A characterization of triangulations of closed surfaces

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Abstract. In this paper we prove that a finite triangulation of a connected closed surface is completely determined by its intersection matrix. The *intersection matrix* of a finite triangulation, K, is defined as $M_K = (\dim(s_i \cap s_j))_{0\le i, 0\le j}^{n-1}$, where $K_2 = \{s_0, \ldots s_{n-1}\}$ is a labelling of the triangles of K.

1 Introduction

Within the theory of convex polytopes, the study of the combinatorial equivalence of k-skeleta of pairs polytopes which are not equivalent themselves has been of interest, this phenomena is referred to in the literature as ambiguity [3].

It is well known that for $k \ge \lfloor \frac{d}{2} \rfloor$ the k-skeleton of a convex polytope is not dimensionally ambiguous, this is, it defines the entire structure of its underlying d-polytope. However for $k < \lfloor \frac{d}{2} \rfloor$ the question is much more intricate.

One of the most interesting results in this direction is the solution to Perle's conjecture by P. Blind and R. Mani [1] and, separately, by G. Kalai [2] which states that the 1-skeleta of convex simple d-polytopes define their entire combinatorial structure. Or, on its dual version, that the dual graph (facet adjacency graph) of a convex simplicial d-polytope determines its entire combinatorial structure.

2 Motivation & contribution

Allured by Perles' conjecture, we decided to explore the extent to which an adequate combination of combinatorial and topological assumptions

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would prove as powerful for characterising certain simplicial complexes. The purpose of this work is to present our first result, product of this exploration.

For topological assumption we will, in this instance, ask for the simplicial complex of study to be a connected closed surface. As for combinatorial assumption, one might be tempted to choose only to have the information provided by its dual graph. However, the dual graph of a triangulation of a closed surface does not provide enough information to characterise it, as there are some dual graphs to triangulations which have been shown in [4] to have combinatorically different polyhedral embeddings.

Therefore, we will need to strengthen the combinatorial hypothesis. In order to do so we will introduce the concept of an intersection preserving mapping of simplices of a simplicial complex.

Definition 2.1. A bijective mapping $f : K_d \to K'_d$ between the sets of d-simplices of two simplicial complexes, K and K', is an intersection preserving mapping if for every pair of simplices $s, t \in K_d$ $\dim(s \cap t) = \dim(f(s) \cap f(t))$.

Throughout this paper we will use the notation K_l to refer to the set of l-dimensional simplices of the complex K. Additionally, we will define two particular triangulations of the projective plane, which are of interest for this work.

Definition 2.2. We define a 10-triangle triangulation of the projective plane, $T\mathbb{P}_{10}$, as the triangulation whose triangles have the vertex sets $(s_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, x\}$, and $(r_i)_0 = \{a_{i \bmod 5}, a_{i+1 \bmod 5}, a_{i-2 \bmod 5}\}$ for $0 \leq i \leq 4$.

Definition 2.3. We define a 12-triangle triangulation of the projective plane, $T\mathbb{P}_{12}$, as the triangulation whose triangles have the vertex sets $(s_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, x\}$, for $0 \leq i \leq 5$ and $(r_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, a_{i+4 \bmod 6}\}$ for $0 \leq i \leq 4$ even, and $(r_i)_0 = \{a_{i \bmod 6}, a_{i+1 \bmod 6}, a_{i+3 \bmod 6}\}$ for $0 \leq i \leq 4$ odd.

We now use the aforementioned definitions to state the main result:

Theorem 2.4. *Let $\|K\|$ and $\|K'\|$ be geometric realizations of finite triangulations which are homeomorphic to connected closed surfaces, and let $f : K_2 \to K'_2$ be an intersection preserving mapping, then one of the following three statements holds:*

(1) *f can be extended into a bijective simplicial mapping between K and K'*