RIGIDITY OF SYMMETRIC PRODUCTS

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Abstract

Given a metric continuum $X$, we consider the following hyperspaces of $X$: $2^X$, $C_n(X)$ and $F_n(X)$ ($n \in \mathbb{N}$). Let $F_1(X) = \{ \{x\} : x \in X \}$. A hyperspace $K(X)$ of $X$ is said to be rigid provided that for every homeomorphism $h : K(X) \to K(X)$ we have $h(F_1(X)) = F_1(X)$. In this paper we study conditions under which a continuum $X$ has a rigid hyperspace $F_n(X)$. Among others, we consider families of continua such as, dendroids, Peano continua, indecomposable continua such that all their proper nondegenerate subcontinua are arcs, hereditarily indecomposable continua and smooth fans.

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1. INTRODUCTION

A continuum is a nondegenerate compact connected metric space. Given a continuum $X$, with metric $d$, we consider the following hyperspaces of $X$.

- $2^X = \{ A \subset X : A$ is nonempty and closed in $X \}$,
- $C_n(X) = \{ A \in 2^X : A$ has at most $n$ components $\}$,
- $F_n(X) = \{ A \in 2^X : A$ has at most $n$ points $\}$,
- $C(X) = C_1(X)$.

All hyperspaces are considered with the Hausdorff metric $H$ [19, Remark 0.4] defined as

$H(A, B) = \max\{ \max\{d(a, B) : a \in A\}, \max\{d(b, A) : b \in B\} \}$,

where $d(a, B) = \min\{d(a, b) : b \in B\}$.

The hyperspace $F_n(X)$ is known as the $n$-th symmetric product of $X$ [2]. The hyperspace $F_1(X)$ is an isometric copy of $X$ embedded in each one of the hyperspaces. We extend the definition of $F_n(X)$ by defining $F_0(X) = \emptyset$. 

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A hyperspace \( K(X) \in \{2^X, C_n(X), F_n(X)\} \) is said to be \textit{rigid} provided that for each homeomorphism \( h : K(X) \to K(X) \), we have, \( h(F_1(X)) = F_1(X) \). The continuum \( X \) is said to have \textit{unique hyperspace} \( K(X) \) provided that the following implication holds: if \( Y \) is a continuum such that \( K(X) \) is homeomorphic to \( K(Y) \), then \( X \) is homeomorphic to \( Y \).

Uniqueness of hyperspaces has been widely studied (see for example [9], [13] and [14] for recent references). A useful technique is to find a topological property that characterizes the elements of \( F_1(X) \) in the hyperspace \( K(X) \). When it is possible to find such a characterization, the hyperspace \( K(X) \) is rigid, this technique has been used in studying uniqueness of hyperspaces, so both topics are closely related. Moreover, the topic of this paper leads us to new results on unique hyperspaces.

In this paper we study rigidity on the symmetric products \( F_n(X) \). Rigidity of hyperspaces was introduced in [10], where the hyperspaces \( C_n(X) \) were studied.

Among others, we consider families of continua such as, dendroids, Peano continua, indecomposable continua such that all their proper nondegenerate subcontinua are arcs, hereditarily indecomposable continua and smooth fans.

\section*{2. Definitions and Conventions}

A \textit{map} is a continuous function. Suppose that \( d \) is a metric for \( X \). Given \( \varepsilon > 0 \), \( p \in X \) and \( A \in 2^X \), let \( B(\varepsilon, p) \) be the \( \varepsilon \)-open ball around \( p \) in \( X \), \( N(\varepsilon, A) = \{ p \in X : \text{there exists } a \in A \text{ such that } d(p, a) < \varepsilon \} \) and \( B^\mathcal{H}(\varepsilon, A) = \{ B \in 2^X : H(A, B) < \varepsilon \} \) (we write \( B_X(\varepsilon, p) \) and \( N_X(\varepsilon, A) \) when the space \( X \) needs to be mentioned). A \textit{simple triod} is a finite graph \( G \) that is the union of three arcs emanating from a single point, \( v \), and otherwise disjoint from one another. The point \( v \) is called the \textit{vertex} of \( G \). Given subsets \( A_1, \ldots, A_m \) of \( X \), let \( (A_1, \ldots, A_m) = \{ B \in 2^X : B \cap A_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\} \text{ and } B \subset A_1 \cup \ldots \cup A_m \} \).

We denote by \( S^1 \) the unit circle in the Euclidean plane. A \textit{free arc} in the continuum \( X \) is an arc \( a \) with end points \( a \) and \( b \) such that \( a - \{a, b\} \) is open in \( X \). A \textit{tail} in a continuum \( X \) is an arc \( a \) with end points \( a \) and \( b \) such that \( a - \{a\} \) is open in \( X \). An \textit{end point} in \( X \) is a point \( p \in X \) such that \( p \) is an end point of every arc containing it.

Given a continuum \( X \), let

\[ G(X) = \{ p \in X : p \text{ has a neighborhood } M \text{ in } X \text{ such that } M \text{ is a finite graph} \} \]

and \( \mathcal{P}(X) = X - G(X) \).
The continuum \( X \) is said to be almost meshed \([9]\) provided that the set \( \mathcal{G}(X) \) is dense in \( X \).

Proceeding as in Lemma 2.1 of \([5]\) and using Lemma 1.48 of \([19]\), the following lemma can be proved.

**Lemma 1.** Let \( X \) be a continuum and let \( \mathcal{A} \) be a connected subset of \( 2^X \) such that \( A \cap C_n(X) \neq \emptyset \). Let \( A_0 = \bigcup \{ A : A \in \mathcal{A} \} \). Then
(a) \( A_0 \) has at most \( n \) components,
(b) if \( \mathcal{A} \) is closed in \( 2^X \), then \( A_0 \in C_n(X) \),
(c) for each \( A \in \mathcal{A} \), each component of \( A_0 \) intersects \( A \).

### 3. WIRED CONTINUA

In this section we present some technical results that will be used later for proving that some hyperspaces are rigid.

A *wire* in a continuum \( X \) is a subset \( \alpha \) of \( X \) such that \( \alpha \) is homeomorphic to one of the spaces \((0, 1), [0, 1), [0, 1]\) or \( S^1 \) and \( \alpha \) is a component of an open subset of \( X \). By \([16, \text{Theorem 20.3}]\), if a wire \( \alpha \) in \( X \) is compact, then \( \alpha = X \). So, if a wire is homeomorphic to \([0, 1]\) or \( S^1 \), then \( X \) is an arc or a simple closed curve. Given a continuum \( X \), let
\[
W(X) = \bigcup \{ \alpha \subset X : \alpha \text{ is a wire in } X \}.
\]

The continuum \( X \) is said to be *wired* provided that \( W(X) \) is dense in \( X \).

Notice that if \( \alpha \) is a free arc of a continuum \( X \) and \( p, q \) are the end points of \( \alpha \), then \( \alpha - \{p, q\} \) is a wire in \( X \). Thus, a continuum for which the union of its free arcs is dense is a wired continuum. Therefore, the class of wired continua includes finite graphs, dendrites with closed set of end points, almost meshed continua \([9]\), compactifications of the ray \([0, \infty)\), compactifications of the real line and indecomposable continua whose proper subcontinua are arcs, these continua will be called indecomposable arc continua (see Lemma 2 of \([10]\)).

Given a continuum \( X \) and \( n \in \mathbb{N} \), let
\[
\mathcal{G}_n(X) = \{ A \in F_n(X) : \text{there is a neighborhood } \mathcal{M} \text{ of } A \text{ in } F_n(X) \text{ such that the component } C \text{ of } \mathcal{M} \text{ that contains } A \text{ is a } 2n\text{-cell} \}.
\]

**Theorem 2.** Let \( X \) be a continuum and \( n \geq 2 \). Then
(a) \( \mathcal{G}_n(X) \subset \{ A \in F_n(X) : A \subset W(X) \} \),
(b) if \( n \geq 4 \), then \( \mathcal{G}_n(X) \subset \{ A \in F_n(X) - F_{n-1}(X) : A \subset W(X) \} \),
(c) \( \{ A \in F_n(X) - F_{n-1}(X) : A \subset W(X) \} \subset \mathcal{G}_n(X) \).
**Proof.** Let $d$ be a metric for $X$. First, we prove (a). Take $A \in \mathcal{G}_n(X)$. Let $\mathcal{M}$ be a neighborhood of $A$ in $F_n(X)$ such that the component $C$ of $\mathcal{M}$ that contains $A$ is a 2$n$-cell. Let $\mathcal{F} = \text{bd}_F(X)(\mathcal{M}) \cap C$. Then $\mathcal{F}$ is a compact subset of $C$ and $A \notin \mathcal{F}$. Let $C_1$ be a 2$n$-cell such that $A \in \text{int}_C(C_1) \subset C_1 \subset C - \mathcal{F}$. Notice that $C_1 \subset \text{int}_F(X)(\mathcal{M})$. Let $\delta > 0$ be such that $B^H(\delta, A) \cap C_1 \subset C_1$. Let $\varepsilon > 0$ be such that $\varepsilon < \delta$, $B^H(2\varepsilon, A) \cap F_n(X) \subset \mathcal{M}$, $N_{F_n(X)}(C_1, \varepsilon) \subset \text{int}_F(X)(\mathcal{M})$, and $\varepsilon < \text{diameter}(X)$.

Let $C_0$ be the component of $\text{cl}_{F_n(X)}(B^H(\varepsilon, A) \cap F_n(X))$ containing $A$. Then $C_0 \subset C \cap B^H(\delta, A)$. So $C_0 \subset C_1$. Let $E = \bigcup\{B : B \in C_1\}$. By [6, Lemma 2.2] and Lemma 1, $E$ is a locally connected compact subset of $X$ with at most $n$ components. Let $E = E_1 \cup \ldots \cup E_m$, where $E_1, \ldots, E_m$ are the different components of $E$. Then, $m \leq n$ and $A \cap E_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}$.

For each $a \in A$, let $C_a$ be the component of $B(\frac{\varepsilon}{2}, a)$ containing $a$. By [16, Theorem 20.3], $C_a$ is nondegenerate. Let $C_a = \{(A - \{a\}) \cup \{x\} : x \in C_a\}$. Then $C_a$ is a nondegenerate connected subset of $F_n(X)$ containing $A$ such that $C_a \subset B^H(\varepsilon, A) \cap F_n(X)$. Thus, $C_a \subset C_0$ and $C_a \subset \bigcup\{B : B \in C_a\} \subset E$. Hence, the component of $E$ containing $a$ is nondegenerate. This proves that each $E_i$ is nondegenerate.

