Phantom points and critical connectivity of matroids

Omar Antolín, Jorge L. Arocha*, Javier Bracho* and Luis Montejano*

August 26, 2005

Abstract

The concept of phantom point of a matroid is introduced. With it, we deduce recursive characterizations of the critically connected matroids contained in a minor-closed class. We give two applications: first, a Helly-type theorem for matroid partitions i.e. submatroids in which each connected component is a flat of the ambient matroid and second, a characterization of projective pseudobases i.e. minimal sets of points in a projective space such that any matroid isomorphism uniquely extends to a projectivity.

1 Fixing sets; a motivation

To fix a sheet of paper to a board, we need two tacks. Mathematically, this means that a set of two points of the plane has the property that any preserving orientation rigid transformation of the plane which is the identity in these two points is the identity in the whole plane. We will say that a set of points A of a space E (a set with a group action) **fixes** if any transformation of E (by the action) which is the identity in A is also the identity in the whole space E.

For example, a linear transformation in a vector space is determined by its values on any basis of the space. Therefore, any basis of a vector space fixes it. The bases of a vector space are not the only fixing sets: a set of vectors fixes a vector space if and only if it spans the space. Therefore, the minimal fixing sets of a vector space are precisely the bases. The same result holds for affine spaces.

*These authors where partially supported by grants CONACYT-U41340-F and DGAPA-IN111702-3

In a projective space things are not so simple. In n-dimensional projective space any set of $n+2$ points in general position (no $n+1$ of them contained in a proper subspace) does fix, but these are not the only minimal fixing sets. For an example, take real three dimensional projective space to be Euclidean space augmented by a plane at infinity. The vertices of a regular octahedron form a minimal fixing set. (That they do in fact fix projective 3-space is easy to prove once one notices that four coplanar vertices of an octahedron fix the plane they span.) However, minimal fixing sets of points in the projective space are a key tool to reveal the structure of the space of configurations of projective points (see [1]). So, we begin this paper with a characterization of such sets.

Proposition 1 *A set of points in a finite-dimensional real projective space fixes it if and only if it is not contained in the union of two complementary subspaces.*

Proof. Let A be a set of projective points. Assume that $A \subset L_1 \cup L_2$ for two complementary subspaces. Consider the linear transformation $\lambda I_1 \oplus \mu I_2$ where λ and μ are two scalars different from each other and from zero, and I_j denotes the identity on the vector subspace corresponding to L_j. This transformation defines a projectivity that fixes both L_1 and L_2, but fixes no other point of the space. Therefore A does not fix L.

Reciprocally, suppose now that A does not fix and let f be a linear transformation corresponding to a projectivity that fixes every point in A but is not the identity. The vectors that represent points of A are eigenvectors of f. If the set of all eigenvectors of f did not span, then A would be contained in a proper projective subspace. Therefore, the map f is diagonalizable. Since f is not a multiple of the identity, it must have at least two eigenvalues. Let λ be any eigenvalue of f. Then the vector subspace of eigenvectors with eigenvalue λ and the vector subspace spanned by the other eigenvectors define two complementary projective subspaces that together contain A. ■

Remark 2 *For simplicity, we stated our result for a projective space over the field of real numbers, but the argument actually works over any field with more than two elements. Since a projective space over $\mathbb{Z}/2\mathbb{Z}$ is simply the set of nonzero vectors of a vector space, then a set of vectors fixes the space if and only if it spans.*

Now, we will translate Proposition 1 to the language of matroid theory. Observe that a set of points in a projective space is a matroid. This matroid structure can be defined in many equivalent ways. For example, the projective closure operator $A \mapsto \langle A \rangle$ can be used to define bases, independent sets, the rank function, circuits, etc. It is precisely in the context of matroid theory, where minimal fixing sets of projective points can be better understood. It is well known that a projective matroid is connected if and only if its points are not contained in the union of two non intersecting projective subspaces. From Proposition 1 we immediately obtain the following.

Theorem 3 *A set of points in a finite-dimensional real projective space fixes the subspace it spans if and only if it is a connected matroid.*

So, minimal fixing sets are exactly the critically connected projective matroids (i.e. connected projective matroids that become disconnected after a removal of any point).

In this paper we characterize all critically connected matroids and give two applications, but first, in the next section, we recall some basic definitions and results in matroid theory. In Section 1 we define the two basic matroid operations needed to generate all critically connected matroids. In Section 4 we give two recursive characterizations of the critically connected matroids which are contained in a fixed but arbitrary minor-closed class of matroids, hereby solving a problem of Murty [5] (for another approach to this problem see [6]). In Section 5 we introduce the partition lattice of a matroid and prove the existence of some special matroid extensions. This result leads in Section 6 to a formula for the Helly number of a partition lattice of "full" matroids. We conclude, in Section 7, characterizing the uniquely representable minimal fixing sets of projective points.

2 Preliminaries

We assume that the reader is at ease with matroid theory. We will use here the geometric language. So, the elements of M will be called **points**. The **dimension** of a set of points is its **rank** minus one. All our matroids will be finite dimensional. The **codimension** of a set of points is equal to the dimension of the whole matroid minus the dimension of the set. The **flats** (**closed sets**) of dimension one will be called **lines** and the flats of codimension one will be called **hyperplanes**. The **corank** of a finite matroid is its number of points minus its rank. Abusing language we will often not distinguish between M and the set of its points.

