Uniqueness of hyperspaces for Peano continua

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Abstract

For a metric continuum $X$ and a positive integer $n$, let $C_n(X)$ be the hyperspace of nonempty closed subsets of $X$ with at most $n$ components. We say that $X$ has unique hyperspace $C_n(X)$ provided that, if $Y$ is a continuum and $C_n(X)$ is homeomorphic to $C_n(Y)$, then $X$ is homeomorphic to $Y$. In this paper we study which Peano continua $X$ have unique hyperspace $C_n(X)$. We find some sufficient and also some necessary conditions for a Peano continuum $X$ to have unique hyperspace $C_n(X)$. Our results generalize all the previous known results on this subject. We also give some significant examples.


Key Words and Phrases: Almost Meshed, Continuum, Dendrite, Hyperspace, Local Dendrite, Meshed, Unique Hyperspace, Peano continuum.

1 Introduction

A continuum is a nondegenerate compact connected metric space. A Peano continuum is a locally connected continuum. For a continuum $X$ and $n \in \mathbb{N}$, consider the following hyperspaces:

$$2^X = \{ A \subset X : A \text{ is closed and nonempty} \},$$

$$C(X) = \{ A \in 2^X : A \text{ is connected} \},$$

$$C_n(X) = \{ A \in 2^X : A \text{ has at most } n \text{ components} \}.$$

All the hyperspaces considered are metrized by the Hausdorff metric $H_X$. Note that $C(X) = C_1(X)$.

We say that a continuum $X$ has unique hyperspace $C_n(X)$ provided that the following implication holds: if $Y$ is a continuum and $C_n(X)$ is homeomorphic to $C_n(Y)$, then $X$ is homeomorphic to $Y$.

Given a continuum $X$, let

$$G(X) = \{ p \in X : p \text{ has a neighborhood } M \text{ in } X \text{ such that } M \text{ is a finite graph} \},$$

and $P(X) = X - G(X)$. 

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A free arc in $X$ is an arc $\alpha \subset X$, with end points $p$ and $q$ such that $\alpha - \{ p, q \}$ is open in $X$. The continuum $X$ is said to be almost meshed provided that the set $G(X)$ is dense in $X$, and an almost meshed continuum $X$ is meshed provided that $X$ has a basis of neighborhoods $B$ such that for each element $U \in B$, $U - P(X)$ is connected. A dendrite is a locally connected continuum without simple closed curves. Let $\mathcal{D}$ denote the class of dendrites with a closed set of end points.

Using the results of R. Duda in [11, 9.1], G. Acosta [1, Theorem 1] observed that finite graphs different from both an arc and a simple close d curve have unique hyperspace $C(X)$. A. Illanes proved in [16] and [17] that finite graphs have unique hyperspaces $C_n(X)$, for each $n \geq 2$.

In [13] D. Herrera-Carrasco showed that if $X$ is in $\mathcal{D}$ and $X$ is not an arc, then $X$ has unique hyperspace $C(X)$. This result was extended in [15], where D. Herrera-Carrasco and F. Macías-Romero proved that if $X \in \mathcal{D}$, then $X$ has unique hyperspace $C_n(X)$ for every $n \geq 3$. The case $n = 2$ has also been solved. It was more difficult so the two papers [14] and [18] were needed to complete its solution. G. Acosta and D. Herrera-Carrasco [2] have shown that if $X$ is a dendrite and $X \notin \mathcal{D}$, then there are uncountable many non-homeomorphic continua $Y$ such that $C(X)$ is homeomorphic to $C(Y)$. Thus, a dendrite $X$ that is not an arc belongs to $\mathcal{D}$ if and only if $X$ has unique hyperspace $C(X)$.

Recently [3], G. Acosta, D. Herrera-Carrasco and F. Macías-Romero have proved that if $X$ is a locally $\mathcal{D}$-continuum (that is, $X$ is a continuum such that each point has a basis of neighborhoods $B$ such that each element in $B$ is an element of $\mathcal{D}$) that is not an arc, then $X$ has unique hyperspace $C(X)$.

On the other hand, the well-known Curtis-Schori theorem (see [9] and [10]) states that if $X$ is a Peano continuum containing no free arcs, then $C(X)$ is homeomorphic to the Hilbert cube. This is why the problem of determining whether a Peano continuum $X$ has unique hyperspace is open only when $X$ contains free arcs.

In this paper we are interested in studying which Peano continua $X$ have unique hyperspace $C_n(X)$. The main results are the following.

A. If a Peano continuum has a nonempty open subset without free arcs (that is, $X$ is not almost meshed), then $X$ does not have unique hyperspace $C_n(X)$ for any $n \in \mathbb{N}$ (Theorem 20). Thus, for a Peano continuum $X$ to have unique hyperspace, we at least need $X$ to be almost meshed.

B. If $X$ is meshed we obtain a completely opposite result (Theorem 37). For $n \neq 1$, $X$ has unique hyperspace $C_n(X)$. If further $X$ is neither an arc nor a simple closed curve, then $X$ has unique hyperspace $C(X)$ (Theorem 37). Recall that if $X$ is either an arc or a simple closed curve, then $C(X)$ is a 2-cell. Thus, the problem of determining if a Peano continuum $X$ has unique hyperspace $C_n(X)$ is open only when $X$ is almost meshed but not meshed.

C. The class of meshed continua contains the following classes: (a) finite graphs, (b) $\mathcal{D}$, (c) locally $\mathcal{D}$ continua. Hence, Theorem 37 covers all the known cases of continua $X$ having unique hyperspace $C_n(X)$.

D. If $X$ is almost meshed and $X - P(X)$ is disconnected, then $X$ does not have unique hyperspace $C(X)$ (Corollary 23).
E. Let \( Z_0 = ([−1, 1] \times \{0\}) \cup (\bigcup\{\{-1/m\} \times [0, 1/m] : m \geq 2\} \). Then \( Z_0 \) plays an important role in this topic:

(a) if a dendrite \( X \) contains \( Z_0 \), then \( X \not\in \mathcal{D} \) and \( X \) does not have unique hyperspace \( C(X) \) ([2]);
(b) \( Z_0 \) is almost meshed, \( \mathcal{P}(Z_0) = \{(0, 0)\}, Z_0 - \mathcal{P}(Z_0) \) is disconnected;
(c) \( Z_0 \) is not meshed (Lemma 3);
(d) The dendrite \( Z_3 = Z_0 \cup (\bigcup\{-1/m\} \times [0, 1/m] : m \geq 2) \) has unique hyperspace \( C_2(Z_3) \) (Example 39);
(e) if we add the segment \( \{0\} \times [0, 1] \) to \( Z_3 \), that is, if \( Z_1 = Z_3 \cup (\{0\} \times [0, 1]) \), then \( Z_1 \) does not have unique hyperspace \( C_2(Z_1) \) (Example 43);
(f) if we add the arc \( L = (\{-1, 1\} \times [0, 1]) \cup ([−1, 1] \times \{1\}) \), that is, if \( Z_2 = Z_0 \cup L \), then \( Z_2 - \mathcal{P}(Z_2) \) is connected, \( Z_2 \) is not meshed and \( Z_2 \) has unique hyperspace \( C(Z_2) \) (Example 38).

\[ \begin{align*}
Z_0 & \quad \quad \quad Z_1 \\
\begin{array}{c}
Z_3 \\
\begin{array}{c}
Z_2
\end{array}
\end{array}
\end{align*} \]

Figure 1

A discussion about uniqueness of other hyperspaces can be found in the introduction of [18].

2 Meshed and almost meshed continua

Given a continuum \( X \) and a subset \( A \) of \( X \), we denote the interior of \( A \) in \( X \) by \( A^\circ \) or \( \text{int}_X(A) \). For \( \varepsilon > 0, p \in X \) and \( A \subset X \), let \( B(\varepsilon, p) \) denote the \( \varepsilon \)-ball around \( p \) in \( X \) and let \( N(\varepsilon, A) = \bigcup\{B(\varepsilon, a) : a \in A\} \). Given \( A \in \mathcal{C}_n(X) \), we denote by \( \dim_A[C_n(X)] \) the dimension of the space \( C_n(X) \) at the element \( A \). Let

\[ \mathcal{F}A(X) = \bigcup\{J^\circ : J \text{ is a free arc in } X\}. \]
Given \( n \in \mathbb{N} \) and a continuum \( X \), let

\[
\mathfrak{F}_n(X) = \{ A \in C_n(X) : \dim_A [C_n(X)] \text{ is finite} \}.
\]

The set \( \mathfrak{F}_1(X) \) is simply denoted by \( \mathfrak{F}(X) \).

Given subsets \( U_1, \ldots, U_m \) of \( X \), let \( \langle U_1, \ldots, U_m \rangle = \{ A \in C_n(X) : A \subset U_1 \cup \ldots \cup U_m \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\} \} \). It is known (see [23, 4.24]) that the family of all sets of the form \( \langle U_1, \ldots, U_m \rangle \), where \( m \in \mathbb{N} \) and each \( U_i \) is open in \( X \), is a basis for the topology in \( C_n(X) \).

We describe some examples in the Euclidean plane \( \mathbb{R}^2 \). Given two different points \( p, q \in \mathbb{R}^2 \), let \( pq \) denote the convex segment joining them.

Let \( Z_0 = ([−1, 1] \times \{0\}) \cup (\bigcup \{ [0, \frac{1}{n}] : m \geq 2 \}) \). Then \( Z_0 \) is a dendrite, \( Z_0 \notin \mathcal{D} \), \( \mathcal{P}(Z_0) = \{(0, 0)\} \), \( Z_0 \) is almost meshed but \( Z_0 \) is not meshed.

Let \( F_ω = \bigcup \{ (0,0)(\frac{1}{m}, \frac{1}{m^2}) : m \in \mathbb{N} \} \). Then \( F_ω \) is a dendrite, \( F_ω \notin \mathcal{D} \), \( \mathcal{P}(F_ω) = \{(0,0)\} \), \( F_ω \) is almost meshed but \( F_ω \) is not meshed.

In [5] it was proved that a dendrite \( X \) is in \( \mathcal{D} \) if and only if \( X \) does not contain a topological copy of neither \( Z_0 \) nor \( F_ω \).

Note that meshed continua do not need to be local dendrites. For example, the continuum \( X \) described in Example 10.38 of [23] (Figure 10.38 (a)) is meshed and \( \mathcal{P}(X) \) is the segment \( A_0 = [0, 1] \times \{0\} \).

The following lemma is easy to prove.

**Lemma 1** Let \( X \) be a continuum. Then \( \text{cl}_X(G(X)) = \text{cl}_X(\mathcal{F}_A(X)) \). Therefore, \( X \) is almost meshed if and only if \( \mathcal{F}_A(X) \) is dense in \( X \).

**Lemma 2** If \( X \) is a meshed continuum, then \( X \) is a Peano continuum.

**Proof.** Let \( B \) be a basis of neighborhoods of \( X \) such that, for each element \( U \in \mathcal{B} \), \( U - \mathcal{P}(X) \) is connected. Since \( X \) is almost meshed, \( \mathcal{P}(X) \supseteq \emptyset \). Thus, for each \( U \in \mathcal{B} \), \( \text{int}_X(U) \subset \text{cl}_X(U - \mathcal{P}(X)) \). Therefore, the family \( \{ \text{cl}_X(U - \mathcal{P}(X)) : U \in \mathcal{B} \} \) is a basis of connected neighborhoods for \( X \). Hence, \( X \) is connected im kleinen and then \( X \) is locally connected.

**Lemma 3** Let \( X \) be a continuum. Then \( X \) is meshed if and only if \( X \) is almost meshed and \( X \) has a basis \( \mathcal{D} \) of open connected subsets of \( X \) such that, for each element \( U \in \mathcal{D} \), \( U - \mathcal{P}(X) \) is connected.

**Proof.** The sufficiency is immediate from the definition of meshed continuum. Now, suppose that \( X \) is meshed. Let \( B \) be a basis of neighborhoods of \( X \) such that, for each element \( U \in \mathcal{B} \), \( U - \mathcal{P}(X) \) is connected. Let \( p \in X \) and \( W \) be an open subset of \( X \) such that \( p \in W \). Let \( U \in \mathcal{B} \) be such that \( p \in \text{int}_X(U) \subset U \subset W \). By Lemma 2, there exists an open connected subset \( Z \) of \( X \) such that \( p \in Z \subset \text{int}_X(U) \). Since \( \mathcal{P}(X) \) is a closed subset of \( X \), for each \( x \in U - \mathcal{P}(X) \) there exists an open and connected subset of \( V_x \) of \( X \) such that \( x \in V_x \subset W - \mathcal{P}(X) \). Let \( V = Z \cup (\bigcup \{ V_x : x \in U - \mathcal{P}(X) \}) \). Clearly \( V \) is an open subset of \( X \) such that \( p \in V \subset W \). Since \( (U - \mathcal{P}(X)) \cup (\bigcup \{ V_x : x \in U - \mathcal{P}(X) \}) \) is a connected subset of \( V - \mathcal{P}(X) \) and \( Z - \mathcal{P}(X) \subset U - \mathcal{P}(X) \), we obtain that
\[ V - \mathcal{P}(X) = (U - \mathcal{P}(X)) \cup (\bigcup \{ V_z : x \in U - \mathcal{P}(X) \} ) \] is an open connected subset of \( X \). Since \( V - \mathcal{P}(X) \subset V \subset \text{cl}_X(V - \mathcal{P}(X)) \), we conclude that \( V \) is connected. This completes the proof of the Lemma. ■

**Theorem 4** Let \( X \) be a Peano continuum, \( n \in \mathbb{N} \) and \( A \in C_n(X) \). Then the following are equivalent.

(a) \( \dim_A[C_n(X)] \) is finite,

(b) there exists a finite graph \( D \) contained in \( X \) such that \( A \subset D^o \),

(c) \( A \cap \mathcal{P}(X) = \emptyset \).

**Proof.** (a) \( \Rightarrow \) (b). Let \( k \) be the number of components of \( A \). In the case that \( k = 1 \), since \( \dim_A[C(X)] \leq \dim_A[C_n(X)] \), we obtain that \( \dim_A[C(X)] \) is finite. Thus, Claim 1 in Lemma 2.2 of [18] guarantees the existence of \( D \). Suppose then that \( k > 1 \). Let \( A_1, \ldots, A_k \) be the components of \( A \). Let \( Z_1, \ldots, Z_k \) be pairwise disjoint subcontinua of \( X \) such that \( A_i \subset Z_i^o \) for each \( i \in \{1, \ldots, k\} \).

Let \( \varphi : C(Z_1) \times \ldots \times C(Z_k) \to (Z_1, \ldots, Z_k) \cap C_k(X) \) be given by \( \varphi(B_1, \ldots, B_k) = B_1 \cup \ldots \cup B_k \). Notice that \( \varphi \) is a homeomorphism. Given \( i \in \{1, \ldots, k\} \),

\[ \dim_{A_i}[C(Z_i)] \leq \dim_{A_1, \ldots, A_i}([C(Z_1) \times \ldots \times C(Z_k)] = \dim_A([Z_1, \ldots, Z_k]) \cap C_k(X)] \leq \dim_A[C_n(X)] < \infty. \]

Since \( C(Z_i) \) is a neighborhood of \( A_i \) in \( C(X) \), \( \dim_A[C(Z_i)] = \dim(A_i[C(Z_i)]) \). Since \( A_i \) is connected, by the first case we considered \( (k = 1) \), there exists a finite graph \( D_{A_i} \) contained in \( X \), such that \( A_i \subset D_{A_i} \). We may assume that \( D_i \subset Z_i \). Since the finite graphs \( D_1, \ldots, D_k \) are pairwise disjoint and \( X \) is arcwise connected [23, 8,23], it is possible to construct a finite number of arcs \( \alpha_1, \ldots, \alpha_r \) in \( X \) such that \( D = D_1 \cup \ldots \cup D_k \cup \alpha_1 \cup \ldots \cup \alpha_r \) is a finite graph. Since \( A \subset D^o \), the proof of (a) \( \Rightarrow \) (b) is finished.

(b) \( \Rightarrow \) (a). Suppose that \( A \subset D^o \) for some finite graph \( D \) in \( X \). Then \( C_n(D) \) is a neighborhood of \( A \) in \( C_n(X) \). Thus, \( \dim_A[C_n(X)] = \dim_A[C_n(D)] \). By the main result in [21], \( \dim_A[C_n(D)] \) is finite (in fact, in Theorem 2.4 of [21] there is an explicit formula for computing \( \dim_A[C_n(D)] \)).

(b) \( \Rightarrow \) (c) is immediate from the definition of \( \mathcal{P}(X) \).

(c) \( \Rightarrow \) (b). Suppose that \( A \cap \mathcal{P}(X) = \emptyset \). For each point \( a \in A \), let \( D_a \) be a finite graph in \( X \) such that \( a \in \text{int}_X(D_a) \). Then there exists a finite graph \( F_a \) in \( X \) such that \( a \in \text{int}_X(F_a) \subset F_a \subset \text{int}_X(D_a) - \mathcal{P}(X) \). By the compactness of \( A \), there exist \( m \in \mathbb{N} \) and \( a_1, \ldots, a_m \in A \) such that \( A \subset \text{int}_X(F_{a_1}) \cup \ldots \cup \text{int}_X(F_{a_m}) \). Let \( F = F_{a_1} \cup \ldots \cup F_{a_m} \). Notice that \( F \) is a subcontinuum of \( X \) such that \( A \subset F^o \).

