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CAROUSELS, ZINDLER CURVES AND THE FLOATING BODY PROBLEM

J. Bracho (México), L. Montejano (México) and
D. Oliveros (Calgary)

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Abstract

A carousel is a dynamical system that describes the movement of an equilateral linkage in which the midpoint of each rod travels parallel to it. They are closely related to the floating body problem. We prove, using the work of Auerbach, that any figure that floats in equilibrium in every position is drawn by a carousel. Of special interest are such figures with rational perimetral density of the floating chords, which are then drawn by carousels. In particular, we prove that for some perimetral densities the only such figure is the circle, as the problem suggests.

1. Introduction

The floating body problem, number 19 of the Scottish Book [4], reads as follows

> *Is a solid of uniform density which will float in water in every position, a sphere?*

It has only been proved true for density 0 by Montejano [5], and false for dimension 2 and density 1/2 by Auerbach [1]. In this latter work some remarkable properties about possible examples in dimension 2 were proved. Namely, that the floating chords have constant length; that the curve of their midpoints has the corresponding chords as tangents, and that these chords divide the perimeter in a fixed ratio α (the perimetral density). Suppose that α is rational. Then, for every point p in the boundary of the figure we have an inscribed equilateral n-gon which moves, as p moves, in such a way that the midpoints of the sides move parallel to them. This is the main motivation for the definition of carousels.

They will be formally defined in Section 2, and this paper may be thought of as the starting point for their study in their own right. Some basic structural properties are established, as well as their existence for n odd. Section 3 is devoted to see how the Auerbach Theorems imply that all figures that float in equilibrium

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in any direction are drawn by special carousels. Also, the closely related notion of Zindler curve is given. The basic properties of carousels are stated in Section 4 and in Section 5, explicit differential equations that rule carousels are derived, and used to prove that for n odd a unique carousel is obtained from any equilateral n-gon. Finally, following [2], we describe the main ideas towards the classification of carousels with five chairs. Such understanding yields, together with previous results, the non existