Chiral polyhedra in 3-dimensional geometries and from a Petrie-Coxeter construction

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Abstract

We study chiral polyhedra in 3-dimensional geometries (Euclidian, hyperbolic and projective) in a unified manner. This extends to hyperbolic and projective spaces some structural results in the classification of chiral polyhedra in Euclidean 3-space given in 2005 by Schulte. Then, we describe a way to produce examples with helical faces based on a classic Petrie-Coxeter construction that yields a new family in $\mathbb{S}^3$, which is described exhaustively.

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1 Introduction

Symmetric structures in space have been studied since antiquity. An example of this is the inclusion of the Platonic solids in Euclid's Elements [14]. In particular, highly symmetric polyhedra-like structures have a rich history including convex ones such as the Archimedean solids (see for example [7, Chapter 21]), star-shaped ones such as the Kepler-Poinsot solids [27], and infinite ones such as the Petrie-Coxeter polyhedra [8].

In recent decades a lot of attention has been dedicated to the so-called skeletal polyhedra, where faces are not associated to membranes. The symmetry of a skeletal polyhedron is measured by the number of orbits of flags (triples of incident vertex, edge and face) under the action of the symmetry group. Those with only one flag-orbit are called regular, and in a combinatorial sense they have maximal symmetry by reflections. Chiral polyhedra are those with maximal (combinatorial) symmetry by rotations but none by reflections and have two flag-orbits.

Regular skeletal polyhedra in Euclidean space $\mathbb{E}^3$ include classical objects like the Platonic solids. They were classified in 1985 by Dress [11, 12] following the work by Grünbaum [16]. The complete list consists of 48 polyhedra, 18 of which are finite.

Since then, regular skeletal polyhedra in other spaces have been explored. Some examples in the hyperbolic space $\mathbb{H}^3$ were described in [15], and some in the projective space $\mathbb{P}^3$ were described in [1, 2]. Regular skeletal polyhedra in higher dimensional spaces have also been considered in [5, 20, 24].

Much less work has been done on chiral skeletal polyhedra. One of the main difficulties of working with chiral skeletal polyhedra is the lack of examples, since the first ones appeared only recently. They first appeared in 2005, when Schulte fully classified those realisable in $\mathbb{E}^3$. They are divided into six families, all containing exclusively infinite polyhedra (see [28, 29]). In recent years the first chiral skeletal polyhedra in $\mathbb{P}^3$ appeared in [3, 4]. To our knowledge there is no published work describing such polyhedra in $\mathbb{H}^3$.

Some aspects of these polyhedra have not been explored in depth. This highlights the relevance of constructing new examples of chiral skeletal polyhedra, relating them to other well-known geometric structures.

In this work we introduce a technique to construct chiral skeletal polyhedra in 3-dimensional spaces from a certain family of regular 4-polytopes that includes the tessellations by convex cells of $\mathbb{P}^3$, $\mathbb{E}^3$ and $\mathbb{H}^3$. This technique is illustrated with the description of all chiral polyhedra in $\mathbb{S}^3$ that can be obtained in this way.

2 Geometric polyhedra

Throughout this section we assume that the ambient space $\mathcal{X}$ is either the Euclidean space $\mathbb{E}^3$, the hyperbolic space $\mathbb{H}^3$ or the projective space $\mathbb{P}^3$ which we understand as elliptic geometry, that is, the 3-sphere $\mathbb{S}^3$ with antipodal points identified. These three spaces are orientable, and any of their isometries is a product of at most 4 plane reflections. Therefore the isometries that are products of 1 or 3 reflections reverse orientation, whereas those that are products of an even number of reflections preserve orientation.

Most of what we do also holds for the 3-sphere $\mathbb{S}^3$, however, some of the proofs would need adjustment, as in contrast to the other three spaces, lines on the sphere intersect in two antipodal points. However, one can recover polygons and polyhedra in $\mathbb{S}^3$ by lifting $\mathbb{P}^3$ to $\mathbb{S}^3$.

Definition 1 *A* skeletal polyhedron *in* $\mathcal{X} \in \{\mathbb{E}^3, \mathbb{H}^3, \mathbb{P}^3\}$ *consists of* vertices *(points of* $\mathcal{X}$*),* edges *(line segments between pairs of vertices) and* faces *(cycles on the graph determined by the vertices and edges) satisfying:*

1. *every edge belongs to exactly two faces;*

2. *the graph determined by the vertices and edges is connected;*

3. *for each vertex* v*, the graph whose vertices are the neighbours of* v*, with two of them joined by an edge whenever they are neighbours of* v *in a face of the polyhedron is a cycle;*

4. *every compact subset of* $\mathcal{X}$ *meets finitely many edges.*

Examples of skeletal polyhedra in $\mathcal{X}$ can be obtained from convex polyhedra in Euclidean space $\mathbb{E}^3$. A convex polyhedron $\mathcal{P}$ in $\mathbb{E}^3$ can be radially projected to a sphere S contained in its interior. This sphere can be identified with a small sphere $S_{\mathcal{X}}$ in $\mathcal{X}$ giving an embedding of the vertices of $\mathcal{P}$ in $\mathcal{X}$. The edge between two vertices can now be taken as the line segment between the corresponding points that is contained in the region of $\mathcal{X} \setminus S_{\mathcal{X}}$ homeomorphic to a ball. The faces then are inherited from those of $\mathcal{P}$. Clearly such a structure satisfies Definition 1. Note that with our definition, skeletal polyhedra in $\mathbb{P}^3$ cannot have loop edges and hence when we lift these polyhedra to $\mathbb{S}^3$, they do not have *digons*. In the remainder of this work we shall refer to a skeletal polyhedron simply as a polyhedron.

Polyhedra in $\mathcal{X}$ defined as above are realisations of abstract polyhedra in the sense of [23, Section 2A]. To any polyhedron $\mathcal{P}$ in $\mathcal{X}$ we may assign the partially ordered set consisting of vertices, edges and faces ordered by inclusion. Two elements are said to be *incident* if they are comparable in the partially ordered set. The ordered set is known as the *abstract polyhedron* associated to $\mathcal{P}$.

The graph determined by the vertices and edges of a polyhedron $\mathcal{P}$ is called its 1-*skeleton*. The graph defined in Item 3 of Definition 1 is called the *vertex-figure* at v.

A *flag* of $\mathcal{P}$ is a triple of incident vertex, edge and face. Two flags are *adjacent* whenever they differ in precisely one element; they are 0-adjacent if they share the edge and face, 1-adjacent if they share the vertex and face, and 2-adjacent if they share the vertex and edge. The i-adjacent flag of a flag Φ is denoted by Φ^i. We extend this notation recursively so that if w is a word in the set $\{0, 1, 2\}$ and $i \in \{0, 1, 2\}$ then $\Phi^{wi} := (\Phi^w)^i$.

We say that $\mathcal{P}$ is *equivelar* whenever all its faces have the same number p of edges and all its vertices have the same degree q in the 1-skeleton. In such cases the *Schläfli type* of $\mathcal{P}$ is $\{p, q\}$. Regular and chiral polyhedra defined below are equivelar.

A *symmetry* of $\mathcal{P}$ is an isometry of $\mathcal{X}$ that preserves $\mathcal{P}$. We denote the group of symmetries of $\mathcal{P}$ by $G(\mathcal{P})$. An *automorphism* of $\mathcal{P}$ is a bijection of the vertices, edges and faces that preserves the incidence; that is, automorphisms are bijections preserving the structure as abstract polyhedron. The group of automorphisms of $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$ and it acts freely on the flags (see [23, Proposition 2A4]). Every symmetry of a polyhedron induces an automorphism, but in general not every automorphism can be realised by a symmetry. However, an automorphism may be realised by more than one isometry.

2.1 Regular and chiral polyhedra

The polyhedron $\mathcal{P}$ is said to be *regular* (resp. *combinatorially regular*) whenever $G(\mathcal{P})$ (resp. $\Gamma(\mathcal{P})$) acts transitively on the flags. It is *orientably regular* if its flags can be partitioned in two sets in such a way that adjacent flags belong to distinct sets. A regular polyhedron that is not orientably regular is said to be *non-orientably regular*. An example is the hemicube (the cube with antipodal points identified) in $\mathbb{P}^2 \subset \mathbb{P}^3$.

If $\mathcal{P}$ is orientably regular we denote by $G^+(\mathcal{P})$ (resp. $\Gamma^+(\mathcal{P})$) the group of symmetries (resp. automorphisms) that preserve the bipartition of flags. The groups $G^+(\mathcal{P})$ and $\Gamma^+(\mathcal{P})$ have index two in $G(\mathcal{P})$ and $\Gamma(\mathcal{P})$, respectively.

The polyhedron $\mathcal{P}$ is said to be *chiral* (resp. *combinatorially chiral*) whenever $G(\mathcal{P})$ (resp. $\Gamma(\mathcal{P})$) induces two orbits on the flags with the property that adjacent flags belong to different orbits. When convenient, we shall refer to the symmetry group and the automorphism group of the chiral polyhedron $\mathcal{P}$ by $G^+(\mathcal{P})$ and $\Gamma^+(\mathcal{P})$, respectively.

It is well known that the group $G(\mathcal{P})$ of a regular polyhedron $\mathcal{P}$ acts transitively on the set of faces, since we can map some flag containing a given face f_1 to any flag containing some other given face f_2. If $\mathcal{P}$ is chiral then $G(\mathcal{P})$ also acts transitively on the faces, since given a flag Φ containing a face f, the 0-adjacent flag Φ^0 also contains f, but Φ and Φ^0 belong to distinct flag orbits; since there are exactly two orbits of flags under $G(\mathcal{P})$, it follows that it acts transitively on faces. Similarly, if $\mathcal{P}$ is regular or chiral then $G(\mathcal{P})$ acts transitively on the vertices and on the edges of $\mathcal{P}$.

Whenever $\mathcal{P}$ is combinatorially regular or chiral, $\Gamma^+(\mathcal{P})$ is generated by two distinguished *abstract rotations*: the element σ_1 mapping the base flag Φ to $\Phi^{1,0}$, and the element σ_2 mapping Φ to $\Phi^{2,1}$ (see [30]). The abstract rotation σ_1 fixes the base face f and can be understood like a one-step rotation along f, while σ_2 fixes the base vertex v and plays the role of a one-step rotation around v.

If $\mathcal{P}$ is combinatorially regular then $\Gamma(\mathcal{P}) = \langle \rho_1 \rangle \ltimes \Gamma^+(\mathcal{P})$, where ρ_1 is an involution mapping Φ to Φ^1 and inverting σ_1 and σ_2 by conjugation. On the other hand, if $\mathcal{P}$ is combinatorially chiral then there is no group automorphism of $\Gamma^+(\mathcal{P})$ mapping σ_i to σ_i^{-1} for $i \in \{1, 2\}$ (see [30]).

The stabiliser in $\Gamma^+(\mathcal{P})$ of the base face (resp. of the base vertex) is $\langle \sigma_1 \rangle$ (resp. $\langle \sigma_2 \rangle$). In particular, the face-stabiliser in $\Gamma^+(\mathcal{P})$ is cyclic and acts regularly on the vertices of the base face. It follows that if we know an element S_1 in $G^+(\mathcal{P})$ acting like σ_1 then the vertices of the base face are the orbit of the base vertex under the group $\langle S_1 \rangle$.

The faces of regular and chiral polyhedra can be of different types, depending on the space its vertices span. If all the vertices of a face lie on a line, then we say that the face is a *linear polygon*, and if all of them lie on a plane, we say it is *planar*. Clearly, linear faces are planar. If the entire polyhedron lies on a plane, we also say that it is planar. Hence, planar polyhedra have planar (and sometimes linear) faces. If there is no plane containing all the vertices of a face, then the face is either skew or helical (the latter were called helicoids in [1]), and we will define them after the proof of Proposition 2.

We first discard all regular or chiral polyhedra with linear faces.