We are going to see that $E$ does not contain simple triods. Suppose to the contrary, for example, that $E_1$ contains a simple triod $T = L_1 \cup L_2 \cup L_3$ with vertex $v$, where each $L_i$ is an arc emanating from $v$. Since $v \in E$ there exists $B \in C_1$ such that $v \in B$. Suppose that $B = \{v, b_2, \ldots, b_r\}$. Since $B \subset E$, and each component of $E$ is a nondegenerate locally connected continuum, for each $i \in \{2, \ldots, r\}$, we can fix an arc $\alpha_i \subset E$ such that $b_i \in \alpha_i$, diameter($\alpha_i$) $< \frac{\varepsilon}{2}$ and, shortening $T$ if necessary, $T, \alpha_2, \ldots, \alpha_r$ are pairwise disjoint and diameter($T$) $< \varepsilon$. In the case that $r < n$, we choose pairwise disjoint subsars $\alpha_{r+1}, \ldots, \alpha_n$ of $L_1 - \{v\}$. Let $T_0$ be a simple subtrioid of $T$ such that $T_0 \cap (\alpha_{r+1} \cup \ldots \cup \alpha_n) = \emptyset$. In the case that $r = n$, define $T_0 = T$. Note that the set $D_0 = (T, \alpha_2, \ldots, \alpha_r) \cap F_n(X)$ is a connected subset of $B(\varepsilon, B)$ (see [17, Lemma 1]) that contains $B$. Since $B(\varepsilon, B) \subset N_{F_n(X)}(C_1, \varepsilon) \subset \text{int}_F(X)(\mathcal{M})$ and $B \subset C_1 \subset C$, $D_0 \subset C$. Note that the set $D_1 = (T_0, \alpha_2, \ldots, \alpha_n) \cap F_n(X)$ is a subset of the set $D_0$ and $D_1$ is homeomorphic to $T_0 \times [0, 1]^{n-1}$. Hence, $T_0 \times [0, 1]^{n-1}$ can be embedded in the $n$-cell $C$. This contradicts the Invariance of Domain Theorem [12, Theorem VI 9] and completes the proof that $E$ does not contain a simple triod. Therefore, each $E_i$ is a locally connected continuum without simple triods. Hence, each $E_i$ is either an arc or a simple closed curve.

Given $a \in A$, $C_a$ is a connected subset of $E$. Since $C_a$ is a component of a proper open subset of $X$, by [19, Theorem 20.3], $C_a$ is not compact, so $C_a$ is a noncompact connected subset of some $E_i$. Thus, $C_a$ is homeomorphic to one of the intervals $(0, 1)$ or $[0, 1)$. Hence, $a \in W(X)$. Therefore, $A \subset W(X)$. This ends the proof of (a).
(b). By (a), we only need to show that if \( A = \{a_1, \ldots, a_m\} \in \mathcal{G}_n(X) \), then \( m = n \). Suppose to the contrary that \( m < n \). We keep the notation of the proof of (a). For each \( i \in \{1, \ldots, m\} \), we can choose an arc \( \beta_i \subset C_{a_i} \) such that \( a_i \in \beta_i \), diameter(\( \beta_i \)) < \( \frac{\varepsilon}{2} \) and \( \beta_1, \ldots, \beta_m \) are pairwise disjoint. In the case that \( m < n - 1 \), we choose pairwise different points \( a_{m+1}, \ldots, a_{n-1} \in \beta_1 - \{a_1\} \) and we choose pairwise disjoint arcs \( \beta_{m+1}, \ldots, \beta_{n-1} \) of \( \beta_1 - \{a_1\} \) such that \( a_{m+1} \in \beta_{m+1}, \ldots, a_{n-1} \in \beta_{n-1} \) and \( a_1 \notin \beta_{m+1} \cup \ldots \cup \beta_{n-1} \). We choose a subarc \( \beta_1' \) of \( \beta_1 - (\beta_{m+1} \cup \ldots \cup \beta_{n-1}) \) such that \( a_1 \in \beta_1' \). Let \( D = \{a_1, \ldots, a_{n-1}\} \). Let \( \mathcal{E}_0 = \langle \beta_1', \beta_2, \ldots, \beta_m \rangle \cap F_n(X) \) and \( \mathcal{E} = \langle \beta_1', \beta_2, \ldots, \beta_{n-1} \rangle \cap F_n(X) \). Clearly, \( A \in \mathcal{E}_0, \mathcal{E} \subset \mathcal{E}_0 \), \( \mathcal{E}_0 \) is connected [17, Lemma1], \( \mathcal{E}_0 \subset B^{H}(\frac{\varepsilon}{2}, A) \cap F_n(X) \subset M \). Thus, \( \mathcal{E} \subset \mathcal{C} \). For each \( i > 1 \), let \( \varphi_i : F_2(\beta_i) \times \beta_1' \times \beta_2 \times \ldots \times \beta_{i-1} \) is an embedding. Since \( F_2(\beta_i) \) is a 2-cell [15, Example K], \( \text{Im} \varphi_i \) is an \( n \)-cell. Note, that if \( i \neq j \), \( \text{Im} \varphi_i \cap \text{Im} \varphi_j \) is the set \( \langle \beta_1', \beta_2, \ldots, \beta_{n-1} \rangle \cap F_n(X) \), which is an \((n-1)\)-cell. Therefore, the sets \( \text{Im} \varphi_1, \ldots, \text{Im} \varphi_{n-1} \) are at least three \( n \)-cells, whose pairwise intersection is the \((n-1)\)-cell \( \langle \beta_1', \beta_2, \ldots, \beta_{n-1} \rangle \cap F_n(X) \). From the Invariance Domain Theorem [12, Theorem VI 9], \( \text{Im} \varphi_1 \cup \ldots \cup \text{Im} \varphi_{n-1} \) cannot be embedded in an \( n \)-cell. This is a contradiction since \( \text{Im} \varphi_1 \cup \ldots \cup \text{Im} \varphi_{n-1} \subset \mathcal{E} \subset \mathcal{C} \). We have shown that \( m = n \).

(c). Suppose that \( A \in F_n(X) - F_{n-1}(X) \) and \( A \subset W(X) \). Let \( A = \{a_1, \ldots, a_n\} \). For each \( i \in \{1, \ldots, n\} \), let \( U_i \) be an open subset of \( X \) such that \( a_i \in U_i \) and the component \( \alpha_i \) of \( U_i \) that contains \( a_i \) is homeomorphic to \((0, 1)\) or \((0, 1] \). Let \( \varepsilon > 0 \) be such that \( B(2\varepsilon, a_i) \subset U_i, 2\varepsilon < \text{diameter}(X) \) and the sets \( B(2\varepsilon, a_1), \ldots, B(2\varepsilon, a_n) \) are pairwise disjoint. For each \( i \in \{1, \ldots, n\} \), let \( M_i = \{p \in X : d(p, a_i) \leq \varepsilon\} \) and \( C_i \) be the component of \( M_i \) that contains \( a_i \). Let \( \mathcal{M} = \{M_1, \ldots, M_n\} \cap F_n(X) \) and \( \mathcal{C} = \{C_1, \ldots, C_n\} \cap F_n(X) \). Then \( \mathcal{M} \) is a neighborhood of \( A \) in \( F_n(X) \). For each \( i \in \{1, \ldots, n\} \), \( C_i \) is a nondegenerate compact connected subset of \( X \). This implies that \( \mathcal{C} \) is an \( n \)-cell. Since \( A \in \mathcal{C} \), to finish the proof of (c), we only need to check that \( \mathcal{C} \) is a component of \( \mathcal{M} \). Let \( \mathcal{D} \) be the component of \( \mathcal{M} \) such that \( A \in \mathcal{D} \). Then \( \mathcal{C} \subset \mathcal{D} \). Let \( D = \bigcup \{E : E \in \mathcal{D} \} \). By Lemma 1, \( D \) is a compact subset of \( X \) with at most \( n \) components and each one of them interesects \( A \). For each \( i \in \{1, \ldots, n\} \), let \( D_i \) be the component of \( D \) that contains \( a_i \). Then \( D_i = D_1 \cup \ldots \cup D_n \). Given a point \( p \in D \), there exists \( E \in \mathcal{D} \subset \mathcal{M} \) such that \( p \in E \). Then there exists \( i \in \{1, \ldots, n\} \) such that \( p \in M_i \). This proves that \( D \subset M_1 \cup \ldots \cup M_n \). Since \( M_1, \ldots, M_n \) are pairwise disjoint, \( D_1, \ldots, D_n \) are connected and each \( D_i \) intersects \( M_i \), we have \( a_i \in D_i \subset M_i \). This implies that \( D_i \subset C_i \). Given \( E_0 \subset D \), \( E_0 \subset D \) and \( E_0 \) intersects each one of the components of \( D \) (Lemma 1). Thus, \( E_0 \subset D \subset C_1 \cup \ldots \cup C_n \) and \( E_0 \) intersects each \( C_i \). This proves that \( \mathcal{D} \subset \mathcal{C} \) and \( \mathcal{C} \) is a component of \( \mathcal{M} \).

**COROLLARY 3.** Let \( n \geq 2 \). Then the continuum \( X \) is wired if and only if \( \mathcal{G}_n(X) \) is dense in \( F_n(X) \).
Proof. Necessity. Suppose that $X$ is wired. Let $U$ be a nonempty subset of $F_n(X)$. It is easy to show that $F_n(X) - F_{n-1}(X)$ is dense in $F_n(X)$, so there exists $A = \{a_1, \ldots, a_n\} \in (F_n(X) - F_{n-1}(X)) \cap U$. Let $\varepsilon > 0$ be such that $B^H(\varepsilon, A) \cap F_n(X) \subset U$ and the sets $B(\varepsilon, a_1), \ldots, B(\varepsilon, a_n)$ are pairwise disjoint. Since $X$ is wired, for each $i \in \{1, \ldots, n\}$, we can choose a point $x_i \in B(\varepsilon, a_i) \cap W(X)$. By Theorem 2 (c), $\{x_1, \ldots, x_n\} \in \mathcal{G}_n(X) \cap U$. Hence, $\mathcal{G}_n(X)$ is dense in $F_n(X)$.

Sufficiency. Let $U$ be a nonempty open subset of $X$. Then $(U) \cap F_n(X)$ is a nonempty open subset of $F_n(X)$. Let $A \in \langle U \rangle \cap F_n(X) \cap \mathcal{G}_n(X)$. Given $a \in A$, by Theorem 2 (a), $a \in U \cap W(X)$. Therefore, $W(X)$ is dense in $X$.

**COROLLARY 4.** Let $n \geq 2$. Let $X$ and $Y$ be continua such that $F_n(X)$ is homeomorphic to $F_n(Y)$. Then $X$ is wired if and only if $Y$ is wired.

**Proof.** Let $h : F_n(X) \rightarrow F_n(Y)$ be a homeomorphism. Since $\mathcal{G}_n(X)$ is defined by a topological property, we have that $h(\mathcal{G}_n(X)) = \mathcal{G}_n(Y)$. Thus, this corollary follows from Corollary 3.

**THEOREM 5.** Suppose that $n \geq 4$, $X$ is a wired continuum, $Y$ a continuum and $h : F_n(X) \rightarrow F_n(Y)$ is a homeomorphism. Then $h(F_1(X)) = F_1(Y)$.

