact sequence $1 \to \mathcal{I}_S \longrightarrow \Gamma_S \xrightarrow{\psi_{\mathrm{Sp}}} \mathrm{Sp}(2g;\mathbb{Z}) \to 1$.

Later in Theorem 6.5 we explain the role of another action of Γ_S in proving that $\mathcal{I}_S$ is torsion free. This implies that the symplectic representation contains all the information about the torsion of Γ_S. On the other hand, we can think of the Torelli group as encoding the information that ψ_{Sp} hides. Some examples of elements in $\mathcal{I}_S$ are Dehn twists about separating curves and *bounding pairs* $T_aT_b^{-1}$, where a and b are homologous non-separating curves. In fact, Birman and Powell showed that $\mathcal{I}_S$ is generated by these kinds of elements and Johnson proved that it is generated by finitely many bounding pairs. We denote by $\mathcal{K}_S$ the subgroup of $\mathcal{I}_S$ generated by Dehn twists about separating curves.

Let $\pi = \pi_1(S_{g,1})$ and π' denote the commutator of π. For a surface $S_{g,1}$ we take the base point of π on the boundary component. Then we get a well-defined action of $\Gamma_{g,1}$ on π. Since $\mathcal{I}_{g,1}$ acts trivially on $H_1(S_{g,1}) \cong \pi/\pi'$, Johnson's idea was to look at its action on the lower central series of π:

$$1 \to \pi'/[\pi,\pi'] \to \pi/[\pi,\pi'] \to \pi/\pi' \to 1.$$

Then he defines $\tau : \mathcal{I}_{g,1} \to \mathrm{Hom}(\pi/\pi', \pi'/[\pi,\pi'])$, the *Johnson homomorphism*, as $f \mapsto \{x \mapsto f(e)e^{-1}\}$, where $e \in \pi/[\pi,\pi']$ that projects on x. Moreover, Johnson proved in [Joh83] that

Theorem 5.1 (Johnson). *The Johnson homomorphism τ surjects on $\Lambda^3(H_1(S_{g,1}))$ and has kernel $\mathcal{K}_{g,1}$.*

6 The action on Teichmüller space

6.1 A definition of Teichmüller space

Let S be a topological surface with $\chi(S) < 0$,which we refer to as the *model surface*. As a set, *Teichmüller space* is defined as $\mathcal{T}_S = \{(X, \phi)\}/\sim$, where

- X is a surface with a complete, finite area hyperbolic metric and geodesic boundary.
- $\phi : S \to X$ is a homeomorphism, called the *marking*.
- $(X_1, \phi_1) \sim (X_2, \phi_2)$ if there is an isometry $I : X_1 \to X_2$ so that $I \circ \phi_1$ is isotopic to ϕ_2.

There is a bijection between $\mathcal{T}_S$ and $\mathrm{DF}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))/\,\mathrm{PSL}_2(\mathbb{R})$, the set of conjugacy classes of discrete and faithful representations of $\pi_1(S)$ on $\mathrm{PSL}_2(\mathbb{R})$. Under this identification we can endow $\mathcal{T}_S$ with the quotient topology from the compact-open topology in $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$.

Fenchel-Nielsen coordinates

The Fenchel-Nielsen coordinates give a parametrization to Teichmüller space that makes it homeomorphic to Euclidean space.

First we consider the simpliest hyperbolic surface which is the pair of pants $P := S_{0,3}$, since each essential simple closed curve in P is homotopic to a boundary component and $\chi(P) = -1$. In this case the marking is equivalent to a labeling of the boundary components, and hyperbolic structures on P are uniquely determined by the lengths of the boundary components. Although we can give coordinates to the Teichmüller space of a surface with boundary or with punctures, here we just sketch the procedure for closed surfaces. Let us consider a maximal collection of non-isotopic disjoint simple closed curves in a hyperbolic surface S_g. By cutting the surface along the $3g - 3$ curves of a such collection, we get a decomposition of S_g in $-\chi(S_g) = 2g - 2 + b$ pairs of pants that we refer as a *pants decomposition* of S.

