Countable dense homogeneity and non-definable spaces

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Rodrigo Hernández-Gutiérrez
rod@xanum.uam.mx

Universidad Autónoma Metropolitana - Iztapalapa

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Homogeneous spaces

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Definition

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- A space $X$ is homogeneous if for every $x, y \in X$ there is a homeomorphism $h : X \rightarrow X$ with $h(x) = y$. 
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Some examples of homogeneous spaces:

$\mathbb{R}$, $\mathbb{R}^n$, $S^n$, the Cantor set, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, the Hilbert cube, all separable manifolds, ...
Definition

A space $X$ is countable dense homogeneous (CDH) if $X$ is separable and every time $D$ and $E$ are countable dense subsets of $X$, there is a homeomorphism $h : X \rightarrow X$ with $h[D] = E$. Theorem (Cantor, Brouwer, Fréchet) $\mathbb{R}$ is CDH.
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Theorem (Cantor, Brouwer, Fréchet)

$\mathbb{R}$ is CDH
Consider two countable dense sets:

\[ D = \{ d_n : n \in \omega \} \]

\[ E = \{ e_n : n \in \omega \} \]

We obtain an order isomorphism between \( D \) and \( E \), extend it to \( \mathbb{R} \).
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More examples

The following examples of homogeneous spaces are also CDH:

- the real line $\mathbb{R}$
- the circumference $\mathbb{S}^1$
- Euclidean spaces $\mathbb{R}^n$
- spheres $\mathbb{S}^n$
- the Cantor set $\{0, 1\}^\omega$
- the set of irrational numbers
- the Hilbert cube $[0, 1]^\mathbb{N}$
- complete Erdős space $\{x \in \ell_2 : \forall n \ (x_n \notin \mathbb{Q})\}$
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The rationals $\mathbb{Q}$ are not CDH: take $D = \mathbb{Q}$ and $E = \mathbb{Q} \setminus \{0\}$. 

More examples
CDH vs homogeneity

Theorem

If $X$ is CDH, then $X$ is a countable free sum of CDH, homogeneous spaces.
CDH vs homogeneity

**Theorem**

*If X is CDH, then X is a countable free sum of CDH, homogeneous spaces.*

**Example:** \( \mathbb{R} \oplus \{0, 1\}^\omega \) is CDH but not homogeneous
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*If $X$ is CDH, then $X$ is a countable free sum of CDH, homogeneous spaces.*

**Example:** $\mathbb{R} \oplus \{0, 1\}^\omega$ is CDH but not homogeneous

A space $X$ is strongly locally homogeneous (SLH) if there is a base of open sets $\mathcal{B}$ such that for $U \in \mathcal{B}$, $x, y \in U$, there is a homeomorphism $h : X \to X$ with $h|_{X \setminus U} = id_{X \setminus U}$ and $h(x) = y$. 
CDH vs homogeneity

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**Theorem (Bennett; Fletcher-McCoy; Anderson-Curtis-van Mill)**

*Any Polish SLH space is CDH*
CDH vs homogeneity

**Theorem**
*If* $X$ *is CDH, then* $X$ *is a countable free sum of CDH, homogeneous spaces.*

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**Theorem (Bennett; Fletcher-McCoy; Anderson-Curtis-van Mill)**
*Any Polish SLH space is CDH*

All of the CDH examples before are SLH, except for complete Erdős space.
An example: the Menger curve

Theorem

If $X$ is a CDH metrizable continuum of dimension 1, then $X$ is either $S^1$ or the Menger curve.

The Menger curve is SLH.

Question (van Mill)

Is every CDH continuum also SLH?
Non CDH but homogeneous

\[ X = \{0, 1\}^\mathbb{N} \times \mathbb{R} \]
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Consider \( D \) and \( E \) dense sets:

Extra: CDH is not preserved under products.
Non CDH but homogeneous

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Consider \( D \) and \( E \) dense sets:
\( D \) intersects every component in at most one point.
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If \( h : X \to X \) is a homeomorphism, \( h[D] \neq E \).
Non CDH but homogeneous

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Extra: CDH is not preserved under products.
Non CDH but homogeneous continua

Let $X$ be any metric, indecomposable continuum.

Consider $D$ and $E$ dense sets:

$D$ intersects every composant in at most one point. 
$E$ intersects some composant in two points.

