# A CLASSIFICATION THEOREM FOR ZINDLER CARROUSELS

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ABSTRACT. The purpose of this paper is to give a complete classification of Zindler Carrousels with five chairs. This classification theorem gives enough evidence to show the non existence of figures, different from the disk, that float in equilibrium in every position for the corresponding perimetral densities.

## 1. Introduction and Carrousels

Zindler Carrousels are analytic dynamical systems. The initial motivation for their study was the following. Auerbach [1] proved that Zindler curves bound figures, different from the disk, that float in equilibrium in every position for the density 1/2 . In general, for figures that float in equilibrium in every position some remarkable facts follow, namely, that the floating chords have constant length; that the curve of their midpoints has the corresponding chords as tangents, and that these chords divide the perimeter in a fixed ratio $\alpha$ (the perimetral density). Suppose that $\alpha$ is rational. Then, for every point $p$ in the boundary of one of this figures, we have an inscribed equilateral $n$-gon which moves, as a linkage with rigid rods, as $p$ moves along the boundary, in such a way that the midpoints of the sides move parallel to them. So, this is the main motivation for the following definition.

A *Carrousel* (with $n$ chairs) is a system which consists of $n$ smooth (not necessarily closed) curves $\{\beta_1(t), \beta_2(t), ..., \beta_n(t)\}$ in $\mathbf{R}^2$ satisfying the following properties for every $t \in \mathbf{R}$ and for all $i = 1, ..., n$, where $\beta_{i+n}(t) = \beta_i(t)$: 1) The length of the interval with end points $\beta_i(t)$ and $\beta_{i+1}(t)$, $|\beta_{i+1}(t) - \beta_i(t)|$, is a non-zero constant 2) The curve of midpoints, $m_i(t) = \frac{\beta_i(t)+\beta_{i+1}(t)}{2}$, of the segments from $\beta_i(t)$ to $\beta_{i+1}(t)$, has tangent vector, $m_i'(t)$, parallel to $\beta_{i+1}(t) - \beta_i(t)$.

A carrousel with $n$ chairs $\{\beta_1(t), ..., \beta_n(t)\}$ is a *Zindler carrousel* if all the curves $\beta_i(t)$ are reparametrizations of the same closed curve.

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Observe as an example, that the circle yields a Zindler carrousel with $n$ chairs, because we can inscribe in it an equilateral $n$-gon such that, when rotating, its vertices describe the original circle and the midpoints of its sides describe a smaller concentric circle. Zindler curves studied in [8] are essentially Zindler carrousels with two chairs, which, according to [5], are in one to one correspondence with curves of constant width.

The purpose of this paper is to give a complete classification of Zindler Carrousels with five chairs. This classification theorem gives enough evidence to show the non existence of figures, different from the disk, that float in equilibrium in every position for perimetral densities $\frac{1}{5}$ and $\frac{2}{5}$. Although the main properties of carrousels were studied in [4], for completeness we summarize them, in this section, without proofs.

**Definition 1.** *Let $\Phi$ be a figure (region bounded by a simple closed curve). A chord system $\{C(p)\}$ for $\Phi$ is a continuous selection of an oriented chord $C(p)$, starting at $p$, for every point $p$ in the boundary of $\Phi$.*

There are three natural kinds of chord systems for a figure $\Phi$:

1) The system $\{C_a(p)\}$ of chords which divide the area in a fixed ratio $\rho$.
2) The system $\{C_p(p)\}$ of chords which divide the perimeter in a fixed ratio $\alpha$.
3) The system $\{C_l(p)\}$ of chords of constant length $\tau$.

Note that for non-convex figures the chord system $\{C_a(p)\}$ is not necessarily well defined, for all $\rho$.

Let $\Phi$ be a figure of area $A$ and let us suppose that the chord system which divides the area of $\Phi$ in a fixed ratio $\rho$, $\{C_a(p)\}$, is well defined. Let $G$ be the mass center of $\Phi$ and $g(p)$ the mass center of the regions of $\Phi$, bounded by $C_a(p)$, of area $\rho A$. Then, according to Archimedes Law, we have the following definition

**Definition 2.** *We say that the figure $\Phi$ floats in equilibrium in a given position $p$, if the line through $G$ and $G(p)$ is orthogonal to $C(p)$. A figure $\Phi$ that floats in equilibrium in every position will be called an Auerbach figure.*

In 1938 Auerbach [1] proved the following theorem;

**THEOREM 1.** *A figure $\Phi$ is an Auerbach figure if and only if the system of chords $\{C_a(p)\}$ is well defined and it is also of the type $\{C_l(p)\}$ of constant length.*

For the prove he used the following facts which will be used later:

**A)** If a system of interior non-concurrent chords, $\{C_i(p)\}$, is of any of the two types $i \in \{a, p, l, \}$, then it is also of the third type.

**B)** The area $A(p)$, of the region of a figure $\Phi$, left to the right by the chord $C(p)$ of a chord system $\{C(p)\}$, is constant if and only if every chord $C(p)$ is tangent to the curve described by the midpoints of $C(p)$.

This motivates the following definition.

**Definition 3.** *If $\Phi$ is an Auerbach figure for the density $\rho$, we say that $\Phi$ has perimetral density $\alpha$ if the chord system which divides the area of $\Phi$ in the ratio $\rho$, $\{C_a(p)\}$, is well defined and divides the perimeter of $\Phi$ in a fixed ratio $\alpha$.*

In what follows, when studying Auerbach figures, we will classify them according with their perimetral density.

**Definition 4.** *Let $\alpha \in \mathbf{R}$, $0 < \alpha < 1$. We say that a figure $\Phi$ is an $\alpha-$Zindler curve, if the system of chords $\{C_p(p)\}$, which divides the perimeter in a fixed ratio $\alpha$, is also a $\{C_l(p)\}$ system of fixed length $\tau$.*

Observe, that the classic Zindler curves [8] are $\frac{1}{2}$-Zindler. The next two theorems relates $\alpha$-Zindler curves, Zindler Carrousels and Auerbach figures.

**THEOREM 2.** *$\{\beta_1, ..., \beta_n\}$ is a Zindler Carrousel with $n$ chairs if and only if there exists an $\alpha = q/n$ (with $q/n$ an irreducible fraction ), for some $q \in \mathbf{Z}$, $1 \leq q \leq \frac{n}{2}$, such that each $\beta_i(t)$ is an $\alpha-$Zindler curve.*

**THEOREM 3.** *Let $\gamma$ be a closed smooth curve such that the system of chords of fixed perimeter $\alpha$ is interior. Then $\gamma$ is an $\alpha-$Zindler curve if and only if the figure bounded by the curve $\gamma$ is an Auerbach curve for some density $\rho$.*

Note now that the existence of Zindler carrousels with interior chords give rise to 2-dimensional bodies that float in equilibrium in every position.

Given a carrousel $\{\beta_1(t), ..., \beta_n(t)\}$, by the first carrousel law [4], $\beta'_{i+1}(t)$ is a reflection of $\beta'_i(t)$ along the line generated by $\beta_{i+1}(t) - \beta_i(t)$. So we may assume that all the curves $\beta_i(t)$ are parametrized by arc length and furthermore that $|\beta_{i+1}(t) - \beta_i(t)| = 2$.

Let $\alpha_i(t)$ denote the angle between the vectors $\beta'_i(t)$ and $\beta_{i+1}(t) - \beta_i(t)$, and let $\theta_i(t)$ be the angle between the $x$-axis and the vector $\beta_{i+1}(t) - \beta_i(t)$, then by the second carrousel law [4], $\theta'_i(t) = \sin(\alpha_i(t))$.

Next theorem exhibits the differential equations of carrousels, where $x_i\,(t)$ denotes the angle between the vectors $\beta_{i+1}(t)-\beta_i(t)$ and $\beta_{i-1}(t)-\beta_i(t)$.

**THEOREM 4.** *Let $\{\beta_1(t),...,\beta_n(t)\}$ be a carrousel with n-chairs. Then, the interior angles $x_i(t)$, $i=1,...,n$, satisfy the following system of constrained differential equations*

$$x_i'(t)=\sin(\alpha_{i-1}(t))-\sin(\alpha_i(t)), \tag{1}$$

*If n is odd then*

$$\alpha_i(t)=x_{i+2}(t)+x_{i+4}(t)+...+x_{i+(n-1)}(t)-(\frac{k-1}{2})\pi, \tag{2}$$

*where k is the integer number such that $\sum_{i=1}^n x_i(0)=k\pi$.*

*Conversely, if n is odd and we have functions $x_i(t)$, $i=1,...,n$, satisfying the system of differential equations (1), (2); and such that the initial conditions $(x_1(0),...,x_n(0))$ are the interior angles of an equilateral n-gon with sides of length 2. Then, there exists a carrousel of n chairs $\{\beta_1(t),...,\beta_n(t)\}$, with the property that $x_i(t)$ is the angle between $\beta_{i+1}(t)-\beta_i(t)$ and $\beta_{i-1}(t)-\beta_i(t)$.*

The following two corollaries will be used in the next section.

**COROLLARY 1.** *Let $X(0)$ be an n-gon with interior angles $(x_1(0),...,x_n(0))$ n odd. Then there exist a unique carrousel $\{\beta_1(t),...,\beta_n(t)\}$ up to orientation, with initial condition $X(0)$.*

**COROLLARY 2.** *Let $\{\beta_1(t),...,\beta_n(t)\}$ be a carrousel with n-chairs, n odd. If there exists $t_0\in\mathbf{R}$ such that $x_i(t_0)=x_{i+1}(0)$ $(i=1,\dots,n)$, then the curves $\beta_1(t)$,...,$\beta_n(t)$ are congruent.*

An interesting propertie of carrousels is given by the following theorem

**THEOREM 5.** *Let $\{\beta_1(t),...,\beta_n(t)\}$ be a carrousel with n-chairs, n odd. Let $X(t)$ be the n-gon with vertices $\{\beta_1(t),...,\beta_n(t)\}$. Then, the area $\mathcal{A}(t)$ of $X(t)$ is constant and the mass center $\mathcal{H}(t)$ of $X(t)$ is a fixed point.*

## 2. Carrousels with Five Chairs

For the study of the carrousel with 5 chairs, we shall consider the space, $\mathcal{P}^5$, of all equilateral pentagons in the plane, one of whose sides is the distinguished interval $[(-1,0),(1,0)]$, and all the other sides have length two.

If $X \in \mathcal{P}^5$ is a pentagon, we can write it as $X = (z_1, ..., z_5)$, where $z_j = e^{ix_j}$ is a complex number and the five real numbers $(x_1, ..., x_5)$ are the interior angles of the equilateral pentagon $X$. Clearly, the $x_i$'s satisfy the following equation:

$$u(0) + u(\pi - x_2) + u(2\pi - (x_2 + x_3)) + ...+ \\ +u(4\pi - (x_2 + x_3 + x_4 + x_5)) = 0.$$

where $u(\theta) = (\cos\theta, \sin\theta)$.

We know [6] that $\mathcal{P}^5$ is an oriented surface of genus 4, embedded in $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ where $\mathbf{S}^1$ is the sphere of dimension 1. So, we can think of Euclidean space $\mathbf{R}^5$ as the covering space of $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ with its natural projection $P : \mathbf{R}^5 \to \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ which sends $(x_1, ..., x_5)$ to the corresponding $(z_1, ..., z_5)$.

It is clear that $\tilde{\mathcal{P}}^5 = P^{-1}(\mathcal{P}^5)$ is also a surface, and $(x_1, ..., x_5)$ is a member of $\tilde{\mathcal{P}}^5$ if it satisfies the following three equations: a) $\sum_{i=1}^{5} x_i = 3\pi + 2\pi k$, where $k$ is an integer.

b) $\cos(x_1) + \cos(x_2) - \cos(x_2 + x_3) - \cos(x_1 + x_5) = 1$.

c) $\sin(x_1) - \sin(x_2) + \sin(x_2 + x_3) - \sin(x_1 + x_5) = 0$.