Proposition 1 *There are no regular or chiral polyhedra in $\mathcal{X} \in \{\mathbb{E}^3, \mathbb{H}^3, \mathbb{P}^3\}$ with linear faces.*

Proof. Assume to the contrary that $\mathcal{P}$ is a regular or chiral polyhedron in $\mathcal{X}$ with linear faces. Note that $G^+(\mathcal{P})$ acts transitively on the faces of $\mathcal{P}$ and hence all faces are linear. Furthermore, Items 2 and 3 of Definition 1 imply that all vertices, edges and faces of $\mathcal{P}$ are in one line Λ. Moreover, $G^+(\mathcal{P})$ acts transitively on the edges of $\mathcal{P}$, implying that all edges have the same length. This forces $\mathcal{P}$ to have only one face, contradicting Item 1 of Definition 1. ■

If $\mathcal{P}$ is planar then there may be more than one isometry acting like the automorphism σ_1 (or σ_2). For example, in the hemicube (naturally realised in $\mathbb{P}^3$) the automorphism σ_1 can be realised by a 4-fold rotation around the line through the centre of the base face f, perpendicular to f. However, the rotatory reflection of order 4 with respect to the plane Π containing the vertices and to the line perpendicular to Π at the centre of the base face also acts like σ_1. The next proposition shows that if $\mathcal{P}$ is non-planar then σ_1 and σ_2 are defined uniquely.

Proposition 2 *Let $\mathcal{P}$ be a non-planar regular or chiral polyhedron in $\mathcal{X}$ with base flag Φ. Then*

- *there is a unique $S_1 \in G(\mathcal{P})$ acting as σ_1 with respect to Φ;*
- *there is a unique $S_2 \in G(\mathcal{P})$ acting as σ_2 with respect to Φ;*

Proof. Since $\mathcal{P}$ is (geometrically) regular or chiral, S_1 and S_2 exist; we need to show that they are unique.

Let f_0 be the face in Φ and assume that S_1 and S_1' are isometries of $\mathcal{X}$ that preserve $\mathcal{P}$ and act as the one-step rotation σ_1 along f_0. Then $S_1(S_1')^{-1}$ is an isometry of $\mathcal{X}$ that preserves $\mathcal{P}$ and that fixes all the vertices of f_0. If f_0 is not a planar face, then $S_1(S_1')^{-1}$ is the identity *id* of $\mathcal{X}$, and hence $S_1 = S_1'$. Otherwise, f_0 is in a plane Π, that is pointwise fixed by $S_1(S_1')^{-1}$. That means that $S_1(S_1')^{-1}$ is either *id* or the reflection through Π.

Now, as an automorphism, $S_1(S_1')^{-1}$ must be the identity, since it fixes the base flag. Hence, all edges of $\mathcal{P}$ are pointwise fixed by $S_1(S_1')^{-1}$. Thus, all edges are contained in Π and thus $\mathcal{P}$ is planar. But this contradicts our assumption that $\mathcal{P}$ is non-planar.

A similar argument shows the uniqueness of S_2. $\blacksquare$

Note that if $\mathcal{P}$ is planar, then the above proof shows that there are exactly two symmetries of $\mathcal{P}$ acting as σ_1 and two symmetries acting as σ_2. In each case, one of the symmetries can be obtained from the other by multiplying it by the reflection on the plane Π (with $\mathcal{P} \in \Pi$). In this case, we shall abuse notation and denote by S_1 and S_2 the symmetries of the plane Π that act as described above. Hence they are also unique (as isometries of Π).

We say that the faces of a regular or chiral polyhedron are *skew* if they are not planar and S_1 is a rotatory reflection. On the other hand, we say that they are *helical* if they are not planar and S_1 is a twist (or screw motion). Hence, all skew faces are finite by Definition 1, while helical faces are infinite in the Euclidean and hyperbolic space, and finite in the sphere and in the projective space. Moreover, a skew face projects orthogonally into a planar polygon in some distinguished plane Π, and to a line segment in a line perpendicular to Π. The vertex set corresponds to the vertex set of a prism or of an antiprism over a planar regular polygon in $\mathcal{X}$. A helical face projects orthogonally to a linear polygon in one axis Λ of the twist S_1, and winds around it. Recall that if $\mathcal{X}$ is Euclidean or hyperbolic then Λ is unique, while if it is spherical then S_1 has two axes. In the latter case, a helical face projects orthogonally to two linear polygons in lines that are polar to each other.

The following lemma provides a criterion to determine when a polyhedron in $\mathcal{X}$ is regular or chiral.

Lemma 3 *Let $\mathcal{P}$ be a regular or chiral polyhedron in $\mathcal{X}$. Then $\mathcal{P}$ is regular if and only if there exists an isometry R of $\mathcal{X}$ preserving the base vertex v, the vertex-figure at v and the base face f, but interchanging the two neighbours of v in f.*

Proof. Clearly if $\mathcal{P}$ is regular, such R exists, as it is the symmetry of $\mathcal{P}$ that sends the base flag to its 1-adjacent flag.

Suppose now that R exists. If we can show that R is a symmetry of $\mathcal{P}$ then it is a symmetry mapping the base flag Φ to Φ^1 so $\mathcal{P}$ cannot be chiral, finishing the proof.

Let S_1 be a symmetry of $\mathcal{P}$ that acts as the one-step rotation of the base face f and let S_2 be a symmetry of $\mathcal{P}$ that fixes the base vertex v acting as the one-step rotation of the vertex-figure of v.

Assume now that the neighbours of v in f are u and w, and that $uS_1 = v$ and $vS_1 = w$. Then RS_1R is an isometry of $\mathcal{X}$ (or of Π, if $\mathcal{P}$ is planar) preserving f, satisfying that $wRS_1R = v$ and $vRS_1R = u$. Since both R and S_1 preserve the face f, then we have that RS_1R is a symmetry of f sending w to v and v to u. Thus, RS_1R acts on f as the inverse of S_1.

If f is skew or helical, since $\mathcal{X}$ is of dimension 3, then there is unique isometry of $\mathcal{X}$ that acts as RS_1R in f, and hence $RS_1R = S_1^{-1}$.

On the other hand, if f is planar then there are two possibilities for RS_1R: either it is S_1^{-1} or it is S_1^{-1} composed by the reflection on the plane containing f. Since $\mathcal{X}$ is an orientable space, we have that S_1 preserves the orientation if and only if any of its conjugates preserves the orientation. On the other hand, S_1 preserves the orientation if and only if S_1^{-1} preserves the orientation if and only if S_1^{-1} composed by the reflection on the plane containing f does not preserve the orientation. Thus, $RS_1R = S_1^{-1}$.

A similar argument shows that $RS_2R = S_2^{-1}$.

Since $G^+(\mathcal{P})$ is generated by S_1 and S_2 and conjugation by R inverts both, we have that $G^+(\mathcal{P}) = RG^+(\mathcal{P})R$ and we are ready to show that R preserves $\mathcal{P}$. Let x be a vertex of $\mathcal{P}$ and assume that $x = vT$ for some $T \in G^+(\mathcal{P})$. Note that

$$x = vT = vR^2TR^2 = (vR)(RTR)R = v(RTR)R,$$

and so $xR = vRTR$. Hence R preserves the vertex set. A similar argument shows that R preserves the edge set and the face set of $\mathcal{P}$. ■

Recall that the vertices of the vertex-figure at the base vertex v are the neighbours of v, and they can be obtained as the orbit of any one of them under the action of $\langle \sigma_2 \rangle$. Since all edges of a regular or chiral polyhedron have the same length, v is the centre of a sphere containing all vertices of the vertex-figure at v. This implies that the vertex-figures of a regular or chiral polyhedron are either planar or skew, since the vertices of helical faces do not lie on a sphere.

In what follows we describe the types of faces and vertex-figures of chiral polyhedra in $\mathcal{X}$.

Lemma 4 *There are no chiral polyhedra with planar faces in $\mathcal{X}$.*

Proof. Suppose there exists a chiral polyhedron $\mathcal{P}$ in $\mathcal{X}$ with planar faces, and let v be the base vertex of $\mathcal{P}$. Let S_1 be a symmetry of $\mathcal{P}$ that acts as the one-step rotation of the base face f and let S_2 be a symmetry of $\mathcal{P}$ that fixes v acting as the one-step rotation of the vertex-figure of v. Then $v\langle S_1 \rangle$ is the set of all vertices of the base face f, while if $u := vS_1$, then $u\langle S_2 \rangle$ are all the vertices of the vertex-figure of v.

Let $w := vS_1^{-1}$ and let Σ be the perpendicular bisector of the line segment between u and w; then Σ is a plane that contains v. Let R be the reflection through Σ. Since f is planar and it is a regular polygon, then its symmetry group contains R.

Since the vertices of the vertex-figure $VF(v)$ of v are the elements of $u\langle S_2 \rangle$, then $VF(v)$ is a regular polygon. If $VF(v)$ is planar, then R is a symmetry of $VF(v)$ that interchanges u and w, while fixing v. Thus, in this case R satisfies all the hypotheses of Lemma 3, implying that $\mathcal{P}$ is a regular polyhedron, which is a contradiction.

Therefore the vertex-figure of v must be skew. In this case, let ℓ be the intersection of the plane that contains f and Σ, and let R' be the 2-fold rotation on ℓ. Then, R' fixes v and preserves f, since R does. Again, the vertex-figure of v is a regular (skew) polygon and the rotation R' is the symmetry of such polygon that fixes v and interchanges u and w. This implies that now R' satisfies all the hypothesis of lemma 3, implying that $\mathcal{P}$ is a regular polyhedron, which is again a contradiction.

Hence, no such chiral polyhedron $\mathcal{P}$ exists. ■

Corollary 5 *There are no planar chiral polyhedra in $\mathcal{X}$.*

Lemma 6 *If $\mathcal{P}$ is a chiral polyhedron with skew faces in $\mathcal{X}$ then its vertex-figures are also skew.*

Proof. Let v be the base vertex of $\mathcal{P}$, let u and w be its two neighbours in the base face, and let Σ be the perpendicular bisector of the line segment between u and v. Then the reflection R_1 with respect to Σ fixes the base face and v, while interchanges u and w.

Suppose to the contrary that the vertex-figure at v is planar, then it lies on a plane Π containing the line through u and w. In particular, Π is perpendicular to Σ because Σ is perpendicular to the line through u and v. Furthermore, the edges of the vertex-figure at v can be chosen so that the edge between u and w is bisected by Σ. In this setting, R_1 is an isometry of $\mathcal{X}$ preserving v, the base face and the vertex-figure at v. By Lemma 3, $\mathcal{P}$ is regular, contradicting our hypothesis. ■

Examples of chiral polyhedra with skew faces and skew vertex-figures can be found in [28].

Lemma 7 *If $\mathcal{P}$ is a chiral polyhedron with helical faces in $\mathcal{X}$ then its vertex-figures are planar. Furthermore, no vertex belongs to the plane determined by its neighbours.*

Proof. Let v be the base vertex of $\mathcal{P}$ and let u and w be its two neighbours in the base face f. Let R_1 be the half-turn with respect to the line Λ containing v, perpendicular to the axis of f. Then R_1 fixes v and f while interchanging u and w. Note that Λ contains the midpoint of one of the line-segments between u and w.

First assume that the vertex-figure is skew. We may choose the line-segment e between u and w intersecting Λ as an edge of the vertex-figure. Then the half-turn about Λ preserves the vertex-figure. Lemma 3 then implies that $\mathcal{P}$ is regular. The same argument proves that $\mathcal{P}$ is regular when the vertex-figure is planar and v is contained in the plane determined by its neighbours, that is, if the planar vertex-figure has v as its centre. ■

Examples of chiral polyhedra with helical faces and planar vertex-figures can be found in [29].

Theorem 8 *Let $\mathcal{P}$ be a chiral polyhedron in $\mathcal{X}$. Then one of the following holds.*

1. *$\mathcal{P}$ has skew faces and skew vertex-figures*
2. *$\mathcal{P}$ has helical faces and planar vertex-figures satisfying that no vertex belongs to the plane determined by its neighbours.*

Proof. Follows directly from Lemmas 4, 6 and 7. ■

We say that a polyhedron $\mathcal{P}$ is *flexible* if there exists a real $\varepsilon > 0$ and a continuous family of different $\mathcal{P}_\alpha$, with $\alpha \in (-\varepsilon, \varepsilon)$, such that $\mathcal{P}_0 = \mathcal{P}$, $G(\mathcal{P}) = G(\mathcal{P}_\alpha)$, for all α, and every $\mathcal{P}_\alpha$ is combinatorially isomorphic to $\mathcal{P}$. For example, if $\mathcal{P}$ is a Platonic solid in $\mathcal{X}$, with vertices embedded in a small sphere, then $\mathcal{P}$ is flexible since we can fix the centre of the small sphere and continuously increase or decrease the radius of the sphere without changing the symmetry group or the combinatorics of $\mathcal{P}$.

