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Nonisomorphic complete triangulations of a surface

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Abstract

This paper is concerned with nonisomorphic triangular embeddings of a complete graph into the same surface. We prove that the minimum order (that is, number of vertices) of such examples is 9 for the nonorientable case, and 12 for the orientable one. We also explore the (nonorientable) case 10, where there are 14 such nonisomorphic triangulations with a remarkable one whose symmetry group is A_5. Finally, we exhibit an infinite family of nonisomorphic nonorientable examples. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

The existence of embeddings of the complete graph K_n into closed surfaces of minimal possible genus was one of the main problems in topological graph theory. Its solution was completed around 1970 and summarized by Ringel [8]. Here, we address the question of how many different embeddings into the same surface are there? It seems hopeless at present to give a precise answer for general n. Bounds other than the classic 'at least 1' are unknown. In fact, the first examples of nonisomorphic embeddings [4,7] are fairly recent. So that settling the question for small values of n is in order.

We shall restrict ourselves to triangular embeddings of K_n into closed surfaces, which we call *3-cycles* for brevity — the origin of this term will be made clear later. They are completely determined by the system of triangles of the embedding. Thus, it is convenient to treat them as 3-graphs, this is, uniform hypergraphs of rank 3. Our motivation also comes from studying the notion of tightness on hypergraphs [3,4], which is a natural generalization of connectedness for graphs. In that context, as the analogues of trees, another type of embeddings of complete graphs into surfaces arose

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naturally. Namely, triangular embeddings of K_n into surfaces with boundary and so that all vertices lie on the boundary. They were called *3-chains*. The smallest examples are the triangle and the unique triangulation of the Moebius Band by K_5. Observe that any 3-cycle minus a vertex is a 3-chain. They seem to play an important role in the study of embeddings of complete graphs. In particular, there is a very natural family of 3-chains called prime surfaces in [3] which unify the small well-known examples. The formal definitions are given in Section 1.

Now, let us be more precise about the results. In these terms, Ringel et al. (cf. [8]) proved that there is at least one nonorientable 3-cycle of order n, whenever $n \equiv 0, 1 \pmod 3$ and $n \neq 7$; and one orientable 3-cycle of order n whenever $n \equiv 0, 3, 4, 7 \pmod{12}$, (see also [10]). However, the question of the existence of 3-cycles which triangulate the same surface but are nonisomorphic as hypergraphs was not addressed in those pioneering works.

More recently, using the notion of tightness, the existence of nonisomorphic nonorientable 3-cycles of the same order was shown in [4]: two of order 30 and three of order 16. In [7] the first nonisomorphic orientable 3-cycles were exhibited: three of order 19. In this paper, we present two nonisomorphic 3-cycles of order 9 (and hence nonorientable), and we observe that two orientable 3-cycles of order 12 which appear in [8] are nonisomorphic. Since, there is only one 3-cycle of orders 6 and 7, which triangulate the Projective Plane and the Torus, respectively, see (1). This settles the question of the least order of nonisomorphic 3-cycles over the same surface for both the nonorientable and the orientable case.

Following [7], let us denote by $\#\mathrm{tri}(n, \mathcal{O})$ (resp. $\#\mathrm{tri}(n, \mathcal{N})$) the number of nonisomorphic orientable (resp. nonorientable) 3-cycles of order n. In Section 3, we prove that $\#\mathrm{tri}(9, \mathcal{N}) = 2$. By means of a deep search algorithm we found that $\#\mathrm{tri}(10, \mathcal{N}) = 14$. The next case, $n = 12$, was too big for a personal computer with our algorithm, so that we just formally know that $\#\mathrm{tri}(12, \mathcal{O}) \geqslant 2$ by Section 2, and that $\#\mathrm{tri}(12, \mathcal{N}) \geqslant 2$ by Section 4. However, the computer says that $\#\mathrm{tri}(12, \mathcal{O}) + \#\mathrm{tri}(12, \mathcal{N}) > 100$.

One of our 3-cycles of order 10 is remarkable because of its symmetry. Its automorphism group is the rotation group of the Dodecahedron (the alternating group on 5 letters A_5). We describe it in Section 3.1.

