UNIQUENESS OF HYPERSPACES OF INDECOMPOSABLE ARC CONTINUA

Rodrigo Hernández-Gutiérrez, Alejandro Illanes and Verónica Martínez-de-la-Vega

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Abstract

Given a metric continuum $X$, we consider the hyperspace $C_n(X)$ of all nonempty closed subsets of $X$ with at most $n$ components. In this paper we prove that if $n \neq 2$, $X$ is an indecomposable continuum such that all its proper nondegenerate subcontinua are arcs and $Y$ is a continuum such that $C_n(X)$ is homeomorphic to $C_n(Y)$, then $X$ is homeomorphic to $Y$ (that is, $X$ has unique hyperspace $C_n(X)$).

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1. INTRODUCTION

A continuum is a nondegenerate compact connected metric space. Given a continuum $X$, we consider the following hyperspaces of $X$.

- $2^X = \{ A \subset X : A \text{ is nonempty and closed in } X \}$, $C_n(X) = \{ A \in 2^X : A \text{ has at most } n \text{ components} \}$,
- $F_n(X) = \{ A \in 2^X : A \text{ has at most } n \text{ points} \}$,
- $C(X) = C_1(X)$.

All hyperspaces are considered with the Hausdorff metric $H$.

The hyperspace $F_n(X)$ is known as the $n$-th symmetric product of $X$. The hyperspace $F_1(X)$ is an isometric copy of $X$ embedded in each one of the hyperspaces.

A hyperspace $K(X) \in \{ 2^X, C_n(X), F_n(X) \}$ is said to be rigid provided that for each homeomorphism $h : K(X) \to K(X)$, we have, $h(F_1(X)) = F_1(X)$. The continuum $X$ is said to have unique hyperspace $K(X)$ provided that the following implication holds: if $Y$ is a continuum such that $K(X)$ is homeomorphic to $K(Y)$, then $X$ is homeomorphic to $Y$.

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Uniqueness of hyperspaces has been widely studied (see for example [3], [4], [6], [7], [8] and [11] for recent references). A detailed survey of what is known about this subject can be found in [12]. In the study of hyperspaces, a useful technique is to find a topological property that characterizes the elements of \( F_1(X) \) in the hyperspace \( K(X) \). When it is possible to find such a characterization, the hyperspace \( K(X) \) is rigid. This technique has been used in studying uniqueness of hyperspaces, so both topics are closely related.

Rigidity of hyperspaces was introduced in [8]. Rigidity of symmetric products was studied in [7].

A continuum \( X \) is indecomposable if it cannot be put as the union of two of its proper subcontinua. The continuum \( X \) is said to be arc continuum if each one of its nondegenerate proper subcontinuum is an arc. Examples of indecomposable arc continua are the Buckethandle continuum and the solenoids [16, 2.8 and 2.9].

As a consequence of Theorem 5 in [7] and Theorem 9 of [3], it follows that if \( X \) is an indecomposable arc continuum and \( n \neq 3 \), then \( X \) has unique hyperspace \( F_n(X) \), the case \( n = 3 \) remains unsolved.

In this paper we prove that if \( X \) is an indecomposable arc continuum, then \( X \) has unique hyperspaces \( C_n(X) \) and \( C_n(X) \) is rigid for every \( n \neq 2 \). The case \( n = 2 \) remains unsolved.

2. DEFINITIONS AND CONVENTIONS

A map is a continuous function. Suppose that \( d \) is a metric for \( X \). Given \( \varepsilon > 0 \), \( p \in X \) and \( A \subseteq 2^X \), let \( B(\varepsilon,p) \) be the \( \varepsilon \)-open ball around \( p \) in \( X \), \( N(\varepsilon,A) = \{ p \in X : \text{there exists } a \in A \text{ such that } d(p,a) < \varepsilon \} \) and \( B^H(\varepsilon,A) = \{ B \subseteq 2^X : H(A,B) < \varepsilon \} \) (we write \( B_X(\varepsilon,p) \) and \( N_X(\varepsilon,A) \) when the space \( X \) needs to be mentioned). A simple \( n \)-od is a finite graph \( G \) that is the union of \( n \) arcs emanating from a single point, \( v \), and otherwise disjoint from one another. The point \( v \) is called the vertex of \( G \). Simple 3-ods are called simple triods. Given subsets \( A_1, \ldots, A_m \) of \( X \), let \( \langle A_1, \ldots, A_m \rangle = \{ B \subseteq 2^X : B \cap A_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\} \text{ and } B \subseteq A_1 \cup \ldots \cup A_m \} \).

We denote by \( S^1 \) the unit circle in the Euclidean plane. A free arc in the continuum \( X \) is an arc \( \alpha \) with end points \( a \) and \( b \) such that \( \alpha - \{a,b\} \) is open in \( X \).

Proceeding as in Lemma 2.1 of [5] and using Lemma 1.48 of [15], the following lemma can be proved.
**LEMMA 1.** Let $X$ be a continuum and let $A$ be a connected subset of $2^X$ such that $A \cap C_n(X) \neq \emptyset$. Let $A_0 = \bigcup\{A : A \in \mathcal{A}\}$. Then
(a) $A_0$ has at most $n$ components,
(b) if $A$ is closed in $2^X$, then $A_0 \in C_n(X)$,
(c) for each $A \in \mathcal{A}$, each component of $A_0$ intersects $A$.

A *wire* in a continuum $X$ is a subset $\alpha$ of $X$ such that $\alpha$ is homeomorphic to one of the spaces $(0, 1)$, $[0, 1)$, $[0, 1]$ or $S^1$ and $\alpha$ is a component of an open subset of $X$. By [15, Theorem 20.3], if a wire $\alpha$ in $X$ is compact, then $\alpha = X$. So, if a wire is homeomorphic to $[0, 1]$ or $S^1$, then $X$ is an arc or a simple closed curve. Given a continuum $X$, let

$$W(X) = \bigcup\{\alpha \subset X : \alpha \text{ is a wire in } X\}.$$ 

The continuum $X$ is said to be *wired* provided that $W(X)$ is dense in $X$.

Notice that if $\alpha$ is a free arc of a continuum $X$ and $p, q$ are the end points of $\alpha$, then $\alpha - \{p, q\}$ is a wire in $X$. Thus, a continuum for which the union of its free arcs is dense is a wired continuum. Therefore, the class of wired continua includes finite graphs, dendrites with closed set of end points, almost meshed continua [6], compactifications of the ray $[0, \infty)$, compactifications of the real line and indecomposable arc continua.

An *m-od* in a continuum $X$ is a subcontinuum $B$ of $X$ for which there exists $A \in C(B)$ such that $B - A$ has at least $m$ components. By [13, Theorem 70.1], a continuum $X$ contains an $m$-od if and only if $C(X)$ contains an $m$-cell. Given $A, B \in 2^X$ such that $A \subsetneq B$, an *order arc* from $A$ to $B$ is a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(s) \subsetneq \alpha(t)$ if $0 \leq s < t \leq 1$. It is known [15, Theorem 1.25] that there exists an order arc from $A$ to $B$ if and only if $A \subsetneq B$ and each component of $B$ intersects $A$.

Given a continuum $X$ and $n \in \mathbb{N}$, let

$$W_n(X) = \{A \in C_n(X) : \text{each component of } A \text{ is contained in a wire of } X\};$$
and
$$Z_n(X) = \{A \in W_n(X) : \text{there is a neighborhood } \mathcal{M} \text{ of } A \text{ in } C_n(X) \text{ such that } \text{the component } \mathcal{C} \text{ of } \mathcal{M} \text{ that contains } A \text{ is a } 2n\text{-cell} \}.$$ 

We will use the following two results of [8].

**LEMMA 2** [8, Lemma 2]. Let $X$ be an indecomposable arc continuum. Then $X$ is a wired continuum.

**THEOREM 3** [8, Theorem 8]. Let $X$ be a continuum and let $n \geq 3$. Then

$$W_1(X) = \{A \in W_n(X) - Z_n(X) : A \text{ has a basis } \mathcal{B} \text{ of neighborhoods of } A \text{ in } C_n(X) \text{ such that for each } \mathcal{U} \in \mathcal{B}, \text{ if } \mathcal{C} \text{ is the component of } \mathcal{U} \text{ that contains } A, \text{ then } \mathcal{C} \cap Z_n(X) \text{ is connected}\}.$$
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THEOREM 4. If \( X \) is an indecomposable arc continuum, then \( X \) has unique hyperspaces \( C_n(X) \) and \( C_n(X) \) is rigid for every \( n \neq 2 \).

Proof. For \( n = 1 \), the uniqueness of \( C(X) \) was shown in Theorem 2.3 of [1]. In the proof of Theorem 3 of [14], it was shown that if \( h : C(X) \to C(X) \) is a homeomorphism, then \( h(F_1(X)) = F_1(X) \). That is, \( C(X) \) is rigid.

Suppose then that \( n \geq 3 \). Let \( Y \) be a continuum such that there exists a homeomorphism \( h : C_n(X) \to C_n(Y) \). Let \( Y_0 \in C_n(Y) \) be such that \( h(X) = Y_0 \).

Claim 1. The only element that arcwise disconnects \( C_n(X) \) is \( X \) and \( C_n(X) - \{X\} \) has uncountably many arc components.

We prove Claim 1. By Corollary 2.2 of [10] \( C_n(X) - \{X\} \) has uncountably many arc components. Let \( A \in C_n(X) - \{X\} \). Let \( C \) be the arc component of \( C_n(X) - \{A\} \) such that \( X \in C \). We claim that \( C = C_n(X) - \{A\} \). Take \( D \in C_n(X) - \{A\} \). If \( D \) is not contained in \( A \), take an order arc \( \alpha \) from \( D \) to \( X \). Notice that for each \( t \in [0,1] \), \( \alpha(t) \neq A \). Then \( \text{Im} \alpha \subseteq C \) and \( D \in C \). Now consider the case that \( D \subseteq A \). Then, we have that \( A \) is not a one-point set. Reasoning as in Theorem 11.3 of [15], it follows that if \( A \) is not connected, then there is an arc joining \( D \) and \( X \) in \( C(X) - \{A\} \). Thus, we assume that \( A \) is connected. Let \( B \in C(X) - \{X\} \) be such that \( A \subseteq B \). Then \( A \) and \( B \) are arcs. Let \( F \) be a finite set containing exactly one point in each one of the components of \( D \). Then \( F \in F_n(X) \subseteq C_n(X) \). Let \( \beta \) be an order arc joining \( F \) and \( D \). Notice that \( \text{Im} \beta \subseteq C_n(X) - \{A\} \). It is easy to show that there exists an arc \( \gamma \) in \( F_n(B) \) joining \( F \) and an element \( E \subseteq B - A \). By the first case, \( E \in C \). Since \( \text{Im} \beta \subseteq C_n(X) - \{A\} \), we conclude that \( D \in C \). We have shown that \( C = C_n(X) - \{A\} \). Hence, \( C_n(X) - \{A\} \) is arcwise connected. This ends the proof of Claim 1.

Claim 2. \( Y_0 \in C(Y) \) and \( Y_0 \) is indecomposable.

To prove Claim 2 observe that if \( Y_0 \) is disconnected, then reasoning as in Theorem 11.3 of [15] it can be proved that \( C_n(Y) - \{Y_0\} \) is arcwise connected. Since \( h \) is a homeomorphism, this contradicts Claim 1. Hence, \( Y_0 \) is connected. Now, suppose that \( Y_0 \) is decomposable. By Lemma 2.4 of [10], \( C_n(Y) - \{Y_0\} \) has at most two arc components. Since \( h \) is a homeomorphism, Claim 1 implies that \( C_n(Y) - \{Y_0\} \) has uncountably many arc components. This contradiction ends the proof of Claim 2.

Claim 3. Let \( k = 2n + 1 \). Then \( C_n(Y) \) does not contain \( k \)-cells.

