Uniqueness of hyperspaces for Peano continua

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What’s a unique hyperspace?

Given a metric continuum $X$ and $n \in \mathbb{N}$, consider the hyperspace $C_n(X)$ of all closed nonempty subsets of $X$ with at most $n$ components, metrized by the Hausdorff metric. Denote $C_1(X) = C(X)$. 
What’s a unique hyperspace?

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**Question:** Given $C_n(X)$, is it possible to know $X$?
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**Question**: Given $C_n(X)$, is it possible to know $X$?

**Formally**: Let $X$ and $Y$ be metric continua, $n \in \mathbb{N}$. Under what conditions does the implication

$$C_n(X) \approx C_n(Y) \Rightarrow X \approx Y$$

hold?
What is known
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• Recall the Curtis-Schori-West Hyperspace Theorem (1978): $C(X)$ is the Hilbert cube if and only if $X$ is a locally connected metric continuum with no free arcs.
What is known

- In 1968, Duda showed that if $X$ is a finite graph, $X \notin \{[0, 1], \mathbb{S}^1\}$ and $Y$ is such that $\mathcal{C}(X) \approx \mathcal{C}(Y)$, then $X \approx Y$.

- Recall the Curtis-Schori-West Hyperspace Theorem (1978): $\mathcal{C}(X)$ is the Hilbert cube if and only if $X$ is a locally connected metric continuum with no free arcs.

- Other results are known for hereditarily indecomposable continua, metric indecomposable continua whose proper subcontinua are arcs, metric compactifications of the ray $[0, 1)$, chainable metric continua, fans...
Our problem

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Let $X$ be a Peano continuum, $n \in \mathbb{N}$ and $Y$ (a Peano continuum) such that $C_n(X) \approx C_n(Y)$. Is $X \approx Y$?
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Let $X$ be a Peano continuum, $n \in \mathbb{N}$ and $Y$ (a Peano continuum) such that $C_n(X) \approx C_n(Y)$. Is $X \approx Y$?

In case the answer to this question is yes, we say that $X$ has unique hyperspace $C_n(X)$. 
Partial results

Let $\mathcal{D}$ be the class of dendrites that have their set of endpoints closed.
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- (G. Acosta, D. Herrera-Carrasco, F. Macías-Romero, 2010) If $X$ is a continuum with a basis of $\mathcal{D}$-continua neighborhoods, then $X$ has unique hyperspace $\mathcal{C}_n(X)$ (for $n = 1$, iff $X \neq [0, 1], \mathbb{S}^1$).
Long nullcomb and $F_\omega$

**Theorem.** (Arévalo, W. Charatonik, Simon, Pellicer-Covarrubias) A dendrite is in $D$ if and only if it does not contain a long nullcomb or a copy of $F_\omega$. 
Almost Meshed continua

Recall a free arc in a continuum $X$ is an arc $A \subset X$ such that by removing the endpoints of $A$, one obtains an open subset of $X$. 
Almost Meshed continua

For a continuum $X$, let

$$\mathcal{FA}(X) = \bigcup \{ J^\circ : J \text{ is a free arc in } X \}$$
Almost Meshed continua

For a continuum $X$, let

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We say a continuum $X$ is *almost meshed* if $\mathcal{F}A(X)$ is dense in $X$. 

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Let $\mathcal{P}(X) = X - \mathcal{G}(X)$. 
Examples of almost meshed Peano continua

The long nullcomb, $F_\omega$ and $M$: 
Examples of almost meshed Peano continua

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Examples of almost meshed Peano continua

The long nullcomb, $F_\omega$ and $M$:

Note that $\mathcal{P}(M)$ is in red and $\mathcal{G}(M)$ in green.
A result on $[0, 1]^{\omega}$

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**Theorem.** (Toruńczyk) Let $Y$ be an AR. If the identity map on $Y$ is a uniform limit of $Z$-maps, then $Y$ is a Hilbert cube.
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Using Toruńczyk’s theorem we proved the following result:

