# The combinatorics of colored triangulations of manifolds

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# THE COMBINATORICS OF COLORED TRIANGULATIONS OF MANIFOLDS

ABSTRACT. Foundations for the topic of crystallizations are proposed through the more general concept of colored triangulations. Classic results and techniques of crystallizations are reviewed from this point of view. A new set of combinatorial invariants of manifolds is defined, and related to the fundamental group and other known invariants. A universal group theoretic approach for this theory is introduced.

A new combinatorial approach to the topology of PL-manifolds has been developing in recent years. It is based on the facts that a graph with colored edges provides precise instructions to construct a space, and that any *manifold* (i.e. PL-manifold) is obtained in this way. Thus, manifolds may be studied through graph theory.

The idea of the construction, due to Pezzana ([18], [19]) and further developed by Gagliardi, Ferri and their group (see the survey [7], and the references), is to take for each vertex of the graph one copy of a standard geometric simplex (whose faces correspond to the colors), and then, each (colored) edge tells us to glue two simplexes along one of their faces (the color says which). Clearly, the space so constructed comes with a rich simplicial structure, which we call a *colored complex*. It is not a classic simplicial complex for two simplexes may meet in more than a single subsimplex (in this sense, it has the flexibility of a pseudocomplex [15] or of a semisimplicial complex [17]). But, on the other hand, it is quite rigid, for every simplex is canonically isomorphic to a standard one (another resemblance to semisimplicial complexes). Thus, the study of manifolds through colored graphs, via colored complexes, is qualitatively different from the classic PL-combinatorial approach, and leads to results of a different nature.

In Section 1 we start from a new general point of view, something like a 'Thinker's Toy' (as in [2]). A colored graph $G$ (Subsection 1.1) is thought of as an 'instructions manual' to build spaces: if we supply a colored space $X$ (a space with a fixed subspace of each color (Subsection 1.2) to act as a 'building block', we obtain a new space $|G;X|$ (Subsection 1.3) by glueing copies as $G$ says. Of course, the main interest lies on $|G|$ (obtained, as above, when $X$ is the standard simplex, (Example 1.4(i)), of which all manifolds are examples (Example 1.4(ii)). But the spaces $|G; X|$ have a lot to do with $|G|$. To talk about links, regular neighborhoods, canonical decompositions and other interesting subspaces of $|G|$, one only has to analyze the basic 'building blocks' and the obvious functoriality of the construction (Section 1-2) takes care of the rest.

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The development of this general point of view makes precise many of the techniques in the 'folklore' of the subject, and leads naturally to the restatement of some known results, throwing new light upon them and placing them in a unified context (due credit is given within the text). In this work, which is basically self-contained, we hope to make the basic combinatorics of this approach clear to the topologist, and the basic topology clear to the combinatorist. Some more geometric consequences of this general point of view are explored in [1] and [2], and in a sequel to this work.

In Section 2 colored complexes are defined and those associated to colored graphs and to manifolds are briefly studied.

Since every manifold $M$ has associated a family of colored graphs $\mathscr{G}(M)$ (its colored triangulations), one obtains invariants of the manifold from graph invariants (see, e.g., [10]). A family of numeric invariants of this type is presented in Section 3. The simplest of them, for example, is the minimum number of vertices among the graphs in $\mathscr{G}(M)$, we call it the *complexity* of $M$. It classifies compact manifolds (of a fixed dimension) up to a finite ambiguity (Theorem 3.12), and in dimension 2 it works just like the Euler characteristic (Theorem 3.13).

In principle, one should be able to compute all the topological invariants of a manifold $|G|$ in terms of the colored graph $G$. But it is important to have precise combinatorial algorithms to do so. Two descriptions of the fundamental group have been proposed ([9] and [21]). In Section 4 we give yet another one, which, in computational and combinatorial terms, is quite simple and natural. It was obtained independently by Donati [3] and Grasselli [8]. It is a presentation with the edges of a fixed color as generators and bicolored cycles as relations, and gives bounds for the rank of the fundamental group in terms of the numeric invariants of Section 3.

Finally, in Section 5 we observe that the study of $n$-manifolds can be reduced to the study of subgroups of the free product of $n + 1$ copies of $\mathbb{Z}_2$ (the integers mod 2). This is because well-colored graphs of dimension $n$ correspond to conjugacy classes of subgroups of this fixed group.

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## 1. Preliminaries

Throughout this work $\mathscr{C}$ will be a *color set*, which is a finite set whose elements are refered to as *colors*. Its dimension is its cardinality minus 1 and will be denoted by $n$ ($n = \dim(\mathscr{C}) = \#\mathscr{C} - 1$).

1-1. *Basic Definitions*

1.1. A *$\mathscr{C}$-colored graph* $G$ (or *$\mathscr{C}$-graph* for short) is a triplet $G = (V(G), E(G), \gamma_G)$ where $V(G)$ (or simply $V$ if no confusion arises) is the set of vertices; $E(G)$ (or $E$) is the set of edges, each of which is *attached to* (or *joins*) a pair of vertices; and $\gamma_G = \gamma: E \to \mathscr{C}$ is an arbitrary but fixed *coloring* of the edges. Multiple edges, i.e. several edges attached to the same pair of vertices, are allowed. But no loops.

1.2. A *$\mathscr{C}$-colored space*, or *$\mathscr{C}$-space*, is a topological space $X$ together with a collection of subspaces $\{X_c\}_{c\in\mathscr{C}}$ indexed by the colors in $\mathscr{C}$.

1.3. Given a $\mathscr{C}$-graph $G$ and $\mathscr{C}$-space $X$, the *realization of $G$ with $X$*, $|G; X|$, is the space obtained by taking one copy of $X$ for each vertex of $G$, and then glueing two copies by the identity along the subspace $X_c$ whenever an edge of color $c$ joins the corresponding vertices. Namely

$$|G; X| = V(G) \times X/\sim,$$

where $\sim$ is generated by the following: for each $e \in E(G)$, let $u, v \in V(G)$ be the vertices to which $e$ is attached, then

$$(u, x) \sim (v, x) \quad \text{for every } x \in X_{\gamma(e)}.$$

Observe that $|G; X|$ comes with a natural $\mathscr{C}$-space structure: $|G; X|_c$ is the image of $V(G) \times X_c$ in $|G; X|$.

1.4. EXAMPLES. (i) *The $\mathscr{C}$-simplex.* Let $\langle\mathscr{C}\rangle$ be the geometric simplex spanned by $\mathscr{C}$, and for each $c \in \mathscr{C}$ let $\langle\mathscr{C}\rangle_c$ be the opposite face to the vertex $c$. That is, if $\bar{c}$ denotes the complement of $\{c\}$ in $\mathscr{C}$, then $\langle\mathscr{C}\rangle_c = \langle\bar{c}\rangle$.

This is the basic example of a $\mathscr{C}$-space (a simplex with its faces painted with different colors), and thus we simplify notation in this case by putting

$$|G| = |G: \langle\mathscr{C}\rangle|,$$

and calling $|G|$ the *realization* of $G$.

(ii) *Manifolds and regular graphs.* Let $M$ be an $n$-dimensional manifold. (We shall work in the PL category; thus *manifold* will mean PL-manifold.) For simplicity of the exposition, we shall assume that $M$ has no boundary. Let $K$ be a simplicial complex triangulating $M$. Let $K'$ be its barycentric subdivision, and denote by $K'_{(r)}$ the set or $r$-simplexes of $K'$.

The 'dimension function' $K'_{(0)} \to \Delta_n = \{0, 1, \ldots, n\}$ which assigns to each barycenter the dimension of its corresponding simplex, induces a natural coloring $\gamma: K'_{(n-1)} \to \Delta_n$ since the vertices of each $(n-1)$-simplex miss exactly one color (element) of $\Delta_n$. Then we define a $\Delta_n$-colored graph $G_K$ by putting $V(G_K) = K'_{(n)}$, $E(G_K) = K'_{(n-1)}$ and $\gamma$ as above. ($\partial M = \emptyset$ implies that every edge is attached to exactly two vertices.) Clearly,

$$|G_K| = |K'| \cong M.$$

Thus, *every manifold is the realization of a colored graph.*

The question arises of which graphs realized closed manifolds. An obviously necessary condition is that it be *regular* – that is, that at each vertex there is exactly one edge of each color, so that, in its realization, each face of each simplex is glued with precisely another. But, of course, this is not sufficient (see [7, §2]).

(iii) *Spheres and projective spaces.* Not all the graphs that realize manifolds are of the type $G_K$ as above. In fact, the interesting ones are those which are not so 'wasteful'. The simplest example of a regular graph, called the *$\mathscr{C}$-dipole* [6], consists of two vertices joined by $n + 1$ edges, one of each color. It realizes the sphere $S^n$. (Observe that it is not a classic simplicial complex.) We give another two examples (for further examples, see [7] or [2]).

First, let $Q(\mathscr{C})$, 'the *$\mathscr{C}$-colored cube*', be the $\mathscr{C}$-graph whose vertices are all subsets of $\mathscr{C}$, and put an edge of color $c$ between vertices (subsets) that differ only by the element $c$ (e.g. between $\emptyset$ and $\{c\}$). $Q(\mathscr{C})$ is clearly regular, and it is easy to see by induction that $|Q(\mathscr{C})| = S^n$.

Now, we form a new graph $P(\mathscr{C})$ by adding to $Q(\bar{c})$, where $c \in \mathscr{C}$ is any color and $\bar{c} = \mathscr{C} - \{c\}$, edges of color $c$ between all pairs $\{\alpha, \bar{c} - \alpha\}$ with $\alpha \subset \bar{c}$. It is a nice and easy exercise to prove that $|P(\mathscr{C})| = \mathbb{R}P^n$, and that $P(\mathscr{C})$ does not depend on the choice of $c \in \mathscr{C}$ (see Example 1.5(i) below).