Theorem 6.1 (Fenchel-Nielsen). *The Teichmüller space $\mathcal{T}_g$, with $g > 1$, is homeomorphic to $\mathbb{R}^{6g-6}$.*

Proof. *(sketch)* The idea is to take a pants decomposition $\mathcal{P}$ of the topological surface S_g and parametrize each point in $[(X, \phi)] \in \mathcal{T}_g$ by two parameters associated to each of the $3g - 3$ curves in the decompositon. For each curve $c_i \in \mathcal{P}$ we consider its length $\ell_X(\phi(c_i))$. The length parameters uniquely determine the hyperbolic structure of each piece of the pants decomposition. The second parameter associated to c_i is called a twist parameter. Roughly speaking, the twist parameters keep track of how the hyperbolic structure X on the topological surface S is recovered by "gluing" together the pairs of pants of the decomposition[5]. Conversely, given such parameters we can construct a marked hyperbolic surface that represents a point in $\mathcal{T}_g$. The continuity of the maps follows from observing how the associated representations of $\pi_1(S)$ in $PSL_2(\mathbb{R})$ change by continuously varying the length and twist parameters. ∎

The Teichmüller metric

The uniformization theorem allows us to establish a bijection between isomorphism classes of marked closed Riemann surfaces of genus $g \geq 2$ and isometry classes of marked closed hyperbolic surfaces. Hence we can think of the points of $\mathcal{T}_S$ as parametrizing marked complex structures on S up to biholomorphism.

We consider the analytic approach to Teichmüller theory in order to endow $\mathcal{T}_S$ with a metric. Given $[(X, \phi)], [(Y, \psi)] \in \mathcal{T}_S$, the *Teichmüller distance* is defined as $d_{\mathcal{T}}(X, Y) = \inf_K\{\frac{1}{2}\log(K)\}$, where K is taken over all the dilations of quasiconformal maps from X to Y which are homotopic to $\psi \circ \phi^{-1}$. The Teichmüller existence and uniqueness theorems imply that this infimum is realized uniquely by a *Teichmüller map* and guarantee that this metric is well defined. It turns out that this endows $\mathcal{T}_S$ with a complete metric that induces the same topology previously defined (see [FM08]). In some sense this "measures" how far two conformal structures are from being conformally equivalent. Moreover, geodesics in the Teichmüller metric are unique.

[5] See [Thu] or [FM08] for a specific definition.

6.2 The action of Γ_S on $\mathcal{T}_S$

There is a natural action of Γ_S on $\mathcal{T}_S$ given by changing the marking: $f \cdot [(X, \phi)] = [(X, \phi \circ \psi^{-1})]$, where ψ is a homeomorphism representing the mapping class f. Moreover, the action of Γ_S preserves K-quasiconformality, therefore Γ_S acts by isometries of the Teichmüller metric on $\mathcal{T}_S$. Observe that isometric surfaces are identified under the action of Γ_S.

The fixed points of the action

Observe that $f \in \mathcal{T}_S$ fixes a point $[(X, \phi)] \in \mathcal{T}_S$ if for a representative diffeomorphism ψ of f, the composition $\phi \circ \psi^{-1} \circ \phi^{-1}$ is isotopic to an isometry on the hyperbolic surface X. But that means that such a representative ψ acts as an isometry of S with the pullback metric induced by $\phi : S \to X$.

Theorem 6.2. *For a closed surface $S = S_g$ of genus $g \geq 2$ endowed with any hyperbolic metric, the group* $\mathrm{Isom}(S)$ *of isometries of S is finite.*

Proof. By looking at a lift on $\mathbb{H}^2$ we see that the unique isometry that is isotopic to the identity is the identity itself. Then $\mathrm{Isom}(S)$ is a discrete group. The Myers-Steenrod Theorem implies that $\mathrm{Isom}(S)$ is a compact group, thus finite. ■

Actually there is a bound (see [FM08]) on the order of the group $\mathrm{Isom}(S)$ that only depends on the genus g of the surface: $|\mathrm{Isom}(S)| \leq 84(g-1)$. Hence, any element $f \in \Gamma_S$ that fixes a point in $\mathcal{T}_S$ has a diffeomorphism representative which is finite order. In particular, torsion free subgroups of Γ_S act freely on $\mathcal{T}_S$. When $f \in \Gamma_S$ has prime order, Smith theory can be applied to its action on $\mathcal{T}_S$ to prove that it has a fixed point. By an induction argument this follows for any $f \in \Gamma_S$ of finite order. From the above remarks we get the following useful result:

Theorem 6.3. *A mapping class of finite order can be represented by a diffeomorphism of finite order.*

The action is properly discontinuous

It turns out that the above action of Γ_S is properly discontinuous. The proof of this fact is based on the description of elements of $\mathcal{T}_S$ in terms of lengths of curves on the surface. The *raw length spectrum* of $[(X, \phi)] \in \mathcal{T}_S$ is the set $rls(X) = \{\ell_X(c) : c \text{ is an isotopy class of simple closed curves}\}$, where $\ell_X(c)$ is the length of the geodesic representative of $\phi(c)$.