Now, consider the function

$$\tilde{f} : \tilde{P}^5 \subset \mathbf{R}^5 \to \mathbf{R}$$

which determines the area of a pentagon.

$$\tilde{f}(x_1, ..., x_5) = \sin(x_1) + \sin(x_2) - \sin(x_1 + x_2) + \sin(x_4),$$

and at the same time let us call $f$ the corresponding area function

$$f : \mathcal{P}^5 \to \mathbf{R}.$$

**LEMMA 1.** *The area function $f : \mathcal{P}^5 \to \mathbf{R}$ has 14 non-degenerate critical points: 2 maxima (which correspond to the area of the positively oriented regular convex pentagon and the negatively oriented regular pentagram), 2 minima (which correspond to the area of the negatively oriented regular convex pentagon and the positively oriented regular pentagram) and 10 saddle points.*

**Proof.** Let us take the following function

$$F = (f_1, f_2, f_3, f_4) : \mathbf{R}^5 \longrightarrow \mathbf{R}^4,$$

where

$$\begin{aligned} f_1(x_1,\ldots,x_5) &:= x_1+x_2+x_3+x_4+x_5. \\ f_2(x_1,\ldots,x_5) &:= \cos(x_1)+\cos(x_2)-\cos(x_2+x_3)-\cos(x_1+x_5). \\ f_3(x_1,\ldots,x_5) &:= \sin(x_1)-\sin(x_2)+\sin(x_2+x_3)-\sin(x_1+x_5). \\ f_4(x_1,\ldots,x_5) &:= \sin(x_1)+\sin(x_2)-\sin(x_1+x_2)+\sin(x_4). \end{aligned}$$

By calculating the determinants of all the $4\times 4$ sub matrices of the matrix $dF_p$ and taking the restriction to $\tilde{\mathcal{P}}^5$ we obtain the following system of equations, whose solution give us the set of critical points of the area function $\tilde{f}:\tilde{\mathcal{P}}^5\to\mathbf{R}$.

$$\left\{ \begin{array}{rcl} \sin(x_3+x_5)-\sin(x_2+x_4) &=& 0 \\ \sin(x_4+x_1)-\sin(x_3+x_5) &=& 0 \\ \sin(x_5+x_2)-\sin(x_4+x_1) &=& 0 \\ \sin(x_1+x_3)-\sin(x_5+x_2) &=& 0 \\ \sin(x_2+x_4)-\sin(x_1+x_3) &=& 0 \end{array} \right\} \tag{2.1}$$

Since $P:\tilde{\mathcal{P}}^5\to\mathcal{P}^5$ is a covering space and the map $\tilde{f}:\tilde{\mathcal{P}}^5\to\mathbf{R}$ is a lift of $f:\mathcal{P}^5\to\mathbf{R}$, then the critical points of $f$ can be obtained by projecting the critical points of $\tilde{f}$.

So, we obtain that the pentagons with interior angles $(x_1,...,x_5)$ which solve the system of equations (2.1) are precisely the two regular convex pentagons, oriented and unoriented, $X_{\frac{1}{5}}$ and $\tilde{X}_{\frac{1}{5}}$, respectively, with interior angles of the form $\frac{3\pi}{5}+2\pi k$, $k$ an integer; the regular pentagrams, $X_{\frac{2}{5}}$ and $\tilde{X}_{\frac{2}{5}}$ with interior angles of the form $\frac{\pi}{5}+2\pi k$, $k$ an integer, and 10 pentagons which are like a triangle, with interior angles of the form: $x_i=x_{i+1}=x_{i+2}=\pi/3+2\pi k_1$, $x_{i+3}=2\pi k_2$ and $x_{i+4}=2\pi k_3$, $k_1,k_2,k_3$ integers and $i=1,...,5$, $(i+5=i)$.

Then the function $f:\mathcal{P}^5\to\mathbf{R}$ has 14 non-degenerate critical points, 2 local maxima, given by $X_{\frac{1}{5}}$ and $\tilde{X}_{\frac{2}{5}}$; 2 local minima given by $X_{\frac{2}{5}}$ and $\tilde{X}_{\frac{1}{5}}$ and 10 more critical points, that by the Euler characteristic, are saddle points and are given by the 10 pentagons which look like a triangle. □

Next, we shall study the Morse Theory of the area function $f:\mathcal{P}^5\to\mathbf{R}$. First note that it has the following 6 critical values: $\{-m,-b,-n,n,b,m\}$, where $m$ is the area of the oriented regular pentagon $X_{\frac{1}{5}}$, $n$ is the area of the regular oriented pentagram $X_{\frac{2}{5}}$ and $b$ is the area of the pentagons which look like a triangle. The set, $\mathcal{P}^5_o$, of oriented pentagons without intersections on their sides is given by

$f^{-1}((b,m])$, which is a connected surface with only one critical point, a maxima, and hence topologically homeomorphic to an open disc. Similarly, the set, $\mathcal{P}_u^5$, of unoriented pentagons without intersections on their sides is given by $f^{-1}([-m,-b))$, which is a connected surface with only one critical point, a minima, and hence topologically homeomorphic to an open disc. $f^{-1}((-b,b))$ is an open surface that consists of three connected components: the set, $\mathcal{R}^5$, of all pentagons with exactly one intersection on their sides, the set, $\mathcal{Q}_o^5$, of oriented pentagrams (with 5 intersections) and the set, $\mathcal{Q}_u^5$, of unoriented pentagrams. Since there are only two critical points in $f^{-1}((-b,b))$, then $\mathcal{Q}_o^5$ and $\mathcal{Q}_u^5$ are homeomorphic to open discs and $\mathcal{R}^5$ is homeomorphic to an open cylinder. Finally, we summarize the situation of the fibers as follows:

a) if $x \in (-m,-b) \cup (-n,n) \cup (b,m)$, then $f^{-1}(x)$ consists of a simple closed curve,
b) if $x \in (-b,-n) \cup (n,b)$, then $f^{-1}(x)$ consists of two simple closed curves,
c) if $x \in \{-m,-n,n,m,\}$, then $f^{-1}(x)$ consists of a single point, and
d) if $x \in \{-b,b\}$, then $f^{-1}(x)$, consists of a chain of 5 simple closed curves in which two consecutive curves have one point in common. Each one of these closed curves represents pentagons in which two consecutive sides coincide.

Note that for the area function $\tilde{f} : \tilde{\mathcal{P}}^5 \to \mathbf{R}$, the kernel $Ker(dF_p) = Ker(d\tilde{f}_p)$, coincides with the system of constrained differential equations given in Theorem 1.4, for $n = 5$. Hence, $Ker(df_p)$ is the set of tangent vectors to the curves $f^{-1}(A)$.