**Proof.** By Corollary 4, $Y$ is wired. In order to show that $h(F_1(X)) = F_1(Y)$, by the symmetry of the roles of $X$ and $Y$, it is enough to show that $h(F_1(X)) \subset F_1(Y)$. Since $X$ is wired, we only need to show that if $p \in W(X)$, then $h(\{p\}) \in F_1(Y)$. Take $p \in W(X)$. Then there exists an open subset $U$ of $X$ such that the component $\alpha$ of $U$ containing $p$ is homeomorphic either to $[0, 1]$ or $(0, 1)$. Let $B = h(\{p\})$. Suppose that $B = \{b_1, \ldots, b_m\}$. We only need to prove that $m = 1$.

Let $\varepsilon > 0$ be such that the sets $B(\varepsilon, b_1), \ldots, B(\varepsilon, b_m)$ are pairwise disjoint. Let $\delta > 0$ be such that $2\delta < \text{diameter}(X)$, $B(2\delta, p) \subset U$ and $h(B^H(2\delta, \{p\}) \cap F_n(X)) \subset B^H(\varepsilon, B) \cap F_n(Y)$. Let $\beta$ be the component of $B(\delta, p)$ containing $p$. Note that $\beta \subset \alpha$. Since $\beta$ is not compact, $\beta$ is homeomorphic to either $[0, 1]$ or $(0, 1)$. Let $C$ be the component of $B^H(\delta, \{p\}) \cap F_n(X)$ containing $\{p\}$. Let $C = \bigcup\{D : D \in C\}$. By Lemma 1, $C$ is a connected subset of $X$, $p \in C$ and $C \subset B(\delta, p)$. This implies that $C \subset \beta$. Since the set $\{x \in \beta\}$ is a connected subset of $B^H(\delta, \{p\}) \cap F_n(X)$ containing $\{p\}$, we have $\{x \in \beta\} \subset C$. Thus $\beta \subset C$ and $C = \beta$. This implies that $C \subset \beta \cap F_n(X)$. Since $\langle \beta \rangle \cap F_n(X)$ is a connected subset of $B^H(\delta, \{p\}) \cap F_n(X)$ containing $\{p\}$, we conclude that $C = \langle \beta \rangle \cap F_n(X)$. Let $\beta_0 = \text{cl}_X(\beta)$. Note that $\beta_0 \subset B(2\delta, p) \subset U$, so $\beta_0 \subset \alpha$ and $\beta_0$ is an arc.

Since $\beta$ is homeomorphic to an interval of the real line, the set $\mathcal{E} = \{A \in F_n(X) - F_{n-1}(X) : A \subset \beta\} \subset C$ is nonempty and arcwise connected. Let $\mathcal{D}$ be
the component of \( h(B^H(\delta, \{p\}) \cap F_n(X)) \) containing \( B \). Then \( D = h(C) \). Thus, by Theorem 2, \( D \cap G_n(Y) = h(C \cap G_n(X)) = h((\beta \cap F_n(X)) \cap G_n(X)) = h(\mathcal{E}) \) is arcwise connected.

Given an element \( E \in \mathcal{E}, h(E) \in B^H(\varepsilon, B) \cap F_n(Y) \cap G_n(Y) \). Thus, \( h(E) \) is a set with exactly \( n \) points (Theorem 2 (b)), \( h(E) \subset W(Y) \) and \( h(E) \in \langle B(\varepsilon, b_1), \ldots, B(\varepsilon, b_m) \rangle \cap F_n(Y) \).

**Claim 1.** \( m = 1 \) or \( m = n \).

In order to prove Claim 1, suppose to the contrary that \( 1 < m < n \).

Let \( \gamma \) be a subarc of \( \beta \) such that \( p \in \gamma \). Fix a subset with exactly \( n \) points \( Q = \{ q_1, \ldots, q_n \} \) of \( \gamma \). Choose a continuous function \( g : [0, 1] \to F_n(\gamma) \) such that \( g(0) = Q, g(1) = \{ p \} \) and \( g(t) \) has exactly \( n \) elements for each \( t \in [0, 1) \). Then for each \( t \in [0, 1), g(t) \in \mathcal{E} \) and \( h(g(t)) \in G_n(Y) \). Let \( G = \bigcup \{ h(g(t)) : t \in [0, 1] \} \).

By [6, Lemma 2.2] and Lemma 1, \( G \) is a locally connected space with at most \( m \) components. By the previous paragraph, \( G \in \langle B(\varepsilon, b_1), \ldots, B(\varepsilon, b_m) \rangle \cap F_n(Y) \).

Thus, \( G \) has exactly \( m \) components. In fact, the components of \( G \) are the sets \( G_1 = G \cap B(\varepsilon, b_1), \ldots, G_m = G \cap B(\varepsilon, b_m) \).

Given \( i \in \{1, \ldots, m\} \), we consider two cases. In the case that \( b_i \in W(Y) \), we take a wire \( \beta_i \) containing \( b_i \) and let \( \lambda_i \) be an arc contained in \( \beta_i \cap B(\varepsilon, b_i) \) such that \( b_i \) is an end point of \( \lambda_i \). In the case that \( b_i \notin W(Y) \), since \( h(g(0)) \subset G \cap W(Y) \), there exists a point \( y \in h(g(0)) \cap B(\varepsilon, b_i) \subset G_i \). Since \( G_i \) is a locally connected continuum, there exists an arc \( \lambda_i \) in \( G_i \) with end points \( y \) and \( b_i \). Since for each \( t < 1, h(g(t)) \subset W(Y) \), \( G_i - \{ b_i \} \subset W(Y) \). Thus, \( \lambda_i - \{ b_i \} \subset W(Y) \). In both cases we have constructed an arc \( \lambda_i \) in \( B(\varepsilon, b_i) \) such that \( \lambda_i - \{ b_i \} \subset W(Y) \) and \( b_i \) is an end point of \( \lambda_i \). Since \( h(B^H(\delta, \{ p \}) \cap F_n(X)) \) is an open subset of \( F_n(Y) \) containing \( B \), shrinking each \( \lambda_i \) if necessary, we can ask that \( (\lambda_1, \ldots, \lambda_m) \cap F_n(Y) \subset h(B^H(\delta, \{ p \}) \cap F_n(X)) \).

Given an element \( K \in \langle F_n(Y) - F_{n-1}(Y) \rangle \cap \langle \lambda_1 - \{ b_1 \}, \ldots, \lambda_m - \{ b_m \} \rangle \), we have that \( K \in G_n(Y) \). Since each \( \lambda_i \) is an arc, it is possible to find a continuous function \( \eta : [0, 1] \to \langle \lambda_1, \ldots, \lambda_m \rangle \cap F_n(Y) \) such that \( \eta(0) = K, \eta(1) = B \) and for each \( t \in [0, 1), \eta(t) \in (F_n(Y) - F_{n-1}(Y)) \cap (\langle \lambda_1 - \{ b_1 \}, \ldots, \lambda_m - \{ b_m \} \rangle \subset h(B^H(\delta, \{ p \}) \cap F_n(X)) \). Hence, \( \Im \eta \) is a subcontinuum of \( h(B^H(\delta, \{ p \}) \cap F_n(X)) \) containing \( B \). Thus, \( \Im \eta \subset D = h(C) \). Given \( t \in [0, 1], \) since \( \eta(t) \in G_n(Y) \), we obtain that \( \eta(t) \in D \cap G_n(Y) = h(\mathcal{E}) \). We have proved that \( (F_n(Y) - F_{n-1}(Y)) \cap (\langle \lambda_1 - \{ b_1 \}, \ldots, \lambda_m - \{ b_m \} \rangle \subset h(\mathcal{E}) \).

We are ready to obtain a contradiction by proving that \( h(\mathcal{E}) \) is not connected. Let \( \mathcal{K}_1 = \{ K \in h(\mathcal{E}) : K \cap B(\varepsilon, b_1) \) has exactly \( n - m + 1 \) elements \} and \( \mathcal{K}_2 = \{ K \in h(\mathcal{E}) : K \cap (B(\varepsilon, b_2) \cup \ldots \cup B(\varepsilon, b_m)) \) has at least \( m \) elements \}.

Given an element \( K \in \mathcal{K}_1 \) and an element \( L \in h(\mathcal{E}) \) which is close enough to \( K \), we have that \( L \cap B(\varepsilon, b_1) \) has at least \( n - m + 1 \) elements. Since \( L \)
intersects each one of the sets $B(\varepsilon, b_1), \ldots, B(\varepsilon, b_m)$, we obtain that $L$ has at least $n - m + 1 + m - 1 = n$ elements. Since $L \in F_n(Y)$, we conclude that $L \cap B(\varepsilon, b_1)$ exactly $n - m + 1$ elements. This proves that $K_1$ is open in $h(\mathcal{E})$. Clearly, $K_2$ is also open in $h(\mathcal{E})$, $h(\mathcal{E}) = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. By the paragraph above we can construct a set $K \in K_1$ by taking $n - m + 1$ elements in $\lambda_1 \setminus \{b_1\}$ and one element in each one of the sets $\lambda_2 \setminus \{b_2\}, \ldots, \lambda_m \setminus \{b_m\}$. Thus, $K_1 \neq \emptyset$. Similarly, $K_2 \neq \emptyset$. We have obtained a separation of the set $h(\mathcal{E})$. Hence, $h(\mathcal{E})$ is not connected. This contradiction finishes the proof of Claim 1.

**Claim 2.** $m = 1$.

In order to prove Claim 2, suppose to the contrary that $m = n$. Let $M = \bigcup \{G : G \in h(\langle \beta_0 \rangle \cap F_n(X))\}$. By [6, Lemma 2.2] and Lemma 1, $M$ is a locally connected connected space with at most $n$ components. Since $\beta_0 \subset B(2\varepsilon, p)$, $h(\langle \beta_0 \rangle \cap F_n(X)) \subset B^{H}(\varepsilon, B) \cap F_n(X)$, so $M \subset (B(\varepsilon, b_1), \ldots, B(\varepsilon, b_n)) \cap F_n(Y)$, $M$ has exactly $n$ components and they are $M_1 = M \cap B(\varepsilon, b_1), \ldots, M_n = M \cap B(\varepsilon, b_n)$. Thus, each $M_i$ is a locally connected continuum. Let $S = \bigcup \{h(A) : A \in \mathcal{E}\}$. Then the components of $S$ are the sets $S_1 = S \cap B(\varepsilon, b_1), \ldots, S_n = S \cap B(\varepsilon, b_n)$ and $S_1 \subset M_1, \ldots, S_n \subset M_n$.