If $h : X \to X$ is a homeomorphism, $h[D] \neq E$. 
Non-Polish examples

Question (Fitzpatrick y Zhou, 1990)

Is there any CDH metrizable space that is not Polish?
Non-Polish examples

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*Is there any CDH metrizable space that is not Polish?*

Theorem (Baldwin, Beaudoin, 1989)

*Martin’s axiom implies the existence of a CDH Bernstein set.*

$X \subset \mathbb{R}$ is Bernstein if for every Cantor set $C \subset \mathbb{R}$, both $X$ and $\mathbb{R} \setminus X$ intersect $C$. 
Borel spaces

A space is **Borel** if it is in the $\sigma$-algebra generated by the open sets of some Polish space.
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**Proposition**

*Every Borel set is either countable or contains a copy of the Cantor set.*
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A space is **Borel** if it is in the $\sigma$-algebra generated by the open sets of some Polish space.

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Thus, a Bernstein set cannot be Borel.

**Theorem (Hrušák and Zamora-Avilés, 2005)**

*Let $X$ be separable and metrizable. If $X$ is CDH and Borel, then $X$ is Polish.*
Non-Borel + CDH

Theorem (Farah, Hrušák, Martínez-Ranero, 2005)

There is a CDH space \( X \subset \mathbb{R} \) of size \( \omega_1 \) that is meager.
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Theorem (HG, Hrušák, van Mill, 2013)

- There is a space $X \subset \mathbb{R}$ that is CDH and meager.
- There is a space $X \subset \mathbb{R}$ that is CDH, non-Polish and $\mathbb{R} \setminus X$ is meager.
Non-Borel \(+\) CDH

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Theorem (Medini, 2015)

There is a space \(X \subset \mathbb{R}\) that is non-Polish and such that \(X^\mathbb{N}\) is CDH.
**Definition**

Let $X \subset K$ (where $K$ is $\mathbb{R}$ or $2^\mathbb{N}$).

- $X$ is a $\lambda$-set if every countable subset of $X$ is a $G_\delta$.
- $X$ is a $\lambda'$-set (of $K$) if $D$ is a $G_\delta$ of $D \cup X$ whenever $D$ is a countable subset of $K$. 

Ways to construct $\lambda$-sets inside the Cantor set:

- Every Hausdorff gap is a $\lambda'$-set (of the Cantor set).
- Every uncountable and strictly almost-increasing family of functions in $\omega^\omega$ is a $\lambda$-set (but it is not a $\lambda'$-set).
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Using the $\lambda$-sets

Let $X \subset \mathbb{R}$ be a $\lambda$-set.
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Y = X \cdot (\mathbb{Q} \setminus \{0\}) + \mathbb{Q}.
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Then:

- $Y$ is CDH and meager (HG, Hrušák, van Mill),
- if $X$ is a $\lambda'$-set, then $[0, 1] \setminus Y$ is CDH, non-meager and non-Polish (HG, Hrušák, van Mill),
- if $X$ is a $\lambda'$-set, then $([0, 1] \setminus Y)$ is CDH (Medini).
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The double arrow space

\[ \mathbb{A} = [(0, 1) \times \{0, 1\}] \cup \{(0, 1), (1, 0)\} \]

We use the lexicographic order: \( p < q < r \). Here, \( r \) is the immediate successor of \( q \).

\( \mathbb{A} \) is linearly ordered, compact, 0-dimensional, first countable, separable and non-metrizable (it has weight \( c \)).
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is homogeneous

\[(0, 1) \rightarrow (1, 0)\]
A is homogeneous

Apparently, the double arrow space has endpoints, but being an endpoint is not a topological property.
is homogeneous
\( \Delta \) is homogeneous
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$A$ is homogeneous
is homogeneous
Compact, non-metrizable and CDH

Theorem (Arkhangelskii, van Mill, 2012)

$\exists$ is not CDH.
Compact, non-metrizable and CDH

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\[ \mathbb{A} \text{ is not CDH.} \]

Question (Arkhangelskii, van Mill, Hrušák)

*Can a modification of \( \mathbb{A} \), for example \( \mathbb{A} \times \{0, 1\}^\omega \) or \( \mathbb{A}^\omega \) be CDH?*
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Theorem (HG, 2013)
Assume that \( X \times Y \) is CDH, of countable \( \pi \)-weight and none of \( X \) or \( Y \) has isolated points. If \( X \) contains a copy of the Cantor set, so does \( Y \).
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Corollary
None of \( \mathcal{A} \times \{0,1\}^\omega \) or \( \mathcal{A}^\omega \) is CDH.
Let $Y \subset (0, 1)$ be a saturated $\lambda'$-set and define.

$$K = (Y \times \{1\}) \cup ([0, 1] \times \{0\}).$$

with the lexicographic order.

Compact, non-metrizable and CDH

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More compact, non-metrizable and CDH?

Theorem (Steprāns, Zhou, 1988; Hrušák, Zamora-Avilés, 2005)

\( \{0, 1\}^\kappa \) is CDH if and only if \( \omega \leq \kappa < p \).

It is consistent that \( p = \omega_1 \), and in this case all CDH Cantor cubes are metrizable.

Question

Is there a compact, CDH space of weight \( c \) (in ZFC)?

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Is there a compact, connected CDH space that is not metrizable?
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Compact, CDH, without Cantor sets

Theorem (HG, 2017)

(CH) There is a compact CDH space that does not have topological copies of the Cantor set.
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The construction is an inverse limit recursion of length $\omega_1$.
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The construction is an inverse limit recursion of length $\omega_1$. For each $\alpha < \omega_1$, $X_\alpha$ is a topological copy of the Cantor set and $\pi_\alpha : X_{\alpha+1} \rightarrow X_\alpha$ is $\leq 2$-to-1 and 1-to-1 in a co-countable set.
Compact, CDH, without Cantor sets

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All Cantor sets appear on a countable step, so we split one point from each one.
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All Cantor sets appear on a countable step, so we split one point from each one.

Given $\alpha < \omega_1$, we will have a group $G_\alpha$ of homemorphisms of $X_\alpha$. Also, $m_\alpha : G_\alpha \to G_{\alpha+1}$ will be a monomorphism of groups.
Compact, CDH, without Cantor sets

These homeomorphisms will witness that the inverse limit is CDH.
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Technical problem: At step $\alpha < \omega_1$, the new homeomorphisms cannot mix points that have been split with those that have not been split yet (as we may do more splitting).
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Definition

Let $X$ be a space. A group $G$ of autohomeomorphisms of $X$ will be called **cofinitary** if every non-identity element of $G$ has only finitely many fixed points.
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Definition

Let $X$ be a space. A group $G$ of autohomeomorphisms of $X$ will be called cofinitary if every non-identity element of $G$ has only finitely many fixed points.

Lemma

Let $G$ be a countable group of autohomeomorphisms of $2^\mathbb{N}$ and $D, E \subset 2^\mathbb{N}$ be two countable dense sets. Then there is a homeomorphism $h : 2^\mathbb{N} \to 2^\mathbb{N}$ such that $h[D] = E$ and the group generated by $G \cup \{h\}$ is cofinitary.
CDH by cofinitary groups

Theorem
Assume MA. Then there is a cofinitary group of homeomorphisms that witnesses that $2^\mathbb{N}$ is CDH.
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Note: We were able to prove the previous theorem about cofinitary groups under MA BUT the proof of our result about compact CDH spaces without Cantor sets only works under CH.
Powers that are CDH

Theorem (HG, 2013)

Assume that \( X \times Y \) is CDH, of countable \( \pi \)-weight and none of \( X \) or \( Y \) has isolated points. If \( X \) contains a copy of the Cantor set, so does \( Y \).
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**Corollary**

If $X$ is of countable $\pi$-weight without isolated points and $X^\mathbb{N}$ is CDH, then $X$ has a topological copy of the Cantor set.
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Corollary

If \( X \) is of countable \( \pi \)-weight without isolated points and \( X^N \) is CDH, then \( X \) has a topological copy of the Cantor set.

Proof.

\( X^N \approx X \times X^N \)
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If $X$ is of countable $\pi$-weight without isolated points and $X^\mathbb{N}$ is CDH, then $X$ has a topological copy of the Cantor set.

Proof.
$X^\mathbb{N} \approx X \times X^\mathbb{N}$

Corollary
If $X$ is a Bernstein set, then $X^\mathbb{N}$ is NOT CDH.
Powers, homogeneity and CDH

Theorem (Sierpiński, 1932)

There exists a subset of $\mathbb{R}$ that is rigid (the only homomorphism is the identity).

Theorem (L. Brian Lawrence 1994; Dow and Pearl 1997)

If $X$ is 0-dimensional and first countable, then $X^\mathbb{N}$ is homogeneous.
Powers, homogeneity and CDH

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There exists a subset of $\mathbb{R}$ that is rigid (the only homomorphism is the identity).

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If $X$ is 0-dimensional and first countable, then $X^\mathbb{N}$ is homogeneous.