**Theorem.** Let $X$ be a Peano continuum and $R$ a closed nonempty subset of $\mathcal{P}(X)$. Then

$$C_n(X, R) = \{ A \in C_n(X) : A \cap R \neq \emptyset \}$$

is a Hilbert cube.
Using Hilbert cubes, I
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**Theorem.** Let $X$ be a Peano continuum that is not almost meshed. Then, for every $n \in \mathbb{N}$, $X$ does not have unique hyperspace $C_n(X)$. 
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Proof: Let $E$ be a continuum such that $E = \text{cl}_X(\text{int}_X(E))$ and $E \subset \mathcal{P}(X)$. Then, $C_n(X, E)$ is a Hilbert cube.
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**Theorem.** Let \( X \) be a Peano continuum that is not almost meshed. Then, for every \( n \in \mathbb{N} \), \( X \) does not have unique hyperspace \( C_n(X) \).

**Proof:** Let \( E \) be a continuum such that \( E = \text{cl}_X(\text{int}_X(E)) \) and \( E \subset \mathcal{P}(X) \). Then, \( C_n(X, E) \) is a Hilbert cube. Let \( Y \) be the result of adjoining some Peano continuum \( D \) without free arcs to \( X \) by some point \( p \in \text{int}_X(E) \), in such a way \( Y \) and \( X \) are not homeomorphic.
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**Proof:** Let $E$ be a continuum such that $E = \text{cl}_X(\text{int}_X(E))$ and $E \subset \mathcal{P}(X)$. Then, $C_n(X, E)$ is a Hilbert cube. Let $Y$ be the result of adjoining some Peano continuum $D$ without free arcs to $X$ by some point $p \in \text{int}_X(E)$, in such a way $Y$ and $X$ are not homeomorphic.

\[
C_n(X) = C_n(X, E) \cup (C_n(X) - C_n(X, E))
\]
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C_n(Y) = C_n(Y, E \cup D) \cup (C_n(X) - C_n(X, E))
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Using Hilbert cubes, I

It can be shown that

$$\delta = \mathcal{C}_n(X, E) \cap cl\mathcal{C}_n(X)(\mathcal{C}_n(X) - \mathcal{C}_n(X, E))$$

$$= \mathcal{C}_n(Y, E \cup D) \cap cl\mathcal{C}_n(X)(\mathcal{C}_n(X) - \mathcal{C}_n(X, E))$$

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is a $Z$-set in the Hilbert cubes $C_n(X, E)$ and $C_n(Y, E \cup D)$. Thus, by “Anderson’s Homogeneity Theorem”, there is a homeomorphism $h : C_n(X, E) \rightarrow C_n(Y, E \cup D)$ such that $h \mid_\delta = id_\delta$. 
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is a $Z$-set in the Hilbert cubes $\mathcal{C}_n(X, E)$ and $\mathcal{C}_n(Y, E \cup D)$. Thus, by “Anderson’s Homogeneity Theorem”, there is a homeomorphism $h : \mathcal{C}_n(X, E) \to \mathcal{C}_n(Y, E \cup D)$ such that $h \upharpoonright \delta = \text{id}_\delta$. Thus, defining $H$ to be the common extension of $h$ and the identity on $\mathcal{C}_n(X) - \mathcal{C}_n(X, E)$, we get a homeomorphism $H : \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$. 
Using Hilbert cubes, I
Using Hilbert cubes, II

**Theorem.** Let $X$ be an almost meshed Peano continuum and $n \in \mathbb{N}$. Suppose that there exist a closed subset $R$ of $\mathcal{P}(X)$ and pairwise disjoint nonempty open sets $U_1, \ldots, U_{n+1}$ such that

- $X - R = U_1 \cup \cdots \cup U_{n+1}$ and
- for each $i \in \{1, \ldots, n+1\}$, $R \subset \text{cl}_X(U_i)$.