Suppose, contrary to Claim 3, that \( C_n(Y) \) contains a \( k \)-cell. Then there exists a \( k \)-cell \( \mathcal{M} \) in \( C_n(X) \). Let \( m = \max \{i \in \{1, \ldots, n\} : \mathcal{M} \cap (C_i(X) - \)
that $C_{i-1}(X) \neq \emptyset$. Since $\mathcal{M} \cap (C_m(X) - C_{m-1}(X))$ is a nonempty open subset of $\mathcal{M}$, there exists $A \in \mathcal{M} \cap (C_m(X) - C_{m-1}(X)) - \{X\}$. Let $\mathcal{N}$ be a $k$-cell such that $A \in \mathcal{N} \subset \mathcal{M} \cap (C_m(X) - C_{m-1}(X)) - \{X\}$ and let $B = \bigcup\{C : C \in \mathcal{N}\}$. Since $A$ has $m$ components, we can take small enough $\mathcal{N}$ in such a way that $B$ has at least $m$-components and $B \neq X$. By Lemma 1, if $C \in \mathcal{N}$, then $C$ intersects each component of $B$. Since $A \subset B$, we have that $B$ has exactly $m$ components. Let $B_1, \ldots, B_m$ be the components of $B$. Then each $B_i$ is an arc or a one-point set. Given $C \in \mathcal{N}$, $C \in \langle B_1, \ldots, B_m \rangle \cap C_n(X)$ and, by the choice of $m$, $C$ has exactly $m$ components. Thus, the components of $C$ are the sets $C \cap B_1, \ldots, C \cap B_m$. Let $\varphi : \mathcal{N} \rightarrow C(B_1) \times \cdots \times C(B_m)$ be given by $\varphi(C) = (C \cap B_1, \ldots, C \cap B_m)$. It is easy to check that $\varphi$ is continuous and one-to-one. Hence, $\mathcal{N}$ can be embedded in $C(B_1) \times \cdots \times C(B_m)$. Since $C([0,1])$ is a 2-cell, we conclude that $\mathcal{N}$ can be embedded in a $j$-cell for some $j \leq 2m \leq 2n$. This implies that $k \leq 2n$. This contradiction proves Claim 3.

**Claim 4.** If $Z \in C(Y) - F_1(Y)$ and $Y_0 \not\in Z$, then $Z$ is decomposable.

Suppose, contrary to Claim 4, that $Z$ is indecomposable. Notice that $Z \neq Y$. Let $\mathcal{B}$ be the arc component of $C_n(Y) - \{Z\}$ such that $Y \in \mathcal{B}$. By Theorem 70.1 of [13] and Claim 3, $Y$ does not contain $(2n+1)$-ods. By Lemma 2.3 of [10], the set $\mathcal{K} = \{K \subset Z : K$ is composant of $Z$ and $\langle K \rangle \cap C_n(Y) \cap \mathcal{B} \neq \emptyset\}$ has at most $2n$ elements. Since $Z$ has infinitely many composants [16, Theorem 11.15], we can take a composant $K_0$ of $Z$ such that $K_0 \not\in \mathcal{K}$. Fix a point $z_0 \in K_0$. Then $\{z_0\} \not\in \mathcal{B}$. This proves that $C_n(Y) - \{Z\}$ is arcwise disconnected. Since $h$ is a homeomorphism, $C_n(X) - \{h^{-1}(Z)\}$ is arcwise disconnected. By Claim 1, $X = h^{-1}(Z)$ and $Z = h(X) = Y_0$, a contradiction. Therefore, $Z$ is decomposable.

**Claim 5.** If $Z \in C(Y) - F_1(Y)$ and $Y_0 \not\in Z$, then $Z$ is an arc.

In order to prove Claim 5, let $W = h^{-1}(C_n(Z))$. Since $Y_0 \notin C_n(Z)$, we have that $X = h^{-1}(Y_0) \notin W$. Let $B = \bigcup\{D : D \in W\}$. By Lemma 1, $B \in C_n(X)$. Let $B_1, \ldots, B_m$ be the components of $B$, where $m \leq n$. By Corollary 2.2 of [10], the arc component of $C_n(X) - \{X\}$ that contains $Z_0 = h^{-1}(Z)$ is a set of the form $\langle K_1, \ldots, K_r \rangle \cap C_n(X)$, where $r \leq n$ and $K_1, \ldots, K_r$ are composants of $X$. Since $C_n(Z)$ is arcwise connected, $W$ is an arcwise connected set and $X \notin W$. Since $Z_0 \in W$, $W \cap \langle K_1, \ldots, K_r \rangle \cap C_n(X)$. This implies that $B \subset K_1 \cup \ldots \cup K_r$ and then $B \neq X$. Hence, each $B_i$ is an arc or a one-point set.

We claim that $Z$ is locally connected.

Suppose to the contrary that $Z$ is not connected im kleinen at some element $z_0 \in Z$. Then there exist an open subset $U$ of $Z$ and a sequence of points $\{z_j\}_{j=1}^{\infty}$ in $U$ such that $z_0 \in U$, $\lim jz_j = z_0$ and if $E_j$ is the component of $U$ containing $z_j$ ($j \in \mathbb{N} \cup \{0\}$), then $E_0, E_1, E_2, \ldots$ are all different. Note that $U \neq Z$. Let $V$
be an open subset of \( Z \) such that \( z_0 \in V \) and \( \text{cl}_Z(V) \subset U \). For each \( j \in \mathbb{N} \), we assume that \( z_j \in V \) and we take the component \( D_j \) of \( \text{cl}_Z(V) \) such that \( z_j \in D_j \). We may assume that \( \lim D_j = D_0 \) for some \( D_0 \subset C(Z) \). Then \( z_0 \in D_0 \subset E_0 \), \( D_j \subset E_j \) and \( D_j \cap \text{bd}_Z(V) \neq \emptyset \) \cite[Theorem 20.3]{15} for each \( j \in \mathbb{N} \). Thus, \( D_0 \cap \text{bd}_Z(V) \neq \emptyset \) and \( D_0 \) is nondegenerate. Fix a nondegenerate continuum \( D \) such that \( z_0 \in D \subset D_0 \cap V \).

Since \( \text{cl}_Z(V) \neq Z \), we can choose pairwise disjoint nondegenerate subcontinua \( G_1, \ldots, G_{n-1} \) of \( Z \) contained in \( Z - \text{cl}_Z(V) \). By Claim 4, each \( G_i \) is decomposable. It is easy to show that the decomposibility of \( G_i \) implies that \( G_i \) contains a 2-od. So, we may assume that each \( G_i \) is a 2-od. For each \( i \in \{1, \ldots, n-1\} \), let \( R_i \subset C(G_i) \) be such that \( G_i - R_i \) is disconnected. By the proof of \cite[Theorem 1.100]{15}, there exists a 2-cell \( g_i \) in \( C(G_i) \) such that \( R_i, G_i \subset g_i \) and for each \( L \subset g_i \), \( R_i \subset L \subset G_i \). Let \( \mathcal{G} = \{ \{y\} \cup L_1 \cup \ldots \cup L_{n-1} \subset C_n(Z) : y \in D \text{ and } L_i \subset g_i \text{ for each } i \in \{1, \ldots, n-1\} \} \). Notice that \( \mathcal{G} \) is homeomorphic to \( D \times g_1 \times \ldots \times g_{n-1} \), so \( \dim(\mathcal{G}) \geq 2n - 1 \) \cite[Remark at the end of section 4 of Chapter III]{9}. Let

\[
\mathcal{M} = h^{-1}(\mathcal{G}).
\]

Then \( \mathcal{M} \) is a subcontinuum of \( C_n(X) \) such that \( \mathcal{M} \subset W \) and \( \dim(\mathcal{M}) \geq 2n - 1 \). Notice that \( X \notin \mathcal{M} \).

Let

\[
m_0 = \max\{i \in \{1, \ldots, n\} : \mathcal{M} \cap (C_i(X) - C_{i-1}(X)) \neq \emptyset\}.
\]

Now we show that \( m_0 = n \). If \( m_0 = 1 \), then \( \mathcal{M} \subset C(X) \cap W \). This implies that each element of \( \mathcal{M} \) is contained in \( B_1 \cup \ldots \cup B_m \). Thus, \( \mathcal{M} \subset C(B_1) \cup \ldots \cup C(B_m) \). Since each \( C(B_i) \) is a one-point set or a 2-cell, we conclude that \( 2n - 1 \leq \dim(\mathcal{M}) \leq 2 \). Hence, \( n = 1 \), contrary to our assumption. Therefore, \( m_0 \geq 2 \).

Let \( M_0 \in \mathcal{M} \cap (C_{m_0}(X) - C_{m_0-1}(X)) \). Let \( M_1, \ldots, M_{m_0} \) be the components of \( M_0 \). Suppose that \( M_0 = h^{-1}(\{y_0\} \cup L_1^{(0)} \cup \ldots \cup L_{n-1}^{(0)}) \), where \( y_0 \in D \) and \( L_i^{(0)} \subset g_i \) for each \( i \in \{1, \ldots, n-1\} \). Let \( \varepsilon > 0 \) be such that the sets \( N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0}) \) are pairwise disjoint. Since \( X \notin \mathcal{M}, M_0 \neq X \), so we can ask that \( X \neq N(\varepsilon, M_1) \cup \ldots \cup N(\varepsilon, M_{m_0}) \).

Since \( C_{m_0-1}(X) \) is closed in \( C_n(X) \) and \( h^{-1} \) is continuous, there exists a nondegenerate continuum \( D' \) of \( D \) and for each \( i \in \{1, \ldots, n-1\} \) there exists a 2-cell \( g_i' \) such that \( L_i^{(0)} \subset g_i' \subset g_i \), \( H(D_0, h^{-1}(L_i)) < \varepsilon \) and \( h^{-1}(L_i) \notin C_{m_0-1}(X) \) for each \( L_i \subset g_i' \). Let \( L_i \subset g_i' \) for each \( i \in \{1, \ldots, n-1\} \).

Given \( L \subset g_i' \), \( h^{-1}(L) \in \mathcal{M} \), then \( h^{-1}(L) \in (N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0})) \), so \( h^{-1}(L) \) has at most \( m_0 \) components and, by definition of \( m_0 \), \( h^{-1}(L) \) has at most \( m_0 \) components. Thus, \( h^{-1}(L) \) has exactly \( m_0 \) components. Since \( h^{-1}(L) \in (N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0})) \cap C_n(X) \), we have that the components
of $h^{-1}(L)$ are the sets $h^{-1}(L) \cap N(\varepsilon, M_1), \ldots, h^{-1}(L) \cap N(\varepsilon, M_{m_0})$. Let $L_0 = \bigcup \{ h^{-1}(L) : L \in \mathcal{G}' \}$. By Lemma 1, $L_0$ has at most $m_0$ components, but $L_0 \in \langle N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0}) \rangle \cap C_n(X)$, so $L_0$ has exactly $m_0$ components and they are $L_0 \cap N(\varepsilon, M_1), \ldots, L_0 \cap N(\varepsilon, M_{m_0})$. This implies that each set $L_0 \cap N(\varepsilon, M_i)$ is an arc or a one-point set. Notice that $\mathcal{G}'$ is homeomorphic to $D^r \times \mathcal{G}'_1 \times \ldots \times \mathcal{G}'_{n-1}$, so $\dim(\mathcal{G}') \geq 2n - 1$ and $\dim(h^{-1}(\mathcal{G}')) \geq 2n - 1$.

Notice that the map $\psi : \mathcal{G}' \rightarrow C(L_0 \cap N(\varepsilon, M_1)) \times \ldots \times C(L_0 \cap N(\varepsilon, M_{m_0}))$ given by $\psi(L) = (h^{-1}(L) \cap N(\varepsilon, M_1), \ldots, h^{-1}(L) \cap N(\varepsilon, M_{m_0}))$ is an embedding. This shows that $\dim(C(L_0 \cap N(\varepsilon, M_1)) \times \ldots \times C(L_0 \cap N(\varepsilon, M_{m_0}))) \geq 2n - 1$. Since for each $i \in \{1, \ldots, m_0\}$, $C(L_0 \cap N(\varepsilon, M_i))$ is either a one-point set or a 2-cell [13, Theorem 5.1], we obtain that $2m_0 \geq \dim(C(L_0 \cap N(\varepsilon, M_1)) \times \ldots \times C(L_0 \cap N(\varepsilon, M_{m_0})))$. Thus, $m_0 \geq n$. Hence, $m_0 = n$.