Theorem (Baldwin, Beaudoin, 1989)
Martin’s axiom implies the existence of a CDH Bernstein set.

Theorem (HG, 2013)
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Spaces of continuous functions

$C_p(X) \subseteq \mathbb{R}^X$ is the space of continuous functions with pointwise convergence.

Question (Vladimir Tkachuk)

Is there a space $X$ such that $C_p(X)$ is CDH?

$C_p(X)$ is separable iff there is an injective and continuous function $f : X \to M$, where $M$ is separable and metrizable.

$C_p(X)$ is metric iff $X$ is countable.
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Countable metric spaces

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Lemma (Fitzpatrick and Zhou, 1992)

*If a CDH metric space is meager, then it is a $\lambda$-set.*
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**Lemma (Fitzpatrick and Zhou, 1992)**

*If a CDH metric space is meager, then it is a $\lambda$-set.*

**Corollary**

*If $X$ is countable, metric and $C_p(X)$ is CDH, then $X$ is discrete.*
Countably compact spaces

Theorem (Steprāns, Zhou, 1988; Hrušák, Zamora-Avilés, 2005)
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\mathbb{R}^\kappa \text{ is CDH if and only if } \kappa < p.
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Theorem (Tkachuk, 2016)

Let \( X \) be uncountable, \( \sigma \)-compact, separable and metric. Then there is a countable dense set of \( C_p(X) \) that has no non-trivial convergent sequences.
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Let \( X \) be uncountable, \( \sigma \)-compact, separable and metric. Then there is a countable dense set of \( C_p(X) \) that has no non-trivial convergent sequences.

Corollary
Let \( X \) be uncountable, \( \sigma \)-compact, separable and metric. Then \( C_p(X) \) is not CDH.
A filter (on \( \mathbb{N} \)) is a subset \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) with the following properties:

- \( \mathbb{N} \in \mathcal{F} \) and \( \emptyset \notin \mathcal{F} \),
- if \( A, B \in \mathcal{F} \), then \( A \cap B \in \mathcal{F} \), and
- if \( A \in \mathcal{F} \) and \( A \subset B \subset \mathbb{N} \), then \( B \in \mathcal{F} \).

Moreover, \( \mathcal{F} \) is free if it contains the cofinite subsets of \( \mathbb{N} \).
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We will say that $\mathcal{F}$ is a $P$-filter if every time $\{A_n : n \in \mathbb{N}\} \subset \mathcal{F}$, there is $A \in \mathcal{F}$ such that for all $n \in \mathbb{N}$ we have $A \setminus A_n$ is finite.
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**Theorem (Hrušák, HG, 2013; Kunen, Medini, Zdomskyy, 2015)**

Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a free filter and consider the Cantor set topology on $\mathcal{P}(\mathbb{N})$. Then $\mathcal{F}$ is CDH iff $\mathcal{F}$ is a non-meager $P$-filter.
Spaces defined by filters

Given \( \mathcal{F} \subset P(\mathbb{N}) \) a free filter, define \( \xi(\mathcal{F}) = \mathbb{N} \cup \{\infty\} \) so that \( \mathbb{N} \) is discrete and a neighborhood of \( \infty \) is of the form \( \{\infty\} \cup A \) with \( A \in \mathcal{F} \).
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**Lemma**

$C_p(\xi(\mathcal{F}), \{0, 1\}^\mathbb{N})$ is homeomorphic to $\mathcal{F}^\omega$. 
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Lemma

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Lemma (Shelah)

Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a free filter. Then $\mathcal{F}$ is a non-meager $P$-filter if and only if $\mathcal{F}^{(\omega)}$ is (homeomorphic to) a non-meager $P$-filter.
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Thus, $C_p(\xi(\mathcal{F}), \{0, 1\}^\mathbb{N})$ is CDH if and only if $\mathcal{F}$ is a non-meager $P$-filter.
Conjecture and half a theorem

Using the techniques of Kunen, Medini, Zdomskyy (2015), the following can be proved.

**Theorem (HG, 2017)**

If \( C_p(\xi(\mathcal{F})) \) is CDH, then \( \mathcal{F} \) is a non-meager \( P \)-filter.

However, we still don't know if the other direction holds.

**Conjecture**

If \( \mathcal{F} \) is a non-meager \( P \)-filter, then \( C_p(\xi(\mathcal{F})) \) is CDH.
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If $F$ is a non-meager $P$-filter, then $C_p(\xi(F))$ is CDH.
Thank you

대단히 감사합니다