Since each point \( p \in F \) belongs to the interior in \( X \) of a finite graph contained in \( X \), it is easy to check that \( F \) satisfies conditions (1) and (2) of Theorem 9.10 in [23]. Thus, \( F \) is a finite graph. This completes the proof of the Theorem. ■

**Theorem 5** For a Peano continuum \( X \) the following are equivalent.

(a) \( X \) is meshed,

(b) for each \( n \in \mathbb{N} \), \( \mathfrak{S}_n(X) \) is dense in \( C_n(X) \),

(c) there exists \( n \in \mathbb{N} \) such that \( \mathfrak{S}_n(X) \) is dense in \( C_n(X) \).
Proof. (a) ⇒ (b). Suppose that $X$ is meshed. Let $n \in \mathbb{N}$, $A \in C_n(X)$ and $\varepsilon > 0$. Let $A_1, \ldots, A_k$ be the components of $A$. We assume that $N(\varepsilon, A_1), \ldots, N(\varepsilon, A_k)$ are pairwise disjoint. For each $a \in A$, by Lemma 3, there exists an open connected subset $U_a$ of $X$ such that $a \subset U_a \subset B(\varepsilon, a)$ and the open set $V_a = U_a - P(X)$ is connected. Notice that $V_a$ is nonempty. Fix a point $b(a)$ in $V_a$. Given $i \in \{1, \ldots, k\}$, by the compactness of $A_i$, there exist $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in A_i$ such that $A_i \subset U_{a_1} \cup \ldots \cup U_{a_m} \subset N(\varepsilon, A_i)$. Let $U = U_{a_1} \cup \ldots \cup U_{a_m}$ and $V = V_{a_1} \cup \ldots \cup V_{a_m}$. Notice that $U$ is connected. We see that $V$ is connected. Suppose to the contrary that $V$ is disconnected. Then, we may assume that there exists $r \in \{1, \ldots, m-1\}$ such that $(V_{a_1} \cup \ldots \cup V_{a_r}) \cap (V_{a_{r+1}} \cup \ldots \cup V_{a_m}) = \emptyset$. Since $U$ is connected, the open set $W = (U_{a_1} \cup \ldots \cup U_{a_r}) \cap (U_{a_{r+1}} \cup \ldots \cup U_{a_m})$ is nonempty. Since $\text{int}_X(P(X)) = \emptyset$, $(V_{a_1} \cup \ldots \cup V_{a_r}) \cap (V_{a_{r+1}} \cup \ldots \cup V_{a_m}) = W - P(X)$ is nonempty, a contradiction. Therefore, $V$ is connected. By Theorem 8.26 of [23], $V$ is arcwise connected. Hence, there exists a tree $T_i \subset V$ such that $\{b(a_1), \ldots, b(a_m)\} \subset T_i$. Clearly, $H_X(A_i, T_i) < 2\varepsilon$ and $T_i \cap P(X) = \emptyset$. Let $T = T_1 \cup \ldots \cup T_k \subset C_n(X)$. Then $H_X(A, T) < 2\varepsilon$ and $T \cap P(X) = \emptyset$. By Theorem 4, $\dim_T[C_n(X)]$ is finite, so $T \in \mathcal{F}_n(X)$.

(b) ⇒ (c) is immediate.

(c) ⇒ (a). Suppose that $\mathcal{F}_n(X)$ is dense in $C_n(X)$. First, we see that $G(X)$ is dense in $X$. Let $p \in X$ and $\varepsilon > 0$. Then there exists $A \in \mathcal{F}_n(X)$ such that $H_X(p, A) < \varepsilon$. By Theorem 4, there exists a finite graph $D$ contained in $X$ such that $A \subset D^\circ$. Fix a point $a \in A$. Then $a \in B(\varepsilon, p)$ and $D$ is a neighborhood of $a$. Thus, $a \in B(\varepsilon, p) \cap G(X)$. Therefore, $G(X)$ is dense in $X$.

Now suppose that $X$ is not meshed. Then there exist $p \in X$ and a neighborhood $W$ of $p$ such that, for each open subset $U$ of $X$ such that $p \in U \subset W$, $U - P(X)$ is not connected. Since $X$ is a Peano continuum, there exists an open connected subset $V$ of $X$ such that $p \in V \subset W$. Then $V - P(X) = S \cup T$, where $S$ and $T$ are disjoint open nonempty subsets of $X$. Fix $x \in T$ and pairwise different points $p_1, \ldots, p_n \in S$. Since $V$ is arcwise connected, there exists an arc $\alpha \subset V$ such that $\alpha$ joins $x$ to a point $p_i$ and $\alpha \cap \{p_1, \ldots, p_n\} = \{p_i\}$. We may suppose that $i = n$. Let $A = \{p_1, \ldots, p_{n-1}\} \cup \alpha \subset C_n(X)$. Let $\varepsilon > 0$ be such that $B(\varepsilon, p_1), \ldots, B(\varepsilon, p_{n-1}), N(\varepsilon, \alpha)$ are pairwise disjoint, $B(\varepsilon, p_1) \cup \ldots \cup B(\varepsilon, p_n) \subset S$, $B(\varepsilon, x) \subset T$ and $N(\varepsilon, \alpha) \subset V$. By the density of $\mathcal{F}_n(X)$, there exists $B \in \mathcal{F}_n(X)$ such that $H_X(B, A) < \varepsilon$. Notice that $B$ is contained in the union of the sets $B(\varepsilon, p_1), \ldots, B(\varepsilon, p_{n-1}), N(\varepsilon, \alpha)$ and intersects each one of them. Thus, the components of $B$ are the sets $B_1 = B \cap B(\varepsilon, p_1), \ldots, B_{n-1} = B \cap B(\varepsilon, p_{n-1})$ and $B_n = B \cap N(\varepsilon, \alpha)$. Notice that $B_n \cap B(\varepsilon, p_n) \neq \emptyset$ and $B_n \cap B(\varepsilon, x) \neq \emptyset$. Thus, $B_n$ is connected, $B_n \subset V$ and $B_n$ intersects $S$ and $T$. This implies that $B_n \cap P(X) \neq \emptyset$ and, by Theorem 4, $B \notin \mathcal{F}_n(X)$, a contradiction. This proves that $X$ is meshed and completes the proof of the theorem. ■

Theorem 6 The class of meshed continua contains the following classes.

(a) finite graphs,
(b) $\mathcal{D}$,
(c) locally $\mathcal{D}$ continua.

**Proof.** Since the class of locally $\mathcal{D}$ continua contains class $\mathcal{D}$ and all the finite graphs, we only need to check that locally $\mathcal{D}$ continua are meshed. Let $X$ be a locally $\mathcal{D}$ continuum. Clearly, $X$ is a Peano continuum. By Theorem 3.9 of [3], $\mathfrak{F}(X)$ is dense in $C(X)$, so Theorem 5 implies that $X$ is meshed. ■

3 Free arcs

A free circle $S$, in a continuum $X$, is a simple closed curve $S$ in $X$ such that there exists $p \in S$ such that $S - \{p\}$ is open in $X$. A maximal free arc is a free arc in $X$ which is maximal with respect to inclusion. Let

$$\mathfrak{A}(X) = \{J \subset X : J \text{ is a maximal free arc in } X\}$$

$$\mathfrak{A}_S(X) = \mathfrak{A}(X) \cup \{S \subset X : S \text{ is a free circle in } X\}.$$

A simple triod is a continuum $T$ homeomorphic to the cone over the discrete space $\{1, 2, 3\}$. The point of $T$ corresponding to the vertex of the cone is called the vertex of $T$.

Given an arc $J$ in a continuum $X$ and points $x, y$ in $J$, let $[x, y]^J$ be the subarc of $J$ joining $x$ and $y$, if $x \neq y$, and $[x, y]^J = \{x\}$, if $x = y$. We also define $[x, y] = [x, y]^J - \{y\}$ and $(x, y) = [x, y]^J - \{x, y\}$.

The following lemma is easy to prove.

**Lemma 7** Let $X$ be a continuum and let $J$ be a free arc in $X$. Then:

(a) no point of $J^\circ$ can be the vertex of a simple triod in $X$,

(b) if $J$ and $K$ are free arcs in $X$ and $J^\circ \cap K^\circ \neq \emptyset$, then $J \cup K$ is a free arc or a free circle in $X$.

**Lemma 8** For a Peano continuum $X$, let $\{J_m\}_{m=1}^\infty$ be a sequence of pairwise different elements of $\mathfrak{A}_S(X)$ and $x_m \in J_m$, for each $m \in \mathbb{N}$. If $\lim x_m = x$ for some $x \in X$, then $\lim J_m = \{x\}$ (in $C(X)$).

**Proof.** Note that $X$ is neither an arc nor a simple closed curve. For each $m \in \mathbb{N}$, $x_m \in \text{cl}_X(J_m)$, so we may assume that $x_m \in J_m$. For each $m \in \mathbb{N}$, $\text{Fr}_X(J_m)$ is a nonempty subset of $X$ with at most two elements. Thus, we can put $\text{Fr}_X(J_m) = \{p_m, q_m\}$. Suppose that the sequence $\{J_m\}_{m=1}^\infty$ does not converge to $\{x\}$ in $C(X)$. Since $C(X)$ is compact, there exists a subsequence of $\{J_m\}_{m=1}^\infty$ that converges to some $A \in C(X)$, where $A \neq \{x\}$. We may assume that $\lim J_m = A$, $\lim p_m = p$ and $\lim q_m = q$, for some $p, q \in X$. Note that $p, q, x \in A$. Since $A \neq \{x\}$, we can choose an element $y \in A - \{p, q\}$. Then there exists a sequence $\{y_m\}_{m=1}^\infty$ in $X$ such that $y_m \in J_m$, for each $m \in \mathbb{N}$ and $\lim y_m = y$. By Lemma 3 of [14], $J_m \cap J_k = \emptyset$, if $m \neq k$. Thus, $y \notin J_m$, for every $m \in \mathbb{N}$. Let $U$ be an open connected (then arcwise connected) set in $X$ such that $y \in U$ and $p, q \notin \text{cl}_X(U)$. Let $m_0 \in \mathbb{N}$ be such that, for each $m \geq m_0$,
Let \( y_m \in U \). For each \( m \geq m_0 \), let \( \alpha_m \) be an arc in \( U \) with end points \( y_m \) and \( y \). Since \( y \notin J^0_p \), \( \alpha_m \) contains one of the points \( p_m \) or \( q_m \). This implies that \( p \in \text{cl}_X(U) \) or \( q \in \text{cl}_X(U) \), a contradiction. This completes the proof of the Lemma. \( \blacksquare \)

**Lemma 9** Let \( X \) be a Peano continuum and \( J \) a free arc with an end point \( e \) such that \( e \in J^0 \). Then there exists a free arc \( K \) such that \( J \subset K \), \( e \) is an end point of \( K \), \( e \in K^0 \) and \( K \) contains every free arc in \( X \) containing \( J \).

**Proof.** We may assume that \( X \) is not an arc. Let \( \mathcal{F} = \{ L \subset X : L \) is a free arc in \( X \) such that \( J \subset L \} \). Given \( L \in \mathcal{F} \) let \( p_L \) and \( q_L \) be the end points of \( L \). We claim that \( e \notin \{ p_L, q_L \} \). Suppose to the contrary that \( e \notin \{ p_L, q_L \} \). Since \( e \in J^0 \), there exist points \( x, y \in L \) such that \( e \in \{ x, y \}_L \subset J \). This is a contradiction since \( e \) is an end point of \( J \). Hence, \( e \notin \{ p_L, q_L \} \) and we may assume that the end points of \( L \) are \( p_L \) and \( e \). Since \( e \in J^0 \), we have that \( e \in L^0 \). Thus, \( L - \{ p_L \} \) is open in \( X \).

By Lemma 7 (a), it follows that if \( L, M \in \mathcal{F} \), then \( L \subset M \) or \( M \subset L \).

Let \( U = \bigcup \{ L - \{ p_L \} : L \in \mathcal{F} \} \) and \( K = \text{cl}_X(U) \). We claim that \( K \neq U \). Suppose to the contrary that \( K = U \). Since \( K \) is compact and \( L - \{ p_L \} \) is open for each \( L \in \mathcal{F} \), by the previous paragraph, there exists \( L \in \mathcal{F} \) such that \( K = L - \{ p_L \} \). This is impossible since \( L - \{ p_L \} \) is not compact. Hence, \( K \neq U \).

Fix a point \( p \in K - U \). Since \( X \) is arcwise connected, there exists an arc \( M \) in \( X \) joining \( p \) and \( e \).

We see that \( K = M \). Let \( L \in \mathcal{F} \) and \( z \in L - \{ e, p_L \} \). Then \( X - \{ z \} = (X - [z, e]_L) \cup ([z, e]_L \setminus \{ e \}) \) is a separation of \( X - \{ z \} \). Thus, \( z \) separates \( p \) and \( e \) in \( X \). Hence, \( z \in M \). We have shown that \( L - \{ e, p_L \} \subset M \). Therefore, \( U \subset M \) and \( K \subset M \). Since \( p, e \in K \), we conclude that \( K = M \). Thus, \( U \) is a connected subset of the arc \( M \), \( e \in U \) and \( p \in \text{cl}_X(U) \). This implies that \( U = M - \{ p \} = K - \{ p \} \). Since \( U \) is open in \( X \), we have that \( K \) is a free arc. Thus, \( K \in \mathcal{F} \). Given \( L \in \mathcal{F} \), since \( K \) is closed in \( X \) and \( L - \{ p_L \} \subset K \), we have \( L \subset K \). This completes the proof of the Lemma. \( \blacksquare \)

**Lemma 10** Let \( X \) be a Peano continuum and let \( J \) be a free arc. Then there exists \( K \in \mathfrak{S}(X) \) such that \( J \subset K \).

**Proof.** We may assume that \( X \) is not a simple closed curve and \( J \) is not contained in a free circle in \( X \). Let \( x, y \) be the end points of \( J \). Fix points \( p, q \in (x, y)_J \) such that \( [x, p]_J \cap [q, y]_J = \emptyset \). Let \( Y = X - (p, q)_J \). Then \( Y \) is a compact subset of \( X \). Let \( X_p \) and \( X_q \) be the components of \( Y \) containing \( p \) and \( q \), respectively. Notice that \( \text{Fr}_X(Y) = \{ p, q \}, [x, p]_J \subset X_p \) and \( [q, y]_J \subset X_q \).

By the Boundary Bumping Theorem (Theorem 5.4 of [23]), each component of \( Y \) contains either \( p \) or \( q \). This implies that \( Y = X_p \cup X_q \) and we have that either \( X_p = X_q = Y \) or \( X_p \cap X_q = \emptyset \). Clearly, \( Y \) is locally connected and each \( X_p \) and \( X_q \) are Peano continua. Notice that \( [x, p]_J \) is a free arc of \( X_p \) and \( p \in \text{int}_{X_p}([x, p]_J) \). By Lemma 9, there exists a free arc \( K_p \) of \( X_p \) such that \( [x, p]_J \subset K_p \) and \( p \) is an end point of \( K_p \). Then \( p \in \text{int}_{X_p}(K_p) \) and \( K_p \) contains every free arc in \( X_p \) containing \( [x, p]_J \). Similarly, \( [q, y]_J \) is a free arc of \( X_q \).
q ∈ int_{X, q}([q, y]_J) and there exists a free arc $K_q$ of $X_q$ such that $[q, y]_J ⊂ K_q$. Let $p_0$ (resp., $q_0$) be the other end point of $K_p$ (resp., $K_q$).

Since $[x, p]_J$ is an open arc of $X_p$ and $p ∈ int_{X_q}([x, p]_J)$, $p$ is an end point of each arc in $X_p$ containing $p$. If $p ∈ (q, q_0)_{K_q}$, then $p ∈ X_p ∩ X_q$ and $X_p = X_q$. This implies that $p$ is not an end point of the arc $[q, q_0]_{K_q} ⊂ X_p$, a contradiction. Hence, $p ∉ (q, q_0)_{K_q}$. Since $Fr_X(X_q) ⊂ \{p, q\}$, we have that $(q, q_0)_{K_q}$ is an open set in $X_q$ such that $(q, q_0)_{K_q} ⊂ int_{X}(X_q)$. Hence, $(q, q_0)_{K_q}$ is open in $X$.

Similarly, $(p, p_0)_{K_p}$ is open in $X$. Thus, $K_p$ and $K_q$ are free arcs in $X$. Since $\emptyset ≠ (x, p)_J ⊂ K_p ∩ [x, q]_J$ and $J$ is not contained in a free circle in $X$, by Lemma 7(b), $K_p ∩ [x, q]_J = K_p ∪ [p, q]_J$ is a free arc in $X$. Similarly, $K_q ∩ [p, q]_J$ is a free arc in $X$. Applying again Lemma 7 (b), $K_p ∪ [p, q]_J ∪ K_q = K_p ∪ J ∪ K_q$ is a free arc in $X$ with end points $p_0$ and $q_0$.

