A Charatheodory Theorem for Closed Semispaces.

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Abstract

Following the spirit of Hadwiger´s transversal theorem we stablish a Caratheodory type theorem for closed halfspaces in which a combinatorial structure is required.

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1 Introduction

Let $F = \{x_1, ..., x_m\}$ be a finite collection of points in S^n. The classic Caratheodory Theorem (see [2]) asserts that F is contained in an open semisphere of S^n if and only if every subset of $n + 2$ points of F is contained in an open semisphere of S^n. The purpose of this paper is to study the same problem but for closed semispheres.

First of all note that a square inscribed in S^1 has the property that any three of its vertices is contained in a closed semicircle of S^1 although the whole square is not contained in a closed semicircle of S^1. As in Hadwiger´s Transversal Theorem [4], it is required an extra combinatorial hypothesis, in this case, the order. That is, suppose $F = \{x_1, ..., x_m\}$ is an ordered collection of points in S^1. Then, F is contained in a closed semicircle of S^1 if and only if every subset A of three points of F is contained in an open semicircle C of S^1 in such a way that the order of A is consistent with the order of the semicircle C. Note that this hypothesis is not satisfied for F, when F consists of the four vertices of an inscribed square. In higher dimensions we follow the spirit of Goodman Pollack Transversal Theorem [3], but instead of using the notion of order type it is more natural to use the notion of separoid developed in [1]. Our main tool is the Borsuk Ulam Theorem [6].

Of course, the following version of Caratheodory Theorem, for closed semispheres, follows straightforward from the $n-1$ dimensional version: if $F = \{x_1, ..., x_m\}$ is a finite collection of points in S^n, then F is contained in a closed semisphere of S^n if and only if there is a point $\Omega \in S^n$ with the property that every subset of $n+1$ points of F is contained in a closed semisphere of S^n which does not contain Ω. Our results are of a complete different character.

2 Separoids

Definition *A separoid S is a set together with a binary relation on its subsets, denoted $\mid$ and called the separation relation, that satisfies the following properties for $\alpha, \beta \subset S$:*

$$\begin{array}{rl} \textbf{i)} & \alpha \mid \beta \Rightarrow \beta \mid \alpha, \\ \textbf{ii)} & \alpha \mid \beta \Rightarrow \alpha \cap \beta = \emptyset, \\ \textbf{iii)} & \alpha \mid \beta \text{ and } \alpha' \subset \alpha \Rightarrow \alpha' \mid \beta. \end{array}$$

When $\alpha \mid \beta$ we call it a separation of S, or say that "α is separated from β". If S further satisfies that $\emptyset \mid S$ then it is called acyclic.

Example 1. Let $a_0, \ldots, a_r$ be points in some Euclidean (or affine) space. They define an acyclic separoid $S(a_0, \ldots, a_r)$ whose underlying set is $\{a_0, \ldots, a_r\}$, and two subsets α and β are separated if there exists a hyperplane that leaves α on one side and β on the other. That is,

$$\alpha \mid \beta \Leftrightarrow cc(\alpha) \cap cc(\beta) = \emptyset,$$

where cc denotes the convex hull. These separoids will be called *point separoids.*

Example 2. Let $a_0, \ldots, a_r$ be points in the sphere S^n. They define a separoid $R(a_0, \ldots, a_r)$ whose underlying set is $\{a_0, \ldots, a_r\}$, and two subsets α and β are separated if there exists a hyperplane through the origin that leaves α on one side and β on the other. These separoids will be called real *separoids.*

Example 3. Let $\mathcal{F} = \{A_0, \ldots, A_r\}$ be a family of convex sets in some Euclidean space. It defines a separoid $S(\mathcal{F})$ with $\mathcal{F}$ as underlying set and, again, with strict separation of subfamilies by hyperplanes as separation relation. If the convex sets are compact, or bounded, then $S(\mathcal{F})$ is acyclic. It is proved in [1] that any acyclic separoid is isomorphic to one of these.

Example 4. Given an oriented matroid, it naturally defines a separoid over the same base set by declaring that the negative part of each covector is separated from its positive part. The topes are then the maximal separations, so that the separoid has all the information of the oriented matroid. Hence separoids generalize oriented matroids. Observe that the oriented matroid is acyclic if and only if its separoid is acyclic.

Definition. Let S and T be separoids, and let $\dot{f} : S \to T$ be a function. Then f is a *comorphism* if $\alpha \mid \beta$ in $S \Rightarrow f(\alpha) \mid f(\beta)$ in T.

3 The Theorems

Our main theorem, whose proof requires the Borsuk Ulam Theorem, is the following one

Theorem 1. Let $F = \{x_1, ..., x_m\}$ be a finite collection of points in S^n. Then F is contained in a closed semisphere of S^n if and only if there is a collection of points $\{y_1, ..., y_m\} \subset R^n$ such that the function $(x_i \to y_i)$ is a separoid comorphism from the real separoid $R\{x_1, ..., x_m\}$ into the point separoid $S\{y_1, ..., y_m\}$.
Proof. Suppose not, suppose that every hyperplane H through the origin separates some subset of F from some other subset of F. If $v \in S^n$, denote by H_v the closed semispace $\{x \in R^{n+1} \mid x * v \leq 0\}$ orthogonal to v. where $*$ denotes the interior product in euclidean space R^{n+1}

Observe that

$$p(v) = \sum_{i=0}^{r} d(x_i, H_v),$$

where $d(x_i, H_v)$ denotes the infimum of the distances of x_i and points in H_v, is never zero and that it depends continuously on $v \in U$. Therefore, we have a continuous map

$$f : S^n \to \mathbf{R}^n$$

$$f(v) = \sum_{i=0}^{m} \frac{d(x_i, H_v)}{p(v)} y_i.$$

By Borsuk-Ulam Theorem (see [6]), there exists $v_0 \in S^n$ for which $f(v_0) = f(-v_0)$.

Let $\phi : \{x_1, ..., x_m\} \to \{y_1, ..., y_m\}$ be the map given by $\phi(x_i) = y_{i.}$ and let $\alpha = \{x_i \in \mathcal{F} \mid x_i \subset H_{v_o}\}$ and $\beta = \{x_j \in \mathcal{F} \mid x_j \subset H_{-v_o}\}$. By definition, α is separated from β and hence, since ϕ is by hypothesis a separoid comorphism, $\phi(\alpha)$ is separated from $\phi(\beta)$. On the other hand, note that $f(v_o)$ is a convex combination of the points $\phi(\beta)$ and also $f(-v_o)$ is a convex combination of the points $\phi(\alpha)$. Hence, $f(v_o) = f(-v_o) \in cc(\phi(\alpha)) \cap cc(\phi(\beta)) \neq \emptyset$, which is a contradiction.

A straightforward application of Kirchberger's Theorem [5] gives the following corollary

Theorem 2. Let $F = \{x_1, ..., x_m\}$ be a finite collection of points in S^n. Then F is contained in a closed semisphere of S^n if and only if there is a collection of points $\{y_1, ..., y_m\} \subset R^n$ such that for every subset $\{x_{i_1}, ..., x_{i_{n+2}}\}$ of F with $n + 2$ points, the function $(x_i \to y_i)$ is a separoid comorphism from the real separoid $R\{x_{i_1}, ..., x_{i_{n+2}}\}$ into the point separoid $S\{y_{i_1}, ..., y_{i_{n+2}}\}$.

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