Chiral polyhedra in $\mathcal{X}$ with helical faces, are also flexible. To see this, let $\mathcal{P}$ be a chiral polyhedron with helical faces in $\mathcal{X}$ and let S_1 and S_2 be the generators of $G(\mathcal{P})$ (by Proposition 2 they are unique). By Lemma 7 its vertex-figures are planar and no vertex belongs to the plane determined by its neighbours. If v is the base vertex of $\mathcal{P}$, then S_2 is a rotation on the line ℓ through v and the centre of its vertex-figure. For each point $x \in \ell$ in an ε-neighbourhood of v we can construct a polyhedron $\mathcal{P}_x$ combinatorially isomorphic to $\mathcal{P}$. The vertices of $\mathcal{P}_x$ are the points of $x\langle S_1, S_2\rangle$. The edges of $\mathcal{P}_x$ are the line segments $e_x\langle S_1, S_2\rangle$, where e_x is the line segment between x and xS_1 ε-close to the basic edge e. Finally, faces of $\mathcal{P}_x$ are the polygons of $f_x\langle S_1, S_2\rangle$, where f_x is the polygon whose vertices are the points $x\langle S_1\rangle$, and whose edges are the line segments $e_x\langle S_1\rangle$. By construction, it is clear that S_1 and S_2 are symmetries of $\mathcal{P}_x$. On the other hand, if we take x in such a way that the distance between v and x is very small, then x and its vertex-figure do not lie on the same plane, implying that $\mathcal{P}_x$ is also a chiral polytope, so we have that $G(\mathcal{P}) = G(\mathcal{P}_x)$, and thus $\mathcal{P}$ is flexible.

The following theorem is a general version of Theorems 5.3, 5.6 and 5.9 in [26].

Theorem 9 *If $\mathcal{P}$ is a chiral polyhedron in $\mathcal{X}$ with helical faces, then it is combinatorially regular.*

Proof. First observe that with the above notation and construction, $\mathcal{P}_x$ makes sense for every $x \in \ell$. However, it may fail to be a polyhedron because two or more vertices can collapse geometrically at a single point (see Figure 7 for an example of this phenomenon). When this is the case, $\mathcal{P}_x$ is still a collection of vertices, edges and faces which we may call a *polygonal complex* on which $G(\mathcal{P})$ acts. Moreover, it always comes with a well defined combinatorial map $\mathcal{P} \to \mathcal{P}_x$. The previous paragraph proves that the set of $x \in \ell$ where this map is an isomorphism is open in ℓ, but moreover, its complement (in ℓ) is discrete because of the discreteness of $G(\mathcal{P})$. The idea of the proof is to try to find a geometrically regular candidate among the family $\mathcal{P}_x$.

Since $\mathcal{P}$ has helical faces and planar vertex-figures, S_1 is a twist and S_2 is a rotation on the line ℓ. Let γ be the axis of S_1(or one of the polar pair of axes, when $\mathcal{X} = \mathbb{P}^3$) and let R be the half-turn along one common perpendicular to ℓ and γ, say m. (In hyperbolic space this line is unique, in projective space there is a polar pair of such lines, but R is unique, and in Euclidian space it is either unique or, even if this case does not happen in this context, there is a continuous family of such.) Then, since R acts as a reflection on both ℓ and γ, we have that $RS_1R = S_1^{-1}$ and $RS_2R = S_2^{-1}$ (compare with the proof of Lemma 3).

Let $\mathcal{Q}$ be the abstract regular polyhedron associated to the string C-group $\langle S_1R, R, RS_2\rangle$ (see [23, Theorem 2E11]). There are two cases to consider.

First, if $R \notin G(\mathcal{P})$, then $G(\mathcal{P})$ is an index 2 subgroup of $\Gamma(\mathcal{Q})$, which in fact coincides with $\Gamma^+(\mathcal{Q})$ and thus $\mathcal{Q}$ is combinatorially isomorphic to $\mathcal{P}$ ([30, Theorem 1]).

The second case, when $R \in G(\mathcal{P})$, splits further in two subcases. If there is a bijection between the edges of $\mathcal{P}$ and the edges of $\mathcal{Q}$, then again $\mathcal{P}$ is combinatorially isomorphic to $\mathcal{Q}$. On the other hand, if the combinatorial map $\mathcal{P} \to \mathcal{P}_x$ is not one-to-one on the edges then we claim that $\mathcal{Q}$ is non-orientable and $\mathcal{P}$ is isomorphic to the orientable double cover of $\mathcal{Q}$, implying that it is also regular (this is also a consequence of [30, Theorem 1]). To see that $\mathcal{Q}$ is non-orientable, recall that v is our base vertex of $\mathcal{P}$, and let $v' = vR$. Then $v' \in \ell$ because R acts as a reflection on ℓ at the point $y = \ell \cap m$. As v moves along ℓ (what we have called x) towards y, v' does likewise but from the other side and in the opposite direction. At y, they colide and their respective vertex figures match in the orthogonal plane to ℓ at y. There, R sends the base flag to its 1-adjacent flag; thus the flag graph of $\mathcal{Q}$ is no longer bipartite because, before the identification, R preserved the bipartition. This proves that $\mathcal{Q}$ is non-orientably regular. Now, $\mathcal{P}$ is combinatorially orientable since it is geometrically chiral, and it is a double cover of $\mathcal{Q}$ since $G(\mathcal{P}) \cong G(\mathcal{Q})$. This concludes the proof. ∎

We comment further on some geometric details of the proof of the last theorem. In the last case, when $R \in G(\mathcal{P})$ and the map $\mathcal{P} \to \mathcal{P}_x$ is not one-to-one on the edges, it may happen that no more vertices merge at y; that is, that the stabiliser of y in $G(\mathcal{P})$ is no more than the diehedral group $\langle S_2, R\rangle$. In this case, $\mathcal{Q} = \mathcal{P}_y$ is rightfully a geometric regular polyhedron in $\mathcal{X}$. As explained in [26, Section 5.3], the family $\{P_3(c, d)\}$ described in [29, Section 6] is an example of this behaviour.

In the first case ($R \notin G(\mathcal{P})$), if the stabiliser of y in $G(\mathcal{P})$ is $\langle S_2\rangle$, then $\mathcal{P}_y$ is a regular polyhedron in $\mathcal{X}$ isomorphic to $\mathcal{Q}$. The families $\{P_1(a, b)\}$ and $\{P_2(c, d)\}$ described in [29, Sections 4, 5] are examples of this case (see also [26, Section 5]).

When the stabiliser of y in $\langle S_1R, R, RS_2\rangle$ is bigger than it should be (that is, than $\langle S_2, R\rangle$) we have that $\mathcal{P}_y$ is not a polyhedron, it is simply a polygonal complex. But nevertheless, it is an unfaithful realisation of the regular abstract polyhedron $\mathcal{Q}$, obtained by continuously "deforming" $\mathcal{P}$. However, we know of no examples in which this collapse at the regular point happens.

2.2 2-orbit poyhedra

A polyhedron is called 2-*orbit* when its symmetry group induces two orbits on flags. Chiral polyhedra are examples of 2-orbit polyhedra.

Let Φ be a flag of a 2-orbit polyhedron $\mathcal{P}$ and assume that Φ and Φ^i are in the same flag-orbit. Then Ψ and Ψ^i are in the same flag-orbit for any flag Ψ of $\mathcal{P}$ (see [18, Lemma 2]). This suggests a classification of 2-orbit polyhedra into seven classes 2_I where $I \subset \{0,1,2\}$ is the set of indices i for which Φ and Φ^i are in the same flag orbit for any flag Φ. The class of chiral polyhedra is then denoted by $2_\emptyset$, or simply by 2. Note that if $I = \{0,1,2\}$ then all flags are in the same flag orbit implying that $\mathcal{P}$ is regular and not 2-orbit.

Of particular interest for this paper are polyhedra in classes $2_{\{0,2\}}$ and $2_{\{1\}}$.

Polyhedra in class $2_{\{0,2\}}$ are vertex- and face-transitive, but not edge-transitive. This is a consequence of the fact that the four flags containing a given edge belong to the same flag-orbit. If all faces of a polyhedron in class $2_{\{0,2\}}$ have four edges then the symmetry group acts transitively on the set of pairs of vertices that are opposite in any face.

Polyhedra in class $2_{\{1\}}$ are vertex-, edge- and face-transitive. Their faces are edge-transitive, but not necessarily vertex-transitive.

The symmetries of polyhedra in class 2_I are summarised in the following remark.

Remark 10 *Let $\mathcal{P}$ be a polyhedron in class 2_I and let w be a word in the set $\{0,1,2\}$. There is a symmetry mapping a flag Φ of $\mathcal{P}$ to Φ^w if and only if w contains an even number of letters in $\{0,1,2\} \setminus I$.*

For more details on combinatorial aspects of 2-orbit polyhedra, see [18].

2.3 Operations

To conclude the section we recall two geometric operations to obtain polyhedra from other polyhedra.

The *halving* operation transforms a polyhedron $\mathcal{P}$ in $\mathcal{X}$ with square faces into a structure $\mathcal{P}^\eta$ whose vertex set is a subset of the vertex set of $\mathcal{P}$, with two vertices being adjacent if and only if they are opposite vertices in a face of $\mathcal{P}$. The faces of $\mathcal{P}^\eta$ are cycles corresponding to vertex-figures of some vertices of $\mathcal{P}$.

Whenever the 1-skeleton of $\mathcal{P}$ is bipartite the vertex set of $\mathcal{P}^\eta$ is one of the parts of the bipartition and the faces of $\mathcal{P}^\eta$ are vertex-figures at the vertices of the other part. If the 1-skeleton of $\mathcal{P}$ is not bipartite then the vertex set of $\mathcal{P}^\eta$ is the same as that of $\mathcal{P}$, and the faces are the vertex-figures at all vertices of $\mathcal{P}$.

In the instances of the halving operation considered in this paper, the structure $\mathcal{P}^\eta$ is indeed a polyhedron. It follows directly from the construction above that the halving operation applied to a polyhedron of type $\{4, q\}$, when it is a polyhedron, has type $\{q, q\}$.

As an example, the halving operation applied to a cube is a tetrahedron.

In this paper we are interested on applying the halving operation only to polyhedra that are either regular or in class $2_{\{0,2\}}$. By convention, if $\mathcal{P}$ is a regular polyhedron in $\mathbb{P}^3$ we choose the edges of $\mathcal{P}^\eta$ so that the faces of $\mathcal{P}^\eta$ are planar inessential (meaning that as curves they are contractible), or skew or helical with inessential orthogonal projection to some distinguished plane. In the instances where $\mathcal{P}$ is in class $2_{\{0,2\}}$ considered in this paper, the faces of $\mathcal{P}^\eta$ are skew and we also choose the edges of $\mathcal{P}^\eta$ so that the projection of a face to some distinguished plane is inessential (although the vertices may belong to two distinct circles as opposed to the regular case).

In principle, the halving operation applied to a polyhedron $\mathcal{P}$ with bipartite 1-skeleton can yield two non-isomorphic polyhedra, depending on the choice of the vertex set of $\mathcal{P}^\eta$. However, if $\mathcal{P}$ is regular or in class $2_{\{0,2\}}$ there is an automorphism of $\mathcal{P}$ interchanging the parts; therefore the two polyhedra obtained from the halving operation are isomorphic.