Finally, in Section 4 we prove that $\#\mathrm{tri}(12s, \mathcal{N}) \geqslant 2$ for all $s \geqslant 1$. The examples come from [8] and one obtained by the *coupling construction*, [4], which yields a 3-cycle of order $2n$ out of a 3-chain of order n. This construction suggests that these numbers grow rapidly. For an independent approach to this problem see [5].

1. Preliminaries

A 3-graph H consists of a vertex set V and a specified collection of triplets of V (called 3-edges, or triangles, of H). Given a vertex u of a 3-graph H, the *trace* of u in H is the graph $\mathcal{T}_H(u)$ with vertex set $V - \{u\}$ and an edge vw whenever uvw is a triangle of H. It is easy to see that an equivalent definition of a 3-cycle, as defined in the introduction, is: a 3-graph all of whose traces are cycles – hence the name. Analogously, we define a 3-*chain* to be a 3-graph all of whose traces are chains. Observe that a simple counting argument yields that if C is a 3-cycle (respectively, a 3-chain) of order n, that is, with n vertices, then $n \equiv 0,1 \pmod 3$ (resp., $n \equiv 0,2 \pmod 3$).

Let C be a 3-chain with vertex set V. The *boundary* of C, ∂C, is the graph with vertex set V and an edge vw whenever the pair $\{v,w\}$ lies in exactly one triangle of C. It is clearly a regular graph of degree 2, and hence a collection of disjoint cycles. When ∂C is connected, the 3-cycle, C^+, is obtained by adjoining a new vertex with triangles to the edges of ∂C.

The *realization* of a hypergraph H, denoted $|H|$, is the topological space obtained by realizing the simplicial complex generated by H. Thus, if C is a 3-cycle (respectively, a 3-chain), $|C|$ is a closed surface (resp. a compact surface with boundary and such that $\partial|C| = |\partial C|$). We say that C is *orientable* or not, according to whether $|C|$ is.

A 3-cycle or a 3-chain may be described by its *trace matrix*: its rows are labeled by the vertices and consist of the corresponding trace. For 3-chains it is defined up to reversal of the rows, and for 3-cycles also up to cyclic permutations. However, for oriented 3-chains and 3-cycles the reversing ambiguity may be eliminated. In [8], Ringel studies the trace matrices in detail.

An important family of 3-chains are the prime surfaces [3], defined as follows. Let n be such that $p = 2n+1$ is prime. Consider the set $Z_p^*/\{1,-1\}$; that is, the set of pairs $[\![x]\!] := \{x, -x\}$ with x in the multiplicative group $Z_p^* = Z_p - \{0\}$. The *prime surface* of order n, $\mathscr{P}_n$ (denoted $\mathscr{L}_p$ in [3,4]), is the 3-chain over this vertex set whose trace matrix has rows

$$[\![k]\!]. \quad [\![2k]\!] \quad [\![3k]\!] \quad \cdots \quad [\![nk]\!].$$

The first three cases have connected boundary and yield the unique 3-cycles of orders 4, 6 and 7:

$$|\mathscr{P}_3^+| = S^2, \quad |\mathscr{P}_5^+| = RP^2, \quad |\mathscr{P}_6^+| = T^2. \tag{1}$$

It is interesting to note that $\partial\mathscr{P}_8$ is not connected.

2. Orientable case 12

After the case 7, where there is only one 3-cycle triangulating the Torus and not for the Klein Bottle, the next orientable 3-cycle appears in order 12.

Theorem 1. *There exist two nonisomorphic orientable* 3*-cycles of order* 12.

Proof. The two examples we present come from [8]. The first one is treated as a special case on p. 82. The other one is constructed in Chapter 11 from the Rotation Group of the Tetrahedron, A_4, and a current graph. By construction its automorphisms

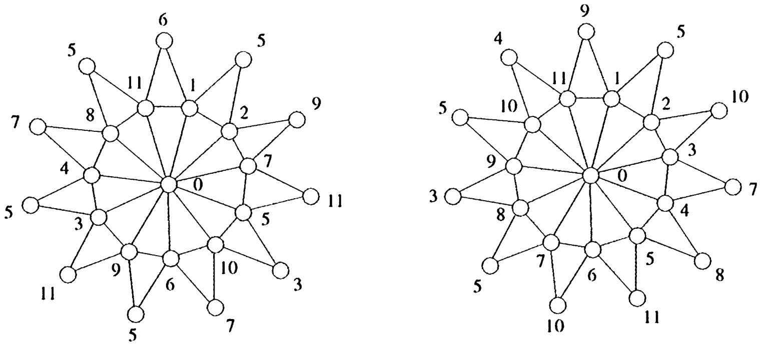

Fig. 1.

are transitive on vertices. When the vertices of the second are relabelled $0,\dots,11$, their trace matrices become, respectively, the following:

0.	5	7	2	1	11	8	4	3	9	6	10	0.	1	2	3	4	5	6	7	8	9	10	11
4.	9	11	6	5	3	0	8	7	1	10	2	1.	3	5	2	0	11	9	7	10	8	4	6
8.	1	3	10	9	7	4	0	11	5	2	6	2.	7	11	4	9	6	8	10	3	0	1	5
1.	8	6	11	0	2	5	9	10	4	7	3	3.	6	9	8	11	7	4	0	2	10	5	1
5.	0	10	3	4	6	9	1	2	8	11	7	4.	10	6	1	8	5	0	3	7	9	2	11
9.	4	2	7	8	10	1	5	6	0	3	11	5.	2	1	3	10	9	11	6	0	4	8	7
2.	3	6	8	5	1	0	7	9	4	10	11	6.	1	4	10	7	0	5	11	8	2	9	3
6.	7	10	0	9	5	4	11	1	8	2	3	7.	5	8	0	6	10	1	9	4	3	11	2
10.	11	2	4	1	9	8	3	5	0	6	7	8.	0	7	5	4	1	10	2	6	11	3	9
3.	2	11	9	0	4	5	10	8	1	7	6	9.	8	3	6	2	4	7	1	11	5	10	0
7.	6	3	1	4	8	9	2	0	5	11	10	10.	11	0	9	5	3	2	8	1	7	6	4
11.	10	7	5	8	0	1	6	4	9	3	2	11.	4	2	7	3	8	6	5	9	1	0	10

Suppose they are isomorphic. Then there exists an isomorphism that sends 0 to 0 because the second is vertex-transitive. Now, call two vertices *opposite* if they are in triangles whose opposite edges coincide. Consider the cyclic arrangements of vertices opposite to 0 in both 3-cycles, and observe that the vertex 5 appears four times in this arrangement for the first 3-cycle but there is no vertex with this property in the other one (Fig. 1). □

3. Non-orientable cases 9 and 10

Theorem 2. *There exist exactly two* 3*-cycles of order* 9.

Proof. They are defined by the triangles in Fig. 2.

The first one was known to Ringel (cf. [8, p. 77]) and the second one was first exhibited in [9]. Their automorphism groups are $Z_3 \times D_3$ (the Z-metacyclic group of order 18) and Z_6, respectively. Thus, they are nonisomorphic. The proof of their

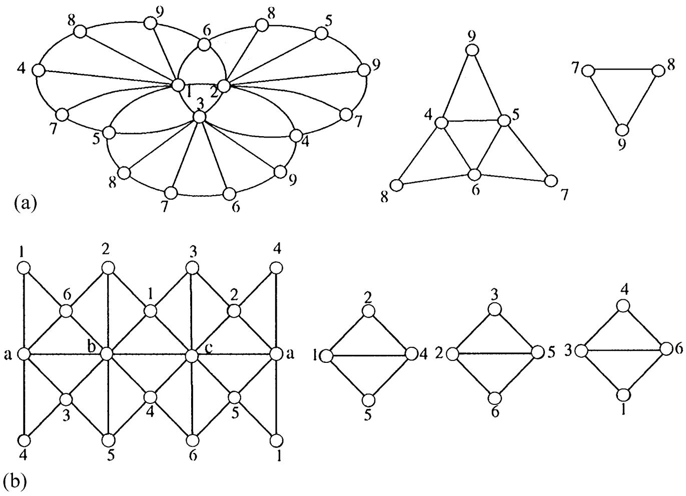

Fig. 2.

uniqueness can still be done by hand in a straightforward case by case way. It is rather long and tedious. Since it adds little to the paper we prefer to omit it. □

Using a deep search algorithm we constructed all 3-cycles of order 10, obtaining the following.

Claim 1. *There exist exactly fourteen* 3-*cycles of order* 10.

A file with the complete list of these 3-graphs and their automorphisms may be obtained by anonymous ftp from gauss.matem.unam.mx/pub/3-cycles. With respect to their automorphism groups, there are four for which it is trivial, four with Z_2 and one with each of the following: Z_3, Z_5, S_3, Z_9, A_4 and A_5. The ones with symmetry Z_5 and Z_9 are $\mathscr{P}_5 \diamond \mathscr{P}_5$ (see Section 4), and $\mathscr{P}_9^+$, respectively, and the one with 60 automorphisms is described below.

3.1. The remarkably symmetric 3-cycle of order 10

Let us denote it $\mathscr{R}_{10}$. To construct it, consider the dual map of $\mathscr{P}_5^+$, which is also obtained by identifying antipodes of the Dodecahedron (see Fig. 3), and hence its automorphism group is the rotation group of the Dodecahedron, A_5. Now, instead of each pentagonal cellular face, glue a Moëbius Band with its cannonical triangulation $\mathscr{P}_5$, whose boundary is a cycle of length 5. It is easy to see that one obtains a 3-cycle. It is $\mathscr{R}_{10}$.

Observe that A_5 acting on the edges of $\mathscr{R}_{10}$ (which are those of K_{10}), has two orbits. One is the Petersen Graph, and the other one consists of the interior edges of the glued $\mathscr{P}_5$'s (which is the edge graph of K_5). If one deletes the edges of the Petersen Graph,

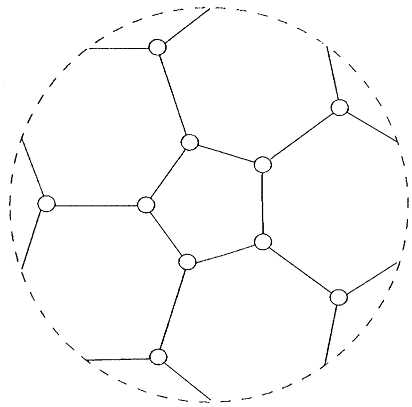

Fig. 3. The regular embedding of the Petersen graph in the Projective Plane.

pairs of triangles match up to form squares. The resulting polyhedron turns out to be regular (flag transitive) of Schläfly Symbol $\{4,6\}$, and its symmetry group grows to S_5 (the symmetric group on 5 letters). Its natural geometric realization is in the Projective (or elliptic) three-dimensional space as an analogue of the Platonic Solids. See [2] for another description and [1] for a combinatorial generalization. Its orientable double cover is the classic regular skew polyhedron in R^4 of type $\{4,6\}$ first discovered by Coxeter [6].

4. Nonisomorphic constructions

This section relies on the coupling construction. Although it can be defined more generally [4], we describe it in a special but most important case.

Let C be a 3-chain of order n. By an *orientation of the boundary*, or a *∂-orientation*, we mean a choice of orientation on each cycle componnent of ∂C. Thus, there are 2^k posible ∂-orientations, where k is the number of cycles in ∂C. Denote by $C^{\rightarrow}$, the 3-chain C together with a fixed ∂-orientation, so that when we write $uv \in \partial C^{\rightarrow}$ we have that $vu \notin \partial C^{\rightarrow}$ and the boundary component of u is oriented from u to v.