Then $X$ does not have unique hyperspace $C_m(X)$ for every $m \leq n$. 
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**Proof.** Let $Y = X \cup_p D$ for some $p \in R$ and $D$ a Peano continuum with no free arcs. Use the same argument as above.
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This looks technical but gives us two interesting results.
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This looks technical but gives us two interesting results.

**Notice:** In this case, a subset of $\mathcal{P}(X)$ **separates** the space.
Using Hilbert cubes, II

**Corollary.** Let $X$ be an almost meshed Peano continuum such that $X - P(X)$ is disconnected. Then $X$ does not have unique hyperspace $C(X)$. 
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**Corollary.** Let $X$ be a dendrite that is not a tree and $k = \sup\{ord_X(p : p \in \mathcal{P}(X))\}$ (notice $k \in \mathbb{N} \cup \{\omega\}$). Then for every $m < k$, $X$ does not have unique hyperspace $C_m(X)$. 
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**Corollary.** Let $X$ be a dendrite that is not a tree and $k = \sup\{\text{ord}_X(p : p \in \mathcal{P}(X))\}$ (notice $k \in \mathbb{N} \cup \{\omega\}$). Then for every $m < k$, $X$ does not have unique hyperspace $C_m(X)$.

So for example, notice $F_\omega$ does not have unique hyperspace $C_n(F_\omega)$ for any $n \in \mathbb{N}$. 
Meshed continua

We will call a continuum $X$ meshed if it is almost meshed and has a basis of neighborhoods $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected.
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**Lemma.** A metric continuum $X$ is almost meshed if and only if $X$ is an almost meshed Peano continuum and for some (all) $n \in \mathbb{N}$, the set

$$\mathcal{F}_n(X) = \{A \in C_n(X) : \dim_A(C_n(X)) < \infty\}$$

is dense in $C_n(X)$. 
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We will call a continuum $X$ meshed if it is almost meshed and has a basis of neighborhoods $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected.

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is dense in $\mathcal{C}_n(X)$.

Notice that the definition depends only on the base space and the equivalence in the lemma is **geometric** and **topological** in nature...
Geometric?

In our last example $M$, we can approximate any given continuum by a finite graph inside $\mathcal{G}(M)$:
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**Proof.** If $h : C_n(X) \approx C_n(Y)$ is a homeomorphism, then $h[\mathcal{F}_n(X)] = \mathcal{F}_n(Y)$ so $\mathcal{F}_n(Y)$ is dense in $C_n(Y)$. 

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Topological?

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**Proof.** If $h : C_n(X) \approx C_n(Y)$ is a homeomorphism, then $h[\mathcal{F}_n(X)] = \mathcal{F}_n(Y)$ so $\mathcal{F}_n(Y)$ is dense in $C_n(Y)$.

We will prove that if $X$ and $Y$ are both almost meshed Peano continua and $C_n(X) \approx C_n(Y)$, then $X \approx Y$. This will prove meshed continua have unique hyperspaces $C_n(X)$. 
Remarks on meshed Continua
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- Finite graphs, $\mathcal{D}$ dendrites and continua which have basis of $\mathcal{D}$-continua neighborhoods are all meshed continua.
Remarks on meshed Continua

- Dendrites that contain a long nullcomb or a $F_\omega$ are not meshed.
- Finite graphs, $D$ dendrites and continua which have basis of $D$-continua neighborhoods are all meshed continua.

So in fact our results will generalize all known
Duda’s method: 2-cells

Recall the classic model of $C([0, 1])$: we identify it with the triangle $\{ (a, b) \in [0, 1] : 0 \leq a \leq b \leq 1 \}$. 
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- $A$ is $\{[0, a] : a \in [0, 1]\} \cup \{[b, 1] : b \in [0, 1]\}$.
- $B$ is $\{\{x\} : x \in [0, 1]\} \approx [0, 1]$.
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Recall the classic model of $C([0, 1])$: we identify it with the triangle $\{(a, b) \in [0, 1]: 0 \leq a \leq b \leq 1\}$.