Take an element $A_0 \in \mathcal{E}$, then $h(A_0) \in W(Y)$ and by Lemma 1 (c), it contains exactly one point in each one of sets $S_1, \ldots, S_n$. Fix $i \in \{1, \ldots, n\}$ for the rest of the proof. Let $y$ be such that $h(A_0) \cap S_i = \{y\}$. Since $y \in W(Y)$, there exists an open subset $V$ of $Y$ such that the component of $V$ containing $y$ is a wire in $Y$. Since $A_0 \subset B^{H}(\delta, \{p\}) \cap F_n(X)$, $h(A_0) \subset h(B^{H}(\delta, \{p\}) \cap F_n(X))$. Thus there exists $\delta_0 > 0$ such that $B^{H}(\delta_0, h(A_0)) \subset h(B^{H}(\delta, \{p\}) \cap F_n(X))$ and $h(\delta_0, y) \subset V \cap B(\varepsilon, b_i)$. Then the component $\eta$ of $B(\delta_0, y)$ containing $y$ is a wire in $Y$. Consider the set $\mathcal{R} = \{(h(A_0) - \{y\}) \cup \{z\} : z \in \eta\}$. Note that $\mathcal{R}$ is connected, $\mathcal{R} \subset W(Y)$ and $h(B^{H}(\delta, \{p\}) \cap F_n(X))$ and each element of $\mathcal{R}$ contains exactly $n$ elements. Thus, $\mathcal{R} \subset G_n(Y)$. Hence, $h^{-1}(\mathcal{R})$ is a connected subset of $G_n(X) \cap B^{H}(\delta, \{p\}) \cap F_n(X)$ containing $A_0$. Since $A_0 \subset h^{-1}(\mathcal{R}) \cap C \subset B^{H}(\delta, \{p\}) \cap F_n(X)$ and $C$ is a component of $B^{H}(\delta, \{p\}) \cap F_n(X)$, we have $h^{-1}(\mathcal{R}) \subset C$. Hence, $h^{-1}(\mathcal{R}) \subset G_n(X) \cap \langle \beta \rangle \cap F_n(X) \subset \mathcal{E}$. We have shown that for each $z \in \eta$, there exists $A \in \mathcal{E}$ such that $h(A) = (h(A_0) - \{y\}) \cup \{z\}$. This implies that $z \in S_i$. Thus, $\eta \subset S_i \subset M_i$. In particular, $M_i$ is nondegenerate. We claim that $\eta$ is a neighborhood of $y$ in $S_i$. Since $M_i$ is locally connected, there exists an open connected subset $Q$ of $M_i$ such that $y \in Q \subset M_i \cap B(\delta_0, y)$. Then $Q \subset \eta \subset S_i \subset M_i$ and $\eta$ is a neighborhood of $y$ in $S_i$. It follows that $S_i$ is open in $M_i$ and each element $y \in S_i$ has a neighborhood in $S_i$ which is homeomorphic to either $[0, 1)$ or $(0, 1)$. This proves that $S_i$ is a 1-dimensional manifold. By [8, Appendix 2, p. 208], $S_i$ is homeomorphic to either $[0, 1)$, $(0, 1)$ or $S^1$. Since $\beta$ is dense in $\beta_0$ and $F_{n-1}(\beta)$ is dense in $F_n(\beta)$, it follows that $S_i$ is dense in $M_i$. Hence, $M_i$ is a locally connected compactification of $S_i$. This implies that $M_i$ is either an arc or a simple closed curve.
Let $\delta_1 > 0$ be such that $B^H(\delta_1, B) \cap F_n(Y) \subset h(B^H(\delta, \{p\}) \cap F_n(X))$, $\delta_1 < \varepsilon$ and $M_i \not\subset B(\delta_1, b_i)$. Let $D_i$ be the component of $B(\delta_1, b_i)$ containing $b_i$. Let $\mathcal{M}_i = \{(B - \{b_i\}) \cup \{z\} \in F_n(Y) : z \in D_i\}$. Then $\mathcal{M}_i$ is a connected subset of $B^H(\delta_1, B) \cap F_n(Y) \subset h(B^H(\delta, \{p\}) \cap F_n(X))$ containing $B$. Thus, $h^{-1}(\mathcal{M}_i)$ is a connected subset of $B^H(\delta, \{p\}) \cap F_n(X)$ containing $\{p\}$. Hence, $h^{-1}(\mathcal{M}_i) \subset C$ and $\mathcal{M}_i \subset h(C) \subset h((\beta) \cap F_n(X)) \subset h((\beta_0) \cap F_n(X))$. This implies that $D_i \subset M_i$. Since $D_i \neq M_i$ and $D_i$ is nondegenerate and connected, we have $D_i$ is homeomorphic either to $(0, 1)$ or $(0, 1)$. This shows that $D_i$ is a wire and $b_i \in W(Y)$.

Since $B$ has $n$ points, $B \in G_n(Y)$ and $\{p\} \in h^{-1}(G_n(Y)) = G_n(X)$. This contradicts Theorem 2 (b) and completes the proof that $m = 1$. ■

**COROLLARY 6.** Let $n \geq 4$ and let $X$ be a wired continuum. Then $X$ has unique hyperspace $F_n(X)$.

**COROLLARY 7.** Let $n \geq 4$ and let $X$ be a wired continuum. Then $F_n(X)$ is rigid.

It is known [3] that finite graphs have unique hyperspace $F_n(X)$ for each $n \geq 2$. It is also known that dendrites with closed set of end points have unique hyperspace $F_n(X)$ for each $n \geq 2$ ([1] and [11]). So Corollary 6 extends these results for $n \geq 4$. Notice that Corollary 6 includes dendrites for which the union of free arcs is dense. This family properly contains the family of dendrites with closed set of end points.

Corollary 7 cannot be extended to $n = 2$ or $n = 3$. The nonrigidity of $F_2([0, 1])$ follows from the fact that there exists a homeomorphism $h$ from $F_2([0, 1])$ onto a 2-cell $D$ that sends $F_1([0, 1])$ onto an arc in the manifold boundary of $D$. For a more general result see Theorem 11. The non-rigidity of $F_3([0, 1])$ follows from the fact that there exists a homeomorphism $h : F_3([-1, 1]) \to D^3$, where $D^3 = \{z \in \mathbb{R}^3 : |z| \leq 1\}$ such that $h\{x\} = (x, 0, 0)$ for each $x \in [-1, 1]$ (see [15, Example K]).

**COROLLARY 8.** Let $n \geq 4$ and let $X$ be compactification of the real line. Then $F_n(X)$ is rigid and $X$ has unique hyperspace $F_n(X)$.

Lemma 2 of [10] and Theorem 5 imply the following.

**COROLLARY 9.** Let $X$ be an indecomposable arc continuum and let $n \geq 4$. Then $F_n(X)$ is rigid and $X$ has unique hyperspace $F_n(X)$.

Let $X$ be a smooth fan (see section 6 for definition) with vertex $v$. Given a point $p \in X - \{v\}$, it is easy to see that $p \in W(X)$. So $X$ is wired, as a consequence we have the following result.
THEOREM 10. If $X$ is a smooth fan, then
(a) $X$ is a wired continuum,
(b) if $n \geq 4$, then $F_n(X)$ is rigid,
(c) if $n \geq 4$, then $X$ has unique hyperspace $F_n(X)$.

In Theorem 20, we will give a characterization of smooth fans $X$ for which $F_2(X)$ is rigid. We do not know what happens for $F_3(X)$, see Question 21.

4. PEANO CONTINUA

THEOREM 11. If a continuum $X$ contains a tail, then $F_2(X)$ is not rigid.

Proof. Let $\alpha$, $a$ and $b$ be as in the definition of a tail. We may assume that $\alpha \not\in X$. Then $a$ is a cut point of $X$, so $E = X - (\alpha - \{a\})$ is a subcontinuum of $X$. Let $\Delta$ be the solid triangle in the Euclidean plane $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$. Given two different points $p, q \in \mathbb{R}^2$, let $pq$ denote the convex segment joining them. By [15, Example K], there exists a homeomorphism $f : F_2(\alpha) \to \Delta$ such that $f(\{a, x\} : x \in \alpha) = (0, 0)(0, 1)$, $f(\{x\} : x \in \alpha) = (0, 0)(1, 1)$ and $f(\{b, x\} : x \in \alpha) = (0, 1)(1, 1)$. Let $k : \Delta \to \Delta$ be a homeomorphism such that $k(0, 0)(0, 1)$ is the identity map and $k(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 1)$. Let $h : F_2(\alpha) \to F_2(X)$ be given by

$$h(A) = \begin{cases} A, & \text{if } A \cap E \neq \emptyset, \\ f^{-1}(k(f(A))), & \text{if } A \subset \alpha. \end{cases}$$

Clearly, $h$ is a homeomorphism and $h(F_1(X)) \not\subset F_1(X)$. □

THEOREM 12. Let $X$ be an almost meshed. Then $F_2(X)$ is rigid if and only if $X$ does not contain tails.

Proof. The necessity was proved in Theorem 11. Now, suppose that $X$ does not contain tails. Let $h : F_2(X) \to F_2(X)$ be a homeomorphism. Let $p \in \alpha - \{a, b\}$, where $\alpha$ is a free arc in $X$ with end points $a$ and $b$. Since the set of such points $p$ is dense in $X$, it is enough to show that $h(\{p\}) \in F_1(X)$. Suppose to the contrary that $h(\{p\}) = \{x, y\}$, where $x \neq y$. Let $U = (\alpha - \{a, b\}) \cap F_2(X)$. Then $U$ is an open subset of $F_2(X)$ containing $\{p\}$. By [15, Example K] there is a homeomorphism from $U$ to the space $C = [0, 1] \times [0, 1]$ that takes $\{p\}$ to the point $(0, 0)$. Hence, $\{x, y\}$ has a neighborhood $\mathcal{M}$ in $F_2(X)$ such that $\mathcal{M}$ is a 2-cell and $\{x, y\}$ belongs to the manifold boundary of $\mathcal{M}$. Let $U$ and $V$ be disjoint open subsets of $X$ such that $x \in U$, $y \in V$ and $(U, V) \cap F_2(X) \subset \mathcal{M}$. Since $(U, V) \cap F_2(X)$ is an open subset of $\mathcal{M}$, there exists a 2-cell $\mathcal{N}$ such that $\{x, y\} \in \mathcal{M} \cap \mathcal{N} \subset \mathcal{M} \cap (\mathcal{U}, V) \cap F_2(X)$. Let $\varepsilon > 0$ be such that $B^H(\varepsilon, \{x, y\}) \cap \mathcal{M} \subset \mathcal{N}$. Since we can ask that $B^H(\varepsilon, \{x, y\}) \subset \mathcal{M}$, we obtain that $B^H(\varepsilon, \{x, y\}) \subset \mathcal{N}$. Let $\pi_1 : \mathcal{N} \to U$ be given by $\pi_1(E)$ is the unique point
in $E \cap U$. It is easy to show that $\pi_1$ is continuous. Let $A = \pi_1(\mathcal{N})$. Then $A$ is a Peano continuum such that $B(x, \overline{x}) \subset A$. Thus, $x \in \operatorname{int}_X(A)$. Similarly, the set $B = \pi_2(\mathcal{N})$, where $\pi_2 : \mathcal{N} \rightarrow V$ is given by $\pi_2(E)$ is the unique point in $E \cap V$, is a Peano continuum such that $y \in \operatorname{int}_X(B)$. Notice that $A \cap B = \emptyset$ and $(A, B) \cap F_2(X) \subset (U, V) \cap F_2(X) \subset \mathcal{M}$.