Suppose that $L$ is a free arc in $X$ such that $K_p ∪ J ∪ K_q ⊂ L$. Suppose that the end points of $L$ are $u$ and $v$ and $[u, p_0]_L ∩ [q_0, v]_L = \emptyset$. Then $[u, p]_L ⊂ X − (p, q)_J$ and $[u, p]_L ⊂ X_p$. By the maximality of $K_p$, $[u, p]_L = K_p = [p_0, p]_L$. This implies that $u = p_0$. Similarly, $v = q_0$. Hence, $L = K_p ∪ J ∪ K_q$. We have shown that $K_p ∪ J ∪ K_q$ is maximal. This ends the proof of the Lemma.

**Lemma 11** Let $X$ be a Peano continuum and $A ∈ C_n(X)$. Then $dim_A[C_n(X)] ≥ 2n$ and, if $dim_A[C_n(X)] = 2n$, then there exist $k ∈ \mathbb{N}$ and elements $J_1, \ldots, J_k ∈ \mathfrak{A}_S(X)$ such that $A ∈ \{J_1^0, \ldots, J_k^0\}$, where each component of $A$ is contained in some $J_i^0$.

**Proof.** We may assume that $dim_A[C_n(X)]$ is finite. Let $A_1, \ldots, A_k$ be the components of $A$. By Theorem 4, there exists a finite graph $D$ contained in $X$ such that $A ⊂ D^0$. Then $C_n(D)$ is a neighborhood of $A$ in $C_n(X)$. Thus, $dim_A[C_n(X)] = dim_A[C_n(D)]$. By Theorem 2.4 of [21],

$$dim_A[C_n(D)] = 2n + \sum_{x ∈ R(D) ∩ A} (ord_D(x) − 2)$$

where $R(D)$ is the set of ramification points of the graph $D$ and $ord_D(x)$ is the order of the point $x$ in $D$. Since $ord_D(x) ≥ 3$ for each $x ∈ R(D)$, $dim_A[C_n(X)] ≥ 2n$ and, if $dim_A[C_n(X)] = 2n$, then $R(D) ∩ A = \emptyset$. Now, assume that $dim_A[C_n(X)] = 2n$, then for each $i ∈ \{1, \ldots, k\}$ there exists a free arc $L_i$ in $D$ such that $A_i ⊂ int_D(L_i)$. Since $A ⊂ D^0$, $A_i ⊂ int_X(L_i)$ so we may assume that $L_i ⊂ D^0$. This implies that $L_i$ is a free arc in $X$. By Lemma 10, there exists $J_i ∈ \mathfrak{A}_S(X)$ such that $L_i ⊂ J_i$. Therefore, $A ∈ \{J_1^0, \ldots, J_k^0\}$.

**4 Continua that are not almost meshed**

Given a continuum $X$ and a nonempty closed subset $K$ of $X$, let

$$C_n^K(X) = \{A ∈ C_n(X) : K ⊂ A\}, \quad \text{and} \quad C_n(X, K) = \{A ∈ C_n(X) : A ∩ K ≠ \emptyset\}.$$
Given $A, B \in 2^X$ such that $A \subseteq B$, an order arc from $A$ to $B$ is a continuous function $\alpha : [0, 1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$ and, if $0 \leq s < t \leq 1$, then $\alpha(s) \subseteq \alpha(t)$. It is known (see Lemma 15.2 of [19]) that if $A \subseteq B$, then there exists an order arc from $A$ to $B$ if and only if each component of $B$ intersects $A$. Given a closed subset $\mathcal{G}$ of $2^X$, we call $\mathcal{G}$ a growth hyperspace provided that for every $A \in \mathcal{G}$ and $B \in 2^X$ such that $A \subseteq B$ and each component of $B$ intersects $A$, we have $B \in \mathcal{G}$ (equivalently, there is an order arc from $A$ to $B$). Note that the sets $C_n(X)$, $C_n^K(X) = \{A \in C_n(X) : K \subseteq A\}$ and $C_n(X, K) = \{A \in C_n(X) : A \cap K \neq \emptyset\}$ are growth hyperspaces. By the comments at the end of section 2 of [8, Section 2], if $X$ is a Peano continuum and $\mathcal{G} \subset 2^X$ is a growth hyperspace, then $\mathcal{G}$ is an AR.

A compactum is a compact metric space. A map is a continuous function. Given a compactum $Y$ with metric $d$, a closed subset $A$ of $Y$ is said to be a Z-set in $Y$ provided that for each $\varepsilon > 0$ there is a continuous function $f_\varepsilon : Y \to Y - A$ such that $d(f_\varepsilon(y), y) < \varepsilon$ for all $y \in Y$. A continuous function between compacta $f : Y_1 \to Y_2$ is called a Z-map provided that $f(Y_1)$ is a Z-set in $Y_2$.

Given two disjoint continua $X$ and $Y$, and points $p \in X$ and $y \in Y$, let $X \cup_p Y$ be the continuum obtained by attaching $X$ to $Y$ (identifying $p$ to $y$).

Given a continuum $X$, a metric $d$ for $X$ is said to be convex provided that for each two points $p, q \in X$, there exists an isometry $\gamma : [0, d(p, q)] \to X$ such that $\gamma(0) = p$ and $\gamma(d(p, q)) = q$. It is known that $X$ is a Peano continuum if and only if $X$ admits a convex metric (see [6] and [22]).

Given a continuum $X$, $\varepsilon > 0$ and $A \in 2^X$, define $C_d(\varepsilon, A)$, the generalized closed d-ball in $X$ of radius $\varepsilon$ about $A$, by $C_d(\varepsilon, A) = \{x \in X : d(x, A) \leq \varepsilon\}$. If $X$ is Peano continuum with a convex metric $d$, then for every $A \in C_n(X)$ and $\varepsilon > 0$, $C_d(\varepsilon, A) \in C_n(X)$.

**Definition 12** Given a Peano continuum $X$ with convex metric $d$ and $\varepsilon > 0$, define $\Phi_\varepsilon : 2^X \to 2^X$ by $\Phi_\varepsilon(A) = C_d(\varepsilon, A)$.

**Remark 13** By [19, Proposition 10.5] $\Phi_\varepsilon$ is a map within $\varepsilon$ of the identity map. Also notice that if $\mathcal{G}$ is a growth hyperspace, $A \in \mathcal{G}$ and $\varepsilon > 0$, then $\Phi_\varepsilon(A) \in \mathcal{G}$.

We will use the following characterization by Toruńczyk of the Hilbert cube ([24], see Theorem 9.3 of [19]).

**Theorem 14** (Toruńczyk’s Theorem). Let $Y$ be an AR. If the identity map on $Y$ is a uniform limit of Z-maps, then $Y$ is a Hilbert cube.

**Lemma 15** Let $X$ be a Peano continuum, $R$ a closed subset of $\mathcal{P}(X)$ and $K \in C(X)$ such that $\text{int}_X(K) \cap R \neq \emptyset$. Then $C_n^K(X)$ is a Z-set of $C_n(X, R)$.

**Proof.** Notice that $C_n^K(X)$ is a closed subset of $C_n(X, R)$. We show that for each $\varepsilon > 0$ there is a map, $g_\varepsilon : C_n(X, R) \to C_n(X, R) - C_n^K(X)$ such that $H_X(g_\varepsilon(A), A) < \varepsilon$ for all $A \in C_n(X, R)$.

Let $\varepsilon > 0$ and fix a point $p \in \text{int}_X(K) \cap R$. We may assume that $X \neq B(\varepsilon, p) \subset \text{int}_X(K)$. By Theorem 8.10 of [23], there exist $m \in \mathbb{N}$ and Peano
subcontinua $X_1, \ldots, X_m$ of $X$ such that for each $i \in \{1, \ldots, m\}$, diameter($X_i$) < $\frac{\epsilon}{2}$ and $X = X_1 \cup \ldots \cup X_m$. We may assume that $\{i \in \{1, \ldots, m\} : p \in X_i\} = \{1, \ldots, r\}$ where $r < m$. Define the star of $p$ by $\text{St}(p) = X_1 \cup \ldots \cup X_r$. Notice that $\text{St}(p) \subset \text{int}_X(K)$.

Let $F = \{ j \in \{1, \ldots, m\} : p \notin X_j \text{ and } X_j \cap \text{St}(p) \neq \emptyset \}$. Since $\text{St}(p) \neq X$ and $X = X_1 \cup \ldots \cup X_m$ is connected, it follows that $F \neq \emptyset$. For each $j \in F$, fix a point $p_j \in X_j \cap \text{St}(p)$. Note that by Proposition 10.7 of [19], $\text{St}(p)$ is a locally connected continuum and therefore it is arcwise connected. Thus, it is possible to construct a tree $T \subset \text{St}(p)$ such that $\{p_j : j \in F\} \subset T$. Hence, $T \cap X_j \neq \emptyset$ for each $j \in F$.

Let $Y = T \cup (\bigcup \{ X_j : j \in F \})$. By Proposition 10.7 of [19], $Y$ is a Peano continuum, since $C(Y)$ a growth hyperspace, $C(Y)$ is an AR. Notice that $Y \subset \text{int}_X(K)$.

Let $Z = Y \cap R$. Notice that $p \in Z$ and $C(Y, Z)$ is an AR $(C(Y, Z)$ is a growth hyperspace).

Define $\alpha : Y \to C(Y)$ by $\alpha(y) = \{ y \}$ and let $\beta : Z \to C(Y, Z)$ be given by $\beta(z) = \{ z \}$. By [19, Theorem 9.1] $\beta$ can be extended to a map $\overline{\beta} : (\text{St}(p) \cup Y) \cap R \to C(Y, Z)$. Notice that $\overline{\beta}|_Z = \alpha|_Z$. Thus, the function $\alpha \cup \overline{\beta} : ((\text{St}(p) \cup Y) \cap R) \cup Y \to C(Y)$ defined by

$$(\alpha \cup \overline{\beta})(x) = \begin{cases} \alpha(x), & \text{if } x \in Y, \\ \overline{\beta}(x), & \text{if } x \in (\text{St}(p) \cup Y) \cap R, \end{cases}$$

is a well-defined map.

By [19, Theorem 9.1], we can extend $\alpha \cup \overline{\beta}$ to a map $\overline{\alpha} : \text{St}(p) \cup Y \to C(Y)$.

Now extend $\overline{\alpha}$ to a function $\gamma : X \to C(X)$ by the formula

$$\gamma(x) = \begin{cases} \overline{\alpha}(x), & \text{if } x \in \text{St}(p) \cup Y, \\ \{ x \}, & \text{if } x \in X \setminus (\text{St}(p) \cup Y), \end{cases}$$

Since $\text{cl}_X(\{x \in (\text{St}(p) \cup Y)) \cap \text{St}(p) \cup Y \cup X_j : j \in F\} \subset Y, \gamma$ is a well-defined map.

Notice that if $x \in R \cap (\text{St}(p) \cup Y)$, then $\gamma(x) = \overline{\alpha}(x) = (\alpha \cup \overline{\beta})(x) = \overline{\beta}(x) \in C(Y, Z)$. Therefore, $\gamma$ has the following property:

$$(*) \text{ For every } x \in R \cap (\text{St}(p) \cup Y), \gamma(x) \cap R \neq \emptyset.$$ 

Define $g_e : C_n(X) \to C_n(X)$ as $g_e(A) = \bigcup \{ \gamma(x) : x \in A \}$. Using [7, Lemma 2.2], it is easy to see that $g_e$ is a well-defined map.

Given $x \in \text{St}(p) \cup Y$, since diameter($\text{St}(p) \cup Y) < \epsilon$ and $\gamma(x) \subset Y$, we have that $H_X(\{x \}, \gamma(x)) < \epsilon$. This implies that $H_X(A, g_e(A)) < \epsilon$ for each $A \in C_n(X)$.

Now we prove that $g_e$ maps $C_n(X, R)$ into $C_n(X, R) - C_n^R(X)$. Let $A \in C_n(X, R)$ and fix a point $a \in A \cap R$. If $a \in X \setminus (\text{St}(p) \cup Y)$, then $\gamma(a) = \{ a \} \subset R$, so $g_e(A) \subset C_n(X, R)$. If $a \in \text{St}(p) \cup Y$, then $a \in R \cap (\text{St}(p) \cup Y)$. By property $(*)$, $\gamma(a) \cap R \neq \emptyset$, so $g_e(A) \subset C_n(X, R)$.

Notice that, by definition of $\mathcal{P}(X)$, $p$ does not have a neighborhood homeomorphic to a finite graph. Since $\text{St}(p) - (\bigcup \{ X_j : j \in F \})$ is an open subset of
Theorem 14 has been verified and we obtain that $C$ is a set.

Hence, $g \in X$ for the assumption of Theorem 14 for $C_\mathcal{D}$. Then $C_\mathcal{D}$ is convex, and the Peano continuum $\mathcal{D}$ is not homeomorphic to $\mathcal{D}_1$. Let $\mathcal{D}_2 = C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$. If $\mathcal{D}_2$ is a Hilbert cube, then $\mathcal{D}_2 = C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$. We only need to show that $\Phi_\mathcal{D}|_{C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)}$ is a Z-map.

Since $R$ is compact, there are finitely many points $p_1, \ldots, p_s$ of $R$ such that $R \subset C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$. For each $i \in \{1, \ldots, s\}$, let $K_i = C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2, \{p_i\})$. Since $\mathcal{D}$ is convex, $K_i$ is a continuum and $p_i \in \text{int}_{\mathcal{D}}(K_i) \cap R$.

Applying Lemma 15, we obtain that $C_{\mathcal{D}^n}(\mathcal{D})$ is a Z-set in $C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$. By [19, Exercise 9.4], the set $\mathcal{G} = C_{\mathcal{D}^n}(\mathcal{D}) \cup \ldots \cup C_{\mathcal{D}^n}(\mathcal{D})$ is a Z-set in $C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$. By the choice of $K_i$, it is easy to see that for each $A \in C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$ there exists $j \in \{1, \ldots, s\}$ such that $\Phi_\mathcal{D}(A) = C_{\mathcal{D}^n}(\mathcal{D}) \subset \mathcal{G}$. Therefore, $\Phi_\mathcal{D}|_{C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)} \subset \mathcal{G}$.

Since a closed subset of a Z-set is a Z-set, we conclude that $\Phi_\mathcal{D}|_{C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)}$ is a Z-map within $\mathcal{D}$ of the identity map. Therefore, the second assumption of Theorem 14 has been verified and we obtain that $C_\mathcal{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$ is a Hilbert cube.

Theorem 17 (Anderson’s Homogeneity Theorem). If $h : A \to B$ is a homeomorphism between Z-sets in a Hilbert cube $\mathcal{Q}$, then $h$ extends to a homeomorphism of $\mathcal{Q}$ onto $\mathcal{Q}$.

The proof of the following result is similar to the proof of Theorem 5.1 of [2].

Theorem 18 Let $X$ be a Peano continuum and $p \in X$. Then there exists an uncountable family $\mathcal{D}$ of pairwise non homeomorphic dendrites such that

(a) for each $D \in \mathcal{D}$, $D$ does not contain free arcs,

(b) the Peano continuum $X \cup_p D$ is not homeomorphic to $X$, and

(c) if $B \neq D$ are elements of $\mathcal{D}$, then $X \cup_p B$ and $X \cup_p D$ are not homeomorphic.

Lemma 19 Let $X$, $Y$ and $D$ be continua and $p$ a point of $Y$ such that $Y = X \cup_D Y$ and $X \cap D = \{p\}$. Suppose that $E$ is a closed subset of $X$ that contains $p$. Then $\text{Fr}_{C_\mathcal{D}(X)}(C_\mathcal{D}(X, E)) = \text{Fr}_{C_\mathcal{D}(Y)}(C_\mathcal{D}(Y, E \cup D)).$
Proof. It follows form the easy to prove following facts: $C_n(Y) - C_n(Y, E \cup D) = C_n(X) - C_n(X, E) \subset C_n(X)$ and $C_n(X) \cap C_n(Y, E \cup D) = C_n(X, E)$. 

Now, we are ready to prove the main results of this section.

Theorem 20 Let $X$ be a Peano continuum that is not almost meshed. Then for every $n \in \mathbb{N}$, $X$ does not have unique hyperspace $C_n(X)$.

Proof. We assume that the metric for $X$ is convex. Since $X$ is not almost meshed, there exist a point $p \in \mathcal{P}(X)$ and an $\varepsilon > 0$ such that $B_{2\varepsilon}(p) \subset \mathcal{P}(X)$. Let $E = C_d(\varepsilon, \{p\})$. Notice that $E$ is a continuum with the properties that $E = \text{cl}_X(\text{int}_X(E))$ and $E \subset \mathcal{P}(X)$. By Theorem 16, $C_n(X, E)$ is a Hilbert cube.