For a point p in M we denote by $M - p$ the **deletion** of p. A **submatroid** of M is any matroid which can be obtained from M by a series of deletions. The inverse operation of a point's deletion is the simple extension. If $M' = M - p$, then we say that M is a **simple extension** of M' by p and we will write $M = M' + p$. The extension M is said to be **free** if p is a **coloop** i.e. if M' is a hyperplane of M. Any two deletion operation commute, therefore for a set of points $N \subseteq M$ the equality $M' = M - N$ makes sense. The inverse operation is $M' + N$. This means that we will use the symbol $+$ to denote a union of two disjoint sets of points.

One of the difficulties of matroid theory is that there is no such thing as a "universal matroid": a matroid which contains as a submatroid any matroid. This leads to the common practice of thinking on matroids as entities which are defined up to isomorphism. We will avoid this practice by thinking that we have a "big" matroid which contains all the points we might need. So, the notation $M + p$ has meaning because the name of the point p "knows" which are the circuits of $M + p$. If p' is another point, then the matroid $M + p'$ can be isomorphic or not to $M + p$, but in any case they are different because they have different sets of points.

If M, N are two matroids with no common points, then their **direct sum** $M \oplus N$ is $M + N$ but with the condition that any **circuit** (a minimal dependent set) of $M \oplus N$ is a circuit of M or is a circuit of N. The equality $\operatorname{rk}(M + N) = \operatorname{rk} M + \operatorname{rk} N$ is necessary and sufficient for the sum to be direct. If $M \oplus N$ is a projective matroid, then this equality means that the projective subspaces spanned by M and N are complementary.

Two points of a matroid are said to be **connected** if there is a circuit containing both. A consequence of the circuit exchange axiom is that the connectivity relation is transitive. Therefore, every matroid is partitioned into classes which are called **connected components**. Any matroid is the direct sum of its connected components. A matroid is **connected** if it has only one connected component. Matroid connectivity was discovered by Tutte in [9] where he proved the fundamental facts cited above. The following facts which we will use below are straightforward or have proofs of routine nature.

Proposition 4 *Any connected matroid with at least* 2 *points has no loops and no coloops.*

Proposition 5 *Every non free simple extension of a connected matroid is connected.*

Proposition 6 *Let p, q be points of a matroid M. If p is a coloop in $M - q$ and it is not a coloop in M, then q is a coloop in $M - p$.*

Proposition 7 *If $M + p$ and $N + p$ are two circuits intersecting exactly in the point p, then $M + N$ is also a circuit.*

Let us call a point p of a matroid M **inessential** if $M - p$ is connected. A connected matroid is said to be **critically connected** if every one of its point is

essential. An obvious example of a critically connected matroid is a single circuit. We will use the acronym CC for "critically connected". To avoid the discussion of trivialities we postulate that *a CC matroid must have at least dimension one.*

3 Subdivisions and Phantom subdivisions

Let us describe the first operation which preserves CC matroids. Let M be a matroid and $\{a, b, p\}$ a 3-circuit such that $\{a, b\}$ is a 2-**cocircuit**. We will say that the matroid $M - \{p\}$ is obtained from $M - \{a, b\}$ by **subdividing** the point p (see Figure 1). We chose this name because this is the usual operation of **edge subdivision** for graphic matroids. To use fewer words, we will say that N is a **subdivision** of M if N is obtained from M by subdividing some point of M.

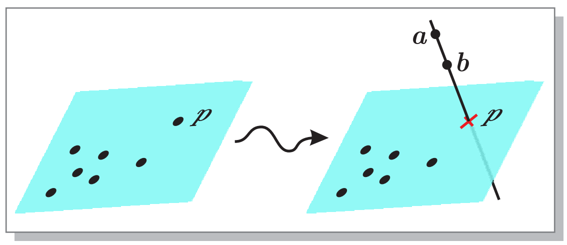

Figure 1: Subdividing a projective matroid

It is easy to see that any circuit of $M - \{p\}$ can be obtained from a circuit of $M - \{a, b\}$ replacing the point p by the points $\{a, b\}$. Therefore, $M - \{p\}$ is connected if and only if $M - \{a, b\}$ is connected. The same property holds for matroids $M - \{p, q\}$ and $M - \{a, b, q\}$ for any $q \in M - \{a, b, p\}$. Since the matroids $M - \{a, p\}$ and $M - \{b, q\}$ are not connected (because they have coloops), hence we obtain that $M - \{p\}$ is CC if $M - \{a, b\}$ is CC. These results first appeared in his full generality in [5] (there, the name "series extension" is used instead of "subdivision"). We state them in two propositions for easy cross referencing:

Proposition 8 (Murty) *Let N be a subdivision of M. Then M is connected if and only if N is connected.*

Proposition 9 (Murty) *Any subdivision of a CC matroid is CC.*

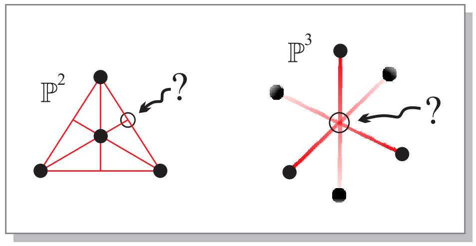

Figure 2: Phantom points of a circuit and an octahedron

Now, let us describe the second operation which preserves CC matroids. When we look at some projective matroids we get the feeling that some points are missing (see Figure 2).

The following is a key definition in this paper. Let $M' = M + p$ be a non-free simple extension. If p is the only inessential element of M', then we call M' a **phantom extension** of M and we call p a **phantom point** of M. Furthermore, if we subdivide the point p in M' we obtain a **phantom subdivision** of M (at the point p). Observe that any phantom subdivision is a two point extension (see again Figure 1).