Since $M_0 \in \mathcal{M} \subset W$, we have $M_0 \subset B$ and by Lemma 1, each $B_i$ intersects $M_0$. Since $B$ is a finite union of arcs or one-point sets, there exist pairwise disjoint subarcs (or one-point sets), $Q_1, \ldots, Q_n$ of $B$ such that for each $i \in \{1, \ldots, n\}$, $M_i \subset \int_B(Q_i)$. Then $M_0 \in C_n(X) \cap W \cap \langle \int_B(Q_1), \ldots, \int_B(Q_n) \rangle$, which is an open subset of $W$. We are going to see that each $Q_i$ is an arc.

Since $C_n(X) - C_{n-1}(X)$ is open in $C_n(X)$ and $M_0 \in \mathcal{M}(C_n(X) - C_{n-1}(X))$, there exists $\varepsilon_0 > 0$ and for each $i \in \{1, \ldots, n - 1\}$ there exists a 2-cell $L_i$ such that $B_Z(\varepsilon_0, y_0) \subset V$, $L_i(0) \subset L_i \subset \mathcal{G}_i$ and $h^{-1}(L) \subset \langle \int_B(Q_1), \ldots, \int_B(Q_n) \rangle \cap C_n(X) \cap W$ for each $L \in \mathcal{L}$, where

$$\mathcal{L} = \{A \cup L_1 \cup \ldots \cup L_{n-1} \in C_n(Z) : H(A, \{y_0\}) < \varepsilon_0 \} \text{ and } L_i \in \mathcal{L}_i \text{ for each } i \in \{1, \ldots, n-1\}.$$

Fix a sequence $\{y_m\}_{m=1}^\infty$ in $Z$ such that $\lim y_m = y_0$ and $y_m \in P_m$ for each $m \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ be such that $y_m \in B_Y(\frac{2}{m}, y_0)$ for each $m \geq N_0$. For each $m \geq N_0$, choose a subcontinuum $P_m$ of $Z$ such that $\dim(P_m) = \frac{2}{m}$ and $y_m \in P_m$. Then $P_m \subset V$, so $P_m \subset D_m$. Taking a subsequence if necessary, we may assume that $\lim P_m = P_0$ for some $P_0 \in C(Z)$ and $\lim C(P_m) = P$ and some $P \subset C(Z)$. Then $y_0 \in P_0$, $\dim(P_0) = \frac{2}{m}$ and $P \subset C(P_0)$. Then $P_0 \subset D_0$. Fix points $p_0, q_0 \in P_0$ such that $p_0 \neq q_0$ and choose sequences $\{p_m\}_{m=N_0}^\infty$, $\{q_m\}_{m=N_0}^\infty$ is $Z$ such that $\lim p_m = p_0$, $\lim q_m = q_0$ and for each $m \geq N_0$, $p_m, q_m \in P_m$. Given $m \geq N_0$, choose order arcs $\alpha_m, \beta_m$ from $\{p_m\}$ to $P_m$ and $\{q_m\}$ to $P_m$, respectively. Let $T_m = \text{Im} \alpha_m$ and $S_m = \text{Im} \beta_m$. We may assume also that $\lim T_m = T_0$ and $\lim S_m = S_0$, for some $T_0, S_0 \in C(C(P_0))$. By [15, Remark 1.34], each of the sets $T_0$ and $S_0$ are images of respective order arcs from $\{p_0\}$ to $P_0$ and $\{q_0\}$ to $P_0$. Notice that $T_1(P_0) \cup T_0 \cup S_0 \subset P$.

Given $m \in \{0, N_0, N_0 + 1, \ldots\}$ and a subcontinuum $A$ of $P_m$, since $A \subset P_m \subset B_Y(\varepsilon_0, y_0)$, $H(A, \{y_0\}) < \varepsilon_0$. Thus, for each choice of elements $L_i \in L_i$ ($i \in \{1, \ldots, n-1\}$), $A \cup L_1 \cup \ldots \cup L_{n-1} \in \mathcal{L}$.

Given $L \in \mathcal{L}$, $h^{-1}(L) \in \langle \int_B(Q_1), \ldots, \int_B(Q_n) \rangle \cap C_n(X) \cap W$. Since $Q_1, \ldots, Q_n$ are pairwise disjoint, we have that $h^{-1}(L)$ has exactly $n$ components.
and they are $h^{-1}(L) \cap Q_1, \ldots, h^{-1}(L) \cap Q_n$. Let $\mathcal{A} = C(Q_1) \times \ldots \times C(Q_n)$. Define $\sigma : \mathcal{L} \to \mathcal{A}$ by $\sigma(L) = (h^{-1}(L) \cap Q_1, \ldots, h^{-1}(L) \cap Q_n)$. Clearly, $\sigma$ is an embedding. By [15, Theorem 2.1], $\dim[C(P_0)] \geq 2$. Since $\mathcal{L}$ contains a topological copy of $C(P_0) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_{n-1}$ and the dimension of this set is $\dim[C(P_0)] + 2(n - 1) \geq 2n$ [9, Remark at the end of section 4 of Chapter III], we have that $\dim[\mathcal{A}] \geq 2n$. Since each $C(Q_i)$ is a one-point set or a $2$-cell, $\dim[\mathcal{A}] \leq 2n$, so $\dim[\mathcal{A}] = 2n$. This implies that each $Q_i$ is an arc and $\mathcal{A}$ is a $2n$-cell.

Since $F_1(P_0) \subset \mathcal{P}$, we have $\dim(\mathcal{P}) \geq 1$. To finish the proof that $Z$ is locally connected, we analyze two cases.