Let $Y = X \cup_p D$, where $D$ is a locally connected continuum without free arcs. By Theorem 18 we can choose $D$ in such a way that $X$ and $Y$ are not homeomorphic.

We show that $C_n(X)$ is homeomorphic to $C_n(Y)$. First notice that $E \cup D$ and $Y$ satisfy the hypothesis of Lemma 16, and therefore $C_n(Y, E \cup D)$ is a Hilbert Cube. Assume also that the metric for $Y$ is convex.

Claim 1. $\text{Fr}_{C_n(X)}(C_n(X, E))$ is a $Z$-set of $C_n(X, E)$ and $\text{Fr}_{C_n(Y)}(C_n(Y, E \cup D))$ is a $Z$-set of $C_n(Y, E \cup D)$.

Let $\delta > 0$ and consider $\Phi_\delta|_{C_n(X, E)} : C_n(X, E) \to C_n(X, E)$ as in Definition 12. By Remark 13, $\Phi_\delta|_{C_n(X, E)}$ is within $\delta$ of the identity map. Since $E = \text{cl}_X(\text{int}_X(E))$, if $A \in C_n(X, E)$, then $\Phi_\delta(A) \cap \text{int}_X(E) \neq \emptyset$. Therefore, $\Phi_\delta(A) \notin \text{Fr}_{C_n(X)}(C_n(X, E))$ and $\Phi_\delta|_{C_n(X, E)} : C_n(X, E) \to C_n(X, E) - (\text{Fr}_{C_n(X)}(C_n(X, E)))$. We have proved that $\text{Fr}_{C_n(X)}(C_n(X, E))$ is a $Z$-set in $C_n(X, E)$. The proof that $\text{Fr}_{C_n(Y)}(C_n(Y, E \cup D))$ is a $Z$-set of $C_n(Y, E \cup D)$ is analogous so the claim is proved.

By Lemma 19, the identity map $\text{id} : \text{Fr}_{C_n(X)}(C_n(X, E)) \to \text{Fr}_{C_n(Y)}(C_n(Y, E \cup D))$ is a well-defined homeomorphism. By Claim 1 and Theorem 17, the identity map $\text{id}$ can be extended to a homeomorphism $h_1 : C_n(X, E) \to C_n(Y, E \cup D)$.

We define a homeomorphism $h : C_n(X) \to C_n(Y)$ as follows.

$$h(A) = \begin{cases} h_1(A), & \text{if } A \in C_n(X, E), \\ A, & \text{if } A \in C_n(X) - C_n(X, E). \end{cases}$$

Hence, $C_n(X)$ is homeomorphic to $C_n(Y)$ and the Theorem is proved.

Corollary 21 Let $X$ be a Peano continuum that is not almost meshed. Then there exists an uncountable family $\mathcal{Y}$ of pairwise non-homeomorphic Peano continua such that:

(A) for each $Y \in \mathcal{Y}$, $X$ is not homeomorphic to $Y$,

(B) for each $n \in \mathbb{N}$ and each $Y \in \mathcal{Y}$, $C_n(X)$ is homeomorphic to $C_n(Y)$.

Proof. Let $\mathcal{D}$ be as in Theorem 18. Fix a point $p \in \text{int}_X(\mathcal{P}(X))$. Let $\mathcal{Y} = \{X \cup_p D : D \in \mathcal{D}\}$. 

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5 Almost meshed continua without unique hyperspace

In this section we show a class of almost meshed Peano continua that do not have unique hyperspace $C_m(X)$.

Theorem 22 Let $X$ be an almost meshed Peano continuum and $n \in \mathbb{N}$. Suppose that there exist a closed subset $R$ of $\mathcal{P}(X)$ and pairwise disjoint nonempty open sets $U_1, \ldots, U_{n+1}$ such that

(a) $X = R = U_1 \cup \ldots \cup U_{n+1}$ and

(b) for each $i \in \{1, \ldots, n+1\}$, $R \subset \text{cl}_X(U_i)$.

Then $X$ does not have unique hyperspace $C_m(X)$ for every $m \leq n$.

Proof. Let $m \leq n$. By Theorem 16, $C_m(X, R)$ is a Hilbert cube.

Fix a point $p \in R$ and let $Y = X \cup_p D$, where $D$ is a locally connected continuum without free arcs. By Theorem 18, we can choose $D$ in such a way that $X$ and $Y$ are not homeomorphic. We show that $C_m(X)$ is homeomorphic to $C_m(Y)$. Notice that $R \cup D$ is a closed subset of $\mathcal{P}(Y)$. By Theorem 16, $C_m(Y, R \cup D)$ is a Hilbert cube. Assume that the metrics for $X$ and $Y$ are convex.

Claim 2. $\text{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$ is a $Z$-set in $C_m(Y, R \cup D)$.

Let $\epsilon > 0$ and consider the map $\Phi_{\epsilon} : C_m(Y, R \cup D) \rightarrow C_m(Y, R \cup D)$ of Definition 12. By Remark 13, $\Phi_{\epsilon} : C_m(Y, R \cup D)$ is within $\epsilon$ of the identity map so we only have to prove that $\Phi_{\epsilon}(C_m(Y, R \cup D)) \cap \text{Fr}_{C_m(Y)}(C_m(Y, R \cup D)) = \emptyset$.

Let $A \subset C_m(Y, R \cup D)$.

Case 1. $A \cap R \neq \emptyset$.

By (b), $\Phi_{\epsilon}(A) \cap U_i \neq \emptyset$, for every $i \in \{1, \ldots, n+1\}$. Consider a sequence $\{A_j\}_{j=1}^{\infty}$ of elements of $C_m(Y)$ such that $\lim A_j = \Phi_{\epsilon}(A)$. Then there exists $M \in \mathbb{N}$ such that for each $j \geq M$ and every $i \in \{1, \ldots, n+1\}$, $A_j \cap U_i \neq \emptyset$.

Given $j \geq M$, since $A_j$ has at most $m$ components and $m < n+1$, we have $A_j \cap (R \cup D) \neq \emptyset$. Thus, $A_j \subset C_m(Y, R \cup D)$ and $\Phi_{\epsilon}(A)$ can not be approached by continua that do not intersect $R \cup D$. Hence, $\Phi_{\epsilon}(A) \notin \text{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$.

Case 2. $A \cap R = \emptyset$.

In this case $p \notin A$ and $\Phi_{\epsilon}(A) \cap (D - \{p\}) \neq \emptyset$. Since $D - \{p\}$ is open in $Y$, we have that $\Phi_{\epsilon}(A) \notin \text{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$.

By Cases 1 and 2, we obtain that $\Phi_{\epsilon} |_{C_m(Y, R \cup D)} : C_m(Y, R \cup D) \rightarrow C_m(Y, R \cup D) - (\text{Fr}_{C_m(Y)}(C_m(Y, R \cup D)))$. This proves Claim 2.

Claim 3. $\text{Fr}_{C_m(X)}(C_m(X, R))$ is a $Z$-set in $C_m(X, R)$.

The proof is similar and easier to the one in Claim 2 since we only need to consider Case 1.

By Lemma 19, the identity map $id : \text{Fr}_{C_m(X)}(C_m(X, R)) \rightarrow \text{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$ is a homeomorphism. By Claims 2, 3 and Theorem 17, the identity map
id can be extended to a homeomorphism \( h_1 : C_m(X, R) \to C_m(Y, R \cup D) \). We define a homeomorphism \( h : C_m(X) \to C_m(Y) \) as follows:

\[
h(A) = \begin{cases} h_1(A), & \text{if } A \in C_m(X, R), \\ A, & \text{if } A \in C_m(X) - C_m(X, R). \end{cases}
\]

Hence, \( C_m(X) \) is homeomorphic to \( C_m(Y) \) and the theorem is proved. \( \blacksquare \)

**Corollary 23** Let \( X \) be an almost meshed Peano continuum such that \( X - \mathcal{P}(X) \) is disconnected. Then \( X \) does not have unique hyperspace \( C(X) \).

**Proof.** Suppose that \( X - \mathcal{P}(X) = U \cup V \), where \( U \) and \( V \) are nonempty open disjoint subsets of \( X \). Since \( X \) is almost meshed, \( \text{int}_X(\mathcal{P}(X)) = \emptyset \). Thus, \( X = \text{cl}_X(U) \cup \text{cl}_X(V) \) and \( R = \text{cl}_X(U) \cap \text{cl}_X(V) \) is a nonempty closed subset of \( \mathcal{P}(X) \). Let \( W = X - \text{cl}_X(U) \) and \( Z = X - \text{cl}_X(V) \). Hence, \( W \) and \( Z \) are nonempty open disjoint subsets of \( X \) such that \( V \subset W, U \subset Z \) and \( R \subset \text{cl}_X(W) \cap \text{cl}_X(Z) \). By Theorem 22, the Corollary follows. \( \blacksquare \)

**Corollary 24** Let \( X \) be an almost meshed Peano continuum satisfying the conditions of Theorem 22. Then there exists an uncountable family \( \mathcal{Y} \) of pairwise non-homeomorphic Peano continua such that:

(A) for each \( Y \in \mathcal{Y} \), \( X \) is not homeomorphic to \( Y \),

(B) for each \( Y \in \mathcal{Y} \) and each \( m \leq n \), \( C_m(X) \) is homeomorphic to \( C_m(Y) \).

**Corollary 25** Let \( X \) be a dendrite that is not a tree and \( k = \sup\{\text{ord}_X(p) : p \in \mathcal{P}(X)\} \), notice \( k \in \mathbb{N} \cup \{\omega\} \). Then for every \( m < k \), \( X \) does not have unique hyperspace \( C_m(X) \).

**Proof.** If \( X \) is not almost meshed, then by Theorem 20, \( X \) does not have unique hyperspace \( C_m(X) \) for every \( m \in \mathbb{N} \). If \( X \) is almost meshed and \( m < k \), there exists a point \( q \in \mathcal{P}(X) \) such that \( \text{ord}_X(q) \geq m + 1 \). Hence, \( X \) and the closed subset \( \{q\} \) satisfy the conditions of Theorem 22 for \( m \) and the Corollary follows. \( \blacksquare \)

6 Meshed continua have unique hyperspaces

Given a continuum \( X \) and \( n \in \mathbb{N} \), let

\[
\mathcal{B}_n(X) = \{ A \in C_n(X) : A \text{ has a neighborhood in } C_n(X) \text{ that is a } 2n\text{-cell} \},
\]

\[
\mathcal{B}_n^0(X) = \{ A \in C_n(X) : A \text{ has a neighborhood } \mathcal{M} \text{ in } C_n(X) \text{ that is a } 2n\text{-cell} \text{ and } A \text{ belongs to the manifold boundary of } \mathcal{M} \},
\]

\[
\Gamma_n(X) = \{ A \in C_n(X) - \mathcal{B}_n(X) : A \text{ has a basis of open neighborhoods } \mathcal{U}_A \text{ in } C_n(X) \text{ such that, for each } U \in \mathcal{U}_A, \dim U = 2n \text{ and } U \cap \mathcal{B}_n(X) \text{ is arcwise connected} \}.
\]
As usual, we denote \( \mathfrak{B}(X) = \mathfrak{B}_1(X) \) and \( \mathfrak{B}^0(X) = \mathfrak{B}_1^0(X) \).

Define

\[
\mathfrak{A}_E(X) = \{ J \in \mathfrak{A}(X) : \text{there exists an end point } p \text{ of } J \text{ such that } p \in J^o \}.
\]

In the case that \( J \in \mathfrak{A}_E(X) \) and \( p \) is an end point of \( J \) such that \( p \in J^o \), \( p \) is said to be an extreme of \( X \).

**Lemma 26** Let \( X \) be a Peano continuum and \( A \in C(X) \). Then the following are equivalent:

(a) \( A \in \mathfrak{B}^0(X) \),

(b) there is \( J \in \mathfrak{A}_S(X) \) such that one of the following two conditions hold:

1. \( A = \{ p \} \), for some \( p \in J^o \),

2. \( J \in \mathfrak{A}_E(X) \) and there exists an extreme \( p \) of \( X \) such that \( p \in A \subset J^o \).

**Proof.** (a) \( \Rightarrow \) (b). Suppose that \( A \in \mathfrak{B}^0(X) \). Then \( \dim_A |C(X)| = 2 \). Lemma 11 implies that there exists \( J \in \mathfrak{A}_S(X) \) such that \( A \subset J^o \). Let \( \mathcal{M} \) be a 2-cell in \( C(X) \) such that \( A \in \text{int}_{C(X)}(\mathcal{M}) \subset \text{int}_{C(X)}(C(J)) \) and \( A \) belongs to the boundary, as manifold, of \( \mathcal{M} \). Thus, \( \mathcal{M} \) is a neighborhood of \( A \) in \( C(J) \). Since \( J \) is either an arc or a simple closed curve, by the geometric models of \( C(J) \) constructed in Examples 5.1 and 5.2 of [19], we obtain that one of the conditions (1) or (2) holds.

(b) \( \Rightarrow \) (a). Let \( J \in \mathfrak{A}_S(X) \) be such that \( A \subset J^o \). Then \( C(J) \) is a neighborhood of \( A \) in \( C(X) \). By the models in Examples 5.1 and 5.2 of [19], in both cases, (1) and (2), there exists a neighborhood \( \mathcal{M} \) of \( A \) in \( C(J) \) such that \( \mathcal{M} \) is a 2-cell, \( A \) belongs to the boundary, as manifold, of \( \mathcal{M} \) and \( \mathcal{M} \subset \text{int}_{C(X)}(C(J)) \). Then \( \mathcal{M} \) is a neighborhood of \( A \) in \( C(X) \). Therefore, \( A \in \mathfrak{B}^0(X) \).

**Theorem 27** Let \( X \) be a Peano continuum that is not an arc. Then there exists a homeomorphism \( h : \text{cl}_X(FA(X)) \to \text{cl}_{C(X)}(\mathfrak{B}^0(X)) \) such that \( h(p) = \{ p \} \) for each \( p \in \text{cl}_X(FA(X)) - \bigcup \{ J^o : J \in \mathfrak{A}_E(X) \} \) and, if \( h(p) \cap P(X) \neq \emptyset \), then \( p \in P(X) \) or \( p \) is an end point of \( J \), for some \( J \in \mathfrak{A}_E(X) \), where \( J \cap P(X) \neq \emptyset \) and \( p \in J^o \).

**Proof.**

By Example 5.2 of [19], we can assume that \( X \) is not a simple closed curve.

Given \( J \in \mathfrak{A}_E(X) \), let \( p_J \) and \( q_J \) be the end points of \( J \), where \( p_J \in J^o \). Since \( X \) is not an arc, \( q_J \notin J^o \). Fix a homeomorphism \( h_J : [0,1] \to J \) such that \( h_J(0) = q_J \) and \( h_J(1) = p_J \).

Let

\[
W = \bigcup \{ J - \{ q_J \} : J \in \mathfrak{A}_E(X) \}.
\]

Then \( W \) is an open subset of \( X \) and \( W \subset FA(X) \).

Define \( h : \text{cl}_X(FA(X)) \to \text{cl}_{C(X)}(\mathfrak{B}^0(X)) \) as follows
Using Lemma 26 it can be shown that $h$ is a well-defined function. Clearly, $h$ is continuous at each point of $W$. Thus, in order to conclude that $h$ is continuous, take a sequence $\{x_m\}_{m=1}^\infty$ of points of $W$ such that $\lim x_m = x$ for some $x \notin W$. We need to show that $\lim h(x_m) = \{x\}$. For each $m \in \mathbb{N}$, let $J_m \in \mathfrak{A}_E(X)$ such that $x_m \in J_m$. We may assume that $J_m \neq J_k$, if $m \neq k$, and that $\lim p_{J_m} = q$, for some $q \in X$. By Lemma 8, $\lim J_m = \{q\}$. Since $h(x_m) \subset J_m$ and $x_m \in J_m$ for each $m \in \mathbb{N}$, we have that $\lim h(x_m) = \{q\}$ and $\lim x_m = q$. Therefore, $q = x$ and $\lim h(x_m) = \{x\}$. This completes the proof that $h$ is continuous.

It is easy to see that $h$ is one-to-one. In order to show that $h$ is onto, note that by Lemma 26, $\mathfrak{B}^0(X) \subset h(cl_X (\mathcal{F}A(X)))$. Hence, $cl_{\mathcal{C}(X)}(\mathfrak{B}^0(X)) \subset h(cl_X (\mathcal{F}A(X)))$. Thus, $h$ is onto.

Finally, take $p \in cl_X (\mathcal{F}A(X))$ such that $h(p) \cap \mathcal{P}(X) \neq \emptyset$. In the case that $h(p) = \{p\}$, we obtain that $p \in \mathcal{P}(X)$. In the case that $h(p) \neq \{p\}$, then $p \in J - \{q\} = J^o$ for some $J \in \mathfrak{A}_E(X)$. Since $h(p) \cap \mathcal{P}(X) \neq \emptyset$, $h(p) \neq J^o$. Hence, $h(p) = J = h_j([0, 1])$ and we are done. 

**Lemma 28** Let $X$ be a Peano continuum and $n \geq 3$. Then $\Gamma_n(X) = \{A \in C_n(X) : A$ is connected and there exists $J \in \mathfrak{A}_S(X)$ such that $A \subset J^o\} = \mathfrak{B}(X)$.

**Proof.** Let $A \in \Gamma_n(X)$. By Lemma 11 and Theorem 4, $\dim_3 [C_n(X)] = 2n$, there exist $k \in \mathbb{N}$, elements $J_1, \ldots, J_k \in \mathfrak{A}_E(X)$ such that $A \in \langle J_1^o, \ldots, J_k^o \rangle$ and a finite graph $D$ in $X$ such that $A \subset D^o$. Then $C_n(D)$ is a neighborhood of $A$ in $C_n(X)$. Thus, we may assume that the basis of open neighborhoods $\mathfrak{B}_J$ in the definition of $\Gamma_n(X)$ satisfies that for each $U \in \mathfrak{B}_J$, $U \subset C_n(D)$. Hence, $\mathfrak{B}_J$ is a basis of neighborhoods of $A$ in $C_n(D)$ such that for each $U \in \mathfrak{B}_J$, $\dim U = 2n$ and $U \subset \mathfrak{B}_n(X)$ is arcwise connected. Given $U \in \mathfrak{B}_J$ and $B \in U \subset \mathfrak{B}_n(X)$, $B$ has a neighborhood $M$ in $C_n(X)$ that is a $2n$-cell. Then there exists an $2n$-cell $N \subset M$ such that $B \in \text{int}_{C_n(X)}(N) \subset U \cap M \subset C_n(D)$. Thus, $N$ is a $2n$-cell that is a neighborhood of $B$ in $C_n(D)$. Hence, $B \in U \cap \mathfrak{B}_n(D)$. We have shown that $U \cap \mathfrak{B}_n(X) \subset U \cap \mathfrak{B}_n(D)$. The other inclusion is easy to prove. Hence, $U \cap \mathfrak{B}_n(X) = U \cap \mathfrak{B}_n(D)$ and $U \cap \mathfrak{B}_n(D)$ is arcwise connected. Since $A \in U - \mathfrak{B}_n(X) = U - \mathfrak{B}_n(D)$, we have proved that $A \in \Gamma_n(D)$. By Lemma 3.6 of [17], $A$ is connected and we may assume that $A \subset J^o$.

Now suppose that $A \in C_n(X)$ is such that $A$ is connected and there exists $J \in \mathfrak{A}_S(X)$ such that $A \subset J^o$. By Lemma 3.6 of [17], $A \in C_n(J) - \mathfrak{B}_n(J)$ and $A$ has a basis of open neighborhoods $\mathfrak{B}_J$ in $C_n(J)$ such that for each $U \in \mathfrak{B}_J$, $\dim U \leq 2n$ (then $\dim U = 2n$, by Lemma 11) and $U \cap \mathfrak{B}_n(J)$ is arcwise connected. Since $A \in \text{int}_{C_n(X)}(C_n(J))$, we can take $U \subset \text{int}_{C_n(X)}(C_n(J))$ so that $U$ is open in $C_n(X)$ for each $U \in \mathfrak{B}_J$. Proceeding as in the previous paragraph, $U \cap \mathfrak{B}_n(X) = U \cap \mathfrak{B}_n(J)$ for each $U \in \mathfrak{B}_J$. This implies that $A \in \Gamma_n(X)$. 


In the case that $J \in \mathfrak{S}(X)$ such that $A \subset J^o$ follows from Examples 5.1 and 5.2 of [19] and Lemma 11.

**Theorem 29** If $X$ and $Y$ are almost meshed Peano continua, $n \geq 3$ and $C_n(X)$ is homeomorphic to $C_n(Y)$, then $X$ is homeomorphic to $Y$.

**Proof.** By Theorem 3.8 of [17], we may assume that $X$ and $Y$ are not arcs. Let $h : C_n(X) \to C_n(Y)$ be a homeomorphism. Notice that the definition of $\Gamma_n(X)$ is given in terms of topological concepts that are preserved under homeomorphisms. Thus, $h(\Gamma_n(X)) = \Gamma_n(Y)$ and $h(\mathfrak{B}(X)) = \mathfrak{B}(Y)$. Note that $\mathfrak{B}(X)$ is an open subset of $C(X)$ and $\mathfrak{B}^g(X) \subset \mathfrak{B}(X)$. Thus, $\mathfrak{B}^g(X) = \{ A \in \mathfrak{B}(X) : A$ has a neighborhood $M$ in $\mathfrak{B}(X)$ that is a 2-cell and $A$ belongs to the manifold boundary of $M \}$.

It follows that $h(\mathfrak{B}^g(X)) = \mathfrak{B}^g(Y)$. Hence, $h|_{\mathfrak{B}(X)}(\mathfrak{B}^g(X)) : cl_{\mathfrak{B}(X)}(\mathfrak{B}^g(X)) \to cl_{\mathfrak{B}(Y)}(\mathfrak{B}^g(Y))$ is a homeomorphism. Theorem 27 implies that $cl_X(\mathcal{F}(A)(X))$ is homeomorphic to $cl_Y(\mathcal{F}(A)(Y))$. By Lemma 1, $cl_X(\mathcal{G}(X))$ is homeomorphic to $cl_Y(\mathcal{G}(Y))$. Since $X$ and $Y$ are almost meshed, we conclude that $X$ is homeomorphic to $Y$.

**Theorem 30** If $X$ and $Y$ are almost meshed Peano continua which are not arcs and $C(X)$ is homeomorphic to $C(Y)$, then $X$ is homeomorphic to $Y$.

**Proof.** Let $h : C(X) \to C(Y)$ be a homeomorphism. Notice that $h(\mathfrak{B}(X)) = \mathfrak{B}(Y)$. Proceeding as in the proof of Theorem 29 we conclude that $X$ is homeomorphic to $Y$.

In Theorem 35 we will extend the conclusions of Theorems 29 and 30 to the case $n = 2$. As in the previous results on finite graphs and class $\mathfrak{D}$, this case is more difficult and requires a different technique. We will use the following conventions.

Given a continuum $X$ that is not a simple closed curve and $J, K \in \mathfrak{A}(X)$, let

$$\mathcal{D}(J, K) = cl_{\mathfrak{C}(X)}(\mathfrak{B}^g_2(X) \cap (J^o, K^o)) \cap cl_{\mathfrak{C}(X)}(\mathfrak{B}^g_2(X) - (J^o, K^o)).$$

In the case that $J$ is an arc, let $p_J$ and $q_J$ be its end points, where $q_J \in Fr_X(J)$. If $J$ is a simple closed curve, let $q_J$ be the unique point in $J$ such that $J - \{q_J\}$ is open. Since $X$ is not a simple closed curve, $q_J \notin J^o$. Given $J \in \mathfrak{A}(X)$, define $\mathcal{E}(J)$ in the following way: If $J$ is an arc, let $\mathcal{E}(J) = C(J)$.

In the case that $J$ is a simple closed curve, let $\mathcal{E}(J) = \{ A \in C(J) : A = J \text{ or } A = \{p\} \text{ for some } p \in J \text{ or } A \text{ is a subarc of } J \text{ such that } q_J \notin A \text{ or } A \text{ is a subarc of } J \text{ such that } q_J \text{ is one of its end points} \}$. Note that, in both cases, $\mathcal{E}(J) = cl_{\mathfrak{C}(X)}((J^o) \cap C(X))$. Let $W_0$ be the continuum obtained as $W_0 = D - \text{int}_\mathbb{R}^2(E)$, where $D$ and $E$ are discs in the plane $\mathbb{R}^2$, $E \subset D$, and $E$ and $D$ are tangents. The following lemma can be easily proved from Examples 5.1 and 5.2 of [19].
Lemma 31. Let $X$ be a continuum that is not a simple closed curve and $J \in \mathcal{A}_S(X)$. Then:

(a) If $J$ is an arc, then $\mathcal{E}(J)$ is a 2-cell,

(b) If $J$ is a simple closed curve, then $\mathcal{E}(J)$ is homeomorphic to $W_0$ (where the point of tangency corresponds to \((q,t)\)).

Lemma 32. Let $X$ be a Peano continuum. Then $\mathcal{B}^2_2(X) = \{A \in \mathcal{B}_2(X) : A$ is connected or $A$ has a degenerate component or $A$ contains an extreme of $X\}$.

Proof. By Lemma 11, $\mathcal{B}_2(X) \subseteq \bigcup \{(J^\circ, K^\circ) : J, K \in \mathcal{A}_S(X)\}$, and by Lemma 2.1 of [18], for every $J, K \in \mathcal{A}_S(Y)$, \((J^\circ, K^\circ)\) is a component of $\mathcal{B}_2(X)$. Using Lemma 7, it can be shown that if $J, K, L, M \in \mathcal{A}_S(X)$ and \(\{J, K\} \neq \{L, M\}\), then \((J^\circ, K^\circ) \cap (L^\circ, M^\circ) = \emptyset\). Thus, the components of $\mathcal{B}_2(X)$ are the sets of the form \((J^\circ, K^\circ)\), where $J, K \in \mathcal{A}_S(X)$.

Given $J \in \mathcal{A}_S(X)$, let $C(J^\circ) = C(X) \cap \langle J^\circ \rangle$ and $\mathcal{B}^0(J^\circ) = \{A \in C(J^\circ) : A$ has a neighborhood $\mathcal{M}$ in $C(J^\circ)$ such that $\mathcal{M}$ is a 2-cell and $A$ is in the manifold boundary of $\mathcal{M}\}$. Notice that $J^\circ$ is homeomorphic to $(0, 1)$ when $J \notin \mathcal{A}_E(X)$ and $J^\circ$ is homeomorphic to $[0, 1]$ when $J \in \mathcal{A}_E(X)$.

By Example 5.1 of [19], $C(J^\circ)$ is homeomorphic to $[0, 1] \times [0, 1]$. In the case that $J \notin \mathcal{A}_E(X)$, $\mathcal{B}^0(J^\circ) = \{\{p\} : p \in J^\circ\}$ and, in the case that $J \in \mathcal{A}_E(X)$, $\mathcal{B}^0(J^\circ) = \{\{p\} : p \in J^\circ\} \cup \{A \in C(J^\circ) : p_J \in A\}$.

If $J \neq K$, then $J^\circ \cap K^\circ = \emptyset$. Let $\varphi : C(J^\circ) \times C(K^\circ) \to \langle J^\circ, K^\circ \rangle$ be given by $\varphi(B, C) = B \cup C$. It is easy to show that $\varphi$ is a homeomorphism and $\mathcal{B}^0_2(X) \cap \langle J^\circ, K^\circ \rangle = \varphi((\mathcal{B}^0(J^\circ) \times C(K^\circ)) \cup (C(J^\circ) \times \mathcal{B}^0(K^\circ))) = \{A \in \langle J^\circ, K^\circ \rangle : A \cap \varphi \subseteq \mathcal{B}^0(J^\circ) \text{ or } A \cap K^\circ \subseteq \mathcal{B}^0(K^\circ)\} = \{A \in \langle J^\circ, K^\circ \rangle : A$ has a degenerate component or $A$ contains an extreme of $X\}$.

If $J = K$, $\langle J^\circ, K^\circ \rangle = \langle J^\circ \rangle = \{A \in C_2(J) : A \subseteq J^\circ\}$. In Lemma 2.2 of [16] the following model (due to R. M. Schori) for $C_2([0, 1])$ was constructed. Let $C_0 = \{A \in C_2([0, 1]) : 0 \in A\}$ and $C_0^1 = \{A \in C_2([0, 1]) : [0, 1] \subset A\} = \{[0, a] \cup [b, 1] : 0 \leq a \leq b \leq 1\}$.

Then $C_0^1$ is homeomorphic to the space obtained by identifying the diagonal of the triangle \([\{a, b\} : 0 \leq a \leq b \leq 1\]) to a point. Thus, $C_0^1$ is a 2-cell and the manifold boundary of $C_0^1$ is the set $\partial(C_0^1) = \{[0] \cup [b, 1] : 0 \leq b \leq 1\} \cup \{[0, a] \cup [1] : 0 \leq a \leq 1\} \cup \{[0, 1]\}$. The function $\eta : \text{cone}(C_0^1) \to C_0$ given by $\eta((A, t)) = (1 - t)A$ is a homeomorphism. Thus, $C_0$ is a 3-cell and its manifold boundary is the set $\partial(C_0^1) = C_0^1 \cup \{(1 - t)A : A \in \partial(C_0^1)\}$ and $t \in [0, 1]$. Finally, the function $\lambda : \text{cone}(C_0) \to C_2([0, 1])$ given by $\lambda((A, t)) = \{t\} + (1 - t)A$ is a homeomorphism. Thus, $C_2([0, 1])$ is a 4-cell and its manifold boundary is the set $\partial(C_2([0, 1])) = C_0 \cup \{(t) + (1 - t)A : A \in \partial(C_0)\}$ and $t \in [0, 1]$. Therefore, $\partial(C_2([0, 1])) = \{A \in C_2([0, 1]) : A$ is connected or $A$ has a degenerate component or $A \cap \{0, 1\} \neq \emptyset\}$.

In the case that $J \notin \mathcal{A}_E(X)$, $J^\circ$ is homeomorphic to $(0, 1)$, so $\mathcal{B}^0_2(X) \cap \langle J^\circ \rangle = \{A \in C_2(J^\circ) : A$ is connected or $A$ has a degenerate component\}, and in the case that $J \in \mathcal{A}_S(X)$, $J^\circ$ is homeomorphic to $[0, 1)$, so $\mathcal{B}^0_2(X) \cap \langle J^\circ \rangle = \{A \in C_2(J^\circ) : A$ is connected or $A$ has a degenerate component or the extreme of $X$ contained in $J$ belongs to $A\}$. Therefore, for all $J \in \mathcal{A}_S(Y)$, $\mathcal{B}^0_2(X) \cap \langle J^\circ \rangle = \{A \in \langle J^\circ \rangle : A$
Hence, \( B \) and nonempty subset of \( B \cap J \). Since \( \exists J \in \mathcal{A}(X) \) that \( \lim E_m = B \), we may assume that each \( E_m \) has two components \( E_m^1 \) and \( E_m^2 \), \( \lim E_m^1 = B_1 \) and \( \lim E_m^2 = B_2 \). Since \( B \in \text{cl}_{C_2}(\mathcal{A}(X)) \), there exists a sequence \( E_m = E_m^1 \cup E_m^2 \) of elements of \( \mathcal{A}(X) \) such that \( \lim E_m^1 = B_1 \) and \( \lim E_m^2 = B_2 \). In the case that \( J = K \), we have that \( E_m \subset J \) for each \( m \in \mathbb{N} \) and \( B \subset J = K \). In the case that \( J \neq K \), \( J \cap K = \emptyset \), so we can assume that \( E_m^1 \subset J^o \) and \( E_m^2 \subset K^o \) for each \( m \in \mathbb{N} \). This implies that \( B_1 \subset J \) and \( B_2 \subset K \). So, in both cases \( (J = K \text{ or } J \neq K) \), we may assume that \( B_1 \subset J \) and \( B_2 \subset K \). Since \( B \in \text{cl}_{C_2}(\mathcal{A}(X)) \), there is also a sequence \( F_m = F_m^1 \cup F_m^2 \) of elements of \( \mathcal{A}(X) \) such that \( \lim F_m^1 = B_1 \) and \( \lim F_m^2 = B_2 \). If \( J \) is an arc, then \( B = \{p\} \cup B_1 \), where \( B_1 \in \mathcal{E}(J) \) and \( p \in \text{Fr}_X(K) \). If \( J \) is a simple closed curve, then \( E_m^1 \subset J^o = \{q_j\} \) for each \( m \in \mathbb{N} \), \( B_1 = \lim E_m^1 \) is either equal to \( J \) or \( B_1 = \{p\} \) for some \( p \in J \). Suppose, for example, that \( \{J, M, m\} \in \mathcal{A}(X) \) such that \( \{J, M, m\} \neq \{J, K\} \) and \( E_m \in \text{cl}_{C_2}(\mathcal{A}(X)) \). Since \( J^o \) and \( K^o \) are open in \( X \), there exists \( m_0 \in \mathbb{N} \) such that for each \( m \geq m_0 \), \( E_m \) intersects \( J^o \) and \( K^o \). If \( L_m \) intersects \( J^o \), then \( L_m = J \). Thus, for each \( m \geq m_0 \) we may suppose that \( L_m = J \) and \( M_m = K \). Hence, \( \{L_m, M_m\} = \{J, K\} \), a contradiction. We have show that \( B \cap J = \emptyset \) or \( B \cap K = \emptyset \). Suppose, for example, that \( B \cap J = \emptyset \). Since \( B \in \{J, K\}, B = (B \cap J) \cup (B \cap K) \) and \( \emptyset \neq B \cap J \). This implies that \( B \cap J \) is a nonempty subset of \( J - J^o \) which consists of at most two elements. Since \( B \cap J \) and \( B \cap K \) are closed in \( B \) and \( B \) is connected, we have that \( B \cap J \subset B \cap K \). Hence, \( B \subset K \). Fix a point \( p \in B \). If \( K \) is an arc, then \( B \) is of the form \( B = \{p\} \cup B \), where \( B \in \mathcal{E}(J) \), \( p \in Fr_X(J) \). Now suppose that \( K \) is a simple closed curve. Since \( B \in \text{cl}_{C_2}(\mathcal{A}(X)) \), there exists a sequence
\[ \{B_m\}_{m=1}^{\infty} \text{ in } (J^o, K^o) \] such that \( \lim B_m = B \). Thus, the components of \( B_m \) are \( B_m \cap J^o \) and \( B_m \cap K^o \), and \( B_m = \lim((B_m \cap J^o) \cup (B_m \cap K^o)) \). We may suppose that the sequences \( \{B_m \cap J^o\}_{m=1}^{\infty} \) and \( \{B_m \cap K^o\}_{m=1}^{\infty} \) are convergent in \( C(X) \).