(Observe that $Q(\mathscr{C})$ and $P(\mathscr{C})$ correspond to the 'ortant triangulations' of $S^n$ and $\mathbb{R}P^n$, respectively.)

## 1-2. *Functoriality*

There are obvious notions for morphisms between colored graphs and between colored spaces. With them, the realization becomes functorial.

Given a *color map* $\rho: \mathscr{C} \to \mathscr{C}'$ between color sets, a *map over* $\rho$ between a $\mathscr{C}$-space $X$ and a $\mathscr{C}'$-space $X'$ is a map $f: X \to X'$ such that $f(X_c) \subset X'_{\rho(c)}$ for each $c \in \mathscr{C}$; and a *morphism over* $\rho$ between a $\mathscr{C}$-graph $G$ and $\mathscr{C}'$-graph $G'$ is a graph morphism $g: G \to G'$ (vertices go to vertices and edges to edges, preserving adjacency) such that $\gamma'(g(e)) = \rho(\gamma(e))$ for each $e \in E(G)$. With these ingredients, one gets naturally a map over $\rho$

$$|g; f|: |G; X| \to |G'; X'|.$$

Maps and morphisms over the identity of $\mathscr{C}$ will be simply denoted *$\mathscr{C}$-maps* and *$\mathscr{C}$-morphisms.*

1.5. EXAMPLES. (i) In Example 1.4(iii) above, one has a $\mathscr{C}$-morphism $a\colon Q(\mathscr{C}) \to Q(\mathscr{C})$ sending a vertex $\alpha \subset \mathscr{C}$ to $a(\alpha) = \bar{\alpha} = \mathscr{C} - \alpha$, which realizes the antipodal map of $S^n$.

Clearly, $a$ induces a $\mathbb{Z}_2$-action on the $\mathscr{C}$-graph $Q(\mathscr{C})$. Its 'orbit $\mathscr{C}$-graph' is precisely $P(\mathscr{C})$ (which was defined by choosing a representative in each orbit).

(ii) *Regular surfaces.* Given a cyclic permutation of $\mathscr{C}$, $\varepsilon$ say, let $P_\varepsilon$ be the $\mathscr{C}$-space consisting of a regular plane polygon with $n+1$ sides colored according to $\varepsilon$. If $G$ is a regular $\mathscr{C}$-graph, one easily sees that $|G; P_\varepsilon|$ is a closed surface. In order to embed $|G; P_\varepsilon|$ in $|G|$, it is enough, by functoriality, to embed $P_\varepsilon$ in $\langle\mathscr{C}\rangle$ in a colored manner. This is done by sending the barycenter of the polygon to the barycenter of $\langle\mathscr{C}\rangle$, $b_{\mathscr{C}}$; the barycenter of the $c$-colored side of $P_\varepsilon$ to the barycenter of $\langle\bar{c}\rangle$, $b_{\bar{c}}$; and the vertex where the sides of colors $c$ and $d$, say, meet to $b_{\overline{\{c,d\}}}$. And then extending linearly. Thus, we obtain $|G; P_\varepsilon| \to |G|$, which is called a *regular surface of* $|G|$.

Regular surfaces were studied by Gagliardi and Ferri ([5], [6], [10] and [11]). They give rise to an invariant of closed oriented manifolds, the *regular genus*, by letting $g(M)$ be the minimum genus among the regular surfaces of $M$ (to be given more precisely in Section 3). Regular genus is a natural generalization of genus and Heegaard genus, and it characterizes spheres; that is, $g(M^n) = 0$ iff $M^n \cong S^n$ ([5], [6]).

(iii) *The geometric graph.* Let $\mathrm{St}(\mathscr{C})$, 'the $\mathscr{C}$-*star*', consists of $n+1$ intervals (1-simplices) with one of their endpoints identified (to a center, say), then color the 'free' endpoints with the colors in $\mathscr{C}$ ($\mathrm{St}(\mathscr{C}) = \mathscr{C} \times [0,1]/\mathscr{C} \times \{0\}$, $\mathrm{St}(\mathscr{C})_c = \{(c,1)\}$). If $G$ is a regular $\mathscr{C}$-graph then $|G; \mathrm{St}(\mathscr{C})|$ coincides with the geometric picture we have of the graph $G$. But

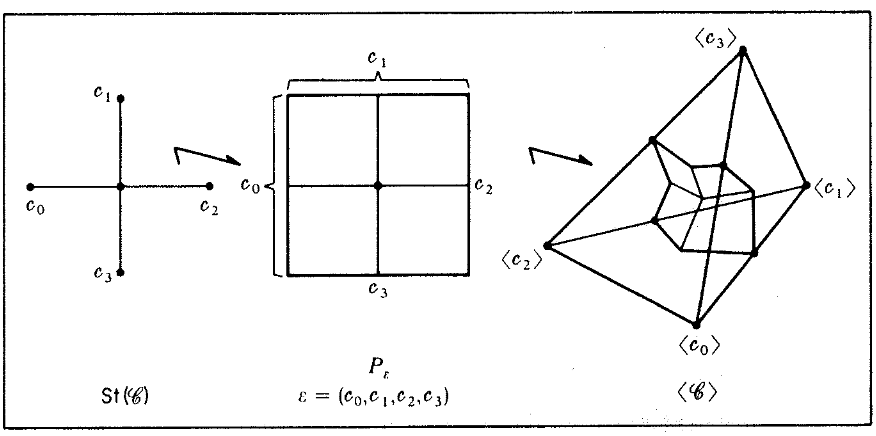



Fig. 1.

we have obvious colored embeddings of St($\mathscr{C}$) into $\langle\mathscr{C}\rangle$ and into $P_\varepsilon$: send the center to the barycenter and each $(c, 1)$ to the barycenter of the corresponding face. Thus, *we may think goemetrically of the graph G as embedded in* $|G|$ (or in $|G; P_\varepsilon|$).

## 2. Colored complexes

Clearly, the space $|G|$ (where $G$ is a $\mathscr{C}$-colored graph as usual) comes with an underlying simplicial structure. It has been well used in the work of Gagliardi and Ferri's group, but it should be made explicit.

It is a good custom to differentiate a simplicial complex from its realization. Thus, we define a *pseudocomplex* $\Delta G$ that captures the simplicial structure of $|G|$. (The only difference between a pseudocomplex and a simplicial complex is that the intersection of two simplices may be a union of common subsimplices instead of a single one; see [15].) However, in our case there is some extra structure.

2.1. A *$\mathscr{C}$-colored complex* is a pseudocomplex together with fixed and compatible isomorphisms of each simplex with some face $\langle\alpha\rangle \hookrightarrow \langle\mathscr{C}\rangle$ $(\alpha \subset \mathscr{C})$. Equivalently, it is a pseudocomplex together with a fixed non-degenerate ($r$-simplices go to $r$-simplices) simplicial map to $\Delta\mathscr{C}$, where $\Delta\mathscr{C}$ is the obvious simplicial complex $(\Delta\mathscr{C} = \{\langle\alpha\rangle \mid \alpha \subset \mathscr{C},\ \alpha \neq \emptyset\})$ whose realization is $\langle\mathscr{C}\rangle$.

For example, the barycentric subdivision of any $n$-dimensional simplicial complex (or pseudocomplex) is a $\Delta_n$-colored complex (see Example 1.4(ii)).

2.2. REMARK. If the colors are ordered (i.e. if an isomorphism $\Delta_n \xrightarrow{\sim} \mathscr{C}$ is given) then a $\mathscr{C}$-colored complex gives rise naturally to a semisimplicial complex. (In fact, one could equivalently define colored complexes as semisimplicial complexes satisfying some extra conditions.) Thus, classic algebraic topology is nearby.

### 2-1. *Residues*

Let $G$ be a $\mathscr{C}$-graph. To define $\Delta G$ combinatorially we have to introduce the notion of residues (see [21]).

$\alpha$ will always denote a subset of $\mathscr{C}$, and $\bar{\alpha}$ its complement.

Let $G_\alpha$ be the maximal $\alpha$-colored subgraph of $G$, that is, $V(G_\alpha) = V(G)$, $E(G_\alpha) = \gamma^{-1}(\alpha)$ and $\gamma_\alpha: E(G_\alpha) \to \alpha$ is the restriction of $\gamma$.

2.3. A *residue* of $G$ is a connected component of $G_\alpha$ for some proper $\alpha \subset \mathscr{C}(\alpha \neq \mathscr{C})$. It is also called an *r-residue* where $r = \#\alpha(\leqslant n)$, or an $\alpha$-

*residue*. The notation $H \lhd G$ stands for '*H is a residue of G*'. And let $\mathrm{Res}_\alpha(G)$ ($\mathrm{Res}_r(G)$) be the set of all $\alpha$-residues ($r$-residues) of $G$. Thus, for example, $\mathrm{Res}_\varnothing(G) = \mathrm{Res}_0(G) = V(G)$ and $G$ is regular iff $E(G) = \mathrm{Res}_1(G)$.

Now, *define* $\Delta G$ to have for each $\alpha$-residue $H$ a copy of $\langle\bar{\alpha}\rangle$ called $\sigma_H$, and make $\sigma_H$ the appropriate face of $\sigma_{H'}(\sigma_H < \sigma_{H'})$ whenever $H' \lhd H$.

Clearly, $\Delta_G$ is a $\mathscr{C}$-colored complex.

2.4. PROPOSITION. *The realization of* $\Delta G$ $|\Delta G|$, *is canonically homeomorphic to* $|G|$; *that is*, $|G| = |\Delta G|$.

*Proof.* First note that the simplices of type $\langle\mathscr{C}\rangle$ in $\Delta G$ correspond to $V(G)$. Then observe, from Subsection 1.3 and from the definition of $|G|$ that given $x \in \text{interior}\ (\langle\bar{\alpha}\rangle) \subset \langle\mathscr{C}\rangle$ and $u, v \in V(G)$ then $(u, x) \sim (v, x)$ iff $u$ and $v$ lie in the same $\alpha$-residue. □

The barycentric subdivision of $\Delta G$ denoted $\Delta' G$, which is a simplicial complex in the usual sense, plays an important role. It has a 0-simplex $b_H$ for each residue $H \lhd G$ ($b_H$ is the barycenter of $\sigma_H$), and an $r$-simplex $\langle b_{H_0}, \ldots, b_{H_r}\rangle$ for each chain $H_r \lhd H_{r-1} \lhd \cdots \lhd H_0$.

The usual notion of link has to be slightly modified for pseudocomplexes so that it captures the classic idea; see, for example, [8]. Given a simplex $\sigma$ of a pseudocomplex $K$, let $\mathrm{link}_K(\sigma)$ be the subcomplex of $K'$, the barycentric subdivision of $K$, generated by all barycenters $b_\tau$ such that $\sigma < \tau$ and $\sigma \neq \tau$. Observe then that the classic link of $b_\sigma$ in $K'$ is precisely the join $\mathrm{link}_K(\sigma) * (\partial\sigma)'$.

2.5. PROPOSITION. *For any residue H of G*:

$$\mathrm{link}_{\Delta G}(\sigma_H) = \Delta' H \subset \Delta' G.$$

*Proof.* The inclusion $\Delta' H \hookrightarrow \Delta' G$ is given by: if $H_0$ is a residue of $H$, it is also a residue of $G$, thus send $b_{H_0}$ to $b_{H_0}$. Now apply the given definition of link. □

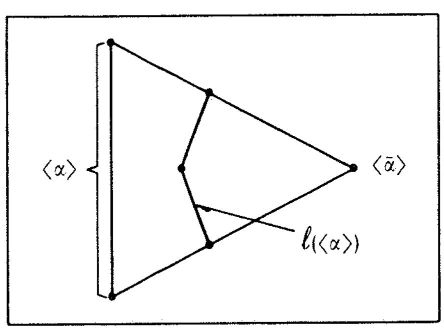



Fig. 2.

This proposition can also be seen with the Tinker-Toy. Suppose $H$ is an $\alpha$-residue. Let $\ell\colon \langle\alpha\rangle \to \langle\mathscr{C}\rangle$ be the colored map over the inclusion $\alpha \hookrightarrow \mathscr{C}$ defined simplicially on the barycentric subdivision by sending the barycenter $b_{\langle\beta\rangle} = b_\beta$, where $\beta \subset \alpha$, to $\ell(b_\beta) = b_{\beta\cup\bar{\alpha}}$. The image of $\ell$ is the link of $\langle\bar{\alpha}\rangle$ in $\langle\mathscr{C}\rangle$.

Together with the inclusion $\iota\colon H \to G$, $\ell$ gives an embedding $|\iota;\ell|\colon |H| \hookrightarrow |G|$, which realizes the link of $\sigma_H$. More precisely, we have commutative diagrams

$$\begin{array}{ccc} |H| & \xrightarrow{|\iota;\ell|} & |G| \\ \uparrow\cong & & \uparrow\cong \\ |\Delta' H| & \hookrightarrow & |\Delta' G| \end{array} \qquad\qquad \begin{array}{ccc} |H| & \xrightarrow{|\iota;\ell|} & |G| \\ \uparrow & & \uparrow \\ \cup & & \cup \\ H & \subset\!\!\longrightarrow & G \end{array}$$

where, in the first diagram, the vertical homeomorphisms are given by Proposition 2.4 and the bottom inclusion by Proposition 2.5, and in the second (assuming that $G$ is regular and hence also $H$), the graphs are thought of geometrically as in Example 1.5(iii). Thus, if $H \lhd G$, this is the natural way to think of $|H|$ in $|G|$, which we simply write, from here on, as $|H| \hookrightarrow |G|$.

## 2-2. *Manifold Graphs*

A $\mathscr{C}$-graph $G$ is a *manifold graph* if for every residue $H$, $|H|$ is a sphere. That is, 'if $\Delta G$ is a combinatorial manifold'. Recall that in general this is not equivalent to '$|G|$ is a topological manifold' [13].

If $G$ is a manifold graph, we have a canonical cell decomposition of $|G|$, dual to $\Delta G$. It consists of an $r$-cell for every $r$-residue. If $H$ is a residue, the corresponding cell is the cone from $b_H$, the barycenter of $\sigma_H$, to its link $|H|$. This we write as

$$(2.6) \qquad b_H * |H| \hookrightarrow |G|,$$

where $*$ denotes join. In particular, observe that the 1-skeleton of the dual cell decomposition corresponds precisely to the geometric graph $G \hookrightarrow |G|$. See also [16].

Observe that for any $\mathscr{C}$-graph $G$ we still have the inclusion of the cone (2.6), but in general it will not be a cell.

## 2-3. *Colored Cores and their Regular Neighborhoods*

A colored complex always comes with distinguished subcomplexes: those spanned by a fixed subset of colors. Furthermore, in the case of the realization of graph, $|G|$, one can also break it into canonical pieces which, analyzed separately, give a powerful technique to study $|G|$. (Used, for example, in [5], [6].) The precise definitions follow.

If $X$ is any $\mathscr{C}$-space, it seems natural to define, for any $\alpha \subset \mathscr{C}$, $x_\alpha = \cap_{c \in \alpha} X_c$ for $\alpha \neq \emptyset$ and $X_\emptyset = X$. And then, if $G$ is a $\mathscr{C}$-graph, $|G|_\alpha$ is the subspace of $|G|$ formed by all simplexes of type $\langle \bar{\alpha} \rangle$ (for $\langle \mathscr{C} \rangle_\alpha = \langle \bar{\alpha} \rangle$), we call it the *$\alpha$-core* of $|G|$. Clearly, $|G|_\alpha$ is the realization of an $\bar{\alpha}$-colored complex, but moreover, it is the realization of an $\bar{\alpha}$-graph. (Namely, take a vertex for each $\alpha$-residue and join two by a $c$-edge, $c \in \bar{\alpha}$, iff there is a $c$-edge between the corresponding residues.)

Now, if $\alpha$ is proper – that is, if $\varphi \neq \alpha \neq \mathscr{C}$ – then $|G| \neq |G|_\alpha \neq \emptyset$, and to describe the regular neighborhood of $|G|_\alpha$ in $|G|$, which we denote $\mathrm{RN}_{|G|}(|G|_\alpha)$, it is enough to analyze the standard pieces. Let $R(\bar{\alpha})$ be the subspace of $\langle \mathscr{C} \rangle$ spanned (in the barycentric subdivision) by all barycenters $b_\beta$ (of $\langle \beta \rangle \hookrightarrow \langle \mathscr{C} \rangle$) such that $\bar{\alpha} \cap \beta \neq \emptyset$ (see Figure 3), and endow $R(\bar{\alpha})$ with the obvious $\mathscr{C}$-space structure, as subspace of $\langle \mathscr{C} \rangle$. (In general, if $Y$ is a subspace of a $\mathscr{C}$-space $X$, put $Y_c = Y \cap X_c$.) Then, since $R(\bar{\alpha})$ is the standard regular neighborhood of $\langle \bar{\alpha} \rangle \hookrightarrow \langle \mathscr{C} \rangle$, we clearly obtain

$$\mathrm{RN}_{|G|}(|G|_\alpha) = |G; R(\bar{\alpha})|. \tag{2.7}$$

Let $T(\alpha) = T(\bar{\alpha}) = R(\alpha) \cap R(\bar{\alpha})$, considered naturally as a $\mathscr{C}$-subspace of $\langle \mathscr{C} \rangle$. (It is spanned by barycenters $b_\beta$ such that $\alpha \cap \beta \neq \emptyset \neq \bar{\alpha} \cap \beta$, and homeomorphic to $\langle \alpha \rangle \times \langle \bar{\alpha} \rangle$.)

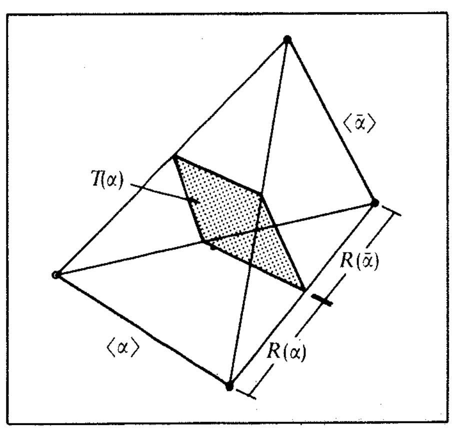



Fig. 3.

And now, glueing the standard pieces, we have a canonical decomposition of $|G|$ into regular neighborhoods of complementary colored cores:

$$(2.8) \qquad |G| = |G; R(\alpha)| \underset{|G;\, T(\alpha)|}{\cup} |G; R(\bar{\alpha})|.$$

Observe that if $|G|$ is a manifold then $|G; T(\alpha)|$ is a submanifold of codimension 1. In fact,

$$|G; T(\alpha)| = \partial|G; R(\alpha)| = \partial|G; R(\bar{\alpha})| = |G; R(\alpha)| \cap |G; R(\bar{\alpha})|.$$

## 3. Numeric invariants

In the first section, we establish some useful notation and terminology. And at the same time, well-known and classic results are translated to our setting. The new invariants are studied thereafter.

### 3-1. *The Euler Characteristic*

Let $\mathcal{M}^n$ be the set of (PL) homeomorphism classes of connected and closed (PL) manifolds of dimension $n$. A basic problem (or the basic problem) of PL topology is to classify $\mathcal{M}^n$. It is only solved in dimension 2 in terms of two invariants (orientability and the Euler characteristic).