**Case 1.** $\dim(\mathcal{P}) \geq 2$.

In this case, let $\mathcal{L}_0 = \{ \mathcal{A} \subset \mathcal{L} \cap \mathcal{L}_i : \mathcal{A} \subset \mathcal{P} \}$, for each $i \in \{1, \ldots, n-1\}$. Since $\mathcal{L}_0$ is homeomorphic to $\mathcal{L} \times \mathcal{P}$, $\dim(\mathcal{L}_0) \geq 2n$. Since $\sigma|_{\mathcal{L}_0} : \mathcal{L}_0 \to \mathcal{A}$ is an embedding, $\dim(\mathcal{L}_0) = 2n$. By [9, Theorem IV 3], $\dim(\mathcal{A})$ is nonempty. Let $\mathcal{L} = A \cap L_1 \cup \ldots \cup L_{n-1} \subset C_0(Z)$ such that $A \subset \mathcal{P}$ and $L_i \subset \mathcal{L}_i$ for each $i \in \{1, \ldots, n-1\}$. Since $A \subset \mathcal{P} = \lim C(P_m)$, there exists a sequence $\{A_m\}_{m=1}^{\infty}$ in $C(Z)$ such that $\lim A_m = A$ and $A_m \subset C(P_m)$ for each $m \in \mathbb{N}$. Then $\dim(\mathcal{A}_m \cup L_1 \cup \ldots \cup L_{n-1}) = \dim(\mathcal{A} \cap L_1 \cup \ldots \cup L_{n-1})$. Since $\mathcal{L}_0$ is one-to-one, $A_m \cap L_1 \cup \ldots \cup L_{n-1} \subset \mathcal{L}_0$. This implies that $A_m \cap L_1 \cup \ldots \cup L_{n-1} = A' \cup L'_1 \cup \ldots \cup L'_{n-1}$, where $A' \subset \mathcal{P}$ and $L'_i \subset \mathcal{L}_i$ for each $i \in \{1, \ldots, n-1\}$. Intersecting these sets with $B_{C_n(Z)}(\varepsilon_0, \{y_0\})$, we obtain that $A_m = A'$. This is a contradiction since $A_m \subset C(P_m)$, $A' \subset \mathcal{P} \subset C(P_0)$ and $P_0 \cap P_m = \emptyset$. Therefore, this case is impossible.

**Case 2.** $\dim(\mathcal{P}) = 1$.

Let $S^+$ (respectively, $S^-$) be the upper (lower) half of $S^1$. Since $F_1(P_0) \cap (T_0 \cup S_0) = \{\{y_0\}, \{q_0\}\}$, by Urysohn’s lemma for metric spaces, there exists a map $f : F_1(P_0) \cup T_0 \cup S_0 \to S^1$ such that $f(F_1(P_0)) = S^-$, $f(\{q_0\}) = \{(-1, 0)\}$, $f(\{y_0\}) = \{(1, 0)\}$ and $f(T_0 \cup S_0) = S^+$. Since $\dim(\mathcal{P}) = 1$, by [9, Theorem VI 4] the map $f$ can be extended to a map (we also call $f$ to the extension) $f : \mathcal{U} \to S^1$, where $\mathcal{U}$ is an open subset of $C(Z)$ such that $\mathcal{P} \subset \mathcal{U}$. Since $\dim(T_m \cup S_m) = \dim(T_0 \cup S_0)$ and $\lim F_1(P_m) = F_1(P_0)$, there exists $m \geq N_0$ such that $C(P_m) \subset \mathcal{U}$, $f(T_m \cup S_m) \subset N_{S^1}(\frac{1}{10}, S^+)$, $f(F_1(P_m)) \subset N_{S^1}(\frac{1}{10}, S^-)$, $f(\{p_m\}) \subset N_{S^1}(\frac{1}{10}, \{(-1, 0)\})$ and $f(\{q_m\}) \subset N_{S^1}(\frac{1}{10}, \{(1, 0)\})$. Lemma 5.12 of [17] and the fact that $F_1(P_m) \cap (T_m \cup S_m) = \{p_m, q_m\}$ imply that $f|F_1(P_m) \cap T_m \cup S_m$ cannot be lifted (that is, there is not a map $f_1 : F_1(P_m) \cup T_m \cup S_m \to \mathbb{R}$ such that $f|F_1(P_m) \cup T_m \cup S_m = (\cos f_1, \sin f_1)$). But, by [2, Lemma 13], $f|C(P_m)$ can be lifted. Since $F_1(P_m) \cup T_m \cup S_m \subset C(P_m)$, we conclude that $f|F_1(P_m) \cup T_m \cup S_m$ can be lifted. This contradiction proves that this case is also impossible.

Therefore, we have shown that $Z$ is locally connected.
Now, suppose that $Z$ contains a simple triod $T$, we may assume that $T \neq Z$, so we can construct arcs $J_1, \ldots, J_{n-1}$ in $Z$ such that $T, J_1, \ldots, J_{n-1}$ are pairwise disjoint. Since $C(T) \times C(J_1) \times \ldots \times C(J_{n-1})$ is naturally embedded in $C_n(Z)$, by [13, Examples 5.1 and 5.4], $C(T) \times C(J_1) \times \ldots \times C(J_{n-1})$ contains a $(2n+1)$-cell. This contradicts Claim 3 and ends the proof that $Z$ does not contain simple triods. Hence, $Z$ is an arc or a simple closed curve. Using an order arc from $Z$ to $Y$, it is possible to construct a subcontinuum $Z_1$ of $Y$ such that $Z \subseteq Z_1$ and $Y_0 \not\subseteq Z_1$. Thus, we can apply what we have proved to $Z_1$ and conclude that $Z_1$ is an arc or a simple closed curve. Therefore, $Z$ is an arc. This completes the proof of Claim 5.

Claim 6. If $D \in C_n(Y)$ and $Y_0 \not\subseteq D$, then $D \in W_n(Y)$. Moreover, $W_n(X) = C_n(X) - \{X\}$.