For a pentagon in $(z_1,...,z_5) \in \mathcal{P}_o^5, \mathcal{P}_u^5, \mathcal{Q}_o^5, \mathcal{Q}_u^5, \mathcal{R}^5$, respectively, we identify $(z_1,...,z_5)$ with $(x_1,...,x_5) \in \tilde{\mathcal{P}}^5$, where $z_j = e^{ix_j}$ and $\sum_{i=1}^5 x_i = 3\pi, 7\pi, \pi, 9\pi, 5\pi$, respectively. Furthermore, $f^{-1}(A)$ is parametrized by $(x_1(t)$ ,...,$x_5(t))$, satisfying the system of differential equations (1),(2), with initial conditions $(x_1(0),...,x_5(0)) \in \mathcal{P}_o^5, \mathcal{P}_u^5, \mathcal{Q}_o^5, \mathcal{Q}_u^5, \mathcal{R}^5$, respectively, which are the interior angles of an equilateral pentagon with sides of length 2, area $A$ and $\sum_{i=1}^5 x_i(0) = 3\pi, 7\pi, \pi, 9\pi, 5\pi$, respectively. Moreover, for every one of this curves, there exists a carrousel of $n$ chairs $\{\beta_1(t),...,\beta_n(t)\}$, with the property that $x_i(t)$ is the angle between $\beta_{i+1}(t) - \beta_i(t)$ and $\beta_{i-1}(t) - \beta_i(t)$. Consequently, carrousels are classified, by real numbers in $[-m,m]$.

Our next purpose is to study $\mathcal{P}_0^5$. First of all, observe that for every $A \in (b,m)$, there exists $t_A$ in $\mathbf{R}$ such that, for every $t$, $x_i(t+t_A) = x_{i+1}(t)$, because $f^{-1}(A)$ consists of a simple closed curve and if $(x_1(0),x_2(0),x_3(0),x_4(0),x_5(0))$ is in the curve $f^{-1}(A)$ then,

$(x_2(0), x_3(0), x_4(0), x_5(0), x_1(0))$ is also in $f^{-1}(A)$. By Corollary 1.6, for the corresponding carrousel $\{\beta_1(t), ..., \beta_n(t)\}$, we conclude that the curve $\beta_i(t + t_A)$ is congruent to $\beta_{i+1}(t)$.

**LEMMA 2.** *Let $A \in (b, m)$ and suppose the curve $f^{-1}(A)$ is parametrized by $(x_1(t), ..., x_5(t))$, satisfying the system of constrained differential equations* (1), (2)*. Let $t_A, \eta_A \in \mathbf{R}$, be the minimum positive numbers such that for every $t \in \mathbf{R}$,*

$$x_i(t + t_A) = x_{i+1}(t) \quad and \quad x_i(t + \eta_A) = x_i(t).$$

*Then,*

$$5t_A = 2\eta_A.$$

**Proof.** Observe that in $\mathcal{P}_o^5 = f^{-1}((b, m))$, which by convention can be thought as a subset of $R^5$, the projection to the first two coordinates is one to one because these equilateral pentagons are determined by two of their angles. Let $g_A \subset R^2$ be the simple closed curve which is the projection of $f^{-1}(A)$ in $R^2$.

First of all we can see that $g_A$ is a simple closed curve symmetric with respect to the line $x = y$, because if the pentagon $(x_1, x_2, x_3, x_4, x_5)$ is in $f^{-1}(A)$ then, the symmetric one, $(x_2, x_1, x_5, x_4, x_3)$, is also in $f^{-1}(A)$. Therefore, $g_A$ intersects the line $x = y$ in exactly two points, lets say $(d, d)$ and $(a, a)$. Then there exists a pentagon $P_1$ of the form $(a, a, b, c, b)) \in f^{-1}(A)$. Let $P_2 = (a, b, c, b, a), P_3 = (b, c, b, a, a), P_4 = (c, b, a, a, b)$ and $P_5 = (b, a, a, b, c)$. They are also in $f^{-1}(A)$. Hence, the projection of these five points, $q_1 = (a, a), q_2 = (a, b), q_3 = (b, c), q_4 = (c, b)$ and $q_5 = (b, a)$ belong to $g_A$. Again, since the curve $g_A$ is a simple closed symmetric curve and since for pentagons of the form $(a, a, b, c, b)$ in these region we have that if $a \leq b$ then $c \leq a$ and if $b \leq a$, then $a \leq c$, we have that there exist only two possibilities for the cyclic order of the points $\{q_i\}$ in the curve $g_A$. Either $\{q_1, q_2, q_4, q_3, q_5\}$ or $\{q_1, q_4, q_2, q_5, q_3\}$. We shall now prove that the first cyclic order is not possible.

Suppose the curve $f^{-1}(A)$ is parametrized by $P(t) = (x_1(t), ..., x_5(t))$, where the $\{x_i(t)\}_1^5$ satisfy the system of constrained differential equations (1), (2). Suppose, without loss of generality, that $P(0) = P_1 = (x_1(0), ..., x_5(0))$. Hence $P(it_A) = P_{i+1} = (x_1(it_A), ..., x_5(it_A))$, $i = 0, 1, 2, 3, 4$.

If the cyclic order of the points $\{q_i\}$ in the curve $g_A$ is $\{q_1, q_2, q_4, q_3, q_5\}$, then the cyclic order of the points $\{P_i\}$ in the curve $f^{-1}(A)$ is $\{P(0), P(t_A), P(3t_A), P(2t_A), P(4t_A)\}$, which implies that there exists $s_0 \in \mathbf{R}$, $t_A < s_0 < 2t_A$ such that $P(2t_A + s_0) = P(4t_A)$. Thus, $P(s_0) = P(2t_A)$, which is impossible because $t_A$ is the minimum positive number such that, for every $t$, $x_i(t + t_A) = x_{i+1}(t)$.

If the cyclic order of the points $\{P_i\}$ in the curve $f^{-1}(A)$ is $\{P(0), P(3t_A),\ P(t_A), P(4t_A), P(2t_A)\}$, then there exists $0 < \epsilon_A < t_A$ such that $x_i(t + \epsilon_A) = x_{i+3}(t)$ for every $t \in \mathbf{R}$. Then, $x_i(t + 5\epsilon_A) = x_i(t + \eta_A)$ and $x_i(t + 10\epsilon_A) = x_i(t + 5t_A) = x_i(t + 2\eta_A)$. Consequently $5t_A = 2\eta_A$ and $2\epsilon_A = t_A$. □

From now on, let $0 < \epsilon_A < t_A$ be the minimum real number such that $x_i(t + \epsilon_A) = x_{i+3}(t)$, for every $t \in \mathbf{R}$. Note that $2\epsilon_A = t_A$.