If $A$ contains a simple triod $T$, since $B$ contains an arc $\beta$, we have that $\mathcal{M}$ contains the set $\{(u, v) : u \in T \text{ and } v \in \beta\}$ which is homeomorphic to the product $T \times \beta$. This contradicts the Invariance of Domain Theorem [12, Theorem VI 9] and proves that $A$ does not contain simple triods. Similarly, $B$ does not contain simple triods. Thus, $A$ and $B$ are arcs or simple closed curves. Taking smaller continua, if necessary, we can assume that $A$ and $B$ are arcs. If $x$ is an end point of $A$, since $x \in \operatorname{int}_X(A)$, then it is possible to find a tail of $X$ inside $A$, a contradiction. Similarly, $y$ is not an end point of $B$. Let $U$ and $V$ open subsets of $X$ such that $U$ and $V$ are homeomorphic to $(0, 1)$, $x \in U \subset A$ and $y \in V \subset B$. Then $(U, V) \cap F_2(X) \subset \mathcal{M}$. Since $(U, V) \cap F_2(X)$ is homeomorphic to $(0, 1) \times (0, 1)$, the Invariance of Domain Theorem implies that $\{x, y\}$ is not in the manifold boundary of $\mathcal{M}$. This contradiction completes the proof of the theorem. ■

**COROLLARY 13.** A finite graph $X$ has rigid hyperspace $F_2(X)$ if and only if $X$ does not have end points.

**THEOREM 14.** If a continuum $X$ contains a free arc, then $F_3(X)$ is not rigid.

**Proof.** Let $\alpha$ be a free arc that joins the points $a$ and $b$. We may assume that $\alpha \neq X$. Let $g : \alpha \rightarrow [-1, 1]$ be a homeomorphism. Let $E = X - (\alpha - \{a, b\})$. Then $E$ is a nonempty compact subset of $X$. By [15, Example K], there exists a homeomorphism $k : F_3(\alpha) \rightarrow D^3$, where $D^3 = \{z \in \mathbb{R}^3 : |z| \leq 1\}$ such that $k(\alpha) = g(\alpha, 0, 0)$ for each $x \in \alpha$ and $k(\{A \in F_3(X) : A \cap \{a, b\} \neq \emptyset\})$ is the manifold boundary of $D^3$. Let $f : D^3 \rightarrow D^3$ be a homeomorphism that is the identity in the boundary of $D^3$ and $f(\frac{1}{2}, 0, 0) \notin \{(t, 0, 0) \in D^3 : t \in [-1, 1]\}$. Let $h : F_3(X) \rightarrow F_3(X)$ be given by

$$h(A) = \begin{cases} A, & \text{if } A \cap E \neq \emptyset, \\ k^{-1}(f(k(A))), & \text{if } A \subset \alpha. \end{cases}$$

Clearly, $h$ is a homeomorphism and $h(F_1(X)) \nsubseteq F_1(X)$. ■

We summarize the results of this section for the particular case when $X$ is a finite graph.

**THEOREM 15.** Let $X$ be a finite graph. Then
(a) if $n \geq 4$, then $F_n(X)$ is rigid,
(b) $F_3(X)$ is not rigid,
(c) $F_2(X)$ is rigid if and only if $X$ does not have end points.

**QUESTION 16.** Suppose that $X$ is a wired continuum. Is it true that $F_3(X)$ is not rigid? It would be interesting to determine if $F_3(X)$ is rigid for the Buckethandle continuum (see [20, 2.8] for definition) or for solenoids (see [20, 2.9] for definition). It is also unknown if $F_2(X)$ is rigid when $X$ is an indecomposable arc continuum.

5. MANIFOLDS

Section 4 has been mainly devoted to wired continua. They have many arcs that are components of neighborhoods. Manifolds of dimension $\geq 2$ are very far from being wired continua. However we can also prove a result about rigidity of symmetric products of $m$-manifolds. We only consider the case that $m \geq 2$, since the case $m = 1$ is included in Theorem 15. Here, the word manifolds refers to compact connected manifolds with or without boundary.

**THEOREM 17.** Let $X$ be an $m$-manifold, $m \geq 2$ and $n \geq 3$. Then $F_n(X)$ is rigid.

**Proof.** Let $D_n(X) = \{ A \in F_n(X) : A$ has a neighborhood in $F_n(X)$ that is an $nm$-cell $\}.

First, we are going to show that $D_n(X) = F_n(X) - F_{n-1}(X)$. Let $A = \{ a_1, \ldots, a_n \} \in F_n(X) - F_{n-1}(X)$. For each $i \in \{1, \ldots, n\}$, let $M_i$ be a neighborhood of $a_i$ such that $M_i$ is an $m$-cell. We may ask that $M_1, \ldots, M_n$ are pairwise disjoint. Since $M_1 \times \ldots \times M_n$ is homeomorphic to $(M_1, \ldots, M_n) \cap F_n(X)$ (under the homeomorphism $(x_1, \ldots, x_n) \rightarrow \{x_1, \ldots, x_n\}$) and $(M_1, \ldots, M_n) \cap F_n(X)$ is a neighborhood of $A$ in $F_n(X)$, we conclude that $A \in D_n(X)$.

Now, take $A \in D_n(X)$ and suppose that $A = \{a_1, \ldots, a_r\}$, where $r < n$. Let $M$ be a neighborhood of $A$ in $F_n(X)$ such that $M$ is an $nm$-cell. Since arbitrarily close to $A$ there are sets with exactly $n - 1$ points, we will assume that $r = n - 1$. For each $i \in \{1, \ldots, n - 1\}$, let $M_i$ be a neighborhood of $a_i$ such that $M_i$ is an $m$-cell. We may ask that $M_1, \ldots, M_{n-1}$ are pairwise disjoint and $\mathcal{N} = (M_1, \ldots, M_{n-1}) \cap F_n(X) \subset M$. Let $\mathcal{R}$ be an $nm$-cell such that $A \in \text{int}_{F_n(X)}(\mathcal{R}) \subset \mathcal{R} \subset \mathcal{N}$. We will see that $\mathcal{S} = (M_1, \ldots, M_{n-1}) \cap F_{n-1}(X) \cap \mathcal{R}$ separates $\mathcal{R}$. Since $(M_1, \ldots, M_{n-1}) \cap F_{n-1}(X)$ is homeomorphic to $M_1 \times \ldots \times M_{n-1}$ which is an $(n - 1)m$-cell and $(n - 1)m \leq nm - 2$, we will have that $\dim[\mathcal{S}] \leq nm - 2$. This will be a contradiction with [12, Theorem IV 4].

Let $\mathcal{K} = \{ B \in \mathcal{R} : B \cap (M_1 \cup \ldots \cup M_{n-2})$ has exactly $n - 1$ elements $\}$ and $\mathcal{L} = \{ B \in \mathcal{R} : B \cap M_{n-1}$ has exactly 2 elements $\}$. Given $B \in \mathcal{R} - \mathcal{S}$,
Hence, \( R - S \) intersects each \( B \). Since \( M_1, \ldots, M_{n-1} \) are pairwise disjoint, \( B \in \mathcal{K} \cup \mathcal{L} \). We have shown that \( \mathcal{R} - \mathcal{S} \subset \mathcal{K} \cup \mathcal{L} \). Since for each \( B \in \mathcal{K} \cup \mathcal{L} \), \( B \) has \( n \) elements, we have that \( B \in R - \mathcal{S} \). Hence, \( R - \mathcal{S} = \mathcal{K} \cup \mathcal{L} \). Clearly, \( \mathcal{K} \) and \( \mathcal{L} \) are open in \( \mathcal{R} \) and \( \mathcal{K} \cap \mathcal{L} = \emptyset \). Since \( M_{n-1} \) is an \( m \)-cell, we can choose a point \( x \in M_{n-1} - \{ a_{n-1} \} \) satisfying \( \mathcal{A} \cup \{ x \} \in \mathcal{R} \). Then \( \mathcal{A} \cup \{ x \} \in \mathcal{L} \). Hence, \( \mathcal{L} \neq \emptyset \). Similarly, \( \mathcal{K} \neq \emptyset \). We have shown that \( \mathcal{S} \) separates \( \mathcal{R} \). With this contradiction, we have proved that \( \mathcal{D}_n(X) \subset F_n(X) - F_{n-1}(X) \). Therefore, \( \mathcal{D}_n(X) = F_n(X) - F_{n-1}(X) \).

Let \( \mathcal{E} = \{ A \in F_n(X) - \mathcal{D}_n(X) : A \) has a basis of neighborhoods \( \mathcal{B} \) in \( F_n(X) \) such that for each \( \mathcal{U} \in \mathcal{B}, \mathcal{U} \cap \mathcal{D}_n(X) \text{ is connected} \} \). We claim that \( \mathcal{E} = F_1(X) \). Given \( A = \{ x \} \in F_1(X) \), let \( M \) be a neighborhood of \( x \) in \( X \) such that \( M \) is an \( m \)-cell. Let \( \mathcal{U} = (M) \cap F_n(X) \). Then \( \mathcal{U} \) is a neighborhood of \( A \) in \( F_n(X) \). Clearly, \( F_n(M) - F_n(A) \) is arcwise connected. Thus, \( \mathcal{U} \cap \mathcal{D}_n(X) \) is connected. Since the family of neighborhoods like \( \mathcal{U} \) is a basis of \( A \) in \( F_n(X) \), we have shown that \( F_1(X) \subset \mathcal{E} \). Now, let \( A \in \mathcal{E} \). Suppose that \( A \notin F_1(X) \). Let \( A = \{ a_1, \ldots, a_r \} \), where \( a_1, \ldots, a_r \) are all different. Then \( 1 < r < n \). Let \( \varepsilon > 0 \) be such that \( B(2\varepsilon, a_1), \ldots, B(2\varepsilon, a_r) \) are pairwise disjoint. Given a neighborhood \( \mathcal{U} \) of \( A \) in \( F_n(X) \) such that \( \mathcal{U} \subset B^R(\varepsilon, A) \cap F_n(X) \), we have \( A \in \mathcal{U} \subset (B(\varepsilon, a_1), \ldots, B(\varepsilon, a_r)) \cap F_n(X) \). Let \( \mathcal{G} = \{ B \in \mathcal{U} \cap \mathcal{D}_n(X) : B \cap B(\varepsilon, a_1) \) has exactly \( n - r + 1 \) elements \} and \( \mathcal{H} = \{ B \in \mathcal{U} \cap \mathcal{D}_n(X) : B \cap B(\varepsilon, a_2) \cup \ldots \cup B(\varepsilon, a_r) \) has at least \( r \) elements \}. Proceeding as in the paragraph above it can be proved that \( \mathcal{U} \cap \mathcal{D}_n(X) = \mathcal{G} \cup \mathcal{H} \) and \( \mathcal{G} \) and \( \mathcal{H} \) are nonempty open disjoint subsets of \( \mathcal{U} \cap \mathcal{D}_n(X) \). Hence, \( \mathcal{U} \cap \mathcal{D}_n(X) \) is not connected. This contradicts the fact that \( A \in \mathcal{E} \) and proves that \( \mathcal{E} \subset F_1(X) \). Hence, \( \mathcal{E} = F_1(X) \).