Let $\mathcal{G}^n$ be the family of finite connected regular $\mathcal{C}$-colored graphs (recall that $n = \#\mathcal{C} - 1$), and let $\mathcal{MG}^n$ be the subclass of manifold graphs. We say that two graphs $G$, $G' \in \mathcal{G}^n$ are equivalent if $|G|$ is (PL) homeomorphic to $|G'|$. And finally, given a manifold $M \in \mathcal{M}^n$ let $\mathcal{G}(M)$ consist of those manifold graphs that realize $M$ (its colored triangulations):

$$(3.1) \qquad \mathcal{G}(M^n) = \{G \in \mathcal{MG}^n \mid |G| \cong M\}.$$

Basic problems are to characterize in combinatorial terms the equivalence relation in $\mathcal{G}^n$, and the subclass $\mathcal{MG}^n$. A good deal has been done in the first direction, see [6]. Observe that the second one is equivalent to characterizing $\mathcal{G}(S^{n-1})$, for $G \in \mathcal{G}^n$ is in $\mathcal{MG}^n$ iff all its $n$-residues are in $\mathcal{G}(S^{n-1})$.

Again, everything is solved in dimension 2: $\mathcal{MG}^2 = \mathcal{G}^2$ (since $\mathcal{G}(S^1) = \mathcal{G}^1$), and equivalence is detected by two combinatorial invariants.

A graph is *bipartite* if its vertices can be partitioned into two disjoint subsets such that every edge goes from one to the other, or, equivalently, if it has no odd cycles [14]. It is easy to see that if $G \in \mathcal{G}^n$ then $|G|$ is orientable iff $G$ is bipartite.

Given a $\mathscr{C}$-colored graph $G$, let

(3.2) $\beta_k(G) = \#\mathrm{Res}_{n-k}(G), \quad 0 \leqslant k \leqslant n.$

Observe that we have chosen the geometric meaning for the subindices. $\beta_k(G)$ is the number of $k$-simplices of $\Delta G$, thus, it is somewhat like a 'Betti number'. Now, the *Euler characteristic* of $G$ is defined as

(3.3) $$\chi(G) = \sum_{k=0}^{n} (-1)^k \beta_k(G),$$

which clearly is the topological Euler characteristic of $|G|$.

The classic classification of compact surfaces gives:

3.4. THEOREM. *For any $G, G' \in \mathscr{G}^2$, we have:*

(i) $|G| = |G'|$ *iff both $G$ and $G'$ are bipartite or not and* $\chi(G) = \chi(G')$.
(ii) $\chi(G) \leqslant 2$ *and equality holds iff* $G \in \mathscr{G}(S^2)$. □

This characterization of $\mathscr{G}(S^2)$ yields the following one for $\mathscr{MG}^3$.

3.5. THEOREM. *Given $G \in \mathscr{G}^3$, then $\chi(G) \geqslant 0$, and equality holds iff $G \in \mathscr{MG}^3$.*

*Proof.* From (ii) of the preceding theorem, we clearly obtain

$$\sum_{H \in \mathrm{Res}_3(G)} \chi(H) \leqslant 2\beta_0(G),$$

and equality holds iff $G \in \mathscr{MG}^3$. Developing the left-hand side, and using that every $i$-residue ($i = 0, 1, 2$) is contained in exactly $4 - i$ 3-residues, we obtain that it equals $2\beta_1(G) - 3\beta_2(G) + 4\beta_3(G)$, which, together with the immediate

3.6. LEMMA. *If $G \in \mathscr{G}^n$ then* $\beta_{n-1}(G) = \frac{1}{2}(n+1)\beta_n(G)$ □

yields

$$-2\chi(G) \leqslant 0 \text{ and equality holds iff } G \in \mathscr{MG}^3.$$ □

Following this line of thought, the immediate problem is to characterize $\mathscr{G}(S^3)$, the graphs that realize the 3-sphere. A work in reference [1] is related to this problem.

### 3-2. *Complexities*

Although the Betti numbers are not topological invariants of graphs (equivalent graphs may have them different), they produce invariants for manifolds.

Given $M \in \mathcal{M}^n$, the *k-complexity* of $M$, $0 \leqslant k \leqslant n$, is

(3.7) $$\kappa_k(M) = \min\{\beta_k(G) \mid G \in \mathcal{G}(M)\}.$$

In words, it is the minimum number of $k$-simplices a colored triangulation of $M$ must have.

Since one always has at least $\binom{n+1}{k+1}$ $k$-simplices,[1] it will be useful to define the *reduced complexities* as

(3.8) $$\tilde{\kappa}_k(M) = \kappa_k(M) - \binom{n+1}{k+1} \quad \text{for} \quad 0 \leqslant k \leqslant n-1$$

$$\tilde{\kappa}_n(M) = \kappa_n(M) - 2.$$

The case $k = n$ is special because any regular graph[1] has at least two vertices.

3.9. THEOREM. *Reduced complexities are subadditive. That is, if $M$, $M' \in \mathcal{M}^n$, $\tilde{\kappa}_k(M \# M') \leqslant \tilde{\kappa}_k(M) + \tilde{\kappa}_k(M')$, where $M \# M'$ is the connected sum.*

*Proof.* This is a consequence of the fact that connected sum can be performed combinatorially (see [6]).

Given $G$, $G' \in \mathcal{G}^n$, and $v \in V(G)$ and $v' \in V(G')$. The connected sum of $G$ and $G'$ along $v$ and $v'$, $G_v \#_{v'} G'$, or $G \# G'$, is the following regular $\mathcal{C}$-graph. Consider the disjoint union of $G - v$ and $G' - v'$ ($G - v$ is obtained by deleting $v$ and all edges incident to it); and then, for each color $c \in \mathcal{C}$, put an edge of color $c$ between $v_c$ and $v'_c$, where $v_c(v'_c)$ is the unique vertex of $G(G')$ $c$-adjacent to $v$ (i.e. there is an edge of color $c$ from $v$ to $v_c$).

Geometrically, $|G \# G'|$ is obtained from $|G|$ and $|G'|$ by deleting the interior of a simplex from each one, and then identifying the (colored) boundaries by the identity. Thus, if $|G|$ and $|G'|$ are manifolds then $|G \# G'| = |G| \# |G'|$.

Moreover:

$$\beta_k(G \# G') = \beta_k(G) + \beta_k(G') - \binom{n+1}{k+1} \quad 0 \leqslant k \leqslant n-1$$

$$\beta_n(G \# G') = \beta_n(G) + \beta_n(G') - 2.$$

The theorem follows. □

Now, we give some results on particular complexities, starting with dimension 0.

3.10. THEOREM. *For any $M \in \mathcal{M}^n$, $\tilde{\kappa}_0(M) = 0$.*

[1] Formally, we should convene that graphs and manifolds are non-void.

This is just another way to write Pezzana's crystallization theorem ([18], [19]). A *crystallization* ([4], [18], [19]) is a manifold graph $G$, such that $G_{\hat{c}}$ is connected for all $c \in \mathscr{C}$; that is, such that $\Delta G$ has a unique vertex of type $\langle c \rangle$ for each $c \in \mathscr{C}$ (which happens iff $\beta_0(G) = n + 1$).

3.10.′ THEOREM (Pezzana's crystallization theorem). *Every connected closed manifold admits a crystallization.*

*Proof.* It is enough to see that any $G \in \mathscr{M}\mathscr{G}^n$ can be reduced to a crystallization. Suppose $G_{\hat{c}}$ is not connected. Since $G$ is connected, there exists a $c$-colored edge $e$ attached to vertices $v$ and $v'$ lying in different components of $G_{\hat{c}}$, $H$ and $H'$ say. Let $G/e$ be the following $\mathscr{C}$-graph: in $G_{\hat{c}}$ change $H \amalg H'$ for $H_v \#_{v'} H'$, then put back all $c$-colored edges but $e$. One easily sees that $G/e \in \mathscr{M}\mathscr{G}^n$ and that $|G| \cong |G/e|$. Then, through a finite sequence of these crystallization moves, $G$ can be reduced to a crystallization $G'$ realizing the same manifold ($|G| \cong |G'|$). □

3.11. COROLLARY. *Given $M \in \mathscr{M}^n$, and $0 \leqslant k \leqslant n$. Then, if $G$ realizes $\kappa_k(M)$ (i.e. if $G \in \mathscr{G}(M)$ and $\kappa_k(M) = \beta_k(G)$) then $G$ is a crystallization.*

*Proof.* In the preceding proof, $\beta_k(G/e) < \beta_k(G)$ for all $0 \leqslant k \leqslant n$. □

Observe that if $\kappa_0$ is generalized to non-connected manifolds it measures the number of components.

Finally, we turn our attention to $\kappa_n$, which deserves to be called simply the *complexity* and to be denoted by $\kappa$. It is the minimum number of simplices one needs to build a manifold in a colored way.

3.12. THEOREM. *$\kappa: \mathscr{M}^n \to 2\mathbb{N}$ is finite to one. And moreover, $\tilde{\kappa}(M^n) = 0$ iff $M^n = S^n$.*

*Proof.* Every regular $\mathscr{C}$-graph has an even number of vertices, and only finitely many isomorphism classes have the same number of vertices, and only one has two. □

3.13. THEOREM. *For any closed surface $M^2$, $\tilde{\kappa}(M) = 4 - 2\chi(M)$.*

*Proof.* Let $G$ realize $\kappa_2(M)$. Then $\chi(M) = \beta_0(G) - \beta_1(G) + \beta_2(G)$, but $\beta_0(G) = 3$ by Corollary 3.11 and $\beta_1(G) = \frac{3}{2}\beta_2(G)$ by Lemma 3.6. The theorem follows. □

Thus, we may regard the complexity as a generalization of the Euler characteristic in dimension 2 (for classification purposes it serves just as well) preserving its nice property of classifying manifolds up to finite ambiguity.