Notice the manifold boundary is $A \cup B$, where

- $A$ is $\{[0, a]: a \in [0, 1]\} \cup \{[b, 1]: b \in [0, 1]\}$.
- $B$ is $\{\{x\}: x \in [0, 1]\} \approx [0, 1]$.

So there is a copy of $[0, 1]$ in the manifold boundary of $C([0, 1])$. 
Free arcs

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- A maximal free arc $A$ that touches the rest in both endpoints $p \neq q$.
- A maximal free arc $J$ that touches the rest in one of its endpoints $p$.
- A free circle $S$ that touches the rest in just one point $p$. 
Free arcs

In Peano continua, we basically have three types of maximal free arcs.

Let $\mathcal{A}_E(X)$ the set of maximal free arcs that have ends:

Call $e$ the end of $J$. 
2-cells in general

Let $\mathcal{B}_\delta(X)$ be the set of elements of $C(X)$ that are in the manifold boundary of a 2-cell contained in $C(X)$. 
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Let $\mathcal{B}_\delta(X)$ be the set of elements of $C(X)$ that are in the manifold boundary of a 2-cell contained in $C(X)$.

**Lemma.** Let $X$ be a Peano continuum and $A \in C(X)$. Then $A \in \mathcal{B}_\delta(X)$ if and only if there is a maximal free arc $J$ and $A = \{p\}$ for some $p \in J$ or $J \in \mathcal{A}_E(X)$ and $A$ is a subarc of $J$ that contains its end.
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That is, we have the same situation that in $[0, 1]$. 
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Recall $\mathcal{F}A(X)$ denotes the set of free arcs of $X$.
2-cells in general

Let $\mathfrak{B}_\delta(X)$ be the set of elements of $\mathcal{C}(X)$ that are in the manifold boundary of a 2-cell contained in $\mathcal{C}(X)$.

**Lemma.** Let $X$ be a Peano continuum and $A \in \mathcal{C}(X)$. Then $A \in \mathfrak{B}_\delta(X)$ if and only if there is a maximal free arc $J$ and $A = \{p\}$ for some $p \in J$ or $J \in \mathcal{A}_E(X)$ and $A$ is a subarc of $J$ that contains its end.

**Theorem.** Let $X$ be a Peano continuum that is not an arc. Then there is a homeomorphism

$$h : \mathcal{FA}(X) \to \mathfrak{B}_\delta(X).$$
Free arcs inside the hyperspace

**Theorem.** Let $X$ be a Peano continuum that is not an arc. Then there is a homeomorphism

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**Theorem.** Let $X$ be a Peano continuum that is not an arc. Then there is a homeomorphism

$$h_X : \overline{\mathcal{FA}(X)} \to \overline{\mathcal{B}_\delta(X)}.$$ 

This results gives a powerful tool: there is a copy of the closure of the free arcs of $X$ in $C(X)$, in the **topological** form of $\mathcal{B}_\delta(X)$. 
Theorem. Let $X$ be a Peano continuum that is not an arc. Then there is a homeomorphism

$$h_X : \overline{\mathcal{FA}(X)} \rightarrow \overline{\mathfrak{B}_\delta(X)}.$$ 

If $X$ is almost meshed, there is a copy of $X$ topologically placed inside $C(X)$. 
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**Theorem.** If $X$ and $Y$ are almost meshed and $h : C(X) \to C(Y)$ is a homeomorphism, then $X \approx Y$. 
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To prove this, just compose $h_Y^{-1} \circ h \circ h_X$. 

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Free arcs inside the hyperspace

**Theorem.** Let $X$ be a Peano continuum that is not an arc. Then there is a homeomorphism

$$h_X : \mathcal{FA}(X) \rightarrow \mathcal{B}_\delta(X).$$

If $X$ is almost meshed, there is a copy of $X$ topologically placed inside $\mathcal{C}(X)$.