- The raw length spectrum of a point $X \in \mathcal{T}_S$ is a closed and discrete subset of $\mathbb{R}$.
- It turns out that any point in $\mathcal{T}_S$ is determined by its marked length spectrum: we can pick a set of simple closed curves $\{c_1, c_2, \ldots, c_{9g-9}\}$ on the surface S so that their length spectrum determines the Fenchel-Nielsen coordinates uniquely, i.e., the map $[(X, \phi)] \mapsto \{\ell_X(c_1), \ell_X(c_2), \ldots, \ell_X(c_{9g-9})\}$ from $\mathcal{T}_S$ to $\mathbb{R}^{9g-9}$ is injective.
- Moreover, Wolpert proved that a K-quasiconformal map $f : X_1 \to X_2$ between two points $X_1, X_2 \in \mathcal{T}_S$ can only increase or decrease the length of an isotopy class of simple closed curves by a bounded amount, namely $\frac{\ell_{X_2}(c)}{K} \leq \ell_{X_1}(c) \leq K\ell_{X_2}(c)$.

From the above remarks it follows that the set $\{Y \in B_r(X) : rls(X) = rls(Y)\}$ is finite for any fixed $r > 0$.

Theorem 6.4 (Fricke). *The action of Γ_S on $\mathcal{T}_S$ is properly discontinuous.*

Proof. Take K any compact subset of $\mathcal{T}_S$ and consider $A = \{f \in \Gamma_S : f(K) \cap K \neq \varnothing\}$. Since Γ_S acts by isometries on $\mathcal{T}_S$ and K is compact it follows that $\bigcup_{f \in A} f(K) \subset B$, where B is a closed Teichmüller ball of diameter $r > 0$. Now we translate the question into the raw length spectrum context: For any $X \in K$, the set $\{f(X) : f \in A\} \subset \{Y \in B : rls(X) = rls(Y)\}$ which is finite. Moreover, $\{g \in \Gamma_S : g \text{ fixes } X\}$ is also finite. Hence A must be finite. ■

6.3 Existence of finite index torsion free subgroups of Γ_S

First we relate the above results regarding the fixed points of the action on $\mathcal{T}_S$ to get more information about the symplectic representation $\psi_{\mathrm{Sp}} : \Gamma_S \to \mathrm{Sp}(2g; \mathbb{Z})$.

Proposition 6.5. *Finite order elements of Γ_S are faithfully represented by ψ_{Sp}.*

Proof. (*Sketch*) A finite order element of Γ_S is represented by a diffeomorphism ϕ of finite order that actually acts as an isometry in S with some hyperbolic metric. Since an isometry is determined by its action on a point and a frame, it follows that the fixed points of ϕ are isolated. An application of the Lefschetz fixed point theorem then implies that $\psi_{\mathrm{Sp}}(\phi)$ cannot act by the identity matrix on $H_1(S_g; \mathbb{Z})$. ∎

For a given natural number m, we can consider the reduction mod m $\mathrm{Sp}(2g; \mathbb{Z}) \to \mathrm{Sp}(2g; \mathbb{Z}/m\mathbb{Z})$. The kernel of this homomorphism, denoted by $\mathrm{Sp}(2g; \mathbb{Z})[m]$, is a finite index normal subgroup of $\mathrm{Sp}(2g; \mathbb{Z})$. It is known that for $m \geq 3$ and $g \geq 1$, this group is torsion free (see [FM08]). With the symplectic representation ψ_{Sp}, for each $m \in \mathbb{N}$ we set the *level m congruence subgroup of* Γ_S to be $\Gamma_S[m] := \psi_{\mathrm{Sp}}^{-1}(\mathrm{Sp}(2g; \mathbb{Z})[m])$. Now we combine this with the Proposition 6.5 to get:

Proposition 6.6. *$\Gamma_S[m]$ is finite index for any $m \in \mathbb{N}$ and it is torsion free for $m \geq 3$ and $g \geq 1$.*

Hence, for $m \geq 3$ and $g \geq 1$, the subgroup $\Gamma_S[m]$ acts freely and properly discontinuously on $\mathcal{T}_S$.