A 2-*hole* of a polyhedron $\mathcal{P}$ is a closed path in the 1-skeleton of $\mathcal{P}$ where e_1, e_3, e_5 are three consecutive edges if and only if there exist edges e_2 and e_4 such that

- e_1, e_2, e_3 share a vertex u, and e_3, e_4, e_5 share a vertex $v \neq u$;
- e_1, e_2 are consecutive edges of a face f_1 of $\mathcal{P}$; e_2, e_3, e_4 are consecutive edges of a face $f_2 \neq f_1$; and e_4, e_5 are consecutive edges of a face $f_3 \neq f_2$

(see Figure 1, where the 2-hole is indicated with a dotted line). In other words, a 2-hole is constructed by traversing an edge to one of its endpoints, skipping the first edge on the left (according to some local orientation) and traversing the

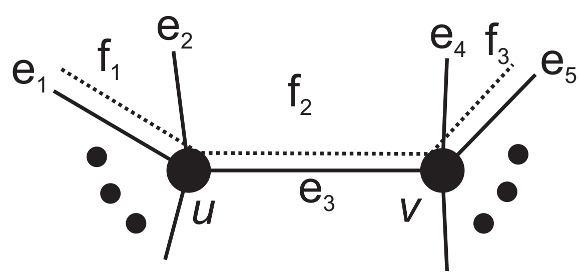

Figure 1: Local view of a 2-hole of a polyhedron

second edge. This procedure is repeated until an edge is traversed twice in the same direction.

The *facetting* (or 2-facetting) operation transforms a polyhedron $\mathcal{P}$ in $\mathcal{X}$ with vertices of degree at least 5 into a structure $\mathcal{P}^\phi$ whose 1-skeleton is contained in the 1-skeleton of $\mathcal{P}$ and whose faces are a subset of the 2-holes of $\mathcal{P}$. Starting from a given triple of incident vertex, edge and hole of $\mathcal{P}$, the 1-skeleton and set of faces of $\mathcal{P}^\phi$ are chosen so that the 1-skeleton is connected and every edge belongs to two 2-holes. That is, it can be constructed from the edges that belong to only one 2-hole and adding at each step the (unique) other 2-hole containing each of them. In each step, new vertices and edges may be added.

As an example, the facetting operation applied to the icosahedron is the great dodecahedron (see [10, Chapter VI]).

The structure $\mathcal{P}^\phi$ needs not satisfy Item 3 of Definition 1, but when it does, it is a polyhedron in $\mathcal{X}$. Here we are only interested in applying the facetting operation to regular polyhedra and to polyhedra in class $2_{\{1\}}$. In these cases there is a unique outcome of the facetting operation since polyhedra in class $2_{\{1\}}$ have only one orbit of vertices and of 2-holes.

More details about the halving and facetting operations in the case when $\mathcal{P}$ is regular can be found in [23, Section 7B], where the operations are considered in the group theoretical setting.

3 The Petrie-Coxeter construction

In this section we provide a construction of a regular or chiral polyhedron in $\mathcal{X}$ from a regular 4-polytope with planar faces in $\mathcal{X}$. The first part of this construction first appeared in [8], and it is addressed for the regular and chiral cases in [22] and [19], respectively. A generalization to higher ranks can be found in [25].

A *regular 4-polytope with planar faces in* $\mathcal{X} \in \{\mathbb{S}^3, \mathbb{P}^3, \mathbb{E}^3, \mathbb{H}^3\}$ is a collection of regular isometric polyhedra, called cells, satisfying:

1. every face of every cell is planar;

2. the vertex set of every cell is contained in a 2-dimensional sphere, and so it has a centre;

3. every face belongs to exactly two cells;

4. given any two cells c and c' there exists a sequence of cells $c_1, \dots, c_k$ where $c_1 = c$, $c_k = c'$, and the cells c_i, c_{i+1} share a face for $i \in \{1, \dots, k-1\}$;

5. the vertex-figure of every vertex is a regular polyhedron,

where the vertex-figure of a vertex v is the polyhedron such that: the vertices are the mid-points of the edges incident to v; the edges are the line segments between two vertices, whenever the corresponding edges are in the same face; and the faces correspond to the cells incident to v.

Regular 4-polytopes with planar faces include the classical regular tilings of $\mathcal{X}$, but also more general structures like the 4-polytope $\{\frac{5}{2}, 5, 3\}$, where every point of $\mathbb{S}^3$ can be understood to belongs to two small stellated dodecahedra, instead of to only one of them (see [10]). It is not our intension here to include all skeletal 4-polytopes as defined elsewhere.

The *dual* of a regular 4-polytope with planar faces in $\mathcal{X}$ can be constructed in the standard way by taking the centres of the (spheres containing the vertex sets of the) cells as vertices, and recursively replacing the roles of edges and faces, and of vertices and cells.

In Section 4 we will be dealing with the classical regular 4-polytopes with planar faces of $\mathbb{E}^4$ projected to $\mathbb{S}^3$, namely the convex and starry regular 4-polytopes. These are examples of regular 4-polytopes with planar faces in $\mathbb{S}^3$. For the remainder of this paper, we shall refer to a regular 4-polytope with planar faces simply as a regular 4-polytope.

Let $\mathcal{T}$ be a regular 4-polytope in $\mathcal{X}$, let d be the distance between a vertex and the centre of any of the cells containing it, and let $0 < \alpha < 1$. For each vertex v and every cell c containing v we define the point $(v, c)_\alpha$ as the point in the line through v and the centre of c that is at distance αd from v and at distance $(1 - \alpha)d$ from the centre of c.

For any given $\alpha \in (0, 1)$, the rank 3 structure $PC_\alpha(\mathcal{T})$ is then defined to have as its vertex set all points $(v, c)_\alpha$ for all cells c of $\mathcal{T}$ and all vertices v in c. Its edge set consists of line segments between $(v, c)_\alpha$ and $(v', c)_\alpha$ where there is an edge of c between v and v', as well as line segments between $(v, c)_\alpha$ and $(v, c')_\alpha$

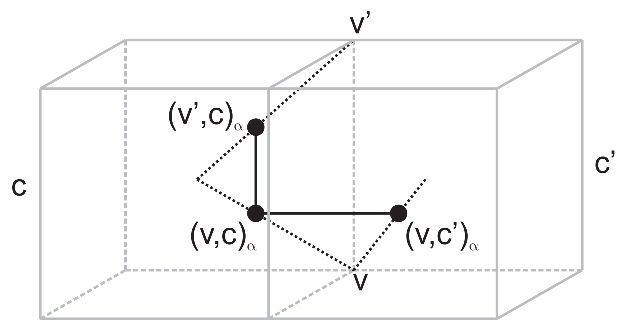

Figure 2: The vertices $(v,c)_\alpha$, $(v',c)_\alpha$ and $(v,c')_\alpha$ from a tessellation by cubes

where c and c' are cells sharing a face that contains v. The edge between the vertices $(v,c)_\alpha$ and $(v',c)_\alpha$ is chosen to be the line segment completely contained in c, whereas the edge between vertices $(v,c)_\alpha$ and $(v,c')_\alpha$ is chosen to be the line segment that intersects the face between c and c' (see Figure 2). Finally, the set of faces of $PC_\alpha(\mathcal{T})$ is the set of squares $((v,c)_\alpha,(v',c)_\alpha,(v',c')_\alpha,(v,c')_\alpha)$, where c and c' are cells of $\mathcal{T}$ sharing a face that contains the edge between v and v'.

Theorem 11 *For any $\alpha \in (0,1)$ and any regular 4-polytope $\mathcal{T}$ of $\mathcal{X}$, $PC_\alpha(\mathcal{T})$ is a polyhedron in $\mathcal{X}$.*

Proof. We need to show that $PC_\alpha(\mathcal{T})$ satisfies Items 1, 2 and 3 of Definition 1.

Let f_1 and f_2 be the two faces in a cell c containing the edge between vertices v and v'. Then the edge between $(v,c)_\alpha$ and $(v',c)_\alpha$ belongs precisely to the two squares $((v,c)_\alpha,(v',c)_\alpha,(v',c_1)_\alpha,(v,c_1)_\alpha)$ and $((v,c)_\alpha,(v',c)_\alpha,(v',c_2)_\alpha,(v,c_2)_\alpha)$, where c_i is the cell sharing face f_i with c ($i \in \{1,2\}$). On the other hand, if e_1 and e_2 are the two edges intersecting at v contained in the face between c and c' then the edge between $(v,c)_\alpha$ and $(v,c')_\alpha$ belongs precisely to the two squares $((v,c)_\alpha,(v,c')_\alpha,(v_1,c')_\alpha,(v_1,c)_\alpha)$ and $((v,c)_\alpha,(v,c')_\alpha,(v_2,c')_\alpha,(v_2,c)_\alpha)$, where e_i joins vertex v with v_i ($i = \{1,2\}$).

The connectivity of the 1-skeleton of $PC_\alpha(\mathcal{T})$ follows from the connectivity of the 1-skeleton of $\mathcal{T}$. In fact, the 1-skeleton of $PC_\alpha(\mathcal{T})$ is obtained from that of $\mathcal{T}$ by replacing every vertex by the 1-skeleton of a Platonic solid, and every edge e by the 1-skeleton of a prism whose top and bottom belong to the Platonic solids originated from the endpoints of e.

Finally, the vertex-figure at a vertex $(v,c)_\alpha$ is the cycle

$$((v_1,c)_\alpha,(v,c_1)_\alpha,(v_2,c)_\alpha,(v,c_2)_\alpha,\ldots,(v_q,c)_\alpha,(v,c_q)_\alpha),$$

where the vertices in cyclic order adjacent to v in c are $v_1,\ldots,v_q$; and the cells sharing a face with c and containing v in cyclic order (around v) are $c_1,\ldots,c_q$.

The labelling is taken so that the edge between v and v_i belongs to the cells c, c_i and c_{i-1} (where the subindices are taken modulo q). ∎

Remark 12 *Given $\alpha \in (0,1)$ and a regular 4-polytope $\mathcal{T}$, if $\mathcal{T}^*$ denotes the dual 4-polytope of $\mathcal{T}$, then $PC_\alpha(\mathcal{T}) = PC_{1-\alpha}(\mathcal{T}^*)$.*

Since $PC_\alpha(\mathcal{T})$ is constructed from $\mathcal{T}$, their symmetry groups are strongly related, as mentioned in the following theorem.

Theorem 13 *For any $\alpha \in (0,1)$ and any regular 4-polytope $\mathcal{T}$ of $\mathcal{X}$, the symmetry group of $\mathcal{T}$ is isomorphic to a subgroup of index at most 2 of $G(PC_\alpha(\mathcal{T}))$. Furthermore, $PC_\alpha(\mathcal{T})$ is either regular or a 2-orbit polyhedron in class $2_{\{0,2\}}$.*

Proof. By construction, any symmetry of $\mathcal{T}$ preserves the vertex, edge and face sets of $PC_\alpha(\mathcal{T})$. Given a flag Φ of $\mathcal{T}$ containing a vertex v and a cell c, we can construct the two flags

$$\{(v,c)_\alpha, \{(v,c)_\alpha, (v',c)_\alpha\}, ((v,c)_\alpha, (v',c)_\alpha, (v',c')_\alpha, (v,c')_\alpha)\},$$
$$\{(v,c)_\alpha, \{(v,c)_\alpha, (v,c')_\alpha\}, ((v,c)_\alpha, (v,c')_\alpha, (v',c')_\alpha, (v',c)_\alpha)\}$$

of $PC_\alpha(\mathcal{T})$, where the edge of Φ joins v with v' and the face of Φ is contained in the cells c and c'. Furthermore, any flag of $PC_\alpha(\mathcal{T})$ is obtained in this way. Hence, there are twice as many flags in $PC_\alpha(\mathcal{T})$ as in $\mathcal{T}$. Since the symmetry group of $\mathcal{T}$ acts transitively on the flags of $\mathcal{T}$, it induces at most two orbits on the flags of $PC_\alpha(\mathcal{T})$.