A few remarks should be made to Theorem 3.12. It gives a function $(\# \kappa^{-1}): 2\mathbb{N} \to \mathbb{N}$. Is it bounded as it is in dimension 2? (Observe that the proof was quite rough and leaves ample room for improvement: among the regular $\mathscr{C}$-graphs with $2m$ vertices, say – whose number could be computed combinatorially – only a 'few' will be manifold graphs; among these, there are representatives of all manifolds with smaller complexities, and several of the 'new' ones could be equivalent.) When is it non-zero? For example, with four vertices one easily sees that, except for $\mathbb{R}P^2$, all manifold graphs give spheres, then $(\#\kappa_n^{-1}(4)) = 0$ for $n \geqslant 3$. Does this fit in a more general and regular pattern?

The notion of complexity gives rise to an interesting family of colored graphs: $G \in \mathscr{MG}^n$ is *minimal* if $\kappa(|G|) = \beta_n(G)$. But very little is known about them.

### 3-3. *Colored Complexities*

Since the $k$-simplices of colored complexes are differentiated by their colors (their type), we can 'refine' the complexities as follows.

Given a $\mathscr{C}$-graph $G$, let

$$\begin{aligned}(3.14)\qquad &\beta_\alpha(G) = \#\mathrm{Res}_{\hat{\alpha}}(G), \quad \alpha \subset \mathscr{C}, \alpha \neq \emptyset,\\ &\mu_k(G) = \min\{\beta_\alpha(G) \mid \#\alpha = k+1\}\\ &\mu_k^+(G) = \max\{\beta_\alpha(G) \mid \#\alpha = k+1\}.\end{aligned}$$

And then given $M \in \mathscr{M}^n$, define for each $0 \leqslant k \leqslant n$:

$$\begin{aligned}(3.15)\qquad &\mu_k(M) = \min\{\mu_k(G) \mid G \in \mathscr{G}(M)\},\\ &\mu_k^+(M) = \min\{\mu_k^+(G) \mid G \in \mathscr{G}(M)\},\end{aligned}$$

and $\tilde{\mu}_k = \mu_k - 1$ for $0 \leqslant k \leqslant n-1$, $\tilde{\mu}_n = \mu_n - 2$ (likewise for $\tilde{\mu}_k^+$).

In words, every colored triangulation of a manifold $M$ has at least $\mu_k(M)$ $k$-simplices of each type, and there exists some type $\langle\alpha\rangle$ (with $\alpha \subset \mathscr{C}$ and $\#\alpha = k+1$) with at least $\mu_k^+(M)$ simplices.

3.16. THEOREM. *Considering the invariants as functions $\mathscr{M}^n \to \mathbb{N}$, we have:*

(i) $\binom{n+1}{k+1}\mu_k \leqslant \kappa_k \leqslant \binom{n+1}{k+1}\mu_k^+$, *and equality holds for* $k = n$, $n-1$; *furthermore, if the first equality holds so does the second.*

(ii) $0 = \mu_0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_n$ *and* $0 = \mu_0^+ \leqslant \mu_1^+ \leqslant \cdots \leqslant \mu_n^+$.

(iii) $\tilde{\mu}_k$ *and* $\tilde{\mu}_k^+$ *are subadditive.*

(iv) *For* $k = n$, $n-1$, *and* $n-2$ *if* $n \geqslant 3$: $\tilde{\mu}_k(M^n) = 0$ *iff* $M^n \cong S^n$.

*Proof.* (i), (ii) and (iii) (following the proof of Theorem 3.9) are simple exercises. For $k = n, n-1$, (iv) follows from (i) and Theorem 3.12. We are then left to prove that $\tilde{\mu}_{n-2}(M^n) = 0$ iff $M^n \cong S^n$, for $n \geqslant 3$. But this is clearly implied by the following theorem. □

3.17. THEOREM. *If* $G \in \mathcal{MG}^n$, $n \geqslant 3$, *contains a Hamiltonian bicolored cycle (i.e. if for some* $\alpha \subset \mathscr{C}$, $\#\alpha = 2$, $G_\alpha$ *is connected) then* $|G| \cong S^n$.

*Proof.* This theorem is an easy consequence of the decomposition technique developed in Section 2-3, with $\alpha$ as above. The hypothesis implies that $|G|_\alpha$ consists of a single $(n-2)$-simplex, and thus, from (2.7), $|G; R(\bar{\alpha})| \cong B^n$ and $|G; T(\alpha)| \cong S^{n-1}$.

On the other hand, $|G|_{\bar{\alpha}}$ is a 1-dimensional complex and the boundary of its regular neighborhood is $S^{n-1}$, but this happens, for $n \geqslant 3$, iff $|G|_{\bar{\alpha}}$ is collapsible and thus $\mathrm{RN}_{|G|}(|G|_{\bar{\alpha}}) = |G; R(\alpha)| \cong B^n$. By (2.8) $|G| \cong S^n$. □

The final result of this section relates $\mu_1$ with the regular genus of Gagliardi [10] that we now state.

Recall Example 1.5(ii), and observe that if $G \in \mathcal{G}^n$, then the surface $|G; P_\varepsilon|$ is orientable iff $G$ is bipartite. Then, we define the *regular genus* of a bipartite $G \in \mathcal{G}^n$ as $g(M) = \min\{\text{genus}|G; P_\varepsilon| \mid \varepsilon \text{ is a cyclic permutation of } \mathscr{C}\}$, and of an orientable $M \in \mathcal{M}^n$ as $g(M) = \min\{g(G) \mid G \in \mathcal{G}(M)\}$. Clearly, this genus is the classic one in dimension 2, but, furthermore, it coincides with Heegaard genus in dimension 3 (see [10]).

Observe that $g(M)$ can be always realized by crystallizations. In fact, the crystallization move does not change the regular surfaces at all.

3.18. THEOREM. *For oriented manifolds:*

$$\tilde{\mu}_1(M^3) = g(M^3) = \text{Heegaard genus } (M^3)$$

*and*

$$\tilde{\mu}_1(M^n) \leqslant g(M^n) \text{ for } n \geqslant 4.$$

*Proof.* For $n = 3$, observe that $P_\varepsilon \hookrightarrow \langle \mathscr{C} \rangle$ corresponds to $T(\alpha)$ where $\alpha$ consists of two non-consecutive colors in $\varepsilon$ (see Figures 1 and 3). Thus, if $G \in \mathcal{MG}^3$ is bipartite, the decomposition (2.8) is a Heegaard splitting of $|G|$ by the regular surface $|G; P_\varepsilon| = |G; T(\alpha)|$ (each side is a handlebody for they are regular neighborhoods of 1-complexes, $|G|_{\bar{\alpha}}$ and $|G|_\alpha$ respectively). This proves that $g(M^3) \leqslant$ Heegaard genus $(M^3)$. For the other inequality one finds appropriate 'triangulations' of Heegaard decompositions [10].

If $G$, as above, is furthermore a crystallization, then $|G|_{\bar{\alpha}}$ consists of two 0-simplices with $\beta_\alpha(G)$ 1-simplices attached to them. And thus, $g(|G; P_\varepsilon|) = \beta_\alpha(G) - 1$. This implies that $g(G) = \tilde{\mu}_1(G)$. And the first equality follows since both invariants for manifolds are realized by crystallizations.

For $n \geqslant 4$, we prove that $\tilde{\mu}_1(G) \leqslant g(G)$ for any bipartite crystallization $G \in \mathcal{MG}^n$.

Given a cyclic permutation of the colors $\varepsilon$, let $\alpha = \{c, d\}$ be any pair of non-consecutive colors in $\varepsilon$. It is sufficient to see that $\beta_\alpha(G) - 1 \leqslant g(|G; P_\varepsilon|)$.

To obtain $|G; P_\varepsilon|$ we first glue with colors not in $\alpha$. $|G_{\bar{\alpha}}; P_\varepsilon|$ is a compact surface with $\beta_\alpha(G)$ connected components, say $S_1, S_2, \ldots, S_k$ where $k = \beta_\alpha(G)$. Since $c$ and $d$ are not consecutive, each $S_i$ has two types of boundary components: those colored with $c$ and those colored with $d$. One can easily find a simple closed curve $\tau_i \to S_i$ that breakes $S_i$ into two connected surfaces $S_i(c)$ and $S_i(d)$, such that all $c$ ($d$) boundary components lie in $S_i(c)$ ($S_i(d)$).

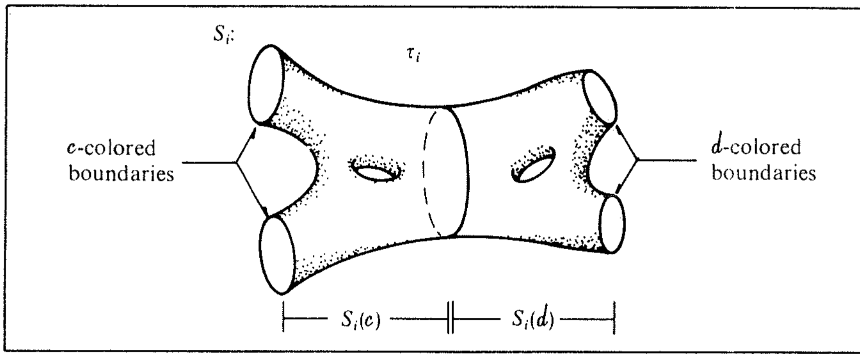



Fig. 4.

Let $S(c)$ (and $S(d)$) be the surface obtained from the $S_i(c)$'s ($S_i(d)$'s) by performing the glueings that $G$ indicates on the $c$ ($d$) -colored edges of their boundaries. Then, $|G; P_\varepsilon|$ is obtained by identifying the $k$ boundary components of $S(c)$, (the $\tau_i$'s), with the corresponding ones in $S(d)$. And, from this description, it is clear that $g(|G; P_\varepsilon|) \geqslant k - 1 = \beta_\alpha(G) - 1$ if both $S(c)$ and $S(d)$ are connected. But this follows from the crystallization hypotesis. □

## 4. The Fundamental Group

First, we give a simple combinatorial way to obtain the fundamental group of a manifold from a crystallization of it. After a couple of examples, a bound for the rank of $\pi_1(M)$ is obtained in terms of complexities. In Section 4-2 another presentation, equivalent to that of Vince [21], is described. It gives, together with the first one, another bound for the rank, which suggests a generalization of the Poincaré conjecture.

### 4-1. $\pi(G, c)$

Let $G$ be a regular $\mathscr{C}$-graph. For any color $c \in \mathscr{C}$, define a group $\pi(G, c)$ as follows:

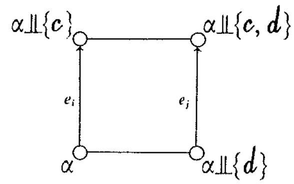
Let $\{e_1, e_2, \ldots, e_k\} = \gamma^{-1}(c)$ be the $c$-colored edges of $G$, and fix an orientation for each of them. $\pi(G, c)$ will be the group generated by the $e_i$'s with one relation for each 2-residue $H$ with color $c$, obtained as follows. $H$ is a bicolored cycle with edges colored alternatively by $c$ and $d$, say; at any vertex and with any direction, start 'reading out' the $c$-colored edges with their orientation. This gives a word of the form $w_c(H) = e_{i_1}^{\pm 1} e_{i_2}^{\pm 1} \cdots e_{i_r}^{\pm 1}$ with all $e_{i_j}$'s different. We may write

(4.1) $\quad \pi(G, c) = \langle\!\langle e_1, \ldots, e_k \mid w_c(H);\ H$ is an $\alpha$-residue, $c \in \alpha$ and $\#\alpha = 2\rangle\!\rangle$,

where $\langle\!\langle$—— $\mid \cdots \rangle\!\rangle$ stands for the group generated by —— with relations $\ldots$.

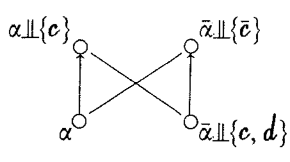
Note that the choices made to 'read' $w_c(H)$ do not alter $\pi(G, c)$, and that the choice of orientation of the edges changes it by canonical isomorphism. In particular, if $G$ is bipartite there are obvious choices for the orientations.

4.2. EXAMPLES. (i) Recall the construction of $Q(\mathscr{C})$ and $P(\mathscr{C})$, (Example 1.4(iii)). First let us compute $\pi(Q(\mathscr{C}), c)$. $Q(\mathscr{C})$ has $2^n$ $c$-colored edges which can be oriented naturally: if $c \notin \alpha \subset \mathscr{C}$, orient the $c$-edge at $\alpha$ to $\alpha \amalg \{c\}$. Now, if $d \neq c$, $d \in \mathscr{C}$, all the $\{c, d\}$-residues are of the form




and thus, their corresponding relation $(e_i e_j^{-1})$ implies an equality of two generators. One easily sees that there are enough 2-residues to imply that all generators are equal. Thus, $\pi(Q(\mathscr{C}), c) \cong \mathbb{Z}$.

Now, to compute $\pi(P(\mathscr{C}), c)$, choose any $d \neq c$, and observe that $P(\mathscr{C})_{\bar{d}} = Q(\bar{d})$. Thus, we may proceed as in the preceding paragraph. But at the end, we have to add the relations corresponding to the $\{c, d\}$-residues, and they are all of the form




which implies that $\pi(P(\mathscr{C}), c) \cong \mathbb{Z}_2$.

(ii) *Lens spaces.* A graph for the lens space $L(p,q)$ is the following $G = G(p,q)$ with four colors $\{c_0, c_1, c_2, c_3\}$ (see [2] or directly Rolfsen [20 p. 237]).

Take two cycles of length $2p$, and let $u_1, \ldots, u_{2p}$ and $v_1, \ldots, v_{2p}$ be their vertices ordered cyclically and with indices in $\mathbb{Z}_{2p}$. Color the edges alternatively with $c_0$ and $c_1$, such that the indices correspond. Then, put edges of color $c_2$ between $u_i$ and $v_i$, and of color $c_3$ between $u_i$ and $v_{i+2q}$.

To compute $\pi(G, c_2)$ orient the $c_2$-edges from $u_i$ to $v_i$. The $\{c_0, c_2\}$ and the $\{c_1, c_2\}$-residues clearly imply that all the generators are equal, to $e$ say. And finally, since $p$ and $q$ are relatively prime, there are exactly two $\{c_2, c_3\}$-residues, and both of them give the relation $e^p$. Then $\pi(G, c_2) \cong \mathbb{Z}_p$. The reader may check that for any other choice of color, one gets the same answer although the presentations may be different.

The following theorem was obtained independently by Donati [3] and Grasselli [8].

4.3. THEOREM. *If $G$ is a manifold graph and $G_{\bar{c}}$ is connected, then*

$$\pi(G, c) \cong \pi_1(|G|).$$

*Proof.* There is only one 0-simplex of type $\langle c \rangle$ in $|G|$, which we call $\langle c \rangle$ without fear of confusion. We will prove that $\pi(G, c) \cong \pi_1(|G|, \langle c \rangle)$.

The regular neighborhood of $\langle c \rangle$ in $|G|$, $\mathrm{RN}_{|G|}(\langle c \rangle)$, is the cone centered at $\langle c \rangle$ and based on its link $|G_{\bar{c}}|$ ($\mathrm{RN}_{|G|}(\langle c \rangle) = \langle c \rangle * |G_{\bar{c}}| \cong B^n$). Clearly, $\pi_1(|G|, \langle c \rangle) = \pi_1(|G|, \mathrm{RN}_{|G|}(\langle c \rangle))$, where we are now using the notion of 'relative homotopy groups'. Recall that to compute such a first homotopy group, it is enough to have a relative CW-complex decomposition and then take the 1-cells modulo the 2-cells.

Consider the dual cell decomposition of $|G|$ (Section 2-2). Observe that the cell corresponding to an $\alpha$-residue $H$ such that $c \notin \alpha$, is entirely contained in $\mathrm{RN}_{|G|}(\langle c \rangle)$, because $H \subset G_{\bar{c}}$. Thus, a CW-complex decomposition of $|G|$ relative to $\mathrm{RN}_{|G|}(\langle c \rangle)$ consists of one $r$-cell for each $\alpha$-residue with $c \in \alpha$ and $\# \alpha = r$. Then the 1-cells are the generators of $\pi(G, c)$ and the 2-cells correspond to the relations. The theorem follows. □

Geometrically, a $c$-colored edge $e$ represents the following loop based at $\langle c \rangle$. Let $u$, $v \in V(G)$ be the vertices such that $e$, with its chosen orientation, goes from $u$ to $v$. Think of the graph geometrically embedded in $|G|$. The vertices $u$ and $v$ lie on the link of $\langle c \rangle$, $|G_{\bar{c}}|$, thus we can go from $\langle c \rangle$ to $u$ radially, then travel through $e$ up to $v$ and come back to $\langle c \rangle$ radially. If we denote by $\langle c \rangle * u$ the radial path from $\langle c \rangle$ to $u$, we have that the isomorphism of the theorem is given by

(4.4) $e \hookrightarrow (\langle c \rangle * u) \cdot e \cdot (v * \langle c \rangle)$.

(See Figure 5.)

4.5. REMARK. It is not hard to see that if the hypothesis of the theorem are loosened in either way, similar results are obtained. If $G_{\hat{c}}$ is not connected, one has to add as relations a minimal set of $c$-edges that connect $G_{\hat{c}}$ to obtain the fundamental group (see Example 4.2(i) above). And if $G$ is not a manifold graph, $\pi(G, c)$ is the fundamental group of the complement of (singular locus $\cap |G|_c$).

Let $\mathrm{rk}(\pi_1(M))$ = minimum number of generators of $\pi_1(M)$. From Theorem 4.3 we obtain an obvious bound.

4.6. COROLLARY. *For any* $M \in \mathcal{M}^n$, $\mathrm{rk}(\pi_1(M)) \leqslant \mu_{n-1}(M) = \frac{1}{2}\kappa(M)$. □

But we can do better. For if we fix another color $d$, then the $d$-relations of $\pi(G, c)$ – that is, those coming from $\{c, d\}$-residues – involve different generators, and each of them allows us to express one generator in terms of others. Thus

4.7. COROLLARY. *If, for a given* $M \in \mathcal{M}^n$, $\mu^+_{n-2}(M)$ *is realized by a minimal graph, then*

$$\mathrm{rk}(\pi_1(M)) \leqslant \tfrac{1}{2}\kappa(M) - \mu^+_{n-2}(M).$$

*In any case,*

$$\mathrm{rk}(\pi_1(M)) \leqslant \min\{\mu_{n-1}(G) - \mu^+_{n-2}(G) \mid G \in \mathcal{G}(M)\}.$$

*Proof.* Given a crystallization $G$ of $M$, choose colors that maximize the number of 2-residues, then, from the observation above, obtain a presentation with $\mu_{n-1}(G) - \mu^+_{n-2}(G)$ generators. □

## 4.2. *Paths in G*

Let $G$ be any connected $\mathscr{C}$-graph. We need to stablish some terminology:

A *walk* in $G$ is a finite sequence of oriented edges $\tau = (e_1, \ldots, e_k)$, such that the endpoint of $e_i$ is the starting point of $e_{i+1}$; if, furthermore, $e_i \neq e_{i+1}$, for each $i = 1, \ldots, k-1$, we call it a *path*. Observe that any walk can be canonically *reduced* to a path. The notion of *closed* walk or path is the obvious one. And finally, a *cycle* will be a closed path with no repetition of the vertices (and thus of the edges) involved.