6.4 The quotient space

The quotient $\mathcal{M}_S = \mathcal{T}_S/\Gamma_S$ is called *moduli space*. Since $\mathcal{T}_S$ is contractible and the action is properly discontinuous, $\mathcal{M}_g$ has the structure of an aspherical orbifold. We can think of moduli space as parametrizing either hyperbolic structures on S up to isometry, complex structures on S up to conformal equivalence, or conformal classes of Rimannian metrics on S.

For each $X \in \Gamma_S$ we can consider the length $\ell(X)$ of the shortest geodesic on X. Moduli space is not compact, since the length parameters of the Fenchel-Nielsen coordinates can be arbitrarily small. However, it has an exhaustion by compact sets:

Theorem 6.7 (Mumford's compactness criterion). *If S_g is a closed surface with genus $g \geq 2$, then $\mathcal{M}_S^{\epsilon} = \{X \in \mathcal{M}_S : \ell(X) \geq \epsilon\}$ is compact for any $\epsilon > 0$.*

It turns out that moduli space has *one end*: for every compact set $K \subset \mathcal{M}_S$, the space $\mathcal{M}_S \backslash K$ has only one component whose closure is not compact. Moreover, any loop in $\mathcal{M}_S$ can be freely homotoped outside K. On the other hand, $\mathcal{M}_S$ is simply connected for $g \geq 1$, but has *orbifold fundamental group* Γ_S.

Earle-Eells proved that for $g \geq 2$ the topological group $\mathrm{Diff}_0(S_g)$ is contractible. From the short exact sequence $1 \to \mathrm{Diff}_0(S_g) \to \mathrm{Diff}^+(S_g) \to \Gamma_g \to 1$ and Whitehead's theorem it follows that

Proposition 6.8. *For $g \geq 2$, $B\mathrm{Diff}^+(S_g)$ is a $K(\Gamma_g, 1)$.*

Here $\mathrm{BDiff}^+(S_g)$ denotes the classifying space of the topological group $\mathrm{Diff}^+(S_g)$. From the theory of classifying spaces we can conclude that given a space B (Hausdorff and paracompact) there is a bijection between isomorphism classes of oriented S_g-bundles over B and conjugacy classes of representations $\rho : \pi_1(B) \to \Gamma_g$. Moreover, $H^*(\Gamma_g; \mathbb{Z}) \cong H^*(\mathrm{BDiff}^+(S_g); \mathbb{Z})$, which means that the elements of $H^*(\Gamma_g; \mathbb{Z})$ are precisely characteristic classes of surface bundles (see [Mor01]).

Moduli space vs mapping class group: Rational cohomology[6]

Now we can put together the information that we have about the action on $\mathcal{T}_S$. Consider a $K(\Gamma_S, 1)$ with universal cover $E\Gamma_S$. Take Γ' a torsion free and finite index normal subgroup of Γ_S. Then the diagonal action of both Γ_S and Γ' on $E\Gamma_S \times \mathcal{T}_S$ is free and properly discontinuous and we get a commutative diagram:

$$\begin{tikzcd} B\Gamma' = E\Gamma_S \times_{\Gamma'} \mathcal{T}_S \arrow[r, "p"] \arrow[d, "\pi"'] & \mathcal{T}_S/\Gamma' \arrow[d, "\pi'"] \\ B\Gamma_S = E\Gamma_S \times_{\Gamma_S} \mathcal{T}_S \arrow[r, "q"] & \mathcal{T}_S/\Gamma_S = \mathcal{M}_S \end{tikzcd} \qquad \begin{tikzcd} H^*(B\Gamma'; \mathbb{Q})^G & H^*(\mathcal{T}_S/\Gamma'; \mathbb{Q})^G \arrow[l, "\cong"'] \\ H^*(B\Gamma_S; \mathbb{Q}) \arrow[u, "\pi^*"', "\cong"] & H^*(\mathcal{M}_S; \mathbb{Q}) \arrow[l, "q^*"'] \arrow[u, "\cong"', "\pi'^*"] \end{tikzcd}$$

[6] For more relations between moduli space and the mapping class group see Harer's survey [Har88].

We denote Γ_g/Γ' by G. The group Γ' acts freely and properly discontinuously on $\mathcal{T}_g$, therefore the map p induced by the projection on $\mathcal{T}_S$ is a G-equivariant homotopy equivalence. On the other hand, since π and π' are G-coverings and G is finite, we can define transfer maps[7] that give us the vertical isomorphisms in rational homology on the right diagram. Therefore $H^*(\mathcal{M}_S;\mathbb{Q}) \cong H^*(\Gamma_S;\mathbb{Q})$.