Proposition 10 *If a matroid has a phantom point, then it is CC.*

Proof. Let p be a phantom point of the matroid M. Since p is inessential, hence M is connected. Let q be any point of M. Since p is the only inessential point in $M+p$, then we know that $M+p-q$ is not connected. Suppose $M-q$ is connected. Then by Proposition 5 the point p has to be a coloop in $M+p-q$ and p is not a coloop in $M+p$. Therefore, by Proposition 6 the point q is a coloop in M. This contradicts that M is connected (using Proposition 4). ∎

Proposition 11 *Any phantom subdivision is CC.*

Proof. Let $M+p$ be a phantom extension of the matroid M. Denote by $N = M + \{a, b\}$ the phantom subdivision of M at p. By the previous proposition M is CC. By propositions 5 and 8 the matroid N is connected.

The matroids $N-a$ and $N-b$ are not connected because both have coloops. Let q be any point of M. Since p is the only inessential point in $M+p$, then $M+p-q$ is not connected. By Proposition 8 the matroid $N-q$ is also not connected. ∎

Let us briefly describe the inverse operations of subdivision and phantom subdivision. First, observe that in both cases two new points $\{a, b\}$ appear. Moreover, in the new matroid $\{a,b\}$ is a 2-cocircuit. So, if we want to apply some inverse operation, then the first thing to do is to find out a 2-cocircuit. However, it is well known (see [5], [8] and [10]) that *any CC matroid has a* 2*-cocircuit* (see also [6] page 49 for historical data).

Let M be some matroid in which $\{a,b\}$ is a 2-cocircuit. It is easy to see that the **contraction** M/a followed by a subdivision (of b) is the identity. However, to get this, we need to define the "contraction" M/a as a matroid whose points are those of $M-a$. On the contrary, we see the contraction as an operation which is well defined only up to matroid isomorphism. For example, for projective matroids, contractions are the projections from a point to a hyperplane, any hyperplane does the job and there is not a canonical one. There is no way to avoid this uncertainty in the general case. However, if we are contracting by an element of a 2-cocircuit $\{a,b\}$ the set of points $M-\{a,b\}$ is a hyperplane of M and we can use it to naturally define our operation. Let $M+p$ be the simple extension of M such that p is in the intersection of the line $\{a,b\}$ and the hyperplane $\langle M-\{a,b\}\rangle$. We will call $M+p-\{a,b\}$ the **quotient** of M by the 2-cocircuit $\{a,b\}$. It is clear that any quotient is isomorphic to a contraction and that the quotient is the inverse operation of the subdivision operation. We will call the point p the **projection** of $\{a,b\}$. Observe that $M+p-\{a,b\}$ is always a non-free extension of $M-\{a,b\}$.

The importance of fine tuning our definitions of inverse operations becomes evident when we consider a class of matroids which are submatroids of a given one. For example, consider the real projective plane with a hole (the Möbius band). If we take four points in a square with its center in the hole, then its quotient by a diagonal of the square is not defined inside our Möbius band. However, the band contains contractions by any of the points (three point lines). This "labelled approach" will be essential only in Section 6.

The inverse operation to phantom extension is easier. Since a phantom extension is a particular case of a two point extension, then its inverse is a two point deletion. We only have to check that the projection of the 2-circuit is a phantom point.

4 Generating all critically connected matroids

Now, we show that subdivisions and phantom subdivisions are the only operations needed to generate all CC matroids.

Lemma 12 *If $\{a,b\}$ is a 2-cocircuit of the CC matroid M, and p is the projection of $\{a,b\}$, then for any $q\in M-\{a,b\}$ the matroid $M+p-\{a,b,q\}$ is not connected.*

Proof. Denote $N = M - \{a, b\}$. We have to show that $N + p - q$ is not connected. If $p \notin \langle N - q \rangle$, then p is a coloop in $N + p - q$ and we are done. If $p \in \langle N - q \rangle$, then $\{a, b\}$ is a 2-cocircuit in $M - q$ and the matroid $N + p - q$ is the quotient of $M - q$ by $\{a, b\}$. Since $M - q$ is not connected, hence Proposition 8 concludes the proof. ■

Proposition 13 *If $\{a, b\}$ is a 2-cocircuit of the CC matroid M, then either the quotient of M by $\{a, b\}$ is CC or the projection of $\{a, b\}$ is a phantom point of $M - \{a, b\}$.*

Proof. Let p be the projection of $\{a, b\}$ and denote $N = M - \{a, b\}$. By Proposition 8 the quotient $N + p$ is connected. If N is connected, then by the lemma p is the only inessential point in $N + p$ i.e. the point p is a phantom point of N. If N is not connected, then by the lemma $N + p$ is critically connected. ■

Theorem 14 *The class of CC matroids coincides with the class of matroids which can be obtained from a 3-point line by a series of subdivisions and phantom subdivisions.*

Proof. The only CC matroid of dimension 1 is the 3 point line. Suppose by induction that the theorem is true for all matroids of dimension at most d. By propositions 9 and 11 any matroid of dimension $d + 1$ obtained by a subdivision or a phantom subdivision of a CC matroid of dimension d is also CC. Reciprocally, if M is a CC matroid of dimension $d + 1$, then by the Murty-Seymour-White theorem it has a 2-cocircuit $\{a, b\}$. By Proposition 13 M is either a subdivision or a phantom subdivision of a matroid of dimension d. ■

Straightforward proofs of several known results can be obtained from this theorem. For example, if we notice that a 4-circuit is the only CC matroid of dimension 2 and observe that phantom subdivisions increase the corank just in one, then we obtain that the corank of a CC matroid is less or equal than the dimension (see [5] and [6]). The fact that this bound is attained only by the d-dimensional octahedron is a consequence of the not difficult to prove fact that this matroid has only one phantom point (for $d \geq 3$). Those are the main results of [5].