The set of closed paths based at a vertex $v_0 \in V(G)$, $C(G, v_0)$, forms a group under juxtaposition followed by canonical reduction. (We shall multiply on the right, i.e. $\tau \cdot \tau'$ means follow $\tau$ and then $\tau'$.) It is the fundamental group of the graph $G$ thought as a 1-complex, and it is free on $\# E - \# V + 1$ generators.

Given a path $\tau$ in $G$, let $\gamma(\tau) \subset \mathscr{C}$ be the set of colors used by $\tau$. Then, let $C_r(G, v_0)$ be the subgroup of $C(G, v_0)$ generated by all elements of the form $\tau\sigma\tau^{-1}$, where $\sigma$ is a cycle (not necessarily based at $v_0$) such that $\#\gamma(\sigma) \leqslant r$. Observe that the same group is obtained if we ask $\sigma$ to be a closed path, and that it is normal.

The following theorem is a consequence of Vince's work [21]. We sketch a proof to emphasize the basic points.

4.8. THEOREM. *For any connected $\mathscr{C}$-graph $G$, and $v_0 \in V(G)$:*

$$\pi_1(|G|, v_0) = \frac{C(G, v_0)}{C_n(G, v_0)}.$$

*Proof.* Even if $G$ is not regular, one has a map of $G$ as a 1-complex into $|G|$, and thus we are thinking of $v_0$ as a point in $|G|$ (the barycenter of its corresponding simplex). Then one proves

4.9. LEMMA. *Every topological loop in $|G|$ based at $v_0$ is homotopic (rel. $v_0$) to a path in $G$.*

Now, let $\tau\sigma\tau^{-1}$ be a generator of $C_n(G, v_0)$, then $\sigma$ lies entirely in a $\gamma(\sigma)$-residue, $H$ say, because $\gamma(\sigma) \neq \mathscr{C}$. And $|H|$ is the link of a simplex, so that using the cone $b_H * |H| \subset |G|$ we may homotope $\sigma$ to its basepoint, and then $\tau\tau^{-1}$ back to $v_0$. Thus $\tau\sigma\tau^{-1}$ is nullhomotopic in $|G|$.

It remains to prove that if $\tau \in C(G, v_0)$ is nullhomotopic in $|G|$, then $\tau \in C_n(G, v_0)$. This is left to the reader. □

Now let us return to our main interest, manifolds. (The following two corollaries also appear in [8].)

4.10. COROLLARY. *If $G$ is a manifold graph, then $C_n(G, v_0) = C_2(G, v_0)$.*

*Proof.* In the dual cell decomposition of $|G|$, the 1-skeleton is $G$ and the 2-cells correspond to the cycles $\sigma$, such that $\#\gamma(\sigma) = 2$. Then

$$\pi_1(|G|, v_0) = \frac{C(G, v_0)}{C_2(G, v_0)}, \qquad (4.11)$$

and the corollary follows. □

*Let $G$ be a crystallization.* How does this presentation of $\pi_1(|G|)$ relate to the one of the preceding section?

Given any path $\tau$ in $G$, let $w_c(\tau)$ be the word on the (previously oriented) $c$-colored edges obtained by 'reading' them as they appear in $\tau$.

4.12. PROPOSITION. *The homeomorphism* $w_c: C(G, v_0) \to \langle\langle \gamma^{-1}(c) \rangle\rangle$ *induces an isomorphism*

$$\frac{C(G, v_0)}{C_2(G, v_0)} \simeq \pi(G, c),$$

*which, in terms of fundamental groups* (4.10) *and* (4.3), *corresponding to topological conjugation with the radial path from* $\langle c \rangle$ *to* $v_0$.

*Proof.* In fact, we prove the last statement and the first follows. Consider $\tau \in C(G, v_0)$, and let $\tau = \tau_0 \cdot e_1 \cdot \tau_1 \cdot e_2 \cdot \cdots \cdot e_k \cdot \tau_k$, where $w_c(\tau) = e_1 e_2 \cdots e_k$ (we are omitting exponents $\pm 1$ for simplicity), and thus, each $\tau_i$ is a path in $G_{\hat{c}}$. The isomorphism $\pi_1(|G|, v_0) \simeq \pi_1(|G|, \langle c \rangle)$ given by conjugation with the radial path $\langle c \rangle * v_0$, sends $[\tau]$ to

$$[(\langle c \rangle * v_0)\tau_0 e_1 \tau_1 e_2 \cdots e_k \tau_k (v_0 * \langle c \rangle)].$$

But, since $\tau_i \subset G_{\hat{c}}$, the cone structure of $\langle c \rangle * |G_{\hat{c}}| \subset |G|$ allows us to homotope $\tau_i$ to $(v_i * \langle c \rangle) \cdot (\langle c \rangle * u_{i+1})$, where $\tau_i$ goes from $v_i$ to $u_{i+1} (u_{k+1} = v_0)$. Then, the image of $[\tau]$ is homotopic to

$$(\langle c \rangle * u_1) \cdot e_1 \cdot (v_1 * \langle c \rangle) \cdots (\langle c \rangle * u_k) \cdot e_k \cdot (v_k * \langle c \rangle)$$

and the proposition follows from (4.4). □

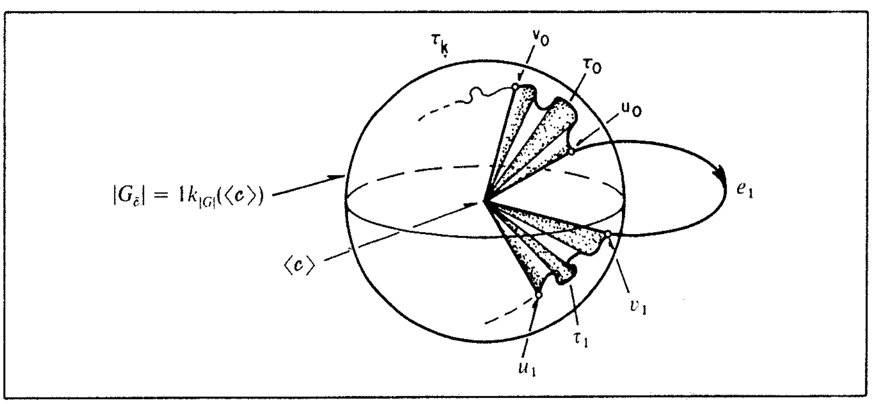



Fig. 5.

4.13. COROLLARY. *If* $\tau$ *is a closed path such that* $\#\gamma(\tau) \leqslant n$, *then* $w_c(\tau)$ *is trivial in* $\pi(G, c)$.

*Proof.* Let $\tau_0$ be a path in $G_{\hat{c}}$ from $v_0$ to the basepoint of $\tau$. Then apply Corollary 4.10 and Proposition 4.12 to $\tau_0 \tau \tau_0^{-1}$. □

Now, we are in a position to give another bound to the rank.

4.14. THEOREM. $\mathrm{rk}(\pi_1(M)) \leqslant \tilde{\mu}_1(M)$.

*Proof.* Let $G$ be a crystallization of $M \in \mathcal{M}^n$ realizing $\mu_1(M)$ with $\alpha = \{c, d\} \subset \mathscr{C}$, that is, $\beta_\alpha(G) = \mu_1(G) = \mu_1(M)$.

$G_{\bar{\alpha}}$ has $\beta_\alpha(G)$ connected components, and since $G_{\hat{c}}$ is connected, we can find a minimal set of $c$-colored edges that connect $G_{\bar{\alpha}}$. Let $\{e_1, e_2, \ldots, e_k\}$ be such a set. Clearly $k = \beta_\alpha(G) - 1 = \tilde{\mu}_1(M)$. We prove that $e_1, \ldots, e_k$ generate $\pi(G, c)$:

Let $e$ be any $c$-colored edge. Since $G_{\bar{\alpha}} \cup \{e_1, \ldots, e_k\}$ is connected, there exists a path in it, $\tau$ say, going from the starting point of $e$ to its endpoint. Since $d \notin \gamma(e^{-1}\tau)$, Corollary 4.13 implies that $e = w_c(\tau)$ in $\pi(G, c)$, and $w_c(\tau)$ is a word in $e_1, \ldots, e_k$. The theorem follows. □

4.15. CONJECTURE. *$M$ is 1-connected iff $\tilde{\mu}_1(M) = 0$.*

One implication is a consequence of the preceding theorem. The other one, which is true in dimension 2, is a restatement of the Poincaré conjecture in dimension 3. For we have proved (Theorem 3.16(iv)) that $\tilde{\mu}_1(M^3) = 0$ iff $M^3 \cong S^3$.

This conjecture can also be seen as a generalization of the crystallization theorem (Theorem 3.10′) which could be stated as: $M$ is 0-connected iff $\tilde{\mu}_0(M) = \tilde{\mu}_0^+(M) = 0$.

A stronger version of the conjecture is obtained if we change $\mu_1$ for the 1-complexity $\kappa_1$. In dimension 3 they are equivalent. But it would be interesting to find examples of manifolds $M^n$, $n \geqslant 4$, with $\tilde{\mu}_1(M) = 0$ and $\tilde{\kappa}_1(M) > 0$. Recent progress with simply connected 4-manifolds makes this seem plausible.