7 Other actions of the mapping class group

7.1 The action on measured foliations

By a *singular foliation* $\mathcal{F}$ we mean a decomposition of S into a disjoint union of 1-dimensional submanifolds (the leaves of $\mathcal{F}$) and a finite set of singularities. A *transverse measure* on $\mathcal{F}$ is a measure μ on arcs that are transverse to $\mathcal{F}$ that is invariant under "leaf-preserving" isotopies. Given $(\mathcal{F},\mu)$ there is an atlas on $S\backslash\{\text{singular points of } \mathcal{F}\}$ with transition maps $(x,y)\mapsto(f(x,y),c\pm y)$, such that $(\mathcal{F},\mu)$ is the pullback of the horizontal foliation on $\mathbb{R}^2$ with transverse measure $|dy|$.

There is a natural action of $\mathrm{Homeo}(S)$ on the set of measured foliations given by $\phi\cdot(\mathcal{F},\mu)=(\phi(\mathcal{F}),\phi_*(\mu))$. This induces an action of Γ_S on $\mathcal{MF}_S$, the set of isotopy classes of measured foliations, where we identify *Whitehead moves*, i.e. collapsing and uncollapsing leaves connecting singularities. Moreover, we get an action of Γ_S on $\mathcal{PMF}_S$, the set of projective classes of measured foliations.

Thurston gave a topology to $\mathcal{MF}_S\cup\mathcal{PMF}_S$ that makes it homeomorphic to a closed ball. By analizing the fixed points of the action of Γ_S on this ball, he gave a classification of the mapping classes that corresponds to Theorem 7.1 stated below.

7.2 The action on geodesic rays

Let X be a Riemann surface of genus g with atlas $\{(U_i,z_i)\}$. A *holomorphic quadratic differential* q on X is specified by a collection $\{\phi_i(z_i)dz_i^2 : \phi_i \text{ is holomorphic}\}$ invariant under change of coordinates. Up to multiplicity, q has exactly $4g-4$ zeros. Let $\mathcal{Z}_q$ denote the set of zeros of q. We can give X a natural atlas so that q is either of the form $z^k dz^2$ near the points in $\mathcal{Z}_q$ or of the form dz^2 elsewhere. We associate to q the *horizontal foliation* $\mathcal{F}_q$ with leaves the smooth paths $\gamma(t)$ on X such that $q(\gamma'(t))>0$ and with singular points the set $\mathcal{Z}_q$. Moreover, we can associate to q an Euclidean length and area that gives a flat metric to $S\backslash\mathcal{Z}_q$. We denote by $\mathcal{QD}(X)$ the $\mathbb{C}$-vector space of holomorphic quadratic differentials on X.

In Section 6.1 we mentioned a map that minimizes the deformation of the complex structure: the *Teichmüller map*. More precisely, it is defined as a homeomorphism $f:(X,q_X)\to(Y,q_Y)$, equipped with a quadratic differential $q_X\in\mathcal{QD}(X)$, such that $f(\mathcal{Z}_{q_X})\subset\mathcal{Z}_{q_Y}$ and is of the form $(x+iy)\mapsto(\sqrt{K}x+i\frac{1}{\sqrt{K}}y)$, in natural coordinates for q_X and q_Y. Such an f has dilation K, which can be thought as the streching factor in the direcion of the foliations.

Given a point $[(X,\psi)]\in\mathcal{T}_S$ and some $q\in\mathcal{QD}(X)$, we can generate a one-parameter family of Teichmüller maps $\{f_K: X\to Y_K\}_{K>0}$ streching in the horizontal direction of the quadratic differential q. Therefore we get a one-parameter family of points $\mathrm{Ray}((X,q),\psi):=\{[(Y_K,f_K\circ\psi)]\}_{K\geq 1}$ in $\mathcal{T}_S$. This corresponds to a *geodesic ray* in the Teichmüller metric that starts at the point $[(X,\psi)]$ and "goes in the direction determined by q". Since Γ_S acts on $\mathcal{T}_S$ by isometries it preserves geodesic rays. Therefore we have an action of Γ_S on the set of geodesic rays defined by $f\cdot\mathrm{Ray}((X,q),\psi)=\mathrm{Ray}((X,q),\psi\circ\phi^{-1})$, where ϕ is a representative of f.