Theorem 14 is perfect for proving (by induction) properties of CC matroids. However, it produces a very long proof that a given matroid is CC. To do this, it is better to have an operation such that, given two CC matroids we obtain another CC matroid. Let A and B be two flats of at least dimension one of the matroid M. It is said that $\{A, B\}$ is a **2-cut** of M if $\langle A \cup B \rangle = M$ and $\operatorname{rk} A + \operatorname{rk} B = \operatorname{rk} M + 1$. By semimodularity of the rank function for any 2-cut we have that $\operatorname{rk} \langle A \cap B \rangle \leq 1$. The set $A \cap B$ will be called the **intersection** of the 2-cut. If M has no parallel points, then $\mathrm{rk}\langle A \cap B\rangle \leq 1$ implies that either the intersection is empty or consists of exactly one point p. It is not difficult to see that in the latter case any circuit which contains p must be contained in A or in B.

Proposition 15 *Any 2-cut $\{A, B\}$ of a CC matroid has empty intersection.*

Proof. Suppose $p = A \cap B$ and denote $M = \langle A \cup B\rangle$. Let us show that $M - p$ is also connected. Let x, y be two points in $M - p$. Since M is connected, then there is a circuit C containing x, y. If C does not contain p, then x, y are connected in $M - p$. If C contains p, then C is contained (say) in A. Since $p \in \langle B - p\rangle$ (otherwise $M = A \oplus (B - p)$ would be not connected), then there exists a circuit C' in B which contains p. By Proposition 7 $(C \cup C') - p$ is a circuit in $M - p$ containing x, y. ■

If $\{A, B\}$ is a 2-cut of the matroid M and $A \cap B$ is empty, then there exist a simple extension $M + p$ (unique up to the name of the new point) in which $\{A + p, B + p\}$ is a modular pair of flats. Intuitively we add the missing point in the intersection. Formally, $M + p$ can be defined as the matroid whose circuits are those of M and those of the form $A' + p$, $B' + p$ where $A' \subseteq A$, $B' \subseteq B$ are non-empty and $A' \cup B'$ is a circuit of M. We will call the point p the **projection** of A to B (also of B to A). The matroid $A + p$ is the **quotient** of M by B. It is clear that if p' is any point in B, then $A + p$ is isomorphic to the contraction $M/(B - p')$. Observe that if $\{a, b\}$ is a 2-cocircuit of a matroid M, then $\{\{a, b\}, M - \{a, b\}\}$ is a 2-cut and our definitions of projections and quotients are just generalizations of the ones given previously.

Proposition 16 *Let $\{A, B\}$ be a 2-cut of the matroid M and p be the projection of A to B. Then M is connected if and only if $A + p$ and $B + p$ are connected.*

Proof. Suppose first that M is connected. If $A + p$ is not connected and $A + p = (A_1 + p) \oplus A_2$, then $M = (B + A_1) \oplus A_2$ and this is a contradiction. So, $A + p$ is connected and by symmetry $B + p$ it is also connected.

Now, suppose that both $A + p$ and $B + p$ are connected. Let x, y be two points in M. If $x, y \in A$, then let C be a circuit of $A + p$ containing x, y. If C does not contain p, then x, y are connected in M. If C contains p, then let C' be a circuit in $B + p$ containing p. By Proposition 7 $(C \cup C') - p$ is a circuit in M containing x, y. If $x \in A$ and $y \in B$, then let C be a circuit of $A + p$ containing x, p and let C' be a circuit of $B + p$ containing y, p. By Proposition 7 $(C \cup C') - p$ is a circuit in M containing x, y. So, in any case we obtain (using symmetry) that x, y are connected in M. ■

Corollary 17 *Let $\{A, B\}$ be a 2-cut of the matroid M with empty intersection and let p be the projection of A to B. Then M is CC if and only if*

1. *$A + p$ is connected and any $a \in A$ is essential in $A + p$,*

2. *$B + p$ is connected and any $b \in B$ is essential in $B + p$.*

Proof. Suppose M is CC. By the previous proposition $A+p$ and $B+p$ are connected. If $A+p-a$ is connected, then by the previous proposition $M-a$ would be connected and this is a contradiction. By symmetry, $B+p-b$ is not connected.

Suppose the properties 1 and 2 hold. By the previous proposition M is connected. If for some $x \in M$ it happens that M is connected, then by the previous proposition $X+p-x$ is connected, where X is A if $x \in A$ or is B if $x \in B$. ■

Let us closely analyze the properties stated in Corollary 17. If $A+p$ is connected and any $a \in A$ is essential in $A+p$, then there are 2 possibilities:

1. p is essential in $A+p$, in which case $A+p$ is CC,

2. p is inessential in $A+p$, in which case A is CC and p is a phantom point of A.