Recall that \( B \cap J \) has at most two elements. If \( q \in B \) and \( q = \lim q_m \), where \( q_m \in B \cap J^o \), for each \( m \in \mathbb{N} \), then \( q \in \text{Fr}_X(J) \). Thus, there at most two points \( q \) of \( B \) of this form. So \( \lim(B_m \cap J^o) \) is a one-point set. This implies that \( B = \lim(B_m \cap K^o) \). Given \( m \in \mathbb{N} \), since \( B_m \cap K^o \) is a connected subset of \( K^o = K \setminus \{q_K\} \), we have that \( B_m \cap K^o \) is an arc such that \( q_K \notin B_m \cap K^o \).

Hence, \( B = \lim(B_m \cap K^o) \in \mathcal{E}(K) \). Therefore, \( B = \{p\} \cup B \), where \( p \in \text{Fr}_X(J) \) and \( B \in \mathcal{E}(K) \).

Finally, we consider the case when \( B \) is connected and \( J = K \). Since \( B \in \text{cl}_{C(X)}(\mathbb{R}^2(X) - \langle J^o \rangle) \), \( B \) is limit of elements in \( \mathbb{R}^2(X) - \langle J^o \rangle \) and \( B \subset J \). Thus, \( B \notin J^o \). Hence, we can fix a point \( p \in B \cap \text{Fr}_X(J) \). If \( J \) is an arc, \( B = \{p\} \cup B \) and \( B \in \mathcal{E}(J) \). If \( J \) is a simple closed curve, let \( B = \lim E_m \), where \( E_m \in \langle J^o \rangle \cap \mathbb{R}^2(X) \) for each \( m \in \mathbb{N} \). For each \( m \in \mathbb{N} \), by Lemma 32, \( E_m \) is connected or \( E_m \) has a degenerate component. In both cases, we can write \( E_m = \{p_m\} \cup F_m \), where \( F_m \in C(J^o) \). Note that \( \lim F_m = B \). Since \( F_m \) is a connected subset of \( J^o = J \setminus \{q_J\} \), we have that \( F_m \) is an arc such that \( q_J \notin F_m \). Hence, \( B = \lim F_m \in \mathcal{E}(J) \). Therefore, \( B = \{p\} \cup B \), where \( p \in \text{Fr}_X(J) \) and \( B \in \mathcal{E}(J) \).

(>) Let \( B = \{p\} \cup A \), where \( p \in \text{Fr}_X(J) \subset \text{cl}_X(\mathcal{F}(X) - J) \) and \( A \in \mathcal{E}(K) \).

Notice that in both cases: \( K \) being an arc and \( K \) being a simple closed curve, \( A = \lim A_m \), where \( A_m \in K^o \) for each \( m \in \mathbb{N} \). Given \( m \in \mathbb{N} \), there exists a point \( p_m \in B(\frac{1}{m}, p) \cap \mathcal{F}(X) - J \). Note that \( \{p_m\} \cup A_m \notin \langle J^o, K^o \rangle \). By Lemma 32, \( \{p_m\} \cup A_m \in \mathbb{R}^2(X) - \langle J^o, K^o \rangle \). Then \( B = \lim \{p_m\} \cup A_m \in \text{cl}_{C(X)}(\mathbb{R}^2(X) - \langle J^o, K^o \rangle) \). On the other hand, since \( p \in \text{Fr}_X(J) \), there exists a sequence \( \{x_m\}_{m=1}^{\infty} \) in \( J^o \) such that \( \lim x_m = p \). Then for each \( m \in \mathbb{N} \), \( \{x_m\} \cup A_m \in \langle J^o, K^o \rangle \) and by Lemma 32, \( \{x_m\} \cup A_m \in \mathbb{R}^2(X) \cap \langle J^o, K^o \rangle \). Hence, \( B \in \text{cl}_{C(X)}(\mathbb{R}^2(X) \cap \langle J^o, K^o \rangle) \). Therefore, \( B \in \mathcal{D}(J, K) \). This completes the proof of the Lemma.

Theorem 34 Let \( X \) and \( Y \) be Peano continua. Let \( J, K \in \mathfrak{X}(X) \) and \( L, M \in \mathfrak{X}(Y) \) be such that \( \text{Fr}_X(J) \subset \text{cl}_X(\mathcal{F}(X) - J) \), \( \text{Fr}_X(K) \subset \text{cl}_X(\mathcal{F}(X) - K) \), \( \text{Fr}_Y(L) \subset \text{cl}_Y(\mathcal{F}(Y) - L) \) and \( \text{Fr}_Y(M) \subset \text{cl}_Y(\mathcal{F}(Y) - M) \). Suppose that \( h : C_2(X) \to C_2(Y) \) is a homeomorphism and \( h(J^o, K^o) = (L^o, M^o) \). Then:

1. if \( J = K \) and \( J \) is a simple closed curve, then \( L = M \) and \( L \) is a simple closed curve,
2. if \( J = K \), \( J \) is an arc and \( J \notin \mathfrak{A}_E(X) \), then \( L = M \), \( L \) is an arc and \( L \notin \mathfrak{A}_E(Y) \),
3. if \( J = K \) and \( J \in \mathfrak{A}_E(X) \), then \( L = M \) and \( L \in \mathfrak{A}_E(Y) \),
4. if \( J \neq L \), then \( M \neq N \),
5. if \( J = K \) and \( p \in J - J^o \), then \( h(\{p\}) \) is a one-point set and \( h(\{p\}) \subset L - L^o \).
Proof. We describe models for the set $D(J,K)$ considering all possibilities for the sets $J$ and $K$ in $\mathcal{A}_S(X)$. These models are illustrated in Figure 2.

(a) $J = K$, $J$ is an arc and $J \notin \mathcal{A}_E(X)$.

According to Lemma 33, $D(J,J) = \{(p_J) \cup A : A \in C(J)\} \cup \{(q_J) \cup A : A \in C(J)\}$. By Example 5.1 of [19], $C(J)$ is a 2-cell. Thus, $D(J,J)$ is the union of two 2-cells intersecting in the elements $\{p_J,q_J\}$ and $J$.

(b) $J = K$, $J \in \mathcal{A}_E(X)$.

Here, $D(J,J) = \{(q_J) \cup A : A \in C(J)\}$ is a 2-cell.

(c) $J = K$ and $J$ is a simple closed curve.

Here, $D(J,J) = \{(q_J) \cup A : A \in E(J)\}$ is homeomorphic to the continuum $W_0$ described in the paragraph previous to Lemma 31.

From now on we assume that $J \neq K$.

(d) Both $J$ and $K$ are arcs and $J, K \notin \mathcal{A}_E(X)$.

Let $D_1 = \{(p_J) \cup A : A \in C(K)\}$, $D_2 = \{(q_J) \cup A : A \in C(K)\}$, $D_3 = \{(p_J) \cup A : A \in C(J)\}$ and $D_4 = \{(q_J) \cup A : A \in C(J)\}$. Note that $D_1, D_2, D_3$ and $D_4$ are 2-cells and $D(J,K) = D_1 \cup D_2 \cup D_3 \cup D_4$. Here, we consider three subcases.

(d.1) $J \cap K = \emptyset$. In this subcase, $D_1 \cap D_2 = \emptyset = D_3 \cap D_4$, $D_1 \cap D_3 = \{(p_J,q_K)\}$, $D_1 \cap D_4 = \{(p_J,q_J)\}$, $D_2 \cap D_3 = \{(q_J,q_K)\}$ and $D_2 \cap D_4 = \{(q_J,q_J)\}$.

(d.2) $J \cap K$ is a one-point set. In this subcase we may assume that $J \cap K = \{q_J\} = \{q_K\}$. Then we have the same equalities as in case (d.1), that is:

$D_1 \cap D_2 = \emptyset = D_3 \cap D_4$, $D_1 \cap D_3 = \{(p_J,p_K)\}$, $D_1 \cap D_4 = \{(p_J,q_K)\}$, $D_2 \cap D_3 = \{(q_J,p_K)\}$ and $D_2 \cap D_4 = \{(q_J,q_K)\}$.

(d.3) $J \cap K$ is a set with exactly two points. We may assume that $p_J = p_K$ and $q_J = q_K$. Then $D_1 \cap D_2 = \{(p_J,q_J)\}$, $D_1 \cap D_3 = \{(p_J,p_K)\}$, $D_1 \cap D_4 = \{(p_J,q_K)\}$, $D_2 \cap D_3 = \{(q_J,q_K)\}$ and $D_2 \cap D_4 = \{(q_J,p_K)\}$.

(e) Both $J$ and $K$ are arcs and $J \notin \mathcal{A}_E(X)$ and $K \in \mathcal{A}_E(X)$.

Let $D_1 = \{(p_J) \cup A : A \in C(K)\}$, $D_2 = \{(q_J) \cup A : A \in C(K)\}$ and $D_3 = \{(q_K) \cup A : A \in C(J)\}$. Note that $D_1, D_2$ and $D_3$ are 2-cells and $D(J,K) = D_1 \cup D_2 \cup D_3$. Here, we consider two subcases.

(e.1) $J \cap K = \emptyset$. In this subcase, $D_1 \cap D_2 = \emptyset$, $D_1 \cap D_3 = \{(p_J,q_K)\}$ and $D_2 \cap D_3 = \{(q_J,q_K)\}$.

(e.2) $J \cap K$ is a one-point set. In this subcase we may assume that $J \cap K = \{q_J\} = \{q_K\}$. Then we have the same equalities as in case (e.1), that is:

$D_1 \cap D_2 = \emptyset$, $D_1 \cap D_3 = \{(p_J,q_K)\}$ and $D_2 \cap D_3 = \{(q_J,q_K)\}$.

(f) $J$ is an arc, $J \notin \mathcal{A}_E(X)$ and $K$ is a simple closed curve.

Let $D_1 = \{(p_J) \cup A : A \in E(K)\}$, $D_2 = \{(q_J) \cup A : A \in E(K)\}$ and $D_3 = \{(q_K) \cup A : A \in C(J)\}$. Note that $D(J,K) = D_1 \cup D_2 \cup D_3$. $D_3$ is a 2-cell while $D_1$ and $D_2$ are homeomorphic to the continuum $W_0$. In both cases, when $J \cap K = \emptyset$ or when $J \cap K$ is a one-point set, we have that $D_1 \cap D_2 = \emptyset$, $D_1 \cap D_3 = \{(p_J,q_K)\}$ and $D_2 \cap D_3 = \{(q_J,q_K)\}$.

(g) $J$ and $K$ are arcs and $J, K \notin \mathcal{A}_E(X)$.

Let $D_1 = \{(q_J) \cup A : A \in C(K)\}$ and $D_2 = \{(q_K) \cup A : A \in C(J)\}$. Then $D(J,K) = D_1 \cup D_2$ and $D_1$ and $D_2$ are 2-cells. Note that $D_1 \cap D_2 = \{(q_J,q_K)\}$. 

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(h) $J \in \mathfrak{A}_E(X)$ and $K$ is a simple closed curve.

Let $D_1 = \{\{q_J\} \cup A : A \in \mathcal{E}(K)\}$ and $D_2 = \{\{q_K\} \cup A : A \in \mathcal{C}(J)\}$. Then $D(J, K) = D_1 \cup D_2$. $D_1$ is a 2-cell and $D_2$ is homeomorphic to $W_0$. Note that $D_1 \cap D_2 = \{\{q_J, q_K\}\}$.

(i) $J$ and $K$ are simple closed curves.

Let $D_1 = \{\{q_J\} \cup A : A \in \mathcal{E}(K)\}$ and $D_2 = \{\{q_K\} \cup A : A \in \mathcal{E}(J)\}$. Then $D(J, K) = D_1 \cup D_2$. $D_1$ and $D_2$ are homeomorphic to $W_0$. Note that $D_1 \cap D_2 = \{\{q_J, q_K\}\}$.

We can observe, in Figure 2, that for different cases we obtain different models.

![Figure 2](image)

If $J = L$ and $J$ is a simple closed curve, then $D(J, J)$ is as in case (c). Hence, $D(L, M)$ is as in case (c). This implies that $L = M$ and $L$ is a simple closed curve. This proves (1). The proofs for (2), (3) and (4) are similar.

In order to prove (5), let $B = h(\{p\})$. Since $p \in \text{Fr}_X(J)$, there exists a sequence $\{p_m\}_{m=1}^\infty$ of points in $J^o$ such that $\lim p_m = p$. Then $\lim h(\{p_m\}) = B$ and $h(\{p_m\}) \subset L^o$ for each $m \in \mathbb{N}$. Thus, $B \subset L$. Take an open subset $U$ of $X$ such that $p \in U$. Since $\text{Fr}_X(J) \subset \text{cl}_X(\mathcal{F}A(X) - J)$, $U \cap \mathcal{F}A(X) - J \neq \emptyset$. This implies that there exists a sequence $\{x_m\}_{m=1}^\infty$ of points of $\mathcal{F}A(X) - J$ such that $\lim x_m = p$. For each $m \in \mathbb{N}$, let $J_m \in \mathfrak{A}_S(X)$ such that $x_m \in J_m^o$. Let
Let $m \in \mathfrak{A}_S(Y)$ be such that $h([J^m]) = [L^m]$. Then $J_m \neq J$, so $L_m \neq L$. Since $h([x_m]) \subset [L^m]$, $h([x_m]) \cap L^o = \emptyset$. Thus, $B = \lim h([x_m]) \subset Y - L^o$. We have shown that $B \subset \text{Fry}(L)$.

By (1) and (3), if $J$ is a simple closed curve or $J \in \mathfrak{A}_E(X)$, then $L$ is a simple closed curve or $L \in \mathfrak{A}_E(Y)$. In these cases, $\text{Fry}(J)$ and $\text{Fry}(L)$ are one-point sets. Then $B$ is a one-point set contained in $\text{Fry}(L)$.

Suppose now that $J$ is an arc and $J \notin \mathfrak{A}_E(X)$. Then $L$ is an arc and $L \notin \mathfrak{A}_E(Y)$. Let $u, v$ be the end points of $L$. Then $u \neq v$ and $\text{Fry}(L) = \{u, v\}$. If $B = \{u\}$ or $B = \{v\}$, we are done. Suppose then that $B = \{u, v\}$. Since $h(D(J, J)) = (L, L)$, by the model described in (a), we obtain that $\{p\}$ is not a local cut point of $D(J, J)$. However $B = h(p) = \{u, v\}$ is a local cut point of $D(L, L)$, a contradiction. This completes the proof of (5) and ends the proof of the theorem.

**Theorem 35** Let $X$ and $Y$ be almost meshed Peano continua. If $C_2(X)$ and $C_2(Y)$ are homeomorphic, then $X$ and $Y$ are homeomorphic.

**Proof.** By Theorem 4.1 of [16], we may assume that $X$ and $Y$ are not connected by a simple closed curve. Let $h : C_2(X) \rightarrow C_2(Y)$ be a homeomorphism. Proceeding as in the beginning of Lemma 32, we have that the components of $\mathfrak{B}_2(X)$ are the sets of the form $\langle J^o, K^o \rangle$ where $J, K \in \mathfrak{A}_S(X)$. Thus, for every $J, K \in \mathfrak{A}_S(X)$, there exist $L, M \in \mathfrak{A}_S(Y)$ such that $h(\langle J^o, K^o \rangle) = \langle L^o, M^o \rangle$. Since $X$ is almost meshed, for each $J \in \mathfrak{A}_S(X)$, $\text{Fr}_X(J) \subset \text{cl}_X(\mathcal{F}A(X) - J)$ and something similar happens for the elements in $\mathfrak{A}_S(Y)$. Hence, we can apply Theorem 34.