Since \( \mathcal{D}_n(X) \) and \( \mathcal{E} = F_1(X) \) are characterized with topological properties in \( F_n(X) \), it follows that for each homeomorphism \( h : F_n(X) \to F_n(X) \), \( h(F_1(X)) \neq F_1(X) \). Therefore, \( F_n(X) \) is rigid.

The following example shows that Theorem 17 cannot be extended to \( n = 2 \).

**EXAMPLE 18.** Let \( X = [0, 1]^2 \). R. Molski [18, Theorem 1] showed that \( F_2(X) \) is a 4-cell. Since \( F_1(X) \) is a 2-cell in \( F_2(X) \), \( F_1(X) \) cannot cover either the manifold boundary of \( F_2(X) \) nor the manifold interior of \( F_2(X) \). Thus, there exists a homeomorphism \( h : F_2(X) \to F_2(X) \) such that \( h(F_1(X)) \neq F_1(X) \). Hence \( X \) is a 2-manifold and \( F_2(X) \) is not rigid.

### 6. SMOOTH FANS

Given a dendroid \( X \) and points \( p, q \in X \), we denote the unique arc joining \( p \) and \( q \) in \( X \) by \( pq \), if \( p \neq q \), and \( pq = \{ p \} \), if \( p = q \). A fan is a dendroid \( X \) with exactly one ramification point \( v \), called the vertex of \( X \). The fan \( X \) with vertex
$v$ is said to be smooth provided that for each sequence $\{x_i\}_{i=1}^{\infty}$ converging to a point $x \in X$ we have $\lim v x_i = v x$. It is known that the class of smooth fans coincides with the class of subcontinua of the Cantor fan that are not arcs [7, Corollary 4].

This section is devoted to characterize those smooth fans $X$ for which $F_2(X)$ is rigid.

Throughout this section the letter $X$ will denote a smooth fan with vertex $v$ and set of end points $E$. We suppose that $X$ is contained in the cone over the Cantor middle-third set contained in the interval $[0,1] \times \{0\}$ and $v = (\frac{1}{2},1)$. Let $\pi : X \to [0,1]$ be the projection on the second coordinate.

Let $K = \{ A \in F_2(X) : v \in A \}$. Given $e_1, e_2 \in E$, let $J(e_1, e_2) = \langle ve_1 - \{v\}, ve_2 - \{v\}\rangle \cap F_2(X)$. In the case that $e_1 = e_2$, we put $J(e_1, e_2) = J(e_1)$. Notice that

$$F_2(X) = K \cup (\bigcup \{ J(e_1, e_2) : e_1, e_2 \in E \}).$$

**Lemma 19.** Let $h : F_2(X) \to F_2(X)$ be a homeomorphism. Then there exists a one-to-one onto function $h_1 : E \to E$ (not necessarily continuous) such that

- (a) $F_2(X) = G_2(X) \cup K$ and $G_2(X) \cap K = \emptyset$,
- (b) $h(K) = K$,
- (c) $h(\{v\}) = \{v\}$,
- (d) for each $e \in E$, $h(J(e)) = J(h_1(e))$,
- (e) for each $e \in E$, $h(F_1(ve)) \cup \{x,e\} : x \in ve \} \subset F_1(ves) \cup \{x,h_1(e)\} : x \in ves \} \cup \{x,h_1(e)\} : x \in ves \}$

**Proof.** (a). Since $X$ is a subcontinuum of the Cantor fan, $X - \{v\} \subset W(X)$. By Theorem 2 (c), if $A \in F_2(X) - F_1(X)$ and $v \notin A$, then $A \in G_2(X)$. In the case that $A = \{x\} \subset F_1(X)$ and $x \neq v$, it is easy to show that $\{x\} \subset G_2(X)$. We have shown that $F_2(X) - G_2(X) \subset K$ and $F_2(X) = G_2(X) \cup K$. Now we check that $G_2(X) \cap K = \emptyset$. Suppose that there exists $A = \{v,x\} \subset G_2(X) \cap K$. Let $M$ be a neighborhood of $A$ in $F_2(X)$ such that the component $C$ of $M$ containing $A$ is a 2-cell. Since $C$ is nondegenerate, we can assume that $x \neq v$. Since $v$ is a ramification point of $X$, there exists a simple triod $T = J_1 \cup J_2 \cup J_3$ such that $v$ is an end point of each $J_i$ and $J_i \cup J_j = \{v\}$, if $i \neq j$. Let $J$ be an arc in $X$ containing $x$. We suppose that $J \cap T = \emptyset$ and $\langle T, J \rangle \cap F_2(X) \subset C$. Since $T \times J$ is homeomorphic to $(T, J) \cap F_2(X)$ (the map $(w,y) \to \{w,y\}$ is a homeomorphism), we have that $T \times J$ is embeddable in the 2-cell $C$. This contradicts the Invariance of Domain Theorem [12, Theorem VI 9] and completes the proof of (a).

(b). Since the definition of $G_2(X)$ is topological, we have that $h(G_2(X)) = G_2(X)$. By (a), we obtain that $h(K) = K$. 

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(c). Recall that $F_2(X) = K \cup (\{J(e_1, e_2) : e_1, e_2 \in E\})$. Notice that, if $e_1, e_2 \in E$, then $J(e_1, e_2)$ is a connected set disjoint from $K$. Since $X$ is contained in the Cantor fan, it follows that the components of $F_2(X) - K$ are the sets of the form $J(e_1, e_2)$, where $e_1, e_2 \in E$. Fix $e_1, e_2 \in E$, where $e_1 \neq e_2$. Then $\{v\} = \text{cl}_{F_2(X)}(J(e_1)) \cap \text{cl}_{F_2(X)}(J(e_2))$, $h(J(e_1)) = J(e_3, e_4)$ and $h(J(e_2)) = J(e_5, e_6)$, for some $e_3, e_4, e_5, e_6 \in E$. This implies that $h(\{v\}) = \text{cl}_{F_2(X)}(J(e_3, e_4)) \cap \text{cl}_{F_2(X)}(J(e_5, e_6))$ contains the set $\{\{\{v, x\} : x \in v\} \} = \{\{v\}\}$. Thus, $e_1 = e_2$. Now, we may define $h_1 : E \to E$ taking $h_1(e)$ to be the only element in $E$ such that $h(J(e)) = J(h_1(e))$. Clearly, $h_1$ is a one-to-one onto function.

(d). Let $e \in E$. Then $h(J(e)) = J(e_1, e_2)$ for some $e_1, e_2 \in E$. We claim that $e_1 = e_2$. Suppose to the contrary that $e_1 \neq e_2$. By [15, Example K], there is a homeomorphism $\varphi : F_2(ve) \to T$, where $T$ is the triangle in the plane $\mathbb{R}^2$ with vertices $(0, 0), (0, 1)$ and $(1, 1)$ such that $\varphi(F_1(ve)) = (0, 0)(1, 1)$ $(0, 0)(1, 1)$ is the convex segment that joins $(0, 0)$ and $(1, 1)$, $\varphi(\{x, e\} : x \in v - \{v\}) = (0, 0) - (0, 1), \varphi(\{\{v\}\} = (1, 1)$ and $\varphi(J(e)) = T - (0, 1)(1, 1)$. On the other hand, there exists a homeomorphism $\psi_0 : (ve_1, ve_2) \cap F_2(X) \to [0, 1]^2$ such that $\psi_0(\{e_1, e_2\}) = (0, 0), \psi_0(\{v\}) = (1, 1)$ and $\psi_0(J(e_1, e_2)) = [0, 1]^2$. Since $h(J(e)) = J(e_1, e_2)$ and $h(\{v\}) = \{v\}$, we conclude that $(T - (0, 1)(1, 1)) \cup \{(1, 1)\}$ is homeomorphic to $[0, 1]^2 \cup \{(1, 1)\}$. Clearly, this is a contradiction. Hence, $e_1 = e_2$. Using the map $\varphi$ defined in the paragraph above, it follows that $F_2(ve)$ is homeomorphic to a solid triangle $T$, the manifold boundary of $F_2(ve)$ consists of three arcs, namely: $F_1(ve), \{\{v, x\} : x \in v\}$ and $\{\{e, x\} : x \in v\}$. Moreover, $J(e) = F_2(ve) - \{\{v, x\} : x \in ve\}$. Thus, $J(e)$ is a manifold and its manifold boundary is $\partial(J(e)) = (F_1(ve) \cup \{\{e, x\} : x \in v\} - \{\{v\}, \{v, e\}\}$. Similarly, $J(h_1(e))$ is a manifold and $\partial(J(h_1(e))) = (F_1(ve_1) \cup \{\{v, x\} : x \in v_1\} - \{\{v\}, \{v, h_1(e)\}\}. Since $h$ is a homeomorphism, $h(\partial(J(e))) = \partial(J(h_1(e)))$ and $h(\text{cl}_{F_2(X)}(\partial(J(e)))) = \text{cl}_{F_2(X)}(\partial(J(h_1(e))))$. This implies that $h(F_1(ve) \cup \{\{e, x\} : x \in v\}) = h(\text{cl}_{F_2(X)}((F_1(ve) \cup \{\{e, x\} : x \in v\} - \{\{v\}, \{v, e\}\}) \subset F_1(ve_1) \cup \{\{v, h_1(e)\} : x \in ve_1\}). Since $h(\text{cl}_{F_2(X)}(\partial(J(e))))$ (respectively, $\text{cl}_{F_2(X)}(\partial(J(h_1(e))))$) is an arc with end points $\{v\}$ and $\{v, e\}$ (respectively, $\{v\}$ and $\{v, h_1(e)\}$). By (e), we conclude that $h(\{v, e\}) = \{v, h_1(e)\}$.

THEOREM 20. $F_2(X)$ is rigid if and only if $E$ does not have isolated points.

Proof. ($\Rightarrow$). Suppose that $e$ is an isolated point of $E$. We claim that $e \in \text{int}_X(ve)$. Suppose to the contrary that there exists a sequence $\{x_m\}_{m=1}^\infty$ of points in $X - ve$ converging to $e$. For each $m \in \mathbb{N}$, let $e_m \in E$ be such that $x_m \in ve_m$. Notice that $e_m \neq e$. We may suppose that $\lim e_m = x_0$ for some
Consider the respective natural orders preserves the orders. Since the minimum. By Lemma 19 (c), and $q \in F_1(x;e;e)$ such that $pe \in \text{int}_X(ve)$. It follows that $pe$ is a tail in $X$. By Theorem 11, $F_2(X)$ is not rigid. This ends the proof of the necessity.