**Theorem.** If $X$ and $Y$ are almost meshed and $h : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is a homeomorphism, then $X \approx Y$.

To prove this, just compose $h_Y^{-1} \circ h \circ h_X$.

$$X = \mathcal{FA}(X) \xrightarrow{h_X} \mathcal{B}_\delta(X) \xrightarrow{h} \mathcal{B}_\delta(Y) \xrightarrow{h_Y^{-1}} \mathcal{FA}(Y) = Y$$
Meshed for \( n \geq 3 \)
Meshed for $n \geq 3$

Let $\mathcal{B}_n(X)$ be the elements of $C_n(X)$ that have a $2n$-cell neighborhood ($\mathcal{B}_1(X) = \mathcal{B}(X)$).
Meshed for $n \geq 3$

Let $\mathfrak{B}_n(X)$ be the elements of $C_n(X)$ that have a $2n$-cell neighborhood ($\mathfrak{B}_1(X) = \mathfrak{B}(X)$).

Let $\Gamma_n(X)$ be the elements $A \in C_n(X) - \mathfrak{B}_n(X)$ that have a local basis $\mathcal{D}$ such that if $U \in \mathcal{D}$, then $\dim(U) = 2n$ and $U \cap \mathfrak{B}_n(X)$ is arcwise connected.
**Meshed for \( n \geq 3 \)**

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Let \( \Gamma_n(X) \) be the elements \( A \in C_n(X) - \mathfrak{B}_n(X) \) that have a local basis \( \mathcal{D} \) such that if \( U \in \mathcal{D} \), then \( dim(U) = 2n \) and \( U \cap \mathfrak{B}_n(X) \) is arcwise connected.

**Theorem.** *Let \( X \) be a Peano continuum and \( n \geq 3 \). Then*

\[
\mathfrak{B}(X) = \Gamma_n(X)
\]
Meshed for $n \geq 3$

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**Theorem.** Let $X$ be a Peano continuum and $n \geq 3$. Then

$$\mathcal{B}(X) = \Gamma_n(X)$$

Thus, the set $\mathcal{B}(X)$ can be found inside $C_n(X)$ in a topological way for $n \geq 3$. 
Meshed for $n \geq 3$

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Let $\Gamma_n(X)$ be the elements $A \in C_n(X) - \mathfrak{B}_n(X)$ that have a local basis $\mathcal{D}$ such that if $U \in \mathcal{D}$, then $\dim(U) = 2n$ and $U \cap \mathfrak{B}_n(X)$ is arcwise connected.

Theorem. Let $X$ be a Peano continuum and $n \geq 3$. Then

$$\mathfrak{B}(X) = \Gamma_n(X)$$

Then, we can also find $\mathfrak{B}_\delta(X)$. 
Meshed for $n \geq 3$

Let $\mathcal{B}_n(X)$ be the elements of $\mathcal{C}_n(X)$ that have a $2n$-cell neighborhood ($\mathcal{B}_1(X) = \mathcal{B}(X)$).

Let $\Gamma_n(X)$ be the elements $A \in \mathcal{C}_n(X) - \mathcal{B}_n(X)$ that have a local basis $\mathcal{D}$ such that if $U \in \mathcal{D}$, then $\dim(U) = 2n$ and $U \cap \mathcal{B}_n(X)$ is arcwise connected.