We prove the first part of Claim 6, the second one can be made with similar arguments. Let $V$ be an open subset of $Y$ such that $D \subseteq V$ and $Y_0 \not\subseteq \text{cl}_Y(V)$. Let $Z$ be a component of $V$. Let $W$ be the component of $V$ containing $Z$. By Claim 5, $Z$ is an arc or a one-point set. Let $B$ be the component of $\text{cl}_Y(V)$ such that $Z \subseteq B$. Then $B$ is nondegenerate. By Claim 5, $B$ is an arc. By [16, Theorem 12.10], $\text{cl}_Y(W) \cap (Y - V) \neq \emptyset$. Thus, $W$ is not compact. Then $W$ is a noncompact connected subset of $B$. Hence, $W$ is homeomorphic either to $(0, 1)$ or $(0, 1)$. That is, $W$ is a wire. This ends the proof of Claim 6.

Claim 7. If $Z \in C(Y) - F_1(Y)$ and $Y_0 \not\subseteq Z$, then $h^{-1}(Z)$ is connected.

We prove Claim 7. Let $A = h^{-1}(Z)$. By Claim 6, $Z \in W_1(Y)$, and by Theorem 3, $Z \not\subseteq Z_n(Y)$. Since $A \neq X$, by Claim 6, $A \in W_n(X)$. Since $h$ is a homeomorphism, $Z \not\subseteq Z_n(Y)$ and the definition of $Z_n(X)$ is given in terms of topological properties that are preserved under homomorphisms, we obtain that $A \not\subseteq Z_n(X)$. By Theorem 3, $Z$ has a basis $\mathcal{B}$ of neighborhoods in $C_n(Y)$ such that for each $V \in \mathcal{B}$, if $C$ is the component of $V$ that contains $Z$, then $C \cap Z_n(Y)$ is connected. Since $Y_0 \not\subseteq Z$, we can ask that for each $V \in \mathcal{B}$ and each $B \in V$, $Y_0 \not\subseteq B$, then by Claim 6, $B \in W_n(Y)$ and $h(V) \not\subseteq V$. Using the fact that $h$ is a homeomorphism and the second part of Claim 6, it is easy to show that if $V \in \mathcal{B}$ and $C$ is the component of $V$ that contains $Z$, then $h^{-1}(C) \cap Z_n(X) = h^{-1}(C \cap Z_n(Y))$. Define $h^{-1}(\mathcal{B}) = \{h^{-1}(V) \in C_n(X) : V \in \mathcal{B}\}$. Then $h^{-1}(\mathcal{B})$ is a basis of neighborhoods of $A$ in $C_n(X)$. Given $V \in \mathcal{B}$ and $C$ the component of $V$ that contains $Z$, the equality $h^{-1}(C) \cap Z_n(X) = h^{-1}(C \cap Z_n(Y))$ implies that $h^{-1}(C) \cap Z_n(X)$ is connected. Hence, we can apply Theorem 3 to conclude that $A \in W_1(X)$. In particular, $A$ is connected. Hence, $h^{-1}(Z)$ is connected.

Claim 8. Let $K_1, \ldots, K_r$ be composants of $X$, where $r \leq n$. Then $C(X) \subseteq \text{cl}_{C_n(X)}((K_1, \ldots, K_r) \cap C_n(X))$.

We prove Claim 8. Since $C(X) - (\{X\} \cup F_1(X))$ is dense in $C(X)$, it is enough to show that $C(X) - (\{X\} \cup F_1(X)) \subseteq \text{cl}_{C_n(X)}((K_1, \ldots, K_r) \cap C_n(X))$. 9
Let \( E \subseteq C(X) \setminus \{ \{X\} \cup F_1(X) \) . Then \( E \) is an arc. Let \( a_1, a_2 \) be the end points of \( E \). Let \( K \) be a composant of \( X \). Given \( i \in \{ 1, 2 \} \), let \( A_i(K) = \{ p \in E : \) there exists a sequence \( \{ B_m \}_{m=1}^\infty \) in \( \langle K \rangle \cap C(X) \) converging to a subcontinuum \( B \) of \( E \) and \( p, a_i \in B \). Since \( K \) is dense in \( X \), \( \{ a_i \} \in A_i(K) \). It is easy to show that \( A_i(K) \) is closed in \( E \) and that if \( p \in A_i(K) \), then the arc \( E \) joining \( a_i \) and \( p \) is contained in \( A_i(K) \). Thus \( A_i(K) \) is a subcontinuum of \( E \).

We claim that \( E = A_1(K) \cup A_2(K) \). Take \( p \in E \setminus \{ a_1, a_2 \} \). Let \( \{ p_m \}_{m=1}^\infty \) be a sequence in \( X \) such that \( \lim p_m = p \). Let \( \mu : C(X) \to [0, 1] \) be a Whitney map, where \( \mu(X) = 1 \) ([13, Theorem 13.4]). Using order arcs, it is possible to find a subcontinuum \( B_m \) of \( X \) such that \( p_m \in B_m \) and \( \mu(B_m) = \mu(E) \), for each \( m \in \mathbb{N} \). We may assume that \( \lim B_m = B \) for some \( B \in C(X) \). For each \( m \in \mathbb{N} \), since \( E \neq X \), we have that \( B_m \neq X \). This implies that \( B_m \in \langle K \rangle \cap C(X) \). Notice that \( p \in B \). Since \( E \) and \( B \) are proper subcontinua of \( X \), \( E \cup B \) is a subcontinuum of \( X \), so \( E \cup B \) is an arc. Since \( \mu(E) = \mu(B) \), it is not possible that \( B \subseteq E \). This implies that \( a_1 \in B \) or \( a_2 \in B \).

For each \( m \in \mathbb{N} \), let \( \alpha_m : [0, 1] \to C(B_m) \) be an order arc from \( \{ p_m \} \) to \( B_m \). We may assume that \( \lim \text{Im} \alpha_m = \gamma \) for some \( \gamma \in C(C(X)) \). By [15, Remark 1.34], \( \gamma \) is the image of an order arc \( \alpha : [0, 1] \to C(X) \) that joins \( \{ p \} \) to \( B \). Let \( s_0 = \{ s \in [0, 1] : \gamma(s) \cap \{ a_1, a_2 \} \neq \emptyset \} \). Given \( s < s_0 \), \( \gamma(s) \cap \{ a_1, a_2 \} = \emptyset \). \( \gamma(s) \) intersects the arc \( E \) and \( \gamma(s) \) is contained in the arc \( E \cup B \). This implies that \( \gamma(s) \in E \). Hence, \( \gamma(s_0) \subseteq E \). Since \( \gamma(s_0) \) belongs to \( \lim \text{Im} \alpha_m \), \( \gamma(s_0) \) satisfies the conditions in the definition of \( A_i(K) \), this allows us to conclude that \( p \in A_1(K) \cup A_2(K) \). We have shown that \( E = A_1(K) \cup A_2(K) \).