**Remark.** The corresponding result for $A \in (n, b)$ and $f^{-1}(A) \subset \mathcal{Q}_o^5$ is that $5t_A = \eta_A$.

## 3. The Classification

In this section we shall classify Zindler carrousels for $n = 5$. So we need to study first some parameters associated to carrousels, and the small pieces of curves that describe them.

Let us do the case in which $A \in (b, m)$ and suppose the curve $f^{-1}(A)$ is parametrized by $(x_1(t), ..., x_5(t))$, satisfying the system of constrained differential equations (1), (2) with initial conditions $(x_1(0), ..., x_5(0)) \in \mathcal{P}_o^5$, which are the interior angles of an equilateral pentagon with sides of length 2, area $A$ and $\sum_{i=1}^5 x_i(0) = 3\pi$. By theorem 1.4, there exists a carrousel $\{\beta_1(t), ..., \beta_n(t)\}$, with the property that $x_i(t)$ is the angle between $\beta_{i+1}(t) - \beta_i(t)$ and $\beta_{i-1}(t) - \beta_i(t)$ and, by Lemma 2.2, such that $x_i(t+\epsilon_A) = x_{i+3}(t)$, for every $t \in \mathbf{R}$, where $2\epsilon_A = t_A$. Consequently, the curves $\beta_i(t + \epsilon_A)$ and $\beta_{i+3}(t)$ are congruent. Furthermore, by Theorem 1.7, assume that the mass center of the pentagons is the origin. Hence, there exists a rotation $R_{\sigma_A}$ of an angle $\sigma_A$ such that, for every $t \in \mathbf{R}$,

$$\beta_i(t + \epsilon_A) = R_{\sigma_A}\beta_{i+3}(t).$$

Let us call $\sigma_A$, the **basic angle** of this carrousel. That is, for every $t \in \mathbf{R}$,

$$\sigma_A = \theta_3(t + \epsilon_A) - \theta_1(t).$$

where $\theta_i(t)$ denotes the angle between the $x$-axes and the vector $\beta_{i+1}(t) - \beta_i(t)$.

We shall classify Zindler carrousels in terms of their basic angles, which in turn, depend on the area $A$ of the carrousel. For that purpose the following definitions are important.

The **period**, $\eta_A$, of the carrousel is the minimum real positive number such that $x_i(t+\eta_A) = x_i(t)$, for every $t \in \mathbf{R}$ and $i = 1, ..., 5$. Note that the pentagons at the time 0 and $\eta_A$ are congruent. So we define the **rotational angle**, $\rho_A$, of the carrousel as the angle between them, that is:

$$\rho_A = \theta_i(\eta_A) - \theta_i(0), \quad i = 1, ..., 5.$$

Remember that $\theta_i'(t) = -\sin(x_{i+2}(t) + x_{i+4}(t))$, hence

$$\rho_A = -\int_0^{\eta_A} \sin(x_2(t) + x_5(t))dt \ .$$

The next Lemma relates the value of the basic angle $\sigma_A$ and the rotational angle $\rho_A$ of a carrousel.

**LEMMA 3.** *Let $A \in (b, m)$. Then,*

$$5\sigma_A - 4\pi = \rho_A.$$

**Proof.** Using the fact that $\theta_3(t) - \theta_1(t) = 2\pi - (x_2(t) + x_3(t))$, and that for $i = 0, 1, 2, 3, 4$,

$$\sigma_A = \theta_3((i+1)\epsilon_A) - \theta_1(i\epsilon_A) =$$

$$= -\int_{i\epsilon_A}^{(i+1)\epsilon_A} \sin(x_5(t) + x_2(t))dt + (\theta_3(i\epsilon_A) - \theta_1(i\epsilon_A)),$$

we obtain, adding this five equalities, we obtain the result. □

**Definition 5.** *Let us call the $i-$**track** of the carrousel, the curve segment $\{\beta_i(s) | 0 \leq s \leq \epsilon_A\}$, $i = 1, ..., 5$.*

**LEMMA 4.** *It is possible to reconstruct the curves $\beta_i(t)$, $t \in \mathbf{R}$, by pasting one after the other, the five $i-$tracks.*

*Proof.* Let us take some $t \in \mathbf{R}$, then we can write $t = m\epsilon_A + \epsilon$, where $m \in \mathbf{Z}$ and $0 < \epsilon < \epsilon_A$. So $\beta_i(t) = \beta_i(m\epsilon_A + \epsilon) = R^m_{\sigma_A}\beta_k(\epsilon)$ where $k \equiv i + 3m \quad mod(5)$. □

Besides, we are interested in finding the index of Zindler carrousels around the mass center, because if the index of the curve has absolute value greater than one, then the curve intersects itself and therefore it is not a figure which floats in equilibrium with perimetral density $\frac{1}{5}$. For that purpose we need to know the angular length of the $i-$tracks.

**Definition 6.** *Let us suppose that the mass center of the pentagons is in the origin. Then we define the angular length of the $i-$track as follows*

$$C_i := \text{the angle between } \beta_i(0) \text{ and } \beta_i(\epsilon_A),$$

*and we call*

$$\gamma_i := \text{the angle between } \beta_i(0) \text{ and } \beta_{i+1}(0).$$

The following theorem classifies $\frac{1}{5}$-Zindler carrousels

**THEOREM 6.** *Any $\frac{1}{5}$-Zindler carrousel of index $d$ has basic angle $\sigma_A = \frac{r2\pi}{m}$, with $m = 5s + 2$, $s$ a natural number, and $1 < r < m$, satisfying for some $K$ the integer equation*

$$5(3s + 1 + r) + 1 + mK = d,$$

*and conversely, if a carrousel with area $A \in (b, m)$ has basic angle $\sigma_A$ satisfying the above integer equation, then it is a $\frac{1}{5}$-Zindler carrousel.*

**Proof.** Let us suppose we have a Zindler Carrousel. So, after a certain time, the curve $\beta_i$ reaches and follows the curve $\beta_{i+1}$. That is, $\beta_i(t + \lambda) = \beta_{i+1}(t)$, for every $t \in \mathbf{R}$. So $x_i(t+\lambda) = x_{i+1}(t)$, which implies that $\lambda = m\epsilon_A$, where $m = 5s + 2$, for $s$ a natural number. Let $0 < \epsilon < \epsilon_A$, so we have $\beta_i(m\epsilon_A + \epsilon) = R^m_{\sigma_A}\beta_{i+1}(\epsilon) = \beta_{i+1}(\epsilon)$. Therefore, $R^m_{\sigma_A}$ must be the identity and hence $\sigma_A = \frac{r2\pi}{m}$, for $1 < r < m$.