($\Leftarrow$). Let $d$ be the Euclidean metric restricted to $X$. Suppose that $E$ does not have isolated points and $F_2(X)$ is not rigid. Then there exist a homeomorphism $h : F_2(X) \to F_2(X)$ and a point $x \in X$ such that $h(\{x\}) \notin F_1(X)$. Let $e \in E$ be such that $x \in ve$ and let $h_1$ be as in Lemma 19. Then $h(\{x\}) = \{y, h_1(e)\}$ for some $y \in vh_1(e) - \{h_1(e)\}$. Let $A = F_1(ve) \cup \{x, e\} : x \in ve$ and $B = F_1(wh_1(e)) \cup \{x, h_1(e)\} : x \in vh_1(e)$. Then $A$ and $B$ are arcs joining $\{v\}$ to $\{v, e\}$ and $\{v\}$ to $\{v, h_1(e)\}$, respectively. By Lemma 19 (e), $h(A) = B$. Consider the respective natural orders $<_A$ and $<_B$ in $A$ and $B$ for which $\{v\}$ is the minimum. By Lemma 19 (c), $h(\{v\}) = \{v\}$. Since $h$ is a homeomorphism, $h$ preserves the orders. Since $\{v\} \leq_A \{x\} \leq_A \{e\} <_A \{v, e\}$, we have $\{v\} <_B \{y, h_1(e)\} <_B h(\{v\})$ for some $u \in B(\varepsilon, h_1(e))$. This implies that $h(\{e\}) = \{u, h_1(e)\}$ for some $u \in vy - \{v\}$. Hence, $\pi(h_1(e)) = \pi(u) = \pi(v) = 1$. Fix numbers $r_1 < r_2 < r_3$ in the open interval $(\pi(h_1(e)), \pi(y))$.

Let $\varepsilon > 0$ be such that $N(\varepsilon, v u) \subset \pi^{-1}((r_3, \infty))$, $v \notin B(\varepsilon, u) and $B(\varepsilon, h_1(e)) \subset \pi^{-1}((\varepsilon, r_1))$. Let $\delta > 0$ be such that $\delta < \varepsilon$ and if $H(A, B) < \delta$, then $H(h(A), h(B)) < \varepsilon$. By hypothesis, there exists $e_1 \in E - \{e\}$ such that $d(e, e_1) < \delta$.

Then $H(\{e\}, \{e, e_1\}) < \delta$, so $h(\{e, e_1\})$ intersects each one of the sets $B(\varepsilon, u)$ and $B(\varepsilon, h_1(e))$. Thus, $h(\{e, e_1\}) = \{p, q\}$ for some $p \in B(\varepsilon, u)$ and $q \in B(\varepsilon, h_1(e))$, in particular, $\varepsilon \notin \{p, q\}$. Then there exist $e_2, e_3 \in E$ such that $p \in ve_2 - \{v\}$ and $q \in ve_3 - \{v\}$. That is, $\{p, q\} \in J(e_2, e_3)$. If $e_2 = e_3$, then $\{p, q\} \in J(e_2)$. Hence, we can apply Lemma 19 (d) to $h^{-1}$ to obtain that $h^{-1}(\{p, q\}) \in J(e_1)$ for some $e_4 \in E$. Thus, $\{e, e_1\} \in ve_4 and $e = e_4 = e_1$, a contradiction. This proves that $e_2 \neq e_3$. Since $J(e_1, e_2)$ and $J(e_2, e_3)$ are components of $F_2(X) - K$ (see the proof of Lemma 19 (c)), $h(K) = K$ and $h(\{e, e_1\}) \in J(e_2, e_3)$, we have $h(J(e, e_1)) = J(e_2, e_3)$. Let $D_1 = \{A \in J(e, e_1) : A \cap \{e, e_1\} \neq \emptyset\}$ and $D_2 = \{A \in J(e_2, e_3) : A \cap \{e_2, e_3\} \neq \emptyset\}$. Notice that $J(e, e_1)$ and $J(e_2, e_3)$ are 2-manifolds with respective manifold boundaries $\partial J(e, e_1) = D_1$ and $\partial J(e_2, e_3) = D_2$. Hence, $h(D_1) = D_2 and h(cl_{F_2(X)}(D_1)) = cl_{F_2(X)}(D_2)$. Notice that $cl_{F_2(X)}(D_1) = \{(z, e) : z \in ve_1\} \cup \{z, e_1 : z \in ve\}$ and $cl_{F_2(X)}(D_2) = \{(z, e_2) : z \in ve_3, \} \cup \{(z, e_3) : z \in ve_2\}$.

Since $\pi(q) < r_1 < r_2 < \pi(v)$, there exists $q_0 \in vq$ such that $\pi(q_0) = r_2$. Since $\{q_0, e_2\} \in cl_{F_2(X)}(D_2)$, there exists $A \in cl_{F_2(X)}(D_1)$ such that $h(A) = \{q_0, e_2\}$. Then there exist $e_4, e_5 \in \{e, e_1\}$ such that $\{e_4, e_5\} = \{e, e_1\}$ and $A = \{z_0, e_5\}$. Since $d(e_4, e) < \delta and $z_0$ belongs to the convex segment $ve_4 in the plane, there exists $z_1 \in ve$ such that $d(z_0, z_1) < \delta$. Since $d(e_5, e) < \delta, H(A, \{z_1, e\}) = H(\{z_0, e_5\}, \{z_1, e\}) < \delta$. Thus, $H((q_0, e_2), h) = H(h(A), h(\{z_1, e\})) = H(h(A), h(\{z_1, e\})) < \varepsilon$. Since $\{e\} \leq_A \{e, z_1\}$, we have $\{u, h_1(e)\} = h(\{e\}) \leq_B h(\{e, z_1\})$. This implies that $h(\{e, z_1\}) = \{w_0, h_1(e)\}$, for some $w_0 \in vu$. Hence,
We have seen that if \( X \) is an indecomposable arc continuum, then \( F_n(X) \) is rigid for each \( n \geq 4 \) (Corollary 7). In this section we obtain partial results for \( n = 2 \). The case \( n = 3 \) is open.

**LEMMA 22.** Let \( X \) be an indecomposable arc continuum. Then the arc components of \( F_2(X) \) are the sets of the form \( (K, L) \cap F_2(X) \), where \( K \) and \( L \) are composants of \( X \).

**Proof.** First we show that if \( K \) and \( L \) are composants of \( X \), then \( (K, L) \cap F_2(X) \) is arcwise connected. Take \( \{p, q\}, \{x, y\} \in (K, L) \cap F_2(X) \), where \( p, x \in K \) and \( q, y \in L \). Then there exist proper subcontinua \( A \) and \( B \) of \( X \) such that \( p, x \in A \subset K \) and \( q, y \in B \subset L \). By hypothesis, each of the sets \( A \) and \( B \) is an arc or a one-point set. Let \( A = \{p, v : v \in B\} \cup \{u, y : u \in A\} \). Clearly, \( A \) is arcwise connected and \( \{p, q\}, \{x, y\} \subset A \subset (K, L) \cap F_2(X) \). Therefore, \( (K, L) \cap F_2(X) \) is arcwise connected.

Now, fix an element \( \{x_0, y_0\} \in (K, L) \cap F_2(X) \), where \( x_0 \in K \) and \( y_0 \in L \). Let \( \mathcal{D} \) be the arc component of \( F_2(X) \) containing \( \{x_0, y_0\} \). Let \( A \in \mathcal{D} \). Let \( \mathcal{A} \) be an arc in \( \mathcal{D} \) connecting \( \{x_0, y_0\} \) and \( A \). Let \( E = \bigcup \{D : D \in \mathcal{A}\} \). By Lemma 1, \( E \) is a closed subset of \( X \) with at most two components \( E_1, E_2 \) such that \( x_0 \in E_1 \) and \( y_0 \in E_2 \), in the case that \( E \) is connected, we take \( E_1 = E_2 \). By \([6, \text{Lemma 2.2}]\), each \( E_i \) is locally connected. Since \( X \) is indecomposable, \( E_1 \) and \( E_2 \) are proper subcontinua of \( X \). Thus, \( E_1 \subset K \) and \( E_2 \subset L \). By Lemma 1, \( A \in (E_1, E_2) \cap F_2(X) \subset (K, L) \cap F_2(X) \). Hence, \( \mathcal{D} \subset (K, L) \cap F_2(X) \). Therefore, \( \mathcal{D} = (K, L) \cap F_2(X) \). \( \blacksquare \)

Let \( \mathcal{K}_n(X) = \{A \in F_n(X) : \text{there is a neighborhood } \mathcal{M} \text{ of } A \text{ in } F_n(X) \text{ such that the component } C \text{ of } \mathcal{M} \text{ that contains } A \text{ is a 2n-cell and } A \text{ belongs to the manifold boundary of } C\} \). Given an indecomposable arc continuum \( X \), let \( E(X) \) be the set of end points of \( X \).

**LEMMA 23.** Let \( X \) be an indecomposable arc continuum. Then

(a) \( \mathcal{K}_2(X) = F_2(X) \),
(b) \( \mathcal{K}_3(X) = F_1(X) \cup \{A \in F_2(X) : A \cap E(X) \neq \emptyset\} \),
(c) if \( \mathcal{A} \) is an arc component of \( \mathcal{K}_2(X) \), then \( \mathcal{A} \) is of one of the following forms:

\( H(\{q_0, e_2\}, \{w_0, h_1(e)\}) < \varepsilon \). This implies that \( q_0 \in N(\varepsilon, wu) \cup B(\varepsilon, h_1(e)) \subset \pi^{-1}((r_3, \infty)) \cup \pi^{-1}((-\infty, r_1)) \). This contradicts the fact that \( \pi(q_0) = r_2 \) and completes the proof of the theorem. \( \blacksquare \)

**QUESTION 21.** Is it true that if \( E \) does not have isolated points, then \( F_3(X) \) is rigid?
(1) $A = F_1(K)$, for some composant $K$ of $X$ such that $K \cap E(X) = \emptyset$.
(2) $A = F_1(K) \cup \{ e, x : x \in K \}$, for some composant $K$ of $X$ such that $K \cap E(X) = \emptyset$.
(3) $A = \{ e, x : x \in K \}$, for some $e \in E(X)$ and some composant $K$ of $X$ such that $K \cap E(X) = \emptyset$.
(4) $A = \{ e_1, x : x \in L \} \cup \{ e_2, x : x \in K \}$, for some composants $K$ and $L$ of $X$ such that $K \neq L$, $K \cap E(X) = \{ e_1 \}$ and $L \cap E(X) = \{ e_2 \}$.
(d) if $A$ is an arc component of $\mathcal{K}_2(X)$, then $B = \text{cl}_{F_2(X)}(A)$ is of one of the following forms:
(5) $B = F_1(X)$,
(6) $B = F_1(X) \cup \{ e, x : x \in X \}$, for some $e \in E(X)$,
(7) $B = \{ e, x : x \in X \}$, for some $e \in E(X)$,
(8) $B = \{ e_1, x : x \in X \} \cup \{ e_2, x : x \in X \}$, for some $e_1, e_2 \in E(X)$ such that $e_1 \neq e_2$.