Now, take $p \in X - \bigcup \{L^o : L \in \mathfrak{A}_S(X)\}$. We claim that $h(\{p\}) = \{y\}$ for some $y \in Y - \bigcup \{K^o : K \in \mathfrak{A}_S(Y)\}$. Since $X = \text{cl}_X(\mathcal{F}A(X))$, there exists a sequence $\{p_m\}_{m=1}^{\infty}$ in $\mathcal{F}A(X)$ such that $\lim p_m = p$. For each $m \in \mathbb{N}$, let $J_m \in \mathfrak{A}_S(X)$ be such that $p_m \in J_m^o$ and choose a point $q_m \in \text{Fr}_X(J_m)$. By Lemma 8, $\lim J_m = \{p\}$. This implies that $\lim q_m = p$. By Theorem 34, for each $m \in \mathbb{N}$, $h(\{q_m\}) = \{w_m\}$, for some $w_m$ in the closed set $Y - \bigcup \{K^o : K \in \mathfrak{A}_S(Y)\}$. Hence $h(\{p\}) = \{y\}$, for some $y \in Y - \bigcup \{K^o : K \in \mathfrak{A}_S(Y)\}$.

We define a map $g : X \rightarrow Y$. Let $F = X - \bigcup \{L^o : L \in \mathfrak{A}_S(X)\}$. Given $p \in F$, let $g(p)$ be such that $h(\{p\}) = \{g(p)\}$. Given $J \in \mathfrak{A}_S(X)$, let $K_J \in \mathfrak{A}_S(Y)$ be such that $h(\langle J^o \rangle) = \langle K_J^o \rangle$.

If $J$ is a simple closed curve, by (5) in Theorem 34, $g(q_J) \in K_J - K_J^o$. Hence, $g(q_J)$ is the only point in $K_J$ such that $K_J - \{g(q_J)\}$ is open in $Y$. Fix a homeomorphism $g_J : J \rightarrow K_J$ such that $g_J(q_J) = g(q_J)$. If $J \in \mathfrak{A}_E(X)$, by Theorem 34, $K_J \in \mathfrak{A}_E(Y)$ and $g(q_J)$ is the only point in the arc $K_J$ such that $K_J - \{g(q_J)\}$ is open in $Y$. Fix a homeomorphism $g_J : J \rightarrow K_J$ such that $g_J(q_J) = g(q_J)$. Finally, if $J$ is an arc and $J \notin \mathfrak{A}_E(X)$, then $K_J$ is an arc in $\mathfrak{A}_S(Y) - \mathfrak{A}_E(Y)$ and $g(p_J)$ and $g(q_J)$ are the end points of $J$. Fix a homeomorphism $g_J : J \rightarrow K_J$ such that $g_J(p_J) = g(p_J)$ and $g_J(q_J) = g(q_J)$.

Now, define $g : X \rightarrow Y$ as the common extension of $g$ (defined in $F$) and the maps $g_J$ for $J \in \mathfrak{A}_S(X)$. Note that $g$ is well-defined and $g$ is continuous in the open set $X - F$. In fact, $g|J$ is continuous for each $J \in \mathfrak{A}_S(X)$. In order
to complete the proof that \( g \) is continuous, take a sequence \( \{p_m\}_{m=1}^{\infty} \) in \( X - F \) such that \( \lim p_m = p \) for some \( p \in F \). For each \( m \in \mathbb{N} \), let \( J_m \in \mathfrak{A}_2(X) \) be such that \( p_m \in J_m^o \). Then \( q_{J_m} \in Fr_X(J_m) \). We may assume that \( J_m \neq J_k \) for \( m \neq k \).

By Lemma 8, \( \lim q_{J_m} = p \). Since \( q_{J_m} \in F \) for each \( m \in \mathbb{N} \), \( \{g(p)\} = h(\{p\}) = \lim h(\{q_{J_m}\}) = \lim \{g(q_{J_m})\} \). Hence, \( \lim g(q_{J_m}) = g(p) \). Given \( m \in \mathbb{N} \), \( g(p_m) = g_{J_m}(p_m) \in K_{J_m} \) and \( g(q_{J_m}) \in K_{J_m} \). By Lemma 8, \( \lim K_{J_m} = \{g(p)\} \). Hence, \( \lim g(p_m) = g(p) \). This completes the proof that \( g \) is continuous.

It is easy to check that \( g \) is one-to-one. In order to see that \( g \) is onto, let \( K \in \mathfrak{A}_2(Y) \). Applying Theorem 34 to \( h^{-1} \), there exists \( J \in \mathfrak{A}_2(X) \) such that \( (J^o) = h^{-1}(K^o) \). This implies that \( K = K_J \), so \( K \subseteq g(X) \). Since \( \bigcup \{K : K \in \mathfrak{A}_2(Y)\} \) is dense in \( Y \), we conclude that \( g \) is onto. Therefore, \( g \) is a homeomorphism. This ends the proof of the theorem. ■

By Theorems 29, 30 and 35, we obtain the following.

**Theorem 36** Suppose that \( X \) and \( Y \) are almost meshed Peano continua and \( C_n(X) \) is homeomorphic to \( C_n(Y) \) for some \( n \in \mathbb{N} \). Then:

(a) if \( n = 1 \) and \( X \) and \( Y \) are neither arcs nor simple closed curves, then \( X \) is homeomorphic to \( Y \);

(b) if \( n \neq 1 \), then \( X \) is homeomorphic to \( Y \).

**Theorem 37** Suppose that \( X \) is a meshed continuum. If \( n \neq 1 \), then \( X \) has unique hyperspace \( C_n(X) \). If \( X \) is neither an arc nor a simple closed curve, then \( X \) has unique hyperspace \( C(X) \).

**Proof.** Suppose that \( C_n(X) \) and \( C_n(Y) \) are homeomorphic. Let \( h : C_n(X) \rightarrow C_n(Y) \) be a homeomorphism. Since \( X \) is meshed, by Lemma 2, \( X \) is a Peano continuum. Then (see Theorem 3.2 of [20]), \( Y \) is a Peano continuum. Note that \( h(\mathfrak{G}_n(X)) = \mathfrak{G}_n(Y) \). By Theorem 5, \( \mathfrak{G}_n(X) \) is dense in \( C_n(X) \). Thus, \( \mathfrak{G}_n(Y) \) is dense in \( C_n(Y) \). By Theorem 5, \( Y \) is meshed. Applying Theorem 36, we conclude the proof of the Theorem. ■

7 An almost meshed continuum with unique hyperspace

Consider the example \( Z_0 = ([-1,1] \times \{0\}) \cup (\bigcup \{\{\frac{1}{m}\} \times [0,\frac{1}{m}] : m \geq 2\}) \) mentioned at the end of the introduction and illustrated in Figure 1. If a dendrite \( Z \) contains a topological copy of \( Z_0 \), then the hyperspace \( C(Z) \) is not unique [2]. Roughly speaking, this happens because there is a Hilbert cube \( C \) near the element \( \{0,0\} \) of \( C(Z) \): Consider the continuum \( W \) that is obtained by attaching a Peano continuum \( D \) without free arcs at \( (0,0) \) to \( Z \), that is, \( W = Z \cup D \). Then \( C(D) \) and the set \( \{A \in C(W) : (0,0) \in A\} \) are Hilbert cubes whose union with \( C \) is again a Hilbert cube and moreover, the homeomorphism obtained can
be extended to the homeomorphism between $C(Z)$ and $C(W)$. One may think local dendrites behave in the same way.

The next example shows that this does not happen. The "simplest" local dendrite $X$ which is not a dendrite and contains a topological copy of $Z_0$ does have unique hyperspace $C(X)$.

**Example 38** There exists a local dendrite $X$ such that $X$ contains a topological copy of $Z_0$, $\mathcal{P}(X)$ is a one-point set, $X - \mathcal{P}(X)$ is connected and $X$ has unique hyperspace $C(X)$.

Let $S = \{(-1, 1) \times [0, 1] \cup ([-1, 1] \times \{0\})\}$. Then $S$ is a simple closed curve.

Let $X = Z_0 \cup S$ and $\theta = (0, 0)$ (X is the continuum $Z_2$ illustrated in Figure 1).

Then $X$ is an almost meshed Peano continuum that contains a simple closed curve $S$, $\mathcal{P}(X) = \{\theta\}$. $X - \mathcal{P}(X)$ is connected and $X$ is not meshed. Observe that $X$ is a local dendrite.

For each $m \geq 2$, let $B_m = \{\frac{1}{m}\} \times [0, 1 \}$, $S_m = S \cup B_2 \cup \ldots \cup B_m$, $A_m = \{\frac{1}{m}\} \times [0, 1]$, and $p_m = (\frac{1}{m}, 0) \in A_m$. We will need the following claim.

**Claim 5.** Let $\alpha : [0, 1] \rightarrow C(X)$ be a map and let $m \in N$ be such that $p_m p_{m+1} \not\subseteq \alpha(0)$ ($p_m p_{m+1}$ denotes the shortest arc in $X$ joining $p_m$ and $p_{m+1}$) and for each $t \in [0, 1]$, $\{p_m, p_{m+1}\} \subseteq \alpha(t)$ and $S \not\subseteq \alpha(t)$. Then $p_m p_{m+1} \not\subseteq \alpha(1)$.

We prove Claim 5. Let $M = \{[-1, 1] \times [0, 1] \cup ([-1, 1] \times \{0\})\} \cup (\{[-1, 1] \times (\frac{1}{m}, 1)\} \cup [\frac{1}{m}, 1] \times \{0\})$. Let $J = \{t \in [0, 1] : p_m p_{m+1} \subseteq \alpha(t)\}$ and $K = \{t \in [0, 1] : t \in M \subseteq \alpha(t)\}$. Then $J$ and $K$ are closed subsets of $[0, 1]$ and $0 \not\in J$. Since $p_m p_{m+1} \cup M = S$ and $S \not\subseteq \alpha(t)$ for any $t \in [0, 1]$, $J \cap K = \emptyset$. Notice that each connected subset of $X$ containing $p_m$ and $p_{m+1}$, contains either $p_m p_{m+1}$ or $M$. Hence, $[0, 1] = J \cup K$. The connectedness of $[0, 1]$ implies that $J = \emptyset$, $1 \not\in J$ and $p_m p_{m+1} \not\subseteq \alpha(1)$. This ends the proof of Claim 5.

In order to prove that $X$ has unique hyperspace $C(X)$, let $Y$ be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. Then $Y$ is a Peano continuum (see Theorem 3.2 of [20]). Let $h : C(X) \rightarrow C(Y)$ be a homeomorphism.

Let $h_X : c_l(C(X)) \rightarrow c_l(C(Y))$, $h_Y : c_l(C(Y)) \rightarrow c_l(C(Y))$ be homeomorphisms with the properties described in Theorem 27. Since $X$ is almost meshed, $X = c_l(C(X))$. Since $h$ is a homeomorphism, $h(c_l(C(Y))) = c_l(C(Y))$ and $c_l(C(Y)) = c_l(C(Y)) = c_l(C(Y))$. Thus, we can consider the map $g : X \rightarrow Y$ given by $g = h_Y^{-1} \circ h(c_l(C(X))) \circ h_X$. Then $g$ is an embedding and $g(X) = c_l(C(Y))$.

In order to prove that $X$ and $Y$ are homeomorphic, we are going to show that $Y = c_l(C(Y))$. Suppose to the contrary that $Y \neq c_l(C(Y))$. Note that $Y - c_l(C(Y)) \in \mathcal{P}(Y)$. We need to show the following claim.

**Claim 6.** If $p \in X$ and $g(p) \in \mathcal{P}(Y)$, then $p \in \mathcal{P}(X)$.

To prove Claim 6, let $y = g(p)$. Then $y \in c_l(C(Y)) - \bigcup \{K^0 : K \in \mathcal{X}_E(Y)\}$. Thus, $h_Y(y) = \{y\}$. By Theorem 4, $\dim_{h_Y(y)}[C(Y)]$ is infinite. Then $\dim_{h_Y(y)}[C(X)]$ is infinite. Applying again Theorem 4 we obtain that $h^{-1}(h_Y(y)) \cap \mathcal{P}(X) \neq \emptyset$. That is, $h_X(p) \cap \mathcal{P}(X) \neq \emptyset$. Given $J \in \mathcal{X}_E(X)$, $J \cap \mathcal{P}(X) = \emptyset$. By the way that $h_X$ was chosen as in Theorem 27, we have that $p \in \mathcal{P}(X)$. This completes the proof of Claim 6.
Since $\mathcal{P}(X) = \{\theta\}$, $\theta$ is the only point $p$ in $X$ for which $g(p) \in \mathcal{P}(Y)$. Thus, $g(X) \cap \mathcal{P}(Y) = \{g(\theta)\}$. Fix a point $y_0 \in Y - g(X)$ and let $\beta : [0,1] \to Y$ be a one-to-one map such that $\beta(0) = g(\theta)$ and $\beta(1) = y_0$. Let $t_0 = \max\{t \in [0,1] : \beta(t) \in g(X)\}$. Then $\beta(t_0) = g(\theta)$. Thus, $t_0 = 0$, $\beta((0,1]) \cap g(X) = \emptyset$ and $\text{Im} \beta \subseteq \mathcal{P}(Y)$.

By Theorem 4, for each $m \geq 2$, $\dim_{S_m}[C(X)] = \infty$ and $S_m \in \text{cl}(C(X))$. Thus, $\dim_{S_m}[C(Y)] = \infty$ and $h(S_m) \in \text{cl}(C(Y))$. This implies that $h(S_m)$ is a subcontinuum of $Y$ contained in $Y - \mathcal{P}(Y)$ and $h(S_m) \cap \mathcal{P}(Y) \neq \emptyset$. Thus, $h(S_m) \subseteq g(X)$ and $g(\theta) \in h(S_m)$. Fix $m_0 \in \mathbb{N}$ such that $m_0 > 4$ and $h(S_{m_0}) \neq \{g(\theta)\}$. Then $h(S_{m_0}) \cap (Y - \mathcal{P}(Y)) \neq \emptyset$.

Let $\mathcal{L} = \{E \in C(X), g(\theta) \in h(E)\}$. The uniform continuity of the map $\beta_0 : \mathcal{L} \times [0,1] \to C(X)$ given by $\beta_0(E,t) = h^{-1}(h(E) \cup \beta([0,t]))$ implies that there exists $s_0 > 0$ such that, if $E \in \mathcal{L}$ and $B_2 \cup B_3 \cup B_4 \subseteq E$, then for each $s \in [0,s_0]$, $A_2 \cup A_3 \cup A_4 \subseteq \beta_0(E,s)$. In particular, since $B_2 \cup B_3 \cup B_4 \subseteq S_{m_0}$, for each $s \in [0,s_0]$, $A_2 \cup A_3 \cup A_4 \subseteq h^{-1}(h(S_{m_0}) \cup \beta([0,s]))$. Let $Y_0 = h(S_{m_0}) \cup \beta([0,s_0])$ and $X_0 = h^{-1}(Y_0)$. Since $\beta(s_0) \in \mathcal{P}(Y) - g(X) \subset \text{int}_Y(\mathcal{P}(Y))$, by Theorem 4, $Y_0 \in \text{int}_Y(C(X) - \overline{g(\theta)})$. Hence, $X_0 \in \text{int}_Y(C(X) - \overline{g(\theta)})$. This implies that $S \subseteq X_0$. Then we can find a point $z_0 \in S - X_0$. Since $A_2 \cup A_3 \cup A_4 \subseteq X_0$, we conclude that $p_2p_3 \subset X_0$ or $p_2p_3 \subset X_0$. We consider the case that $p_2p_3 \subset X_0$, the other one is similar. Note that $z_0 \notin p_2p_3$.

Let $\varepsilon > 0$ be such that, if $A \in C(X)$ and $H_X(A,X_0) < \varepsilon$, then $z_0 \notin A$. Let $\delta > 0$ be as in the definition of the uniform continuity of $h^{-1}$ for the number $\varepsilon$. Let $x,y \in p_2p_3 - \{p_2,p_3\}$ be such that $x \neq y$ and let $K$ be the subarc of $p_2p_3$ joining $x$ and $y$, notice $K^0 = K - \{x,y\}$. We choose $x$ and $y$ close enough to each other in such a way that $H_Y(h(S_{m_0} - K^0),h(S_{m_0})) < \delta$, we also ask that $h(S_{m_0} - K^0) \cap (Y - \mathcal{P}(Y)) \neq \emptyset$. Since $\theta \in S_{m_0} - K^0$, by Theorem 4, $\dim_{S_{m_0} - K^0}[C(X)]$ is infinite, so $\dim_{S_{m_0} - K^0}[C(Y)]$ is infinite and $h(S_{m_0} - K^0) \cap \mathcal{P}(Y) \neq \emptyset$. Hence, $g(\theta) \in h(S_{m_0} - K^0)$.

Define $\alpha, \gamma : [0,1] \to C(X)$ by $\alpha(t) = h^{-1}(h(S_{m_0} - K^0) \cup \beta([0,t_0]))$ and $\gamma(t) = h^{-1}(h(S_{m_0}) \cup \beta([0,t]))$. Then $\alpha$ and $\gamma$ are continuous, $\alpha(0) = S_{m_0} - K^0$, $\alpha(1) = h^{-1}(h(S_{m_0} - K^0) \cup \beta([0,t_0]))$, $\gamma(0) = S_{m_0}$, and $\gamma(1) = X_0$. Since $H_Y(h(S_{m_0} - K^0),h(S_{m_0})) < \delta$, $H_Y(h(S_{m_0} - K^0) \cup \beta([0,t]),h(S_{m_0} \cup \beta([0,t])) < \delta$ for each $t \in [0,1]$. Thus, $H_X(\alpha(t),\gamma(t)) < \varepsilon$ for each $t \in [0,1]$. Hence, $H_X(\alpha(1),X_0) < \varepsilon$. This implies that $z_0 \notin \alpha(1)$.