Let $\Phi = ((v,c)_\alpha, e, f)$ be the base flag of $PC_\alpha(\mathcal{T})$, where e is the edge between $(v,c)_\alpha$ and $(v',c)_\alpha$ for some vertex v' of $\mathcal{T}$, and f is the square $((v,c)_\alpha, (v',c)_\alpha, (v',c')_\alpha, (v,c')_\alpha)$ for some cell c' of $\mathcal{T}$. The symmetry of $\mathcal{T}$ interchanging v with v' while preserving c and both faces of c containing v and v' maps Φ to Φ^0. Similarly, the symmetry of $\mathcal{T}$ preserving v and v' while interchanging both faces of c containing v and v' maps Φ to Φ^2. It follows that if $PC_\alpha(\mathcal{T})$ is not regular then it is a 2-orbit polyhedron in class $2_{\{0,2\}}$. ∎

Since the faces of $PC_\alpha(\mathcal{T})$ are squares we can apply the halving operation to it. Furthermore, if the cells of $\mathcal{T}$ have type $\{p,q\}$ then $PC_\alpha(\mathcal{T})$ has type $\{4,2q\}$ and $PC_\alpha(\mathcal{T})^\eta$ has type $\{2q,2q\}$. Since $q \geq 3$, we can apply the facetting operation to $PC_\alpha(\mathcal{T})^\eta$. For simplicity, we set

$$H_\alpha(\mathcal{T}) := (PC_\alpha(\mathcal{T})^\eta)^\phi.$$

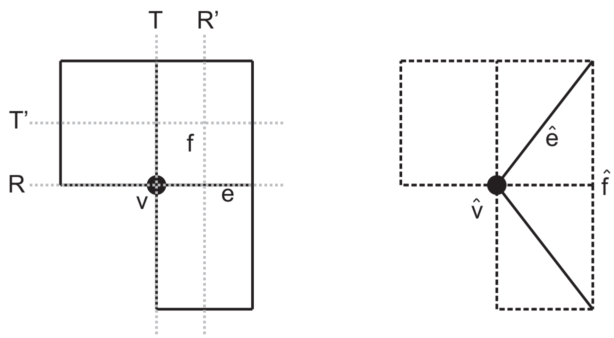

Figure 3: The symmetries R, T, R' and T' of $PC_\alpha(\mathcal{T})$ and the flag $\hat{\Phi}$ of $PC_\alpha(\mathcal{T})^\eta$

Proposition 14 *Assume that for some $\alpha \in (0,1)$ and some regular 4-polytope $\mathcal{T}$ of $\mathcal{X}$, $H_\alpha(\mathcal{T})$ is a polyhedron. Then $H_\alpha(\mathcal{T})$ is either regular or chiral.*

Proof. We need to show that there are symmetries S_1 and S_2 of $H_\alpha(\mathcal{T})$ as in Proposition 2.

We know from the definitions of the operations η and ϕ that the vertices of $H_\alpha(\mathcal{T})$ are vertices of $PC_\alpha(\mathcal{T})$, and that the edges of $H_\alpha(\mathcal{T})$ are diagonals of the 4-gonal faces of $PC_\alpha(\mathcal{T})$.

There are symmetries R and T that map a base flag $\Phi = \{v, e, f\}$ of $PC_\alpha(\mathcal{T})$ to Φ^2 and to Φ^{121}, respectively (see Remark 10 and Figure 3). Let $\hat{\Phi} = \{\hat{v}, \hat{e}, \hat{f}\}$ be the flag of $(PC_\alpha(\mathcal{T}))^\eta$ containing $v = \hat{v}$, the diagonal of f and the vertex-figure at the vertex of e other than v. Then $\hat{\Phi}R$ has the same vertex and face as $\hat{\Phi}$, and so $\hat{\Phi}R = \hat{\Phi}^1$. Similarly, $\hat{\Phi}T = \hat{\Phi}^{212}$, and hence $\hat{\Phi}RT = \hat{\Phi}^{2121}$. Finally, let $\tilde{\Phi}$ be the flag of $H_\alpha(\mathcal{T})$ containing the vertex and edge of $\hat{\Phi}$, and the 2-hole containing the edge of $\hat{\Phi}$ and its image under TR. Then $\tilde{\Phi}RT = \tilde{\Phi}^{21}$; that is, RT is the desired symmetry S_2.

Let R, T, $\hat{\Phi}$ and $\tilde{\Phi}$ be as above, and let R' and T' be the symmetries of $PC_\alpha(\mathcal{T})$ that map the base flag Φ to Φ^0 and to Φ^{101}, respectively (see Figure 3). Recall that $\hat{\Phi}TR = \hat{\Phi}^{1212}$ and note that $\hat{\Phi}T'R' = \hat{\Phi}^{20}$. Hence $\hat{\Phi}T'R'TR = \hat{\Phi}^{1210}$. This implies that $\tilde{\Phi}RTR'T' = \tilde{\Phi}^{10}$; that is, $T'R'TR$ is the desired symmetry S_1.

■

Corollary 15 *When $PC_\alpha(\mathcal{T})^\eta$ is a polyhedron, it is either regular or in class $2_{\{1\}}$.*

In our definition of $PC_\alpha(\mathcal{T})$ we required $0 < \alpha < 1$ because for these values, the vertex set of $PC_\alpha(\mathcal{T})$ can be easily visualised as the vertices obtained by shrinking every cell of $\mathcal{T}$. However, we are mainly interested in $H_\alpha(\mathcal{T})$, and it is important to note that our definitions naturally extend to consider values of α in all of $\mathbb{R}$. Let m be the line through the vertex v and the centre of the cell c that contains v. Then, if $\mathcal{X} \in \{\mathbb{E}^3, \mathbb{H}^3\}$, the unique parametrization of m, proportional to distance, which yields v for $\alpha = 0$ and the center of c for $\alpha = 1$, defines $(v, c)_\alpha \in m$ by the same formula using directed distances. This idea extends naturally to $\mathbb{S}^3$ and $\mathbb{P}^3$.

Some values of α (particularly 0 and 1, but maybe more) result in distinct pairs (v, c) and (v', c') of incident vertex and cell of $\mathcal{T}$, having the points $(v, c)_\alpha$ and $(v', c')_\alpha$ coincide and the "polyhedron" $PC_\alpha(\mathcal{T})$ (or $H_\alpha(\mathcal{T})$), degenerates to a structure not satisfying Definition 1. However, for all other values of α, the polyhedra $PC_\alpha(\mathcal{T})$ (respectivelly, $H_\alpha(\mathcal{T})$) are isomorphic and have the same symmetry group, regardless of whether $\alpha \in (0, 1)$ or not.

The polyhedrality of $H_\alpha(\mathcal{T})$ will be discussed in the next section.

To conclude this section we remark that the family $\{P_1(a, b)\}$ explained in [29, Section 4] is precisely $\{H_\alpha(\mathcal{T})\}$ where $\mathcal{T}$ is the tessellation $\{4, 3, 4\}$ by cubes of $\mathbb{E}^3$ (see also [26, Section 5.2]).

4 Chiral polyhedra in $\mathbb{P}^3$ and $\mathbb{S}^3$

In this section we work with each of the 6 convex regular polytopes of $\mathbb{R}^4$, and the 10 starry ones (see [10]). Given a finite regular 4-polytope $\mathcal{T}$ in $\mathbb{R}^4$ (centered at the origin), we first project it to $\mathbb{S}^3$ and we work with the projection that, abusing notation, we also denote by $\mathcal{T}$. Given $\alpha \in \mathbb{R}$, we shall describe $H_\alpha(\mathcal{T})$ in the spirit of Wythoff's construction. We denote by $V(\mathcal{T}), E(\mathcal{T}), F(\mathcal{T})$ and $C(\mathcal{T})$ the set of vertices, edges, faces and cells of $\mathcal{T}$, respectively.

Note that the vertices of $H_\alpha(\mathcal{T})$ are some of the vertices of $PC_\alpha(\mathcal{T})$ and thus they are points in the set

$$\{(v, c)_\alpha \mid v \in V(\mathcal{T}), c \in C(\mathcal{T}) \text{ and } v \text{ is incident to } c\} \tag{1}$$

described in Section 3. Since $\mathcal{T}$ is regular, its symmetry group is transitive on the vertex-cell incident pairs, and hence the vertices in the set in (1) can be described as the orbit under $G(\mathcal{T})$ of the vertex $\bar{v} := (v, c)_\alpha$ corresponding to the vertex and cell of a base flag $\Phi := \{v, e, f, c\}$ of $\mathcal{T}$. Since $G(H_\alpha(\mathcal{T})) \leq \langle S_1, S_2 \rangle$, where S_1 and S_2 are as in the proof of Proposition 14 (the "rotations" of the face and vertex, respectively), the vertices of $H_\alpha(\mathcal{T})$ are the orbit of $\bar{v}$ under $G := \langle S_1, S_2 \rangle$.

Likewise, the edges and faces of $H_\alpha(\mathcal{T})$ are the orbits of the base edge $\bar{e}$ and the base face $\bar{f}$ of a base flag $\Psi = \{\bar{v}, \bar{e}, \bar{f}\}$, where $\bar{e}$ and $\bar{f}$ are as follow. Since

$\Phi S_1^{-1} = \Phi^{01}$, then S_1^{-1} sends $\bar{v}$ to the other vertex of $\bar{e}$, and thus $\bar{e}$ is the line segment between $\bar{v}$ and $\bar{v}S_1^{-1}$, while $\bar{f}$ is the orbit of $\{\bar{v}, \bar{e}\}$ under $\langle S_1 \rangle$.

Since $\mathcal{T}$ is regular, there are symmetries of $\mathcal{T}$ fixing all but one of the elements of the base flag Φ. More precisely, there are symmetries R_0, R_1, R_2 and R_3 sending the base flag $\Phi = \{v, e, f, c\}$ to the flags $\{v', e, f, c\}$, $\{v, e', f, c\}$, $\{v, e, f', c\}$ and $\{v, e, f, c'\}$, respectively. Following the proofs of Theorem 13 and Proposition 14, one can see that S_1 and S_2 can then be written in the following way:

$$\begin{aligned} S_1 &= R_0 R_1 R_3 R_2, &(2)\\ S_2 &= R_2 R_1. &(3) \end{aligned}$$

It is well known that the symmetries R_0, R_1, R_2 and R_3 are hyperplane reflections. Thus, S_1 is a twist (or screw motion) and its only fixed point (in $\mathbb{R}^4$) is the centre of the (finite) polytope but it is fixed-point free in $\mathbb{S}^3$. On the other hand, S_2 is a rotation on a line (great circle) of $\mathbb{S}^3$, which can be also seen as a translation on its *polar* line (the corresponding 2-planes are orthogonal in $\mathbb{R}^4$). In particular, this means that since we know the line of fixed points of S_2 (it is the line through v and the centre of c), it can be described by only one parameter: the angle of translation $\frac{2\pi}{q}$, where q is a rational number. On the other hand, since S_1 is a twist, it fixes (set wise) two polar lines of $\mathbb{S}^3$ (great circles as far away as possible) and it can be described by such a pair of lines (called the *core* of the twist), together with the angles of translation $2\pi\frac{p_1}{p}$ and $2\pi\frac{p_2}{p}$, where p, p_1 and p_2 are integers (note that if either p_1 or p_2 was zero we would get a simple rotation as for S_2). According to this, we say that $H_\alpha(\mathcal{T})$ has type $\{\frac{p}{p_1,p_2}, q\}$ or equivalentely $\{\frac{p}{p_2,p_1}, q\}$ (this is a particular case of the notation in [21, Section 4], where $2\pi\frac{p_1}{p}$ and $2\pi\frac{p_2}{p}$ are the angles of rotation on the two 2-planes that are fixed under the twist S_1, seen as a transformation of $\mathbb{R}^4$).

In what follows, we describe $H_\alpha(\mathcal{T})$, for each classic regular 4-polytope $\mathcal{T}$ in $\mathbb{S}^3$. The symmetries R_0, R_1, R_2 and R_3 of $\mathcal{T}$, which will be presented as matrices, are determined as the reflections on the facets of a *basic tetrahedron* given by points $v_0, v_1, v_2, v_3 \in \mathbb{S}^3$ which are the centroids of the elements of the base flag. Using these matrices we can obtain the type of each $H_\alpha(\mathcal{T})$ and the core of S_1. Moreover, with the help of Lemma 7 and Theorem 9 we determine the values of α for which $H_\alpha(\mathcal{T})$ is geometrically regular, otherwise it is chiral or a polyhedral complex. For computational simplicity, we linearize the meaning of the parameter α, so that what we called $(v, c)_\alpha$ in terms of distance (which is angular in $\mathbb{S}^3$ and $\mathbb{P}^3$) will now be $(v, c)_\alpha = [\,(1-\alpha)v_0 + \alpha v_3\,]$; where, by $[\,x\,]$ we denote the normalization of x, i.e.,

$$[\,x\,] := (1/\|x\|)x\,.$$

In each case it is convenient to work with various values of α, however, There are four important cases: for $\alpha = 0, 1, 1/2, \infty$, $(v, c)_\alpha$ will respectively mean $[\,v_0\,], [\,v_3\,], [\,[\,v_0\,] + [\,v_3\,]\,], [\,[\,v_3\,] - [\,v_0\,]\,]$.