Now, it is clear how to build new CC matroids from smaller ones. Let $A+p$ and $B+p$ be two matroids intersecting just in the point p. Denote by $A \circledast B$ the matroid on $A+B$ whose circuits are those of A and B and those of the form $A'+B'$ where $A'+p$ and $B'+p$ are circuits of $A+p$ and $B+p$ respectively. We have the following:

Theorem 18 *The matroid $A \circledast B$ is CC if and only if for any $X \in \{A, B\}$ either $X+p$ is CC or p is a phantom point of X. Any CC matroid can be obtained by a series of these operations starting from the 3-point line.*

Proof. The first statement of the theorem is a reformulation of Corollary 17. For the proof of the second, observe that, if $B+p$ is a 3-point line, then $A \circledast B$ is a subdivision of A or a phantom subdivision of A. ■

The circuit matroid of a graph (with at least 3 vertices) is connected if and only if the graph is biconnected. Biconnected graphs are also called **blocks**. A block is **minimal** if after removal of any edge the result is not a block i.e. minimal blocks are exactly those graphs whose circuit matroid is CC. The study of minimal blocks started with a paper of Dirac [3] (see also [7]). However, we did not find in the literature a precise answer to the construction of all minimal blocks.

A **non-edge** of a graph is a pair of vertices which is not an edge of it. We will say that a non-edge p of a minimal block G is a **phantom edge** if for any edge e of G, the graph $G+p-e$ is not a block. A **subdivision of a phantom edge** p is the operation which consist of adding the phantom edge and subdividing it. From Theorem 14 we obtain the following:

Theorem 19 *Minimal blocks are exactly the graphs which can be obtained from the complete graph $\mathbb{K}_3$ with a series of edge and phantom edge subdivisions.*

In [4] Hedetniemi proved that any minimal block can be obtained from the complete graph $\mathbb{K}_3$ with a series of edge and non-edge subdivisions. However, he noted that not every non-edge subdivision produces a minimal block and asked which do. Theorem 19 is the answer to his question[1].

A **minor** of a matroid M is a matroid that can be obtained from M by a series of deletions and contractions. A **minor-closed class** is a class of matroids such that if a matroid is in the class, then any of its minor is also in the class. It is easy to see (by taking a matroid and subdividing each of its points) that the only minor-closed class which is closed by subdivision and phantom subdivisions is the class of all matroids. If the minor-closed class is closed by subdivisions (this is the case for graphic, linear, etc. matroids), then to obtain all CC matroids inside the class we only need to worry about when a phantom extension is inside the class. So, theorems similar to Theorem 19 can be obtained for such classes of matroids: for graphs the phantom point must be a phantom edge, for projective matroids the phantom point must be a projective point, etc.

5 The partition lattice of a matroid

Let M be a matroid and N be a set of its points. Since N is a submatroid of M, then N splits into the direct sum of its connected components $N_1, \ldots, N_t$. We will call $[N] = \langle N_1 \rangle \cup \ldots \cup \langle N_t \rangle$ the **closure by components** of N. It is easy to see that the operator $N \mapsto [N]$ is increasing, monotone and idempotent, i.e. it is a closure operator. However, it is not a matroidal closure operator, i.e. in the general case the **exchange closure property** does not hold. For example, if N is a circuit and $N + p = M$ is a phantom extension, then for any $q \in N$ we have that $q \notin [N - q + p]$ but $p \in [N] \setminus [N - q]$.

A set of points which is closed under the connected closure operator will be called a **partition** of the matroid. Any partition is a union of flats. A union of flats $F_1 \cup \ldots \cup F_t$ is a partition if and only if $F_1 \cup \ldots \cup F_t = F_1 \oplus \ldots \oplus F_t$. The set of all partitions is naturally ordered by inclusion and it is a lattice with meet $A \wedge B = A \cap B$ and join $A \vee B = [A \cup B]$. We will call it the **partition lattice** of the matroid M. It is easy to see that the partition lattice of a matroid splits into the direct product of the partition lattices of its connected components. In particular,

[1]We thank Eduardo Rivera-Campo for bringing our attention to this paper.

if M is a matroid with no circuits, then its partition lattice is the boolean algebra 2^M i.e. for any $N \subseteq M$ the equality $N = [N]$ holds. The following is an interesting characterization of CC matroids.

Proposition 20 *The partition lattice of a connected matroid M is the boolean algebra 2^M if and only if M is CC.*

Proof. If M is not CC, then there is $q \in M$ such that $M - q$ is connected and therefore $q \in [M - q]$. Now, suppose that M is CC and for some $q \notin N \subset M$ we have that $q \in [N]$. Let N' be the connected component of N such that $q \in [N']$. Since M is CC, then $M - q = M_1 \oplus M_2$. Since N' is connected, we can suppose that $N' \subseteq M_1$. Therefore $q \in [N'] \subseteq [M_1] = M_1 \subseteq M_1 \oplus M_2 = M - q$ and this is a contradiction. ■

Remark 21 *We have to define an independent point as the only CC matroid with less than 3 points to make the previous proposition true.*

Let M be a CC matroid and $M + p$ a simple non-free extension of M. We will say that p is a **strong phantom point** of M if for every $q \in M$ the point p belongs to $[M - q]$. In this case, we notice that $M + p$ is connected and for any $q \in M$ the matroid $M - q + p$ is not connected. This means that every strong phantom point is a phantom point.

Proposition 22 *A CC matroid has a strong phantom point if and only it is not a circuit.*

Proof. Let C be a circuit. For any $q \in C$ the set $C - q$ is independent and therefore $[C - q] = C - q$. So, the set $\bigcap_{q \in C} [C - q] = \bigcap_{q \in C} C - q$ is empty.