**Theorem.** Let $X$ be a Peano continuum and $n \geq 3$. Then

$$\mathcal{B}(X) = \Gamma_n(X)$$

**Theorem.** Let $X$ and $Y$ be an almost meshed Peano continuum and $n \geq 3$. If $\mathcal{C}_n(X) \approx \mathcal{C}_n(Y)$, then $X \approx Y$. 
Meshed for $n = 2$

For $n = 2$, we cannot localize a copy of $X$ so easily. For example, in $C_2([0, 1])$, singulars have boundary-4-cells-neighborhoods, as well as the neighborhoods of sets like $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$: 
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\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (5,0);
\node at (0,0) [below] {0}; \node at (1,0) [below] {1/3}; \node at (2,0) [below] {2/3}; \node at (3,0) [below] {1};
\draw[red] (1,0) -- (1,0.5);
\draw[blue] (2,0) -- (2,0.5);
\end{tikzpicture}
\end{center}
Meshed for $n = 2$

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\[
\begin{array}{cccc}
U & V \\
0 & 1/3 & 2/3 & 1
\end{array}
\]

\[
\langle U, V \rangle \approx C(U) \times C(V) \approx C([0, 1]^2) \approx [0, 1]^4
\]
Meshed for $n = 2$

For a Peano continuum, define $\mathcal{B}_2^\delta(X)$ to be the set of elements of $C_2(X)$ that lie on the manifold boundary of some 4-cell neighborhood.
Meshed for $n = 2$

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Recall the $\mathcal{A}_E(X)$, the maximal free arcs with ends:
Meshed for $n = 2$

For a Peano continuum, define $\mathcal{B}_2^\delta(X)$ to be the set of elements of $\mathcal{C}_2(X)$ that lie on the manifold boundary of some 4-cell neighborhood.

**Theorem.** Let $X$ be a Peano continuum, $A \in \mathcal{C}_2(X)$ and $A = A_1 \cup A_2$ with $A_1$ and $A_2$ connected. Then $A \in \mathcal{B}_2^\delta(X)$ if and only if there are maximal free arcs $J_1, J_2$ such that $A_i \subset J_i$ ($i \in \{1, 2\}$) and either one of $A_1, A_2$ is one point or $J_j \in \mathcal{A}_E(X)$ and $A_j$ contains the end of $J_j$ for some $j \in \{1, 2\}$. 
Meshed for $n = 2$

Thus, elements of $\mathcal{B}_2^\delta(X)$ are like this
Meshed for $n = 2$

Thus, elements of $B^\delta_2(X)$ are like this
Meshed for $n = 2$

Thus, elements of $\mathcal{B}_2^\delta(X)$ are like this

or like this
Meshed for $n = 2$

Thus, elements of $\mathcal{B}_2^\delta(X)$ are like this

or like this
**Meshed for** \( n = 2 \)

Thus, elements of \( \mathcal{B}_2^\delta(X) \) are like this or like this

It turns out that for almost meshed continua, all the possible cases (remember there may be circles) for \( A_1 = A_2 \) can be characterized topologically.
Meshed for $n = 2$

Notice that if $A_1 = A_2$ we either have a singular or an arc with an end...
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Meshed for \( n = 2 \)

Notice that if \( A_1 = A_2 \) we either have a singular or an arc with an end... just like in case \( n = 1 \)!!! We can then identify \( \mathcal{FA}(X) \) (the free arcs of \( X \)) in \( C_2(X) \) in a topological way.

**Theorem.** If \( X \) and \( Y \) are almost meshed continua and \( C_2(X) \approx C_2(Y) \), then \( X \approx Y \).
Meshed have unique hyperspace

**Corollary.** If $X$ is a meshed Peano continuum, then $X$ has unique hyperspace $C_n(X)$ for all $n \in \mathbb{N}$, except for $n = 1$ and $X \in \{[0,1], S^2\}$. 
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**Corollary.** If $X$ is a meshed Peano continuum, then $X$ has unique hyperspace $C_n(X)$ for all $n \in \mathbb{N}$, except for $n = 1$ and $X \in \{[0, 1], S^2\}$.