By the choice of $s_0$, since $B_2 \cup B_3 \cup B_4 \subseteq S_{m_0} - K^0$, we obtain that $A_2 \cup A_3 \cup A_4 \subseteq \alpha(t)$ for each $t \in [0,1]$. In particular, $\{p_2,p_3\} \subseteq \alpha(t)$ for each $t \in [0,1]$.

Given $t > 0$, $\beta(t_0) 

\in (h(S_{m_0} - K^0) \cup \beta([0,t_0])) \cap \text{int}_Y(\mathcal{P}(Y))$, Theorem 4 implies that $(h(S_{m_0} - K^0) \cup \beta([0,t])) \in \text{int}_Y(C(Y) - \overline{g(\theta)})$. Hence, $\alpha(t) \in \text{int}_Y(C(X) - \overline{g(\theta)})$. If $S \subseteq \alpha(t)$, then there exists a sequence of elements in $C(X)$ which do not contain $\theta$ and converge to $\alpha(t)$, so $\alpha(t) \notin \text{int}_Y(C(X) - \overline{g(\theta)})$, a contradiction. Therefore, $S \notin \alpha(t)$.

We have shown that $\alpha$ satisfies the hypothesis in Claim 5, so $p_2p_3 \notin \alpha(1)$. But $z_0$ is a point in $S$ such that $z_0 \notin p_2p_3$, $z_0 \notin \alpha(1)$ and, since $p_2,p_3 \in \alpha(1)$ we contradict the connectedness of $\alpha(1)$. This contradiction completes the proof that $X$ has unique hyperspace $C(X)$.
8 Dendrites not in class $\mathfrak{D}$ and hyperspace $C_2(X)$

For a dendrite $W$, it is known ([2] and [13]) that $C(W)$ is unique if and only if $W$ is in class $\mathfrak{D}$. This is not true for $C_2(W)$ as we see in this section. We prove that the continuum $Z_3 = (\{-1, 1\} \times \{0\}) \cup (\bigcup \{[\frac{1}{m}, 1] : m \geq 2\}) \cup (\bigcup \{[\frac{1}{m}, 1] : m \geq 2\})$ has unique hyperspace $C_2(Z_3)$. We emphasize that $Z_3$ does not have unique hyperspace $C(Z_3)$ (see [2] or Corollary 14). Let $\theta = (0, 0)$.

**Example 39** The continuum $Z_3$ has unique hyperspace $C_2(Z_3)$.

Note that $Z_3 \notin \mathfrak{D}$. We see that $Z_3$ has unique hyperspace $C_2(Z_3)$. Suppose that $Y$ is a continuum such that $C_2(Z_3)$ and $C_2(Y)$ are homeomorphic. Let $h : C_2(Z_3) \to C_2(Y)$ be a homeomorphism. By Theorem 4.1 of [16], $Y$ is not a finite graph.

Let $J, K \in A_S(Z_3)$. Notice that $\theta \notin J, K$ and $J$ and $K$ are arcs. By Theorem 4, $\dim_{\mathbb{R}}[C_2(Z_3)]$ and $\dim_{\mathbb{R}}[C_2(Z_3)]$ are finite. By the first paragraph in the proof of Lemma 32, there exist $L, M \in A_S(Y)$ such that $h((J^o, K^o)) = (L^o, M^o)$. Thus, $h(cl_{C_2(Z_3)}((J^o, K^o))) = cl_{C_2(Y)}((L^o, M^o))$. Since $L \cup M \in cl_{C_2(Y)}((L^o, M^o))$, there exists $A \in cl_{C_2(Z_3)}((J^o, K^o))$ such that $h(A) = L \cup M$. Since $A \subset J \cup K$, by Theorem 4, $\dim_{\mathbb{R}}[C_2(Z_3)]$ is finite. Thus, $\dim_{\mathbb{R}}[C_2(Y)]$ is finite and $(L \cup M) \cap P(Y) = \emptyset$. By Theorem 4 there exists a finite graph $D$ in $Y$ such that $L \cup M \subset int_D(D)$. This implies that $Fr_Y(L) \subset cl_Y(F_A(Y) - L)$ and $Fr_Y(M) \subset cl_Y(F_A(Y) - M)$. Since Fr$_Z_3(J) \subset cl_{Z_3}(F_A(Z_3) - J)$ and Fr$_Z_3(K) \subset cl_{Z_3}(F_A(Z_3) - K)$, we can apply Theorem 34. In particular, if $J = K$, then $L = M$ and $L$ is an arc, moreover, for each $p \in J \neq J^o$, $h(\{p\})$ is a one-point set and $h(\{\theta\})$ is also a one-point set in $Y - \bigcup \{L^o : L \in A_S(Y)\}$.

We define a map $g : Z_3 \to Y$. Let $F = Z_3 - \bigcup \{L^o : L \in A_S(Z_3)\}$. Given $p \in F$, let $g(p) \in Y$ be such that $h(\{p\}) = \{g(p)\}$, which exists by Theorem 34. Given $J \in A_S(Z_3)$, let $K_J \in A_S(Y)$ be such that $h((J^o)) = (K_J^o)$. Note that $J$ is not a simple closed curve.

If $J \in A_E(Z_3)$, let $q_J$ and $p_J$ be the end points of $J$, where $p_J \in J^o$. Then $q_J$ is the only point in $J$ such that $J - \{q_J\}$ is open in $Z_3$. By Theorem 34, $K_J \in A_E(Y)$. Note that $q_J \in F$ and $g(q_J) \in Y - \bigcup \{K^o : K \in A_S(Y)\}$. Thus, $\{q_J\} \in cl_{C_2(Z_3)} ((J^o))$ and $\{g(q_J)\} \in cl_{C_2(Y)} ((K_J^o))$. Hence, $g(q_J) \in K_J - K_J^o$. Therefore, $g(q_J)$ is the only point in $K_J$ such that $K_J - \{g(q_J)\}$ is open in $Y$.

Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(q_J) = g(q_J)$. If $J$ is an arc and $J \notin A_E(X)$, let $q_J$ and $p_J$ be the end points of $J$. Then $q_J$ and $p_J$ are the only points in $J$ such that $J - \{p_J, q_J\}$ is open in $X$. By Theorem 34, $K_J$ is an arc in $A_S(Y) - A_E(Y)$. Proceeding as before, $g(p_J)$ and $g(q_J)$ are the only points in the arc $K_J$ such that $K_J - \{g(p_J), g(q_J)\}$ is open in $Y$. Hence, $g(p_J)$ and $g(q_J)$ are the end points of $K_J$. Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(p_J) = g(p_J)$ and $g_J(q_J) = g(q_J)$.

Now define $g : Z_3 \to Y$ as the common extension of $g$ (defined in $F$) and the maps $g_J$ for $J \in A_S(Z_3)$. Proceeding as in Theorem 35, it can be shown that $g$ is a well-defined embedding from $Z_3$ into $Y$. Given $J \in A_S(Z_3)$,
$g(J) \subset \text{cl}(FA(Y))$. Then $g(Z_3) = g(\text{cl}_{Z_3}(FA(Z_3))) \subset \text{cl}(g(FA(Z_3))) \subset \text{cl}(FA(Y))$. Hence, $g(Z_3) \subset \text{cl}(FA(Y))$. Given $K \in \mathcal{A}_S(Y)$, fix a point $q \in K^\circ$. Then $\{q\} \in \mathcal{B}_2(Y)$ and $h^{-1}(\{q\}) \in \mathcal{B}_2(Z_3)$. Hence, there exist $J, L \in \mathcal{A}_S(Z_3)$ such that $h^{-1}(\{q\}) \in \langle J^\circ, L^\circ \rangle$. If $J \neq L$, proceeding as in the first paragraph of the proof of Theorem 32 and using Theorem 34, we obtain that there exist $M, N \in \mathcal{A}_S(Y)$ such that $M \neq N$ and $h((J^\circ, L^\circ)) = (M^\circ, N^\circ)$. Thus, $\{q\} \in (M^\circ, N^\circ)$, a contradiction. Hence, $J = L = K = K_j$. This proves that $K \subset g(Z_3)$, for every $K \in \mathcal{A}_S(Y)$. Hence, $\text{cl}(FA(Y)) \subset g(Z)$. Therefore, $g(Z) = \text{cl}(FA(Y))$.

In order to prove that $Z_3$ and $Y$ are homeomorphic, we are going to show that $Y = \text{cl}(FA(Y))$. Suppose to the contrary that $Y \neq \text{cl}(FA(Y))$. Note that $Y - \text{cl}(FA(Y)) \subset P(Y)$.

We need to show the following claim.

**Claim 7.** If $p \in Z_3$ and $g(p) \in P(Y)$, then $p \in P(Z_3)$.

To prove Claim 7, let $y = g(p)$. Then $y \in \text{cl}(FA(Y)) = \bigcup\{K^\circ : K \in \mathcal{A}_E(Y)\}$. Thus, $p \in Z_3 - \bigcup\{J^\circ : J \in \mathcal{A}_E(Z_3)\}$. Hence, $h(\{p\}) = \{g(p)\} = \{y\}$. By Theorem 4, $\text{dim}_h(\{p\}) |C_2(Y)|$ is infinite. So $\text{dim}(\{p\} |C_2(Z_3)|)$ is infinite. Thus, $p \in P(Z_3)$. So Claim 7 is proved.

Since $P(Z_3) = \{\theta\}$, $\theta$ is the only point $p$ in $X$ for which $g(p) \in P(Y)$. Thus, $g(Z_3) \cap P(Y) = \{g(\theta)\}$. This implies that $P(Y)$ is a subcontinuum of $Y$.

We are going to obtain a contradiction by proving that the set $\mathcal{I}_{Z_3} = \text{int}_{C_2(Z_3)}(C_2(Z_3) - \mathcal{I}_2(Z_3))$ is disconnected and the set $\mathcal{I}_Y = \text{int}_{C_2(Y)}(C_2(Y) - \mathcal{I}_2(Y))$ is pathwise connected.

Take $A \in \mathcal{I}_{Z_3}$. Then $\theta \in A$. If $A$ is connected, then $A$ is the limit of elements $A_m$ in $C_2(Z_3)$ such that $\theta \notin A_m$. This implies that $A_m \in \mathcal{I}_2(Z_3)$ and $A \notin \text{int}_{C_2(Z_3)}(C_2(Z_3) - \mathcal{I}_2(Z_3))$. This contradiction proves that $A$ has two components $A_1$ and $A_2$. We may assume that $\theta \in A_1$. Let $\pi: Z_3 \to [-1,1]$ be the projection on the first coordinate. Then $\mathcal{I}_{Z_3} \subset \{A_1 \cup A_2 \in C_2(X) : A_1 \cup A_2 = \emptyset, \theta \in A_1 \text{ and } \pi(A_2) \subset [-1,0) \cup \{A_1 \cup A_2 \in C_2(X) : A_1 \cup A_2 = \emptyset, \theta \in A_1 \text{ and } \pi(A_2) \subset (0,1]\}$. It follows that $\mathcal{I}_{Z_3}$ is disconnected.

Take $B \in \mathcal{I}_Y - \{\theta\}$. If $B \notin g(Z_3)$, then $B \cap \text{int}_Y(P(Y)) \neq \emptyset$. Let $\alpha: [0,1] \to C_2(Y)$ be an order arc from $B$ to $Y$. Then for each $t \in [0,1]$, $\alpha(t) \cap \text{int}_Y(P(Y)) \neq \emptyset$. This implies that $\alpha(t) \in \mathcal{I}_Y$. Therefore, $B$ can be connected to $Y$ by a path in $\mathcal{I}_Y$. Now suppose that $B \subset g(Z_3)$. Since $\text{dim}_{\beta}(C_2(Y))$ is infinite, $B \cap P(Y) \neq \emptyset$. Thus, $g(\theta) \in B$. Let $\beta: [0,1] \to \text{cl}(\{g(\theta)\})$ be an order arc from $\{g(\theta)\}$ to $P(Y)$. Let $\alpha: [0,1] \to C_2(Y)$ be given by $\alpha(t) = B \cup \beta(t)$. Then $\alpha$ is continuous, $\alpha(0) = B$, $\alpha(1) = B \cup P(Y)$, and for each $t > 0, \emptyset \neq \beta(t) \cap \text{int}_Y(P(Y)) \subset (\alpha(t) \cap \text{int}_Y(P(Y)))$. Hence, $\alpha(t) \in \mathcal{I}_Y$. Therefore, $B$ can be connected to $B \cup P(Y)$ by a path in $\mathcal{I}_Y$. Since $P(Y) \cap \text{int}_Y(P(Y)) \neq \emptyset$, we have reduced the problem to the first case. Hence, $\mathcal{I}_Y$ is pathwise connected.

Therefore, $\mathcal{I}_{Z_3}$ is disconnected and $\mathcal{I}_Y$ is connected. This contradicts the fact that $h$ is a homeomorphism. This contradiction completes the proof that $Z_3$ and $Y$ are homeomorphic. Therefore, $Z_3$ has unique hyperspace $C_2(Z_3)$.

**Problem 40** Characterize dendrites $X$ with unique hyperspace $C_2(X)$. 

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Problem 41 Does there exist a Peano continuum X such that X has unique hyperspace $C(X)$ but X does not have unique hyperspace $C_2(X)$?

Problem 42 Let X be an almost meshed Peano continuum such that $X - B(X)$ is connected. Does X have unique hyperspace $C(X)$?

9 Other examples

Example 43 Let $Z_1 = Z_3 \cup \{(0) \times [0, 1]\}$, then $Z_1$ does not have unique hyperspace $C_2(Z_1)$. To see this, notice that the point $(0, 0)$ satisfies the conditions of Corollary 25. Recall that, by Example 39 $Z_3$ has unique hyperspace $C_2(Z_3)$.

Example 44 Let X be a Peano continuum that contains a homeomorphic copy of dendrite $F_o$. Suppose that there is a point $q \in F_o$ such that $F_o \setminus \{q\}$ is open in X. Then X does not have unique hyperspace $C_n(X)$ for any $n \in \mathbb{N}$. To see this, notice that the vertex of $F_o$ satisfies the conditions of Corollary 25.

Example 45 Let X be a local dendrite. Suppose that X contains a homeomorphic copy of dendrite $F_o$. Then X does not have unique hyperspace $C_n(X)$ for any $n \in \mathbb{N}$.

Proof. Let $d$ be a metric for X. Let $F_o = \bigcup \{\theta p_m : m \in \mathbb{N}\}$, where $\theta, p_m \in X$, each $\theta p_m$ is arc in X, joining $\theta$ and $p_m$, $\lim \theta p_m = \{\theta\}$ (in $C(X)$) and $\theta p_m \cap \theta p_k = \{\theta\}$, if $m \neq k$. In order to apply Theorem 22 we only need to prove that $X - \{\theta\}$ has infinitely many components. Suppose to the contrary that $X - \{\theta\}$ has only finitely many components. Then we may suppose that there exists a component $W$ of $X - \{\theta\}$ such that $\theta p_m - \{\theta\} \subset W$ for each $m \in \mathbb{N}$. Let M be a dendrite in X such that $\theta \in M^o$ and let $\varepsilon > 0$ be such that $B(2\varepsilon, \theta) \subset M$. We may assume that $F_o \subset B(\varepsilon, \theta)$ for each $m \in \mathbb{N}$ and $W - M \neq \emptyset$. Fix a point $w \in W - M$. Given $m \in \mathbb{N}$, since W is arcwise connected, there exists an arc $\alpha_m \subset W$ which joins $p_m$ and $w$. Then we can choose a point $q_m \in \alpha_m$ such that $d(\theta, q_m) = \varepsilon$ and the subarc $\beta_m$ of $\alpha_m$ joining $p_m$ and $q_m$ is contained in $\{x \in X : d(x, \theta) \leq \varepsilon\}$. We may assume that $\lim q_m = q$ for some $q \in X$ such that $d(\theta, q) = \varepsilon$. Let $U$ be an open connected subset of X such that $q \in U \subset M$ and $\theta \notin U$. Let $w_0 \in \mathbb{N}$ be such that $q_{m_0}, q_{m_0+1} \in U$. Then there exists an arc $\gamma$ in $U$ joining $q_{m_0}$ and $q_{m_0+1}$. Thus, $p_{m_0}$ and $p_{m_0+1}$ can be joined by a path in $\beta_{m_0} \cup \gamma \cup \beta_{m_0+1} \subset M - \{\theta\}$. This is a contradiction since the unique arc in M joining $p_{m_0}$ and $p_{m_0+1}$ is $\theta p_{m_0} \cup \theta p_{m_0+1}$. Therefore, $X - \{\theta\}$ has infinitely many components.

References


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