Define the *coupling* of $C^{\rightarrow}$, denoted $C^{\rightarrow} \diamond C^{\rightarrow}$, to be the following 3-graph, which turns out to be a 3-cycle of order $2n$ [4]. First, the vertices of $C^{\rightarrow} \diamond C^{\rightarrow}$ are two disjoint copies of those of C; if V is the vertex set of C, we will denote these two disjoint copies by V and V', so that $u \in V$ corresponds to $u' \in V'$. Second, for every triangle uvw of C, there are four triangles in $C^{\rightarrow} \diamond C^{\rightarrow}$, namely: uvw, $uv'w'$, $u'vw'$ and $u'v'w$. And third, for every edge $uv \in \partial C^{\rightarrow}$, we add two more triangles to $C^{\rightarrow} \diamond C^{\rightarrow}$: uvv' and $u'v'v$. As the main example, and unique exception to the following proposition, observe that the coupling of the triangle yields the complete triangulation of the projective plane (Fig. 4).

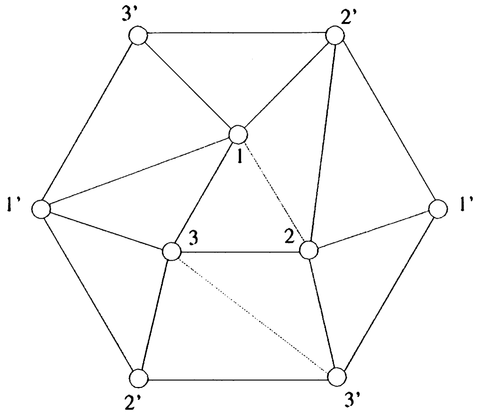

Fig. 4. $\mathscr{P}_3^{\rightarrow} \diamond \mathscr{P}_3^{\rightarrow} = \mathscr{P}_5^{+}$.

Proposition 1. *Let C be a 3-chain of order $n > 3$. Then, for any vertex of $C^{\rightarrow} \diamond C^{\rightarrow}$, its stabilizer does not act transitively on its trace.*

Proof. The stabilizer of a vertex in a 3-cycle is a subgroup of the automorphisms of the trace cycle. If it is transitive, it must contain all the rotations of the trace. We must consider two cases. The trace in $C^{\rightarrow} \diamond C^{\rightarrow}$ of a vertex in V' is a cycle of length $2n-1$, whose vertices lie alternatively in V and V', except for two consecutive ones in V (see [4]). And the trace in $C^{\rightarrow} \diamond C^{\rightarrow}$ of a vertex in V contains a path of length $n-1$ in V (in fact, its trace in C).

Suppose that $C^{\rightarrow} \diamond C^{\rightarrow}$ has a vertex with transitive stabilizer. In any of the two cases, there exists an automorphism $f \in \mathrm{Aut}(C^{\rightarrow} \diamond C^{\rightarrow})$ which sends $n-1$ vertices of V into V'. Indeed, the generating rotation works for the first case, and $n-1$ times the generator works for the second.

For $n > 3$, any subset of $n-1$ vertices of C supports a triangle α (i.e., α has all its vertices in the given set). But then $f(\alpha) \notin C^{\rightarrow} \diamond C^{\rightarrow}$, because there is no triangle with all of its vertices in V'. This is a contradiction, and the proposition is proved. □

With this proposition one may prove that there are nonisomorphic nonorientable 3-cycles for many orders. The first example occurs for $n = 10$. Since $\partial\mathscr{P}_9$ is connected (because 2 is a generator of the multiplicative group Z_{19}^*, see [4]), we obtain $\mathscr{P}_9^{+}$ of order 10 with a transitive stabilizer of the adjoined vertex. By the proposition, it is nonisomorphic to $\mathscr{P}_5^{\rightarrow} \diamond \mathscr{P}_5^{\rightarrow}$. In fact, they are respectively the 3-cycles with automorphism groups Z_9 and Z_5 in Claim 1.

Finally, with one of Ringel's constructions we exhibit an infinite family of examples of nonisomorphic nonorientable 3-cycles. Consider the 3-cycle of order $n = 12s$ defined as Case 0 in Section 8.2 of Ringel [8]. Observe that the special vertex x has a transitive stabilizer. Since there exist 3-chains of order $6s$, then their couplings are not isomorphic to it by Proposition 1.

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