We provide pictures, using the stereographic projection of $\mathbb{S}^3$ to $\mathbb{R}^3$ of $H_\alpha(\mathcal{T})$, for some of the classic regular 4-polytope $\mathcal{T}$. The faces are depicted by ribbons which result from the geometric Wythoff's construction applied to the cyclic group $\langle S_1 \rangle$ of rotations of the face, using as fundamental region the quadrilateral surface with opposite edges given by the basic edge and its orthogonal projection to the closest core-component of the generating twist. Thus, the ribbons (topologically, $\mathbb{S}^1 \times [\,0, 1\,]$) have as boundary the polygon on one side and a circular core component of the twist on the other side. Therefore, the polyhedra are depicted by surfaces whose boundary components (in the general case) correspond to the faces.

The core of the basic face are the common perpendiculars (perpendicular in $\mathbb{E}^3$ and $\mathbb{H}^3$) to the lines v_2v_0' and v_1v_3 in the 0-adjacent fag of the basic tetrahedron (see Figure 4). To prove this, recall equation (2) and observe that R_0R_1 and R_3R_2 can also be expressed as compositions of two half-turn rotations along lines. Namely, rotations on v_2v_0' and then on v_1v_2 gives R_0R_1 and on v_2v_1 followed by v_1v_3 yields R_3R_2. Composing these four half-turn rotations on lines gives half-turning on v_2v_0' followed by v_1v_3, because the middle ones cancel out. The common perpendiculars to these two lines are fixed and thus it is the core of S_1.

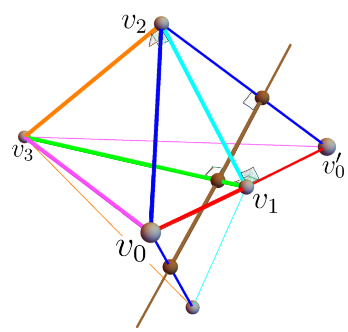

Figure 4: The basic tetrahedron of $\mathcal{T}$ with its 0- and 2-adjacent ones, and the core of the basic face of $H_\alpha(\mathcal{T})$ in brown.

Chiral polyhedra from the regular polytope $\{3, 3, 3\}$

The simplex $\{3, 3, 3\}$ is a regular polytope whose cells are regular tetrahedra, three of them around each edge. In contrast to the other finite regular polytopes of $\mathbb{R}^4$, the easier way to work with $\{3, 3, 3\}$ is in the 4-dimensional subspace of

$\mathbb{R}^5$ with coordinate sum equal zero: it is obtained by translating the standard affine simplex there. Hence, the matrices used in this case are 5×5 permutation matrices, while in all of the other cases we study they will be 4×4 matrices.

For the basic tetrahedron

$$\begin{array}{ll} v_0 = [\,(4,-1,-1,-1,-1)\,], & v_1 = [\,(3,3,-2,-2,-2)\,], \\ v_2 = [\,(2,2,2,-3,-3)\,], & v_3 = [\,(1,1,1,1,-4)\,], \end{array}$$

arising from the natural order on the canonical basis, the matrices for R_0, R_1, R_2 and R_3 are the four that transpose consecutive basis vectors of $\mathbb{R}^5$. Then, the generators of $G(H_\alpha(\{3,3,3\}))$ are

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The type of $H_\alpha(\{3,3,3\})$ is $\{\frac{5}{1,2},3\}$. It is combinatorially isomorphic to the dodecahedron, $\{5,3\}$, and it takes over the 20 vertices of $PC_\alpha(\{3,3,3\})$. For $\alpha = \frac{1}{2}, \infty$, $H_\alpha(\{3,3,3\})$ is regular, and for all other $\alpha \neq 0,1$ it is a chiral polyhedron (see Figure 5). For $\alpha = 0,1$ it collapses: groups of 4 vertices merge, corresponding to the two oriented families of 5 embedded tetrahedra in $\{5,3\}$.

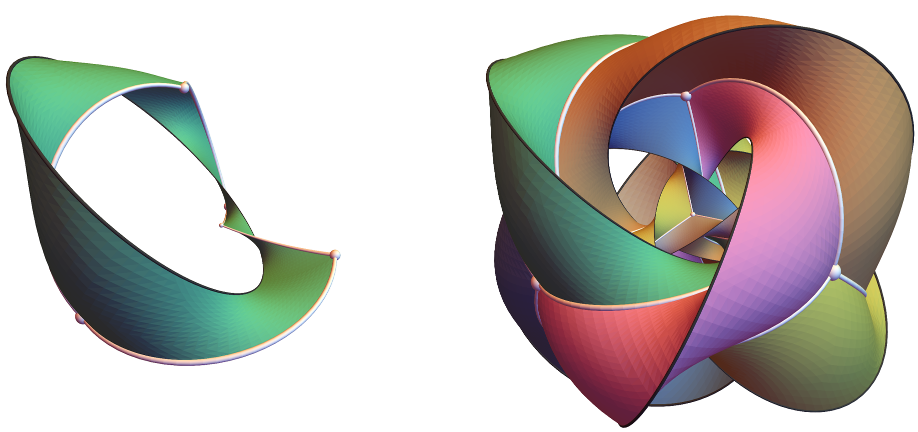

Figure 5: One face of $\{\frac{5}{1,2}\}$ and the chiral dodecahedron $H_{1/5}(\{3,3,3\})$. Notice that opposite faces of the dodecahedron (with the same color) share their (black) core because they share their rotation group.

The basic vertices for the regular cases are $[(1,0,0,0,-1)]$ (for $\alpha = 1/2$) and

$[(3,-2,-2,-2,3)]$ (for $\alpha = \infty$) with respective "reflection" matrices

$$R = \pm\frac{1}{5}\begin{pmatrix} 2 & 2 & 2 & 2 & -3 \\ 2 & 2 & -3 & 2 & 2 \\ 2 & -3 & 2 & 2 & 2 \\ 2 & 2 & 2 & -3 & 2 \\ -3 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Chiral polyhedra from $\{4,3,3\}$ and $\{3,3,4\}$

Taking as the basic tetrahedron for $\{4,3,3\}$:

$$v_0 = [(1,1,1,1)], v_1 = [(0,1,1,1)], v_2 = [(0,0,1,1)], v_3 = (0,0,0,1)\,;$$

the generators of G become:

$$S_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The general $H_\alpha(\{4,3,3\})$ has type $\{\frac{8}{1,3},3\}$. In this case, for $\alpha = 0$ no vertices come together, so we still have a chiral polyhedron $H_0(\{4,3,3\})$, which was taken as facet of a chiral 4-polytope in $\mathbb{S}^3$ (or $\mathbb{R}^4$) in [3]. See Figure 6.

Figure 6: $H_0(\{4,3,3\}/2)$ is a chiral polyhedron in $\mathbb{P}^3$ with the combinatorial structure of the cube $\{4,3\}$. Antipodal points on the 2-sphere are to be identified.

The basic vertices for the regular cases are $[(1,1,1,\sqrt{3})]$ and $[(1,1,1,-\sqrt{3})]$

with respective "reflection" matrices

$$R = \pm\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Chiral polyhedra from $\{3,4,3\}$

A basic tetrahedron for the selfdual regular polytope $\{3,4,3\}$ is:

$$v_0 = (1,0,0,0), v_1 = [(3,1,1,1)], v_2 = [(2,1,1,0)], v_3 = [(1,1,0,0)]\,.$$

The type of $H_\alpha(\{3,4,3\})$ is $\{\frac{12}{1,5},4\}$, it becomes regular at $\alpha = 1/2, \infty$ (see Figure 7) and the generators of G are:

$$S_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix},\ S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The basic vertices for the regular cases are $[(1+\sqrt{2},1,0,0)]$ and $[(1-\sqrt{2},1,0,0)]$ with respective "reflection" matrices

$$R = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Chiral polyhedra from $\{5,3,3\}$ and $\{3,3,5\}$

A basic tetrahedron for the regular polytope $\{5,3,3\}$ is:

$$\begin{aligned} v_0 &= [\,(\phi^2,1,-\phi^{-2},0)\,], \quad v_1 = [\,(\phi,\phi^{-1},0,0)\,], \\ v_2 &= [\,(2+\phi,1,0,\phi^{-1})\,], \quad v_3 = (1,0,0,0)\,, \end{aligned}$$

where ϕ denotes the golden ratio. The generators of G, which is of order 1440, are:

$$S_1 = \frac{1}{2}\begin{pmatrix} \phi & 1 & 0 & -\phi^{-1} \\ 1 & -1 & -1 & 1 \\ 0 & 1 & \phi^{-1} & \phi \\ \phi^{-1} & -1 & \phi & 0 \end{pmatrix},\ S_2 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \phi & -1 & \phi^{-1} \\ 0 & -1 & -\phi^{-1} & \phi \\ 0 & -\phi^{-1} & -\phi & -1 \end{pmatrix}.$$

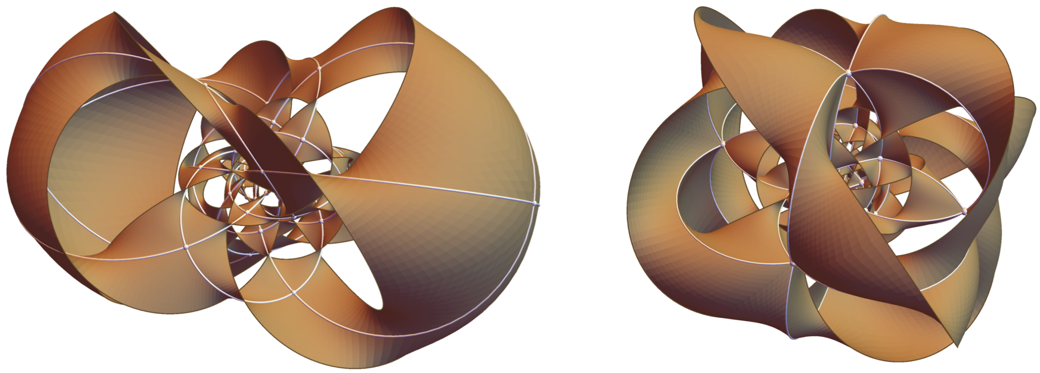

Figure 7: The regular $H_{1/2}(\{3,4,3\})$ and $H_1(\{3,4,3\})$ which is not a polyhedron because pairs of vertices collapse.

The type of $H_\alpha(\{5,3,3\})$ is $\{\frac{30}{1,11},3\}$, and for $\alpha=0$ no vertices identify, so that $H_0(\{5,3,3\})$ can be taken as facet of a chiral 4-polytope, but not $H_1(\{5,3,3\})$ in which four vertices come together.

The basic vertices for the regular cases are

$$[(1-2\phi+2\sqrt{2},-\phi,\phi^{-1},0)] \quad \text{and} \quad [(1-2\phi-2\sqrt{2},-\phi,\phi^{-1},0)]\,,$$

with respective "reflection" matrices

$$R=\pm\frac{1}{2\sqrt{2}}\begin{pmatrix} 1-2\phi & -\phi & \phi^{-1} & 0 \\ -\phi & 2 & -\phi^{-1} & 1 \\ \phi^{-1} & -\phi^{-1} & \phi^{-1} & \phi^2 \\ 0 & 1 & \phi^2 & -\phi^{-2} \end{pmatrix}.$$

Chiral polyhedra from the starry regular polytopes

There are 10 regular starry regular polytopes, two of which are self-dual. In this section we give the details of the construction of only one of each dual pair.

For $\{3,5,5/2\}$ the type of H_α is $\{\frac{20}{1,9},5\}$. Vertices that collapse at $\alpha=0,1$ are $2,2$ respectivelly. The group G is of order 1200. For the basic tetrahedron $\{v_0{=}[(\phi^2,1,-\phi^{-2},0)], v_1{=}[(\phi,\phi^{-1},0,0)], v_2{=}[(2+\phi,1,0,\phi^{-1})], v_3{=}(1,0,0,0)\}$, the generators S_1, S_2 of G and the reflection matrices that extend the regular case are

$$\frac{1}{2}\begin{pmatrix} \phi & 1 & 0 & -\phi^{-1} \\ 1 & -1 & -1 & 1 \\ 0 & 1 & \phi^{-1} & \phi \\ \phi^{-1} & -1 & \phi & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \phi & -1 & \phi^{-1} \\ 0 & -1 & -\phi^{-1} & \phi \\ 0 & -\phi^{-1} & -\phi & -1 \end{pmatrix}, \pm\frac{1}{2\sqrt{2}}\begin{pmatrix} 1-2\phi & -\phi & \phi^{-1} & 0 \\ -\phi & 2 & -\phi^{-1} & 1 \\ \phi^{-1} & -\phi^{-1} & \phi^{-1} & \phi^2 \\ 0 & 1 & \phi^2 & -\phi^{-2} \end{pmatrix};$$

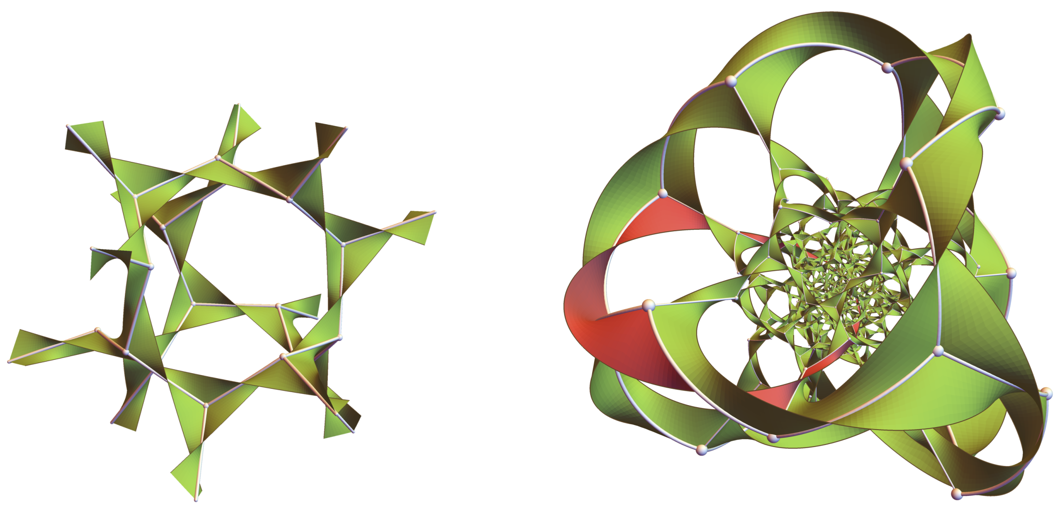

Figure 8: The part of $H_0(\{5,3,3\})$ that touches the (central) dodecahedral facet of $\{5,3,3\}$, and an outside view of it with a face highlighted in red.

with respective basic vertices for the regular cases: $[(1-2\phi+2\sqrt{2},-\phi,\phi^{-1},0)]$ and $[(1-2\phi-2\sqrt{2},-\phi,\phi^{-1},0)]$.

For $\{5,5/2,5\}$ the type of H_α is $\{\frac{15}{1,4},5/2\}$. Vertices that collapse at $\alpha=0,1$ are $12,12$ respectively. The group G is of order 7200. For the basic tetrahedron $\{v_0=(1,0,0,0), v_1=[(2+\phi,1,0,\phi^{-1})], v_2=[(2,1,-\phi^{-1},-\phi^{-2})], v_3=[(\phi,1,0,-\phi^{-1})]\}$, the generators S_1,S_2 and the reflection matrices that extend the regular case are

$$\frac{1}{2}\begin{pmatrix} \phi & 0 & \phi^{-1} & -1 \\ 1 & \phi^{-1} & 0 & \phi \\ 0 & -\phi & 1 & \phi^{-1} \\ \phi^{-1} & -1 & -\phi & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -\phi^{-1} & -\phi \\ 0 & \phi^{-1} & -\phi & 1 \\ 0 & -\phi & -1 & -\phi^{-1} \end{pmatrix}, \pm\frac{1}{2}\begin{pmatrix} \phi & 1 & 0 & -\phi^{-1} \\ 1 & -1 & 1 & 1 \\ 0 & 1 & -\phi^{-1} & \phi \\ -\phi^{-1} & 1 & \phi & 0 \end{pmatrix};$$

with respective basic vertices for the regular cases: $[(2+\phi,1,0,-\phi^{-1})]$ and $[(2-\phi,-1,0,\phi^{-1})]$.

For $\{5,3,5/2\}$ the type of H_α is $\{\frac{12}{1,5},3\}$; it remains a polyhedron at $\alpha=0,1$. The group G is of order 144. For the basic tetrahedron $\{v_0=(1,0,0,0), v_1=[(2+\phi,1,0,\phi^{-1})], v_2=[(2\phi,\phi,-1,-\phi^{-1})], v_3=[(1,1,1,-1)]\}$, the generators S_1,S_2 of G and the reflection matrices that extend the regular case are

$$\frac{1}{2}\begin{pmatrix} \phi & -\phi^{-1} & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & -\phi & -1 & \phi^{-1} \\ \phi^{-1} & 0 & -1 & -\phi \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \pm\frac{1}{2\sqrt{6}}\begin{pmatrix} 3\phi & \phi^{-2} & \phi^{-2} & -\phi^{-2} \\ \phi^{-2} & -3-\phi & 1 & 2\phi^{-1} \\ \phi^{-2} & 1 & -2\phi^{-1} & 3+\phi \\ -\phi^{-2} & 2\phi^{-1} & \phi-3 & 1 \end{pmatrix};$$

with respective basic vertices for the regular cases: $[(3+3\phi+2\sqrt{6}\phi,\phi^{-1},\phi^{-1},-\phi^{-1}))]$ and $[(3+3\phi-2\sqrt{6}\phi,\phi^{-1},\phi^{-1},-\phi^{-1}))]$. See Figure 9.

For $\{3,3,5/2\}$ the type of H_α is $\{\frac{30}{7,13},3\}$. Vertices that collapse at $\alpha=0,1$ are $4,1$ respectivelly. The group G is of order 1440. For the basic tetrahedron

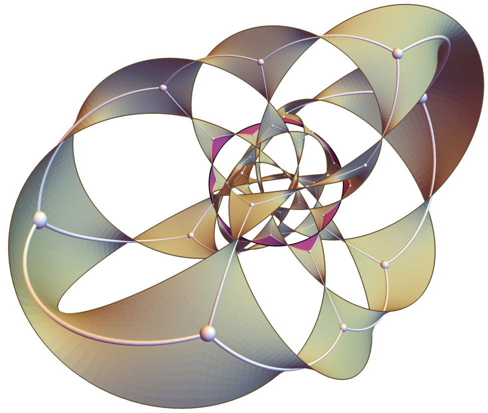

Figure 9: One regular $H_\alpha(\{5,3,5/2\})$ with a highlighted face at the ecuator.

$\{v_0{=}[\,(1,1,1,-1)\,], v_1{=}[\,(2\phi,\phi,-1,-\phi^{-1})\,], v_2{=}[\,(2\phi-1,-1,-1,1-2\phi)\,], v_3{=}[\,(\phi^{-1},-1,\phi,2)\,]\}$, the generators S_1, S_2 and the reflection matrices that extend the regular case are

$$\frac{1}{2}\begin{pmatrix} -\phi^{-1} & \phi & 0 & -1 \\ -1 & 0 & \phi & \phi^{-1} \\ -\phi & -\phi^{-1} & -1 & 0 \\ 0 & -1 & \phi^{-1} & -\phi \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -\phi^{-1} & \phi & -1 \\ \phi & 0 & -\phi^{-1} & -1 \\ 1 & 1 & 1 & 1 \\ \phi^{-1} & -\phi & 0 & 1 \end{pmatrix}, \pm\frac{1}{2\sqrt{2}}\begin{pmatrix} \phi & 1-2\phi & 0 & \phi^{-1} \\ 1-2\phi & -\phi & \phi^{-1} & 0 \\ 0 & \phi^{-1} & \phi & 2\phi-1 \\ \phi^{-1} & 0 & 2\phi-1 & -\phi \end{pmatrix};$$

with respective basic vertices for the regular cases: $[(1-\phi+\sqrt{2},-\phi+\sqrt{2},\sqrt{2},-1+2\phi-\sqrt{2})]$ and $[(1-\phi-\sqrt{2},-\phi-\sqrt{2},-\sqrt{2},-1+2\phi+\sqrt{2})]$.

For $\{3,5/2,5\}$ the type of H_α is $\{\frac{20}{3,7},5/2\}$. Vertices that collapse at $\alpha=0,1$ are $2,2$ respectivelly. The group G is of order 1200. For the basic tetrahedron $\{v_0{=}[\,(1,1,1,-1)\,], v_1{=}[\,(2\phi,\phi,-1,1-\phi)\,], v_2{=}[\,(2\phi-1,-1,-1,1-2\phi)\,], v_3{=}[\,(\phi,-1,0,1-\phi)\,]\}$, the generators S_1, S_2 and the reflection matrices that extend the regular case are

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & \phi & \phi^{-1} \\ -1 & -1 & 1 & -1 \\ -1 & -\phi^{-1} & 0 & \phi \\ -1 & \phi & \phi^{-1} & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -\phi^{-1} & 0 & -\phi \\ -1 & 1 & 1 & -1 \\ 1 & \phi & -\phi^{-1} & 0 \\ -1 & 0 & -\phi & -\phi^{-1} \end{pmatrix}, \pm\frac{\sqrt{2+\phi}}{10}\begin{pmatrix} 2-4\phi & 3\phi-4 & 2\phi-1 & 3-\phi \\ 3\phi-4 & 3\phi-4 & \phi+2 & -\phi-2 \\ 2\phi-1 & \phi+2 & 3-\phi & 6-2\phi \\ 3-\phi & -\phi-2 & 6-2\phi & 2\phi-1 \end{pmatrix};$$

with basic vertices for the regular cases: $[(\sqrt{2+\phi}-1,2\phi+\sqrt{2+\phi},\phi+\sqrt{2+\phi},1-\phi-\sqrt{2+\phi})]$ and $[(-\sqrt{2+\phi}-1,2\phi-\sqrt{2+\phi},\phi-\sqrt{2+\phi},1-\phi+\sqrt{2+\phi})]$, respectivelly. See Figure 10.

For $\{5/2,5,5/2\}$ the type of H_α is $\{\frac{15}{2,7},5\}$. Vertices that collapse at $\alpha=0,1$ are $12,12$ respectivelly. The group G is of order 7200. For the basic tetrahedron $\{v_0{=}[\,(1,1,1,-1)\,], v_1{=}[\,(2\phi,\phi,-1,-\phi^{-1})\,], v_2{=}[\,(\phi+2,1,0,\phi^{-1})\,], v_3{=}[\,(\phi^{-1},\phi,0,1)\,]\}$, the generators S_1, S_2 and the reflection matrices that extend the regular case are

$$\frac{1}{2}\begin{pmatrix} 1 & \phi & -\phi^{-1} & 0 \\ -\phi^{-1} & 0 & -1 & -\phi \\ 0 & -\phi^{-1} & -\phi & 1 \\ \phi & -1 & 0 & -\phi^{-1} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi & \phi^{-1} \\ 1 & \phi & -\phi^{-1} & 0 \\ -1 & 1 & 1 & -1 \\ -1 & \phi^{-1} & 0 & \phi \end{pmatrix}, \pm\frac{1}{2}\begin{pmatrix} 0 & \phi & -1 & -\phi^{-1} \\ \phi & \phi^{-1} & 1 & 0 \\ -1 & 1 & 1 & 1 \\ -\phi^{-1} & 0 & 1 & -\phi \end{pmatrix};$$

with respective basic vertices for the regular cases: $[(1,\phi,\phi^{-1},0)]$ and $[(\phi^{-1},-1,\phi,-2\phi)]$.