Since $R_{\sigma_A}$ sends the 4-track to the set $\{\beta_1(\epsilon_A + t)|0 < t < \epsilon_A\}$, then $\sigma_A = \gamma_4 + \gamma_5 + C_1 + 2l\pi$ for some integer $l$. We already know that $\sigma_A$ must be of the form $\frac{r2\pi}{m}$, with $m = 5s+2$, therefore, $C_1$ must be of the form

$$C_1 = \gamma_1 + \gamma_2 + \gamma_3 + (\frac{r}{m} + k_1)2\pi,$$

for some $k_1 \in \mathbf{Z}$.

Similarly

$$C_i = \gamma_i + \gamma_{i+1} + \gamma_{i+2} + (\frac{r}{m} + k_i)2\pi.$$

Then, $C_i + C_{i+3} + ... + C_{i+3(m-1)} = \gamma_i + (3s + 1)2\pi + r2\pi + (k_i + k_{i+3} + ... + k_{i+3(m-1)})2\pi$, where $C_i + C_{i+3} + ... + C_{i+3(m-1)}$ is the angular length of the curve $\{\beta_i(t)|0 < t < m\epsilon_A\}$.

For a Zindler carrousel with index $d$ we have that

$$\sum_{i=1}^{5}(C_i + C_{i+3} + ... + C_{i+3(m-1)}) = 2\pi d,$$

that is

$$5(3s + 1 + r) + mK = d - 1,$$

where $K = \sum_{i=1}^{5} k_i$.

Therefore, the integer solutions of the preceding equation with $1 < r < m$ give rise all the possible angles $\sigma_A$ for $\frac{1}{5}$-Zindler carrousels with index $d$ and, by construction, if a carrousel with five chairs has basic angle determined by this equation, then it is a $\frac{1}{5}$-Zindler carrousel. With this discussion we have finished the proof. □

**COROLLARY 3.** *Any $\frac{1}{5}$-Zindler carrousel of index 1 has basic angle of the form $\frac{4s+2}{5s+2}\pi$, for $s > 0$, a natural number and conversely, if a carrousel with area $A \in (b, m)$ has basic angle $\sigma_A = \frac{4s+2}{5s+2}\pi$, for $s > 0$ a natural number, then it is a $\frac{1}{5}$-Zindler carrousel.*

*Proof.* The natural solutions of the equation $5(3s + 1 + r) + mK = 0$, with $m = 5s + 2$ and $1 < r < m$ are precisely the natural numbers for which $\frac{2r}{m} = \frac{4s+2}{5s+2}$ . □

Figure 1 and 2 shows carrousels with basic angles $\frac{6\pi}{7}$ and $\frac{10\pi}{12}$ respectively, the basic angles $\frac{4s+2}{5s+2}\ \pi$ tends to $\frac{4\pi}{5}$, when $s$ tends to infinity, which correspond to the basic angle of the carrousel shown in Figure 4. Figure 3 corresponds to the carrousel of area zero whose center of mass is at infinity.

**THEOREM 7.** *A $\frac{1}{5}$-Zindler carrousel with index* 1 *and interior chords must have a period $\eta_A < 2.4002$.*

*Proof.* Let us suppose we have a $\frac{1}{5}$-Zindler carrousel parametrized by $(x_1(t), ..., x_5(t))$, satisfying the system of constrained differential equations (1), (2) with initial conditions $(x_1(0), ..., x_5(0)) \in \mathcal{P}_o^5$, which are the interior angles of an equilateral pentagon of area $A \in (b, m)$, and $\sum_{i=1}^{5} x_i(0) = 3\pi$. Let us assume that our carrousel has basic angle $\sigma_A = \frac{4s+2}{5s+2}\ \pi$, with $s$ a natural number greater than zero. Suppose that all chords are interior. Then for $0 < t < (5s + 2)\epsilon_A$, we have that

$0 < \theta_1(t) < \theta_2(0)$. Writing $\theta_1(\epsilon_A)$ in terms of $\sigma_A = \theta_3(\epsilon_A)$, we obtain:

$$\theta_3(\epsilon_A) + x_1(0) + x_5(0) = \theta_1(\epsilon_A),$$

$$2\theta_3(\epsilon_A) + x_1(0) + x_5(0) + x_4(0) + x_3(0) = \theta_1(2\epsilon_A),$$

$$3\theta_3(\epsilon_A) + x_1(0) + x_5(0) + x_4(0) + x_3(0) + x_2(0) + x_1(0) = \theta_1(3\epsilon_A),$$

and so on. Therefore, taking $\sigma_A = \frac{4s+2}{5s+2}\,\pi = \theta_3(\epsilon_A)$, we have that $x_1(0) + x_5(0) > \frac{6s+2}{5s+2}\,\pi$, $\quad x_2(0) < \frac{3s+2}{5s+2}\,\pi$ and $x_1(0) > \frac{3s}{5s+2}\,\pi$. Since this follows for every initial condition, we have the following inequalities:

$$x_i(t) + x_{i+1}(t) > \frac{6s+2}{5s+2}\,\pi, \quad \text{and}$$

$$\frac{3s}{5s+2}\,\pi < x_i(t) < \frac{3s+2}{5s+2}\,\pi,$$

which in turn give rise to the following inequality, for $s \geq 2$

$$-\sin(\frac{6s}{5s+2}\,\pi) < \ -(\sin(x_i(t) + x_{i+2}(t))) < -\sin(\frac{6s+4}{5s+2}\,\pi).$$

Using now Lemma 3.1, we obtain, for $s \geq 3$, the following bound for $\eta_A$,

$$\frac{2\pi sec\frac{s+2}{5s+2}\,\pi)}{(5s+2)} < \eta_A < \frac{2\pi sec(\frac{s-2}{5s+2}\,\pi)}{(5s+2)} < 2.4002.$$

Finally, the carrousels with basic angle $\frac{6\pi}{7}$ and $\frac{10\pi}{12}$ are shown in the figures 1 and 2, respectively. In both of them their chords are not interior. □

The case $\frac{2}{5}$ can be studied in a similar way. One proves that, for $A \in (n, b)$, $5\sigma_A - 6\pi = \rho_A$. It is also possible to obtain a classification of $\frac{2}{5}$-Zindler carrousels of index $d$. In particular, we have the corresponding theorem.