Now, let M be a CC matroid different from a circuit and let m be a point in M. The matroid $M - m$ is not connected and therefore it splits into its connected components $M - m = M_1 \oplus \cdots \oplus M_t$. Since M is not a circuit not all of the M_i are single point matroids, therefore at least one of them (say M_1) has 3 points. Denote $N = M_2 \oplus \cdots \oplus M_t$. We see that the pair $\{M_1, N + m\}$ is a 2-cut of M and therefore $M_1 \cap N + m$ is empty. Let p be the projection of M_1 to $N + m$. We will show that p is an strong phantom point of M.

Indeed, let q be a point in M. If $q = m$, then we have $p \in \langle M_1 \rangle \subseteq [M - m]$. If $q \in N$, then it is not hard to see that one of the components of $M - q$ (say M_1') contains M_1 and therefore $p \in \langle M_1 \rangle \subseteq \langle M_1' \rangle \subseteq [M - q]$. Finally, suppose that $q \in M_1$, then (since M_1 is connected) $p \in \langle M_1 - q \rangle$ and $p \in \langle N + m \rangle$. Let $C_1 \subseteq M_1 - q$ and

$C_2 \subseteq N + m$ such that $C_1 + p$ and $C_2 + p$ are circuits. By Proposition 7 the matroid $C_1 + C_2$ is a circuit and therefore $p \in \langle C_1 + C_2 \rangle = [C_1 + C_2] \subseteq [M_1 - q + N + m] = [M - q]$. ■

Remark 23 *The strong phantom extension built in the proof of the previous proposition often does not get us outside of a given class of matroids. For example, if the matroid is projective, then this strong phantom point is a projective point (because it is the intersection of two projective flats). The situation for graphic matroids is not so nice because the projection of a 2-cut is not always a pair of vertices of a graph (i.e. the extension may not be graphic). However, an easy analysis of the deletion of an edge from a minimal block shows that in all cases the projection of M_1 to $N + m$ from the previous proof is always a non-edge of M.*

6 The Helly number of a lattice

Let L be a lattice. In order to make our exposition clearer, we will suppose (all the time) that L is complete and atomic i.e. it has a minimum element, every element of L is a join of a set of atoms and every set of atoms has a join. This means that L is defined by a closure operator (denoted by $\langle\rangle$) on the set of atoms. So, the elements of L will be called closed sets and its minimum can be identified with the empty set. The join, is the closure of the union and the meet is equal to the intersection.

We will say that L is **k-Helly** if for any finite family $T \subseteq L$ the condition that every subfamily of T of cardinality k has non-empty intersection is sufficient to conclude that the whole family has non-empty intersection. If L is k-Helly, then it is k'-Helly for any $k' \geq k$. The **Helly number** of L is the minimum number h such that the lattice is h-Helly. Of course, L may have or not Helly number. If every chain of L has bounded length, then it has (see [2]). The classic Helly theorem states that the lattice of all convex sets (ordered by inclusion) of an affine space of dimension d has Helly number $d + 1$. From now on, in all cases, the existence of the Helly number will be obvious and our task will be to find its value. Therefore, we will not make any provisions for the case that the Helly number does not exist. We will denote by $h(L)$ the Helly number of L. For a finite set A of atoms we will call $Z(A) = \bigcap_{a \in A} \langle A - a \rangle$ the **center** of A. Denote by $h^*(L)$ the maximum cardinality of a finite set of atoms of L with empty center.

Proposition 24 *$h(L) = h^*(L)$.*

Proof. Let A be a set of atoms with empty center. The family $T = \{\langle A - a \rangle\}_{a \in A}$ has the property that any subfamily of cardinal $\#A - 1$ has non-empty intersection but the whole family has empty intersection. Therefore L is not $(\#A-1)$-Helly and so $h(L) \geq \#A$. This shows that $h(L) \geq h^*(L)$.

Now, let T be any finite family with empty intersection and such that any of its subfamilies of cardinality $h(L)-1$ has non-empty intersection. By the definition of the Helly number, there must exist $T' \subseteq T$ with empty intersection and such that $\#T' = h(L)$. Any subfamily of T' of cardinality $h(L)-1$ has non-empty intersection and there are $h(L)$ such subfamilies. For each such subfamily we choose an atom in the intersection, thus obtaining a set of atoms A of cardinality $h(L)$ (no two of the chosen atoms can be equal). It is clear that the center of A is contained in the intersection of T' and therefore is empty. So, $h(L) \leq h^*(L)$. ■

Corollary 25 $h(L_1 \times L_2) = h(L_1) + h(L_2)$

Proof. Let A and B be sets of atoms of L_1 and L_2 respectively. Denote $A' = \{(a, \varnothing) \mid a \in A\}$ and $B' = \{(\varnothing, b) \mid b \in B\}$. The set $C = A' + B'$ is a set of atoms of $L_1 \times L_2$. Reciprocally, let C be a set of atoms of $L_1 \times L_2$. Any atom of $L_1 \times L_2$ is of the form $(a, \varnothing)$ or is of the form $(\varnothing, b)$. Therefore, there exist A and B sets of atoms of L_1 and L_2 respectively such that $C = A' + B'$.