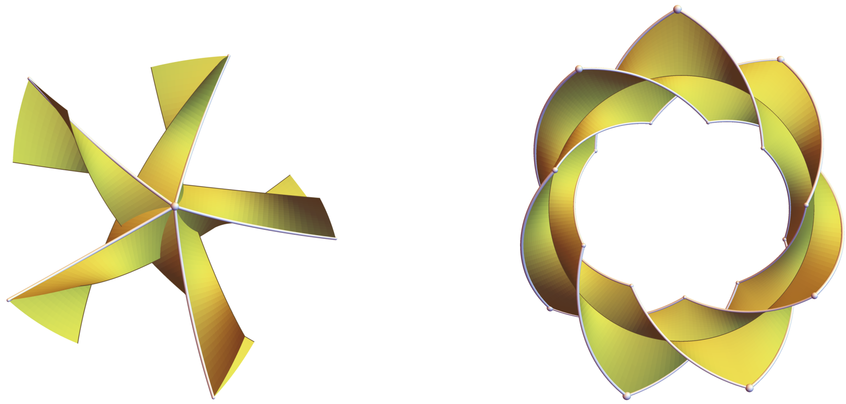

Figure 10: Around a vertex and a face of $H_{3/5}(\{3,5/2,5\})$ of type $\{\frac{20}{3,7},5/2\}$.

Summary of chiral polyhedra from the regular polytopes

Polytope $\mathcal{T}$	Type of $H_\alpha(\mathcal{T})$	$\#(G)$	$[G:\Gamma^+]$	Colapses at $\alpha=(0,1)$
$\{3,3,3\}$	$\{\frac{5}{1,2},3\}$	60	1	$(4,4)$
$\{4,3,3\}$	$\{\frac{8}{1,3},3\}$	48	4	$(1,2)$
$\{3,4,3\}$	$\{\frac{12}{1,5},4\}$	192	3	$(2,2)$
$\{5,3,3\}$	$\{\frac{30}{1,11},3\}$	1440	5	$(1,4)$
$\{3,5,5/2\}$	$\{\frac{20}{1,9},5\}$	1200	6	$(2,2)$
$\{5,5/2,5\}$	$\{\frac{15}{1,4},5/2\}$	7200	1	$(12,12)$
$\{5,3,5/2\}$	$\{\frac{12}{1,5},3\}$	144	50	$(1,1)$
$\{3,3,5/2\}$	$\{\frac{30}{7,13},3\}$	1440	5	$(4,1)$
$\{3,5/2,5\}$	$\{\frac{20}{3,7},5/2\}$	1200	6	$(2,2)$
$\{5/2,5,5/2\}$	$\{\frac{15}{2,7},5\}$	7200	1	$(12,12)$

The groups G given in the above table refer to the isometry groups of the chiral polyhedra $H_\alpha(\mathcal{T})$. As we showed, these polyhedra are combinatorially regular, and thus the groups G are their rotational subgroups. For each polytope $\mathcal{T}$, we shall denote by $\mathcal{P}_\mathcal{T}$ the (regular) abstract polytope $H_\alpha(\mathcal{T})$. In other words, $H_\alpha(\mathcal{T})$ is a realisation in $\mathbb{E}^4$ of $\mathcal{P}_\mathcal{T}$, and while for all α the elements of $\Gamma^+(\mathcal{P}_\mathcal{T})$ are symmetries of $H_\alpha(\mathcal{T})$, only for specific values of α we get that all the elements of the full automorphism group $\Gamma(\mathcal{P}_\mathcal{T})$ are symmetries of $H_\alpha(\mathcal{T})$.

One can observe that when $\mathcal{T}=\{3,3,3\}$ we obtain that $\mathcal{P}_\mathcal{T}$ is a regular polyhedron of Schläfli type $\{5,3\}$ with $\Gamma^+(\mathcal{P}_\mathcal{T})$ having 60 elements. Moreover, the isometry $R\notin\Gamma^+(\mathcal{P}_\mathcal{T})$ and thus, $\mathcal{P}_\mathcal{T}$ is a dodecahedron, implying that $H_\alpha(\mathcal{T})$ is a chiral realisation of the dodecahedron in $\mathbb{E}^4$.

The regular polyhedra $\mathcal{P}_{\mathcal{T}}$ arising from the polytopes $\mathcal{T} = \{4,3,3\}$, $\mathcal{T} = \{3,4,3\}$ and $\mathcal{T} = \{5,3,\frac{5}{2}\}$ can be found in Michael Hartely's Atlas of Regular Abstract Polytopes ([17]). In fact, they correspond to the polytopes $\{8,3\} * 96$, $\{12,4\} * 384e$ and $\{12,3\} * 288$ of such atlas, respectively. In the atlas, the interested reader can find different permutation representations of the groups of these polyhedra.

The groups of the polyhedra $\mathcal{P}_{\mathcal{T}}$ corresponding to the polytopes $\mathcal{T} = \{5,3,3\}$, $\mathcal{T} = \{3,5,\frac{5}{2}\}$, $\mathcal{T} = \{3,3,\frac{5}{2}\}$ and $\mathcal{T} = \{3,\frac{5}{2},5\}$ can be derived from Marston Conder's list of regular and chiral maps [6], were one can find generators and relations for the groups of each $\mathcal{P}_{\mathcal{T}}$. In fact, the regular polyhedra arising from $\mathcal{T} = \{5,3,3\}$ and $\mathcal{T} = \{3,3,\frac{5}{2}\}$ are isomorphic of type $\{30,3\}$ and are listed as $R97.9$. The polyhedra from the polytopes $\mathcal{T} = \{3,5,\frac{5}{2}\}$ and $\mathcal{T} = \{3,\frac{5}{2},5\}$ are of type $\{20,5\}$ and also isomorphic, listed as $R.151.11$.

Finally, in the cases when the symmetry group G of the chiral polyhedra $H_\alpha(\mathcal{T})$ has size $7,200$, we note that G is precisely the orientation preserving subgroup of the Coxeter group $[3,3,5]$. In the two choices of $\mathcal{T}$ where this occurs, the symmetry group of the regular member of the family $\{H_\alpha(\mathcal{T})\}_\alpha$ is $\langle G, R_0\rangle$, where the symmetry R_0 maps the base flag of $H_\alpha(\mathcal{T})$ to its 0-adjacent flag. Since S_1 is a twist, R_0 must be a half-turn, and hence it preserves orientation. According to the classification in [13, Section 21], there is no finite group of orientation preserving isometries of $\mathbb{E}^4$ that contains the orientation preserving subgroup of $[3,3,5]$ as a subgroup of index 2. Hence in those two cases $R_0 \in G$, and the polyhedra are non-orientably regular. The chiral members of these family are orientable double covers of the regular ones, which is the same phenomenon that occurs with the chiral polyhedra in $\mathbb{E}^3$ in the family $P_3(c,d)$ (see [26]).

The 3D pictures and computations were done with *Mathematica* under UNAM's license. The interested reader can find the source code for computing the matrices for each of the polyhedra, together with more pictures of them in the website `https://www.matem.unam.mx/~roli/PetrieCoxeter.html`

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References

[1] J. L. Arocha, J. Bracho, L. Montejano, Regular projective polyhedra with planar faces, I, *Aequationes Math.* **59** (2000), 55–73.

[2] J. Bracho, Regular projective polyhedra with planar faces, II, *Aequationes Math.* **59** (2000), 160–176.

[3] J. Bracho, I. Hubard, D. Pellicer, A finite chiral 4-polytope in $\mathbb{E}^4$, *Discrete Comput. Geom.* **52** (2014), 799–805.

[4] J. Bracho, I. Hubard, D. Pellicer, Realising equivelar toroids of type $\{4,4\}$, *Discrete Comput. Geom.* **55** (2016), 934–954.

[5] H. Burgiel, D. Stanton, Realizations of regular abstract polyhedra of types $\{3,6\}$ and $\{6,3\}$. *Discrete Comput. Geom.* **24** (2000), 241–255.

[6] Marston Conder, Regular maps and hypermaps of Euler characteristic -1 to -200, *J. Combinatorial Theory, Series B*, **99** (2009), 455-459. `https://www.math.auckland.ac.nz/~conder/OrientableRegularMaps301.txt`

[7] J. H. Conway, H. Burgiel, C. Goodman-Strauss, *The symmetries of things*, A K Peters, Ltd., Wellesley, MA, 2008.

[8] H. S. M. Coxeter, Regular skew polyhedra in three and four dimensions, and their topological analogues, *Proc. London. Math. Soc.* **43** (1937), 33–62.

[9] H. S. M. Coxeter, *Non-Euclidean Geometry* (5th ed.), University of Toronto Press, Toronto, 1968.

[10] H. S. M. Coxeter, *Regular Polytopes* (3rd edition), Dover, New York, 1973.

[11] A. W. M. Dress, A combinatorial theory of Grünbaum's new regular polyhedra. I. Grünbaum's new regular polyhedra and their automorphism group, *Aequationes Math.* **23** (1981), 252–265.

[12] A. W. M. Dress, A combinatorial theory of Grünbaum's new regular polyhedra. II. Complete enumeration, *Aequationes Math.* **29** (1985), 222–243.

[13] P. Duval, *Homographies, quaternions and rotations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[14] Euclid, *Euclid's Elements*, Green Lion Press, Santa Fe, NM, 2002. All thirteen books complete in one volume, The Thomas L. Heath translation, Edited by Dana Densmore.

[15] C. Garner, Regular skew polyhedra in hyperbolic three-space, *Canad. J. Math.* **19** (1967), 1179–1186.

[16] B. Grünbaum, Regular polyhedra—old and new, *Aequationes Math.* **16** (1977), 1–20.

[17] Hartley, M. An Atlas of Small Regular Abstract Polytopes. *Period Math Hung* **53** (2006), 149–156. `https://www.abstract-polytopes.com/atlas/index.html`

[18] I. Hubard, Two-orbit polyhedra from groups. *European Journal of Combinatorics*. **31** (2010), 943–960.

[19] I. Hubard, E. Schulte, A. I. Weiss, Petrie–Coxeter maps revisited, *Beiträge Algebra Geom.* **47** (2006), 329–343.

[20] P. McMullen, Four-dimensional regular polyhedra, *Discrete Comput. Geom.* **38** (2007), 355–387.

[21] P. McMullen, Regular polytopes of nearly full rank, *Discrete Comput. Geom.* **46** (2011), 660–703.

[22] P. McMullen, E. Schulte, Self-dual regular 4-polytopes and their Petrie–Coxeter–polyhedra, *Results Math.,* **12** (1987), 366–375.

[23] P. McMullen, E. Schulte , *Abstract Regular Polytopes* (1st ed.), Cambridge University Press, (December 2002).

[24] B. Monson, A. I. Weiss, Realizations of regular toroidal maps of type $\{4, 4\}$, *Discrete Comput. Geom.* **24** (2000), 453–465.

[25] D. Pellicer, Cleaved abstract polytopes, *Combinatorica* **38** (2018), 709–737.

[26] D. Pellicer, A. I. Weiss, Combinatorial structure of Schulte's chiral polyhedra, *Discrete Comput. Geom.* **44** (2010), 167–194.

[27] L. Poinsot, Mémoire sur les polygones et les polyèdres, *J. École Polytech.* **10** (1810), 16–48.

[28] E. Schulte, Chiral polyhedra in ordinary space, I, *Discrete Comput. Geom.* **32** (2004), 55–99.

[29] E. Schulte, Chiral polyhedra in ordinary space, II, *Discrete Comput. Geom.* **34** (2005), 181–229.

[30] E. Schulte and A. I. Weiss, Chiral polytopes, *Applied Geometry and Discrete Mathematics (The "Victor Klee Festschrift")*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **4** (1991), (eds. P. Gritzmann and B. Sturmfels), 493–516.