The Nielsen-Thurston Classification Theorem

The group Γ_S acts on $\mathcal{T}_S$ by isometries. We can give a classification of mapping classes in terms of the *translation distance* $\tau(f):=\inf_{X\in\mathcal{T}_S} d_{\mathcal{T}}(X,f(X))$, where $f\in\Gamma_s\subset\mathrm{Isom}(\mathcal{T}_S)$. We say that f is *elliptic* if $\tau(f)=0$ and the infimum is achieved, *parabolic* if $\tau(f)$ is not achieved, and *hyperbolic* if $\tau(f)>0$ and it is achieved. Bers proved this is precisely the Nielsen-Thurston classification.

Theorem 7.1. *(Nielsen-Thurston classification) Any element $f\in\Gamma_S$ satisfies one of the following:*

1. *f is periodic: $f^n=1$ for some $n\in\mathbb{N}$.*
2. *f is reducible: f fixes a collection of disjoint isotopy classes of simple closed curves in S.*
3. *f is pseudo-Anosov: There are two transverse measured foliations $(\mathcal{F}^s,\mu_s)$ and $(\mathcal{F}^u,\mu_u)$ and some $\lambda>0$ such that $f\cdot(\mathcal{F}^s,\mu_s)=(\mathcal{F}^s,\lambda\mu_s)$ and $f\cdot(\mathcal{F}^u,\mu_u)=(\mathcal{F}^u,\lambda^{-1}\mu_u)$.*

[7] See [Hai95] and [Bro94].

Proof. (*sketch*) Being elliptic corresponds to fixing a point in $\mathcal{T}_S$; therefore by Theorem 6.2, elliptic mapping classes are periodic. Let f be parabolic and $\{X_n\} \subset \mathcal{T}_S$ be such that $d_{\mathcal{T}}(X_n, f(X_n))$ converges to $\tau(f)$. The proper discontinuity of the action of Γ_s on $\mathcal{T}_S$ implies that the projection of this sequence to $\mathcal{M}_S$ leaves every compact set. In particular, this is true for Mumford's exhaustion of $\mathcal{M}_S$ by compact sets, thus and $\lim_{n\to\infty} \ell(X_n) = 0$. Together with the collar lemma, this condition will yield a collection of closed curves on S that is fixed by f. Finally, we consider the hyperbolic case and take $[(X, \phi)] \in \mathcal{T}_S$ so that $d_{\mathcal{T}}(X, f(X)) = \tau(f)$. Let γ be the unique geodesic in $\mathcal{T}_S$ from X to $f \cdot X$. The minimality of $\tau(f)$ implies that f fixes γ. Translating this into the action of Γ_S on geodesic rays, we get that f has the local expression of a pseudo-Anosov. ■

Periodic and reducible mapping classes fix, up to a power, at least an isotopy class of simple closed curves. On the other hand, under the iteration of a pseudo-Anosov the lenght of any simple closed curve grows. As a consequence, we get from Theorem 7.1 a useful criteria: $f \in \Gamma_S$ is pseudo-Anosov if f is not reducible neither periodic. Some interesting dynamical properties of pseudo-Anosov are given in terms of density: any leaf of $\mathcal{F}^s$ or $\mathcal{F}^u$ is dense in the surface, any pseudo-Anosov has dense orbit, and, moreover, its set of periodic points is dense in the surface.

References

[BP92] Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992.

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.

[FM08] Benson Farb and Dan Margalit, *A primer on mapping class groups*, 2008, Working Draft version 3.0.

[Hai95] Richard M. Hain, *Torelli groups and geometry of moduli spaces of curves*, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), Math. Sci. Res. Inst. Publ., vol. 28, Cambridge Univ. Press, Cambridge, 1995, pp. 97–143.

[Har85] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985), no. 2, 215–249.

[Har88] ———, *The cohomology of the moduli space of curves*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Math., vol. 1337, Springer, Berlin, 1988, pp. 138–221.

[Hat91] Allen Hatcher, *On triangulations of surfaces*, Topology Appl. **40** (1991), no. 2, 189–194.

[Iva02] Nikolai V. Ivanov, *Mapping class groups*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 523–633.

[Joh83] Dennis Johnson, *A survey of the Torelli group*, Low-dimensional topology (San Francisco, Calif., 1981), Contemp. Math., vol. 20, Amer. Math. Soc., Providence, RI, 1983, pp. 165–179.

[Kat92] Svetlana Katok, *Fuchsian groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992.

[Mor01] Shigeyuki Morita, *Geometry of characteristic classes*, Translations of Mathematical Monographs, vol. 199, American Mathematical Society, Providence, RI, 2001, Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics.

[Thu] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35.