The center is a monotone function and therefore $Z(A') \subseteq Z(C)$ and $Z(B') \subseteq Z(C)$. On the other hand, if c is an atom in $Z(C)$, then $c = (a, \varnothing)$ and $c \in Z(A')$ or $c = (\varnothing, b)$ and $c \in Z(B')$. So, $Z(C) = Z(A') + Z(B')$. This equality with an easy reasoning about maximum cardinalities concludes the proof. ■

Proposition 24 is a powerful tool to compute the Helly number of a lattice. For example, in the lattice L of convex sets of the affine space of dimension d any affine independent set of points has empty center. On the other hand, if the set of points A is not affinely independent, then it is very well known (the so called Radon theorem) that there is a partition $A_1 + A_2 = A$ such that $\langle A_1 \rangle \cap \langle A_2 \rangle \neq \varnothing$ (their convex closures intersect). Any point in this intersection is in the center of A. So, the maximal sets with empty center are the affine bases and therefore $h(L) = d+1$ which gives us a proof of the classic Helly theorem. Other easy examples are the lattices of flats of matroids (semimodular atomic lattices). Any independent set has empty center and if $p \in \langle A \rangle \setminus A$, then $A + p$ has p in its center. This shows that here, the maximal sets of points with empty center are the bases and therefore, the Helly number is the rank of the matroid. A more complicated example will be studied bellow.

Let M be a matroid (without loops and parallel elements) and $\pi(M)$ its partition lattice. We will discuss the problem of computing the Helly number of $\pi(M)$. The atoms of $\pi(M)$ are just the points of M. If M is not connected, then $\pi(M)$ splits into the direct product of the partition lattices of its connected components and by Corollary 25 its Helly number is the sum of the Helly numbers of the partition lattices of its connected components.

Let A be a set of points of M. The center of A in $\pi(M)$ depends not only of A but also of M (the ambient). However, no mater what M it is, if A is not connected and $A = A_1 \oplus A_2$, then a straightforward computation (which uses the definition of the closure by components and the distributive properties between union and intersection) gives us that

$$\bigcap_{a \in A} [A - a] = \bigcap_{a \in A_1} [A_1 - a] \cup \bigcap_{a \in A_2} [A_2 - a]$$

and therefore A has empty center in $\pi(M)$ if and only if both A_1 and A_2 have empty center in $\pi(M)$.

So, we have to find out which connected submatroids of M have empty center in $\pi(M)$. If A is connected but not CC, then there is $p \in A$ such that $A - p$ is connected and therefore $p \in \langle A - p \rangle = [A - p]$. This implies that p is in the center of A in $\pi(M)$. Hence our chances to find out connected submatroids with empty center in $\pi(M)$ are limited to single points and CC submatroids. If A is a single point, then its center in $\pi(M)$ always is empty. In Section 5 we called the elements of the center in $\pi(M)$ of a CC matroid "strong phantom points" and proved that circuits do not have any (i.e. circuits have empty center no matter which M is). We also proved that any CC matroid different from a circuit has a strong phantom point. However, its strong phantom points may belong to M or not. So, the situation heavily depends on which CC submatroids of M do not have strong phantom points inside M. We will say that the matroid M is **full** if any of its CC submatroids different from a circuit has a strong phantom point inside M. For example, we know from Remark 23 that any projective space (over any field) is a full matroid and that the circuit matroid of the complete graph is also full. The direct sum of full matroids is full.

Theorem 26 *If M is a full matroid, then its partition lattice is $\lfloor 3 \operatorname{rk}(M)/2 \rfloor$-Helly.*

Proof. If M is full, then all we have to do to find $h(\pi(M))$ is to solve a maximization problem: how big can the number of points be in a submatroid of M which is a direct sum of circuits and points. Let $N = N_1 \oplus \cdots \oplus N_n$ be such submatroid. Let s be the number of N_i which are points. For $j \geq 3$ let t_j the number of N_i which are j-circuits. We have $\#N = s + \sum j t_j$ and the inequality

$$\operatorname{rk}(M) \geq \operatorname{rk}(N) = s + \sum (j-1) t_j = \#N - \sum t_j = \#N + s - n$$

must hold in any matroid M. From this, we conclude that we must make $n - s$ as big as possible. This can be achieved when $t_j = 0$ $(j \geq 4)$, $t_3 = \lfloor \operatorname{rk}(M)/2 \rfloor$ and s

is the residue of the division $\operatorname{rk}(M)/2$. In this case $\#N = s + 3t_3 = \lfloor 3\operatorname{rk}(M)/2 \rfloor$. Thus $h(\pi(M)) \leq \lfloor 3\operatorname{rk}(M)/2 \rfloor$ and this implies the theorem. ■

Remark 27 *Observe from the proof of the previous theorem that the equality* $h(\pi(M)) = \lfloor 3\operatorname{rk}(M)/2 \rfloor$ *holds for full matroids which contain as a submatroid a direct sum of* 3-*cycles (and a point when its rank is odd). This is the case of any projective space: take some lines in it whose direct sum is the whole space (or perhaps is a hyperplane) and choose three different points in each line (and perhaps a point outside the hyperplane). This is also the case of the complete graph: take a chain of triangles (and perhaps an extra edge) each sharing a vertex with each (one or two) of its neighbors.*

For the projective space, partitions are collections of projective subspaces such that each of them do not intersect the join of the others and we will call them **linear partitions**. For any family of linear partitions in $\mathbb{P}^d$ if each subfamily of cardinality $\lfloor 3(d+1)/2 \rfloor$ has non-void intersection, then the whole family has non-void intersection. The number $\lfloor 3(d+1)/2 \rfloor$ is the smallest such that this implication holds. This result is used in [2] to obtain a theorem about flat transversals to flats in the real projective space.

For complete graphs, partitions are graphs whose biconnected components are complete and they are called **cacti**. Consider a family of cacti and let us color the vertices of each cactus in the family with different colors. Suppose that the total number of colors we used is n. If any subfamily of cardinality $\lfloor 3(n-1)/2 \rfloor$ has a pair of colors which occur as an edge in every one of its cacti, then the whole family also has this property. The number $\lfloor 3(n-1)/2 \rfloor$ is the smallest such that this implication holds.