## 5. THE UNIVERSAL $\mathscr{C}$-GRAPH

The universal cover of a connected regular $\mathscr{C}$-graph $G \in \mathscr{G}^n$, thought of as a 1-complex, depends only on the color set $\mathscr{C}$ and not on the particular graph. It clearly has the structure of a regular $\mathscr{C}$-colored graph, which we call the *universal $\mathscr{C}$-graph*, $U^n$, because any $G \in \mathscr{G}^n$ is a quotient of it. But moreover, the vertex set of $U^n$ has a natural group structure which can be identified with the $\mathscr{C}$-automorphisms of $U^n$, so that the covering transformations for any $G \in \mathscr{G}^n$ become a subgroup. Thus, regular $\mathscr{C}$-graphs, and therefore manifolds, correspond to certain subgroups of a fixed group. This opens a possibility of studying (PL) manifolds through concrete group theory.

5.1. DEFINITIONS (Recall that $\mathscr{C}$ is a fixed color set of dimension $n$). (i) Let $W^n = W(\mathscr{C}) = \langle\langle \mathscr{C} \mid c^2;\ c \in \mathscr{C} \rangle\rangle$. That is, $W^n$ is the group of words with the colors in $\mathscr{C}$ as letters and no exponents (or repetitions). The product is right juxtaposition followed by *reduction* (all repetitions of a letter, $\ldots cc \ldots$, are cancelled). In fact, $W^n$ is isomorphic to the free product of $n + 1$ copies of $\mathbb{Z}_2$, one for each color:

$$W^n = \underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{n+1}$$

(ii) Let $U^n$ be the regular $\mathscr{C}$-colored graph with vertex set $V(U^n) = W^n$ and with an edge of color $c$ attached to each pair $\{w, wc\}$, $w \in W^n$. $U^n$ is the Cayley graph of the given presentation of $W^n$ (see [22]).

5.2. REMARK. The group $W^n$ can be identified with the group of $\mathscr{C}$-graph automorphisms of $U^n$. Indeed, given $w \in W^n$, left multiplication $\mu_w: W^n \to W^n$, $\mu_w(w') = ww'$, extends naturally to a $\mathscr{C}$-graph isomorphism $\mu_w: U^n \to U^n$. And it is easy to see that any $\mathscr{C}$-graph morphism of $U^n$ to itself is of this form.

Now, let $G$ be any regular $\mathscr{C}$-graph.

There is natural 1–1 correspondence between the set of paths in $G$ and $V(G) \times W^n$. Given $(v, w) \in V(G) \times W^n$, with $w = c_1 c_2 \cdots c_k$, let $\tau(v, w)$ be the path in $G$ which starts at $v$, then leaves by the (unique) edge of color $c_1$ at $v$, then takes the edge of color $c_2$, and so on. The relations $c^2 = \mathbb{1}$ correspond to the definition of path given in Section 4-2. And conversely, given a path, take its starting point and the word of the colors used by it.

With this identification, the endpoint map $t: V(G) \times W^n \to V(G)$ is a right action of the group $W^n$ on the set $V(G)$. And many of the notions introduced above can be stated in terms of this action; for example, the $\alpha$-residues of $G$ are the orbits of the subgroup $W(\alpha) \subset W(\mathscr{C}) = W^n$. In general, this action does not extend to a $\mathscr{C}$-graph action on $G$. In fact, this happens iff every 2-residue has at most four vertices, and iff the action is 'abelian' (i.e. for all $w$, $w' \in W^n$ and $v \in V(G)$, $v \cdot (ww') = t(v, ww') = t(v, w'w) = v \cdot (w'w)$).

Now, given a basepoint $v \in V(G)$, the evaluation $t_v: W^n \to V(G)$ extends to a $\mathscr{C}$-graph morphism $p_v: U^n \to G$, which, if $G$ is connected (if the action is transitive), is the universal cover of $G$. The isotropy group of $v$, $t_v^{-1}(v) \subset W^n$ (which is isomorphic to closed paths at $v$, $C(G,v)$, see Section 4-2), corresponds to the covering transformations, see Remark 5.2. Thus, we have:

5.3. THEOREM. *For any connected regular $\mathscr{C}$-graph $G$, there exists a sub-*

*group* $\Gamma$ *of* $W^n$, *unique up to conjugation, for which*

$$G \cong \Gamma \setminus U^n. \qquad \square$$

Observe that the left action of $\Gamma$ on $U^n$ induces, by functoriality, a left action on $|U^n|$; and that $|\Gamma\setminus U^n| = \Gamma\setminus|U^n|$. So that:

5.4. COROLLARY. *For any closed connected n-manifold M there exists a subgroup* $\Gamma$ *of* $W^n$ *for which* $M \cong \Gamma\setminus|U^n|$. $\square$

As examples, we give the subgroups corresponding to the three graphs defined in Example 1.4(iii), and thus to $S^n$ (twice) and to $\mathbb{R}P^n$ respectively:

The subgroup of $W^n$ corresponding to the $\mathscr{C}$-dipole consists of all words of even length. To the $\mathscr{C}$-cube, $Q(\mathscr{C})$, corresponds the commutator $[W^n, W^n]$. And to obtain the subgroup corresponding to $P(\mathscr{C})$, add the generator $c_0c_1\cdots c_n$ (a word with each color appearing once) to the commutator.

The obvious question to ask after Theorem 5.3 is if all subgroups of $W^n$ give regular graphs. This is not so. But first, the general notion of 'quotient' needs to be stated clearly.

Given any subgroup $\Gamma \subset W^n$, $\Gamma$ acts on the left of $U^n$ by $\mathscr{C}$-graph morphisms, see Remark 5.2. We define $\Gamma\setminus U^n$ to be the $\mathscr{C}$-graph with vertex set the right cosets of $\Gamma$ in $W^n$, $V(\Gamma\setminus U^n) = \{\Gamma w\} = \Gamma\setminus W^n$, and then put an edge of color $c$ between $\Gamma w$ and $\Gamma wc$ whenever $\Gamma w \neq \Gamma wc$ (remember that we allowed no loops; see Subsection 1.1).

Thus, in general, a quotient of $U^n$ is not regular. It is only a *well-colored* graph (i.e. no two edges of the same color are incident). And to have a 'quotient map' $U^n \to \Gamma\setminus U^n$, we must extend our notion of $\mathscr{C}$-graph morphism to *virtual* $\mathscr{C}$-morphism, in which an edge may be collapsed to a vertex.

5.5. THEOREM. *There is a one-to-one correspondence between conjugacy classes of subgroups of* $W^n$ *and* $\mathscr{C}$*-isomorphism classes of connected well-colored* $\mathscr{C}$*-graphs.*

*Proof.* For any well-colored graph $G$, we still have a right action of $W^n$ on $V(G)$. It is best defined on the generators: given $c \in \mathscr{C}$ and $v \in V(G)$, let $v\cdot c$ be the other vertex on the $c$-edge at $v$ if such an edge exists and $v$ if it does not. Clearly $(v\cdot c)\cdot c = v$ for all $v \in V(G)$, so that this extends to an action of $W^n$. Now, if $\Gamma$ is the isotropy group of some $v \in V(G)$, and $G$ is connected, one easily sees that $\Gamma\setminus U^n \cong G$. $\square$

To characterize those subgroups that give regular graphs, we introduce a natural filtration. (Compare with the definition of $C_r(G, v)$ given in Section 4-2.)

Given any subgroup $\Gamma$ of $W^n$, let $\Gamma_r$, $0 \leqslant r \leqslant n+1$, be the subgroup of $\Gamma$ generated by all its elements of the form $w \cdot w_0 \cdot w^{-1}$ where $w_0$ is a word, of any length, that uses at most $r$ different characters (letters). We then have a filtration of normal subgroups

$$\mathbb{1} = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n \subset \Gamma_{n+1} = \Gamma,$$

which has, at least, the following properties:

5.6. PROPOSITION.

(i) $\Gamma \backslash U^n$ *is regular iff* $\Gamma_1 = \Gamma_0$.
(ii) $C_r(\Gamma \backslash U^n, \Gamma) = \Gamma_r / \Gamma_1$.
(iii) $\Gamma_1 \backslash U^n$ *is the universal cover of* $\Gamma \backslash U^n$.
(iv) $\pi_1(|\Gamma \backslash U^n|, \Gamma) = \Gamma / \Gamma_n$.
(v) *If* $\Gamma \backslash U^n$ *is a manifold graph then* $\Gamma_2 = \Gamma_n$.

*Proof.* (i) By definition, $\Gamma \backslash U^n$ is regular iff $\Gamma w \neq \Gamma w c$ for all choices of $w$ and $c$. But this happens iff $\Gamma$ has no elements of the form $wcw^{-1}$.

(ii) A word $w \in W^n$ produces (as before) a walk in $\Gamma \backslash U^n$ from $\Gamma$ to $\Gamma w$. This gives epimorphisms $\Gamma_r \twoheadrightarrow C_r(\Gamma \backslash U^n, \Gamma)$ for $0 \leqslant r \leqslant n+1$. A generator of $\Gamma_1$, $wcw^{-1} \in \Gamma$, produces the walk also defined by $ww^{-1}$ and is therefore reduced to the trivial path. On the other hand, the reduction step for paths can be easily interpreted as deleting a generator of $\Gamma_1$ from the defining word. Thus, $\Gamma_1$ is the kernel of the epimorphism.

(iii) From (ii), $\Gamma_1 \backslash U^n$ is simply connected, and it is clearly a covering of $\Gamma \backslash U^n$.

(iv) follows from (ii) and Theorem 4.8, and (v) follows from (i), (ii) and Corollary 4.10. □

To conclude, we observe the obvious. The results in this section are far from being exhaustive. They seem to be only the prelude to a possible fruitful interplay between group theory, combinatorics and topology.

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