**THEOREM 8.** *Any $\frac{2}{5}$-Zindler carrousel of index* 1 *has basic angle of the form $\frac{6s+4}{5s+3}\,\pi$, for $s$ a natural number.*

**COROLLARY 4.** *A $\frac{2}{5}$-Zindler carrousel with index* 1 *must have a period $\eta_A <$* 2.5.

**Proof.** Is easy to see that for $X = (x_1, ..., x_5) \in \mathcal{Q}_o^5$, we have $\sin(x_3 + x_5) \geq \sin(4\arcsin(\frac{1}{4})) \geq .847214$, which implies that, for any $A \in (n, b)$, the rotational angle of a carrousel is $\rho_A \geq .847\eta_A$. Therefore, a $\frac{2}{5}$-Zindler carrousel with basic angle $\sigma_A = \frac{6s+4}{5s+3}\,\pi$, must have a period $\eta_A < 2.5$. □

## 4. Some Consequences of the Theory

Using the fact that $f^{-1}(m)$,where $f : \mathcal{P}^5 \to \mathbf{R}$, is an isolated singular point of the vector field and a non-degenerate center, that is, the linear part of the vector field has eigenvalues $\pm i\omega$, $\omega > 0$, we may prove, using the Classical Poincaré-Lyapunov Center Theorem [2],[7], that the limit of the period function $\eta : (b, m) \to R$, when $A \to m$, is $2\pi/\omega$ which, after the corresponding calculations, gives $\eta(m) = 2\pi/\omega \sim 2.4002$.

**CONJECTURE 1.** *The period function $\eta : (b, m) \to R$ is an decreasing function. In fact, $\eta(A) \le \eta(m)$, for every $A \in (b, m)$.*

There is clear evidence of this fact given by the computer. The graph 1(a) shows the values, obtained with a computer, for $\eta_A$ and $\sigma_A$. Note that $\sigma_A < 2.9132$ and $\eta_A > 2.4002$, for every $A \in (b, m)$. In [4] it was proved that there are no figures that float in equilibrium in every position with perimetral density $\frac{1}{3}$ *and* $\frac{1}{4}$, although there are with perimetral density $\frac{1}{2}$. This time we show that *there are no figures that float in equilibrium in every position with perimetral density $\frac{1}{5}$ and $\frac{2}{5}$, different from the circle.*

To see this, let us suppose that there exist a figure that float in equilibrium in every position with perimetral density $\frac{1}{5}$, which give rise to a $\frac{1}{5}$-Zindler carrousel. If the index is 1, by Theorem 3.5, $\eta_A < 2.4002$, which is a contradiction. The same ideas can be analogously applied to study $\frac{1}{5}$-Zindler carrousels of index $-1$ to conclude that there are no figures that float in equilibrium in every position with perimetral density $\frac{1}{5}$.

Furthermore, by Corollary 3.7, a $\frac{2}{5}$-Zindler carrousel of index 1 must have a period $\eta_A < 2.5$. It is possible to verify, using the previous discussion of the Poincaré-Lyapunov Center Theorem and graph 1(b), that the period of any carrousel with area $A \in (n, b)$ is greater than 2.5. Therefore, there are no $\frac{2}{5}$-Zindler carrousels of index 1 and an analogous discussion shows the same for index $-1$. $\square$

## 5. Acknowledgments

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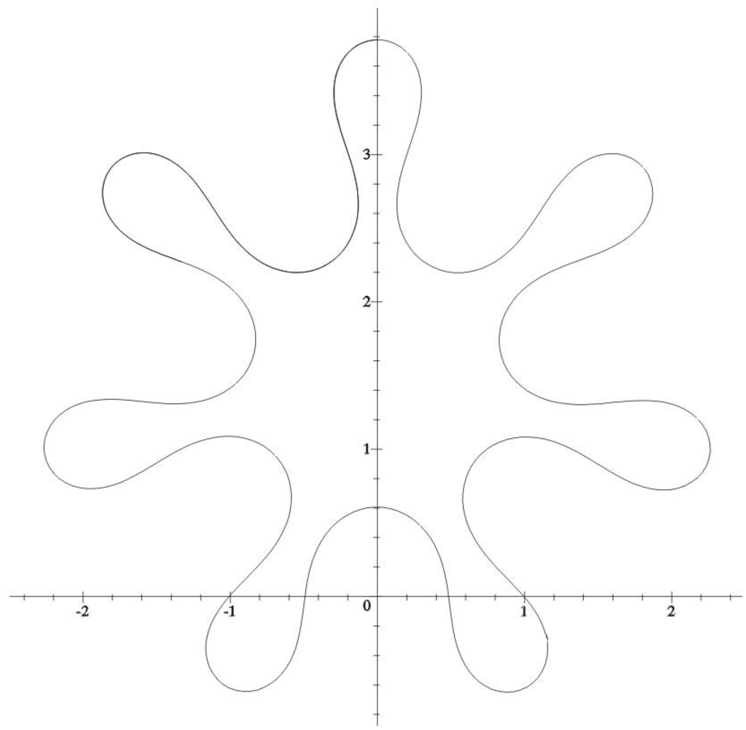