7 Projective pseudobases

A classic theorem in projective geometry is the following: If A and B are two projective spanning circuits, then for any bijection $A \leftrightarrow B$ there exist exactly one projectivity which extends it.

Any projectivity is a matroid isomorphism, i.e. it must preserve the rank of any set of projective points. A projective matroid A will be called **uniquely representable** if for any other set of points B and any matroid isomorphism $A \leftrightarrow B$ there exist a projectivity which extends it. If moreover, A is a fixing set (see Section 1), then any matroid isomorphism extends to a unique projectivity. We will call a set of projective points A **pseudobasis** if it is uniquely representable and it is a minimal fixing set (of the subspace it spans). For a circuit, any bijection is an isomorphism and therefore the theorem cited above may be formulated as: Any circuit is a projective pseudobasis. In this section we give a recursive characterization of projective pseudobases.

The following proposition is straightforward and we will omit its proof.

Proposition 28 *A set of points $A + p$ is uniquely representable if and only if the subdivision of p is uniquely representable.*

Let A be a set of points in a projective space and let p be another projective point. The **envelope** of p with respect to A is the intersection of all the projective subspaces F containing p such that $F = \langle I \rangle$ for some independent set $I \subseteq A$. We will say that the extension $A+p$ is **anchored** if the envelope of p with respect to A is the point p.

Proposition 29 *Let A be a uniquely representable set of real projective points A. The extension $A + p$ is uniquely representable if and only if $A + p$ is anchored.*

Proof. Let $\{C_i\}$ be the family of all circuits of $A + p$ containing p. The envelope of p is equal to $\bigcap \langle C_i - p \rangle$. If it has dimension cero, then there is a unique point in the projective space which extends A to a matroid isomorphic to $A+p$ (because a matroid is defined by its circuits). If moreover, A is uniquely representable, then so is $A + p$. On the contrary, if the envelope T of p has dimension at least 1, then for almost all points $q \in T$ (all but a finite number) we have that $A + q$ is isomorphic to $A + p$ (the isomorphism is the identity in A and maps p to q). If moreover A is uniquely representable, then this matroid isomorphism does not extend to a projectivity. ∎

From Theorem 14 and the two previous propositions we easily obtain the following characterization of projective pseudobases.

Theorem 30 *The class of real projective pseudobases coincides with the class of sets of projective points which can be obtained from 3-point lines by a series of subdivisions and anchored phantom subdivisions.*

Remark 31 *Again, we stated our theorem for real projective spaces, however, all proofs also work for any infinite field.*

Since any graphic extension of a connected matroid is anchored (being defined by a 2-cut of a circuit), then we obtain that a set of projective points whose matroid is the matroid of a minimal block is a pseudobasis and that any graphic connected matroid is uniquely representable. It is easy to see that a matroid is uniquely representable if and only if each of its connected component is uniquely representable. Therefore, we have the following:

Corollary 32 *The projective realization of any graphic matroid is unique up to a projectivity.*

References

[1] J. L. Arocha, J. Bracho and L. Montejano, *On configurations of flats I: Manifolds of points in the projective line*, Discrete Comp. Geometry, **34** (2005), 111-128.

[2] J. L. Arocha, J. Bracho and L. Montejano, *Flat transversals to flats and convex sets of a fixed dimension*, (in preparation).

[3] G. A. Dirac, *Minimally* 2*-connected graphs*, J. Reine Angew Math., **228** (1967), 204-216.

[4] S. T. Hedetniemi, *Characterizations and constructions of minimally* 2*-connected graphs and minimally strong digraphs*, Proceeding of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing, 257-282, Lou. State Univ., Baton Rouge, La., 1971.

[5] U. S. R. Murty, *Extremal critically connected matroids, Discrete Math.*, **8** (1974), 49-58.

[6] J. G. Oxley, *On connectivity in matroids and graphs*, Trans. Amer. Math. Soc., **265** (1981), 47-58.

[7] M. D. Plummer, *On minimal blocks*, Trans. Amer. Math. Soc., **134** (1968), 85-94

[8] P. D. Seymour, *Matroid representation over GF* (3), J. Combin. Theory Ser. B, **26** (1979), 159-173.

[9] W. T. Tutte, *Connectivity in matroids*, Canad. J. Math. **18**, (1966), 1301-1324.

[10] N.L. White, *The bracket ring of a combinatorial geometry. II: Unimodular geometries*, Trans. Amer. Math. Soc., **214** (1975), 233-248.

Keywords and frases:

Critically connected matroids, Helly type theorems, Projectivities, Minimal blocks.

AMS Mathematics Subject Classification 2000:

05B35, 05C40, 52A35,

Addresses:

Omar Antolín
Departamento de Matemáticas,
Facultad de Ciencias, UNAM,
Ciudad Universitaria, México D.F. 04510
e-mail: `omar@arce0.fciencias.unam.mx`

Jorge L. Arocha
Instituto de Matemáticas, UNAM,
Ciudad Universitaria, México D.F. 04510
e-mail: `arocha@math.unam.mx`

Javier Bracho
Instituto de Matemáticas, UNAM,
Ciudad Universitaria, México D.F. 04510
e-mail: `roli@math.unam.mx`

Luis Montejano
Instituto de Matemáticas, UNAM,
Ciudad Universitaria, México D.F. 04510
e-mail: `luis@math.unam.mx`