In the case that \( r = 1 \), by the connectedness of \( E \), we conclude that there exists a point \( p \in A_1(K_1) \cap A_2(K_1) \). Let \( \{ B_m \}_{m=1}^\infty \) and \( \{ C_m \}_{m=1}^\infty \) be sequences in \( \langle K_1 \rangle \cap C(X) \) converging to respective subcontinua \( B \) and \( C \) of \( E \) satisfying \( p, a_1 \in B \) and \( p, a_2 \in C \). Then \( B \cup C \) is a subcontinuum of \( E \) containing \( a_1 \) and \( a_2 \). Thus, \( E = B \cup C \). Hence, \( E = \lim B_m \cup C_m \). Since \( B_m \cup C_m \in \langle K_1 \rangle \cap C_n(X) \) for each \( m \in \mathbb{N} \), we conclude that \( E \in eC_n(X)(\langle K_1 \rangle \cap C_n(X)) \).

In the case that \( r \geq 2 \), take the natural order in \( E \) such that \( a_1 < a_2 \). By the connectedness of \( E \), we can choose points \( p_1 \in A_1(K_1) \cap A_2(K_1) \) and \( p_2 \in A_1(K_2) \cap A_2(K_2) \). We can assume that \( p_1 \leq p_2 \). Let \( \{ B_m \}_{m=1}^\infty \) and \( \{ C_m \}_{m=1}^\infty \) be sequences in \( \langle K_1 \rangle \cap C(X) \) and \( \langle K_2 \rangle \cap C(X) \), respectively, converging to subcontinua \( B \) and \( C \), respectively, of \( E \) satisfying \( p_1, a_1 \in B \) and \( p_2, a_2 \in C \). Thus, \( E = B \cup C \) and \( E = \lim B_m \cup C_m \). For each \( i \in \{ 3, \ldots, r \} \), choose a sequence \( \{ x_m^{(i)} \}_{m=1}^\infty \) in \( K_i \) such that \( \lim x_m^{(i)} = a_i \). For each \( m \in \mathbb{N} \), let \( E_m = B_m \cup C_m \cup \{ x_m^{(3)}, \ldots, x_m^{(r)} \} \). Then \( E_m \in \langle K_1, \ldots, K_r \rangle \cap C_n(X) \) and \( \lim E_m = E \). This ends the proof of Claim 8.

**Claim 9.** \( Y_0 = Y \).

Since \( C_n(X) \setminus \{ X \} \) has uncountably many arc components (Claim 1), \( C_n(Y) \setminus \{ Y_0 \} \) has uncountably many arc components. Let \( G \) be an arc component of
$C(Y) - \{Y_0\}$ such that $Y \notin \mathcal{G}$. Given $G \in \mathcal{G}$, if $G \not\subseteq Y_0$, then an order arc from $G$ to $Y$ is a path connecting $G$ to $Y$ without passing through $Y_0$, a contradiction. Thus, $G \subset Y_0$ and $\mathcal{G} \subset C_n(Y_0)$. By [10, Corollary 2.2] $h^{-1}(\mathcal{G})$ is of the form $h^{-1}(\mathcal{G}) = (K_1, \ldots, K_r) \cap C_n(X)$ for some $r \leq n$ and composants $K_1, \ldots, K_r$ of $X$. Suppose that $Y_0 \neq Y$. Take a point $y \in Y - Y_0$. Then there exists a nondegenerate subcontinuum $Z$ of $Y$ such that $y \in Z \subset Y - Y_0$. Let $E = h^{-1}(Z)$. By Claim 7, $E$ is a subcontinuum of $X$. By Claim 8, $E \in \text{cl}_{C_n(X)}((K_1, \ldots, K_r) \cap C_n(X))$. Then $Z \in \text{cl}_{C_n(Y)}(h((K_1, \ldots, K_r) \cap C_n(X))) = \text{cl}_{C_n(Y)}(\mathcal{G}) \subset C_n(Y_0)$ and $Z \subset Y_0$. This contradicts the choice of $Z$ and completes the proof of Claim 9.

We have shown that $Y$ is an indecomposable continuum (Claim 2) such that each one of its nondegenerate proper subcontinua are arcs (Claim 5). Moreover, $h^{-1}(Z) \in C(X)$ for each $Z \in C(Y)$ (this follows from Claim 7). Thus, $Y$ satisfies the initial conditions we had for $X$. By symmetry, we can conclude that $h(W) \in C(Y)$ for each $W \in C(X)$. Hence, $h|C(X) : C(X) \to C(Y)$ is a homeomorphism. By the proof of Theorem 3 of [14], $h(F_1(X)) = F_1(Y)$. This proves that $X$ has unique hyperspace $C_n(X)$ and $C_n(X)$ is rigid. ■

**QUESTION 5.** Suppose that $X$ is a wired continuum. Is it true that $C_2(X)$ is not rigid? It would be interesting to determine if $C_2(X)$ is rigid for the Buckethandle continuum (see [16, 2.9] for a description), the solenoids (see [16, 2.8] for a description) or the cone over the Cantor set.

**QUESTION 6** [12, Problem 23]. Suppose that $X$ is an indecomposable arc continuum. Does $X$ have unique hyperspace $C_2(X)$? It would be interesting to solve this question for the case that $X$ is the buckethandle or a solenoid.

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R. Hernández-Gutiérrez
Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, A.P. 61-3, Xangari, Morelia, Michoacán, 58089, México

A. Illanes and V. Martínez-de-la-Vega
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, México, 04510, D.F.

E-mail addresses:
rod@matem.unam.mx,
ilanes@matem.unam.mx,
vmvm@matem.unam.mx

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