Figure 1
The Zindler Carrousel with basic angle $6\pi/7$

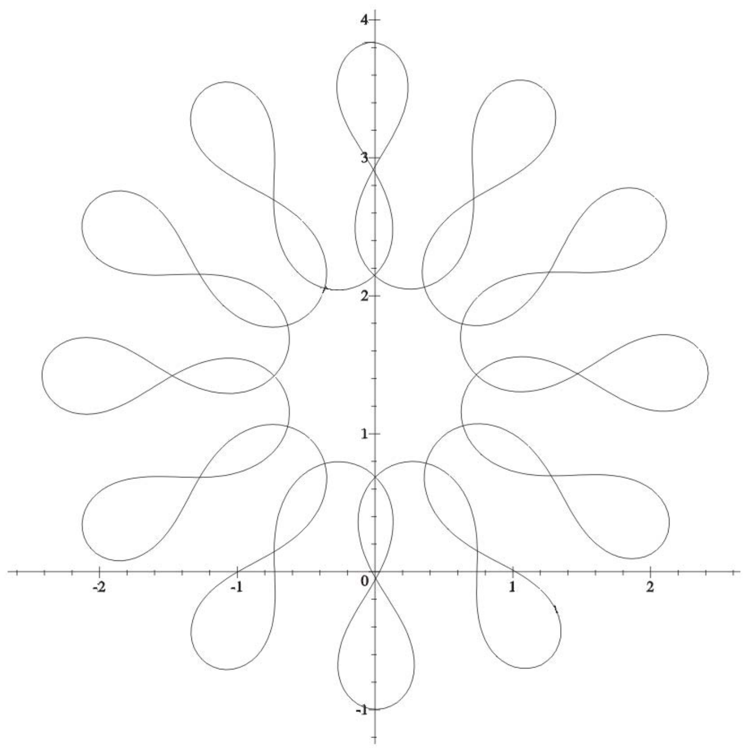



Figure 2
The Zindler carrousel with basic angle $10\pi/6$

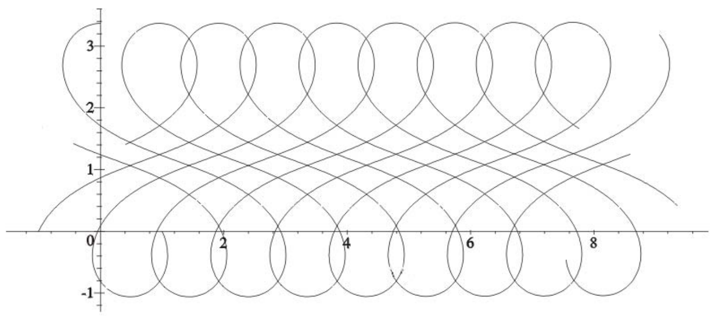



Figure 3
The unique carrousel of area zero and center of mass at infinity

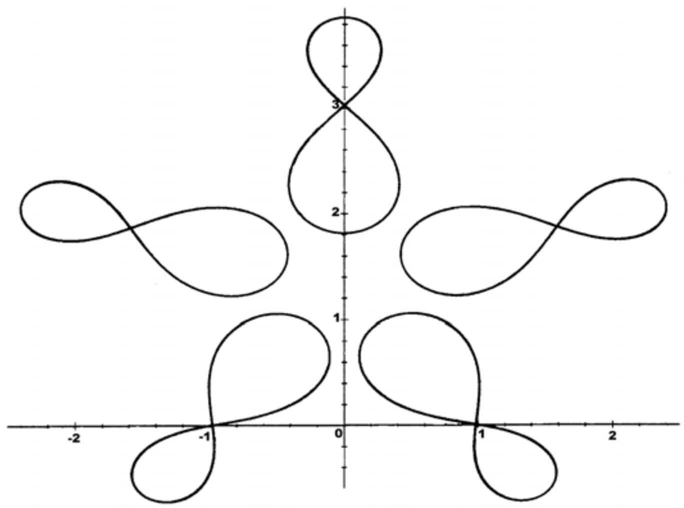
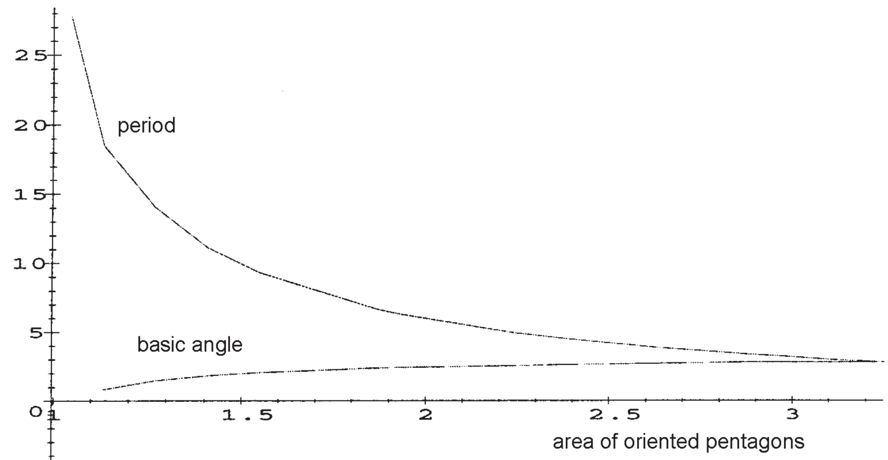
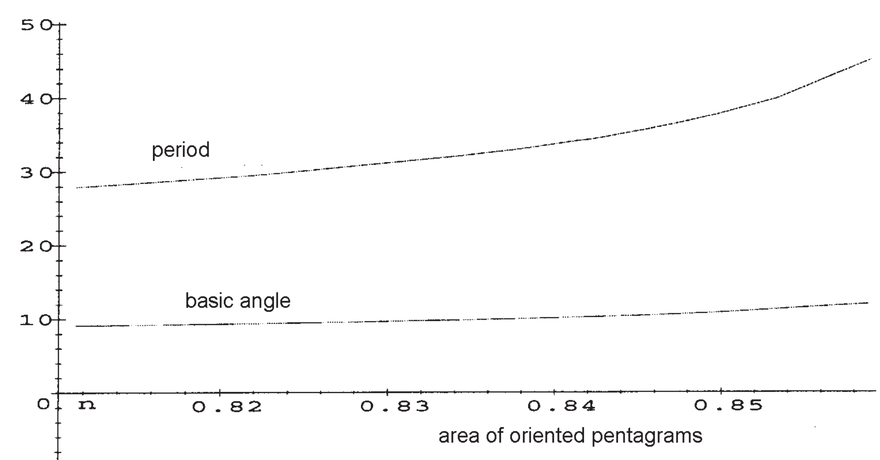



Figure 4
Carrousel with area 2.1523 and basic angle $4\pi/5$




Graph 1 (a)




Graph 1 (b)