Proof. By Lemma 2 of [10], $X = W(X)$. By Theorem 2 (e), $F_2(X) - F_1(X) \subset \mathcal{G}_2(X)$. Let $x \in X$. Since $x \in W(X)$, there exists an open set $U$ of $X$ such that $x \in U$ and the component $C$ of $U$ containing $x$ is homeomorphic to $(0, 1)$ or $[0, 1)$. Let $V$ be an open subset of $X$ such that $x \in V$ and $\text{cl}_{X}(V) \subset U$. Let $D$ be the component of $\text{cl}_{X}(V)$ containing $x$. Then $D$ is a nondegenerate subcontinuum of $C$. Hence, $D$ is an arc. Let $\mathcal{M} = (\text{cl}_{X}(V)) \cap F_2(D)$. Then $\mathcal{M}$ is a neighborhood of $\{ x \}$ in $F_2(X)$. Let $\mathcal{D}$ be the component of $\mathcal{M}$ such that $\{ x \} \in \mathcal{D}$. Let $E = \bigcup \{ G : G \in \mathcal{D} \}$. By Lemma 1, $E \subset C(X)$. Since $E \subset \text{cl}_{X}(V)$, $E \subset D$. Thus, $\mathcal{D} \subset (D) \cap F_2(X) = F_2(D)$. Since $F_2(D)$ is connected and $F_2(D) \subset \mathcal{M}$, we conclude that $\mathcal{D} = F_2(D)$. By [15, Example K], $F_2(D)$ is a 2-cell. Notice that $\{ x \}$ belongs to the manifold boundary of $F_2(D)$. We have shown that $\{ x \} \in \mathcal{G}_2(X)$ and $\{ x \} \in \mathcal{K}_2(X)$. This completes the proof of (a) and we also have $F_1(X) \subset \mathcal{K}_2(X)$.

Let $A = \{ a_1, a_2 \} \subset F_2(X)$, where $a_1 \in E(X)$ and $a_1 \neq a_2$. Let $U_1, U_2$, $\varepsilon > 0$, $M_1, M_2, C_1, C_2$, $\mathcal{M} = \langle M_1, M_2 \rangle \cap F_2(X)$ and $\mathcal{C} = \langle C_1, C_2 \rangle \cap F_2(X)$ be as in the proof of Theorem 2 (e) for $n = 2$. As we saw in that proof, $\mathcal{M}$ is a neighborhood of $A$ in $F_2(X)$ and $\mathcal{C}$ is the component of $\mathcal{M}$ containing $A$. Since $a_1 \in C_1 \cap E(X)$ and $C_1$ is an arc, $a_1$ is an end point of $C_1$. Notice that $C_1 \times C_2$ is homeomorphic to $\mathcal{C}$ under the homeomorphism $(x, y) \mapsto \{ x, y \}$. Thus, $A$ belongs to the manifold boundary of $\mathcal{C}$. Hence, $A \in \mathcal{K}_2(X)$. We have shown that $F_1(X) \cup \{ A \in F_2(X) : A \cap E(X) \neq \emptyset \} \subset \mathcal{K}_2(X)$.

Now, take $A = \{ x, y \}$, where $x \neq y$ and $x, y \notin E(X)$. Let $\mathcal{M}$ be a neighborhood of $A$ in $F_2(X)$ and let $\mathcal{C}$ be the component of $\mathcal{M}$ containing $A$. Then there exist arcs $\alpha$ and $\beta$ such that $x \in \alpha$, $y \in \beta$, $x$ is not an end point of $\alpha$ and $y$ is not an end point of $\beta$ and $\langle \alpha, \beta \rangle \cap F_2(X) \subset \mathcal{M}$, we may assume that $\alpha \cap \beta = \emptyset$. Since $\langle \alpha, \beta \rangle \cap F_2(X)$ is a 2-cell that contains $A$ in its manifold interior, $\langle \alpha, \beta \rangle \cap F_2(X) \subset \mathcal{C}$ so $\mathcal{C}$ cannot be a 2-cell having $A$ in its manifold interior. Hence, $A \notin \mathcal{K}_2(X)$. This completes the proof of (b).
(c). Since \( X \) is not an arc, then for each composant \( K \) of \( X \), \( K \cap E(X) \) has at most one element. Let \( A \) be an arc component of \( \mathcal{K}_2(X) \). By Lemma 22, there exist composants \( K \) and \( L \) of \( X \) such that \( A \subset (K,L) \cap F_2(X) \). We consider two cases.

In the case that \( K = L \), by (b), \( A \subset (K \cap F_2(X)) \cap \mathcal{K}_2(X) = ((K \cap F_1(X)) \cup (\{ e \}) \cap \{ x \in E(X) : A \cap E(X) \neq \emptyset \}) \cup \mathcal{K}_2(X) \). If \( K \cap E(X) = \emptyset \), then \( A \subset F_1(X) \subset \mathcal{K}_2(X) \). Since \( F_1(X) \) is homeomorphic to \( K \) and \( K \) is arcwise connected, we obtain that \( A = F_1(X) \). If \( K \cap E(X) = \{ e \} \), then \( A \subset F_1(X) \cup \{ e \} \subset F_2(X) : x \in K \}. By (b), \( F_1(X) \cup \{ e \} \subset F_2(X) : x \in K \} \subset \mathcal{K}_2(X) \). Since \( F_1(X) \cup \{ e \} \subset F_2(X) : x \in K \) is arcwise connected, we have that \( A = F_1(X) \) and \( \{ e \} \subset F_2(X) \). Finally, (d) is obtained just by taking the closure of \( A \) in each one of the cases (1) - (4) in (c).

**THEOREM 24.** If \( X \) is an indecomposable arc continuum and \( h : F_2(X) \to F_2(X) \) is a homeomorphism such that \( h(F_1(X)) \not\subset F_1(X) \), then there exists an end point \( x \) of \( X \) such that \( h(F_1(X)) = \{ x \} \).

**Proof.** Let \( h : F_2(X) \to F_2(X) \) be a homeomorphism. Since \( \mathcal{K}_2(X) \) is defined in topological terms, \( h(\mathcal{K}_2(X)) = \mathcal{K}_2(X) \). Suppose that there exists \( p_0 \in X \) such that \( h(\{ p_0 \}) \not\subset F_1(X) \). Let \( K_0 \) be the composant of \( X \) containing \( p_0 \). Since \( K_0 \) is dense in \( X \), there exists \( p_1 \in K_0 - \{ p_0 \} \) such that \( h(\{ p_1 \}) \) is close enough to \( h(\{ p_0 \}) \) in such a way that \( h(\{ p_1 \}) \not\subset F_1(X) \). Since \( K_0 \) has at most an element of \( E(X) \), one of the points \( p_0 \) or \( p_1 \) does not belong to \( E(X) \). Thus, we may assume that \( p_0 \not\subset E(X) \). Let \( A_0 \) be the arc component of \( \mathcal{K}_2(X) \) containing \( \{ p_0 \} \). Then \( A_0 \) is of one of the forms (1) - (4) in Theorem 23. Since \( \{ p_0 \} \in A_0 \) and \( p_0 \not\subset E(X) \), \( A_0 \) is not of the form (3) or (4). Thus, \( F_1(X) \subset A_0 \). By Theorem 23 (b), there exist \( e_0 \in E(X) \) and \( x_0 \in X - \{ e_0 \} \) such that \( h(\{ p_0 \}) = \{ e_0, x_0 \} \).

Notice that \( A = h(A_0) \) is an arc component of \( \mathcal{K}_2(X) \). Let \( B = \text{cl}_{F_2(X)}(A) \). Then \( h(F_1(X)) \subset B \). By Theorem 23, \( B \) is of one of the forms described in (5) - (8). Since \( h(\{ p_0 \}) \in B = F_1(X) \), \( B \not\subset F_1(X) \).

If \( B \) is of the form described in (6), \( B = F_1(X) \cup \{ e, x \} : x \in X \}, for some \( e \in E(X) \), let \( C = \{ e, x \} : x \in X \}. Notice that \( C \) is homeomorphic
to X and \( C \cap F_1(X) = \{\{e\}\} \). Thus \( B \) is separated by \( \{e\} \). Since \( h(\{p_0\}) \in h(F_1(X)) - F_1(X) \subset B - F_1(X), h(F_1(X)) \cap (C - \{\{e\}\}) \neq \emptyset \). If \( h(F_1(X)) \cap (F_1(X) - \{\{e\}\}) \neq \emptyset \), then \( \{e\} \) separates \( h(F_1(X)) \). This is a contradiction since \( h(F_1(X)) \) is homeomorphic to X and then \( h(F_1(X)) \) is indecomposable. We have shown that \( h(F_1(X)) \cap (F_1(X) - \{\{e\}\}) = \emptyset \). Thus, \( h(F_1(X)) \subset C \). Since \( h(F_1(X)) \) is indecomposable and the nondegenerate proper subcontinua of \( C \) are arcs, we conclude that \( h(F_1(X)) = C \). Therefore, \( h(F_1(X)) = \{\{e, x\} : x \in X\} \).

If \( B = \{\{e_1, x\} : x \in X\} \cup \{\{e_2, x\} : x \in X\} \) for some \( e_1, e_2 \in E(X) \) such that \( e_1 \neq e_2 \), proceeding as in the paragraph above it can proved that \( h(F_1(X)) = \{\{e_i, x\} : x \in X\} \) for some \( i \in \{1, 2\} \).

Finally, if \( B = \{\{e, x\} : x \in X\} \) for some \( e \in E(X) \) ((7)), then \( B \) are homeomorphic to X and \( h(F_1(X)) \subset B \). Thus, \( h(F_1(X)) = B \) .

**COROLLARY 25.** If \( X \) is a solenoid (see [20, 2.8] for definition), then \( F_2(X) \) is rigid.

**Proof.** It follows from Theorem 24 and the fact that solenoids do not have end points. ■

**QUESTION 26.** Is \( F_2(X) \) rigid for each indecomposable arc continuum \( X \)? We do not even know if \( F_2(X) \) is rigid when \( X \) is the Buckethandle continuum (see [20, 2.8] for definition).

**QUESTION 27.** Has every indecomposable arc continuum unique hyperspace \( F_2(X) \)?

**QUESTION 28.** Let \( X \) be a hereditarily indecomposable continuum. Is it true that \( F_n(X) \) is rigid.

A particular interesting case of Question 28 is when \( X = P \) is the Pseudo-arc and \( n = 2 \). For solving this particular case it could be useful the following recent result [4]: if \( e : P \times P \to P \times P \) is an embedding, then there exist embeddings \( e_1, e_2 : P \to P \) such that for each \( (x, y) \in P \times P \), \( e(x, y) = (e_1(x), e_2(y)) \) or for each \( (x, y) \in P \times P \), \( e(x, y) = (e_1(y), e_2(x)) \).

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