**Corollary.** If $X$ and $Y$ are Peano continua, $X$ is almost meshed, $X$ is not homeomorphic to $Y$ and $C_n(X) \approx C_n(Y)$ for some $n \in \mathbb{N}$, then
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- $Y$ is not almost meshed.
Meshed have unique hyperspace

**Corollary.** If $X$ is a meshed Peano continuum, then $X$ has unique hyperspace $C_n(X)$ for all $n \in \mathbb{N}$, except for $n = 1$ and $X \in \{[0, 1], S^2\}$.

**Corollary.** If $X$ and $Y$ are Peano continua, $X$ is almost meshed, $X$ is not homeomorphic to $Y$ and $C_n(X) \approx C_n(Y)$ for some $n \in \mathbb{N}$, then

- $Y$ is not almost meshed.
- the subset $\mathcal{FA}(Y)$ of $Y$ is homeomorphic to $X$. 
Example 1

There exist an Almost meshed, not meshed Peano continuum $X$ with unique hyperspace $C(X)$. 
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![Diagram of X and P(X)]
Example 1

There exist an Almost meshed, not meshed Peano continuum $X$ with unique hyperspace $C(X)$.

Notice $X - \mathcal{P}(X)$ is connected but $\mathcal{P}(X)$ disconnects locally.
Example 2

There exists a dendrite $D$ not in $\mathcal{D}$ such that $D$ has unique hyperspace $\mathcal{C}_2(D)$ and moreover, $D - \mathcal{P}(D)$ is disconnected.
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Summary of results:

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- If \( X \) is not almost meshed, \( C_n(X) \) is not unique for any \( n \in \mathbb{N} \).
Summary of results:

For Peano continua:

• If $X$ is not almost meshed, $C_n(X)$ is not unique for any $n \in \mathbb{N}$.

• If $\mathcal{P}(X)$ cuts $X$, $C(X)$ is not unique.
Summary of results:

For Peano continua:

- If $X$ is not almost meshed, $C_n(X)$ is not unique for any $n \in \mathbb{N}$.
- If $\mathcal{P}(X)$ cuts $X$, $C(X)$ is not unique.
- If $X$ is a dendrite and there is $p \in \mathcal{P}(X)$ with $ord_X(p) > n$, then $C_n(X)$ is not unique.
Summary of results:

For Peano continua:

• If $X$ is not almost meshed, $C_n(X)$ is not unique for any $n \in \mathbb{N}$.
• If $\mathcal{P}(X)$ cuts $X$, $C(X)$ is not unique.
• If $X$ is a dendrite and there is $p \in \mathcal{P}(X)$ with $\text{ord}_X(p) > n$, then $C_n(X)$ is not unique.
• If $X$ and $Y$ are not homeomorphic but have the same hyperspace $C_n(X)$ and $X$ is almost meshed, then $Y$ is not almost meshed and contains a copy of $X$. 
Summary of results:

For Peano continua:

- If $X$ is not almost meshed, $C_n(X)$ is not unique for any $n \in \mathbb{N}$.
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- If $X$ is a dendrite and there is $p \in \mathcal{P}(X)$ with $ord_X(p) > n$, then $C_n(X)$ is not unique.
- If $X$ and $Y$ are not homeomorphic but have the same hyperspace $C_n(X)$ and $X$ is almost meshed, then $Y$ is not almost meshed and contains a copy of $X$.
- If $X$ is meshed, $X$ has unique hyperspace $C_n(X)$ for all $n \in \mathbb{N}$.
Problems

• Characterize dendrites $X$ with unique hyperspace $C_2(X)$. 
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• Does there exist a Peano continuum $X$ with unique hyperspace $C(X)$ but not unique hyperspace $C_2(X)$?
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- Characterize dendrites $X$ with unique hyperspace $C_2(X)$.
- Does there exist a Peano continuum $X$ with unique hyperspace $C(X)$ but not unique hyperspace $C_2(X)$?
- Let $X$ be a Peano continuum such that $X - \mathcal{P}(X)$ is connected. Does $X$ have unique hyperspace $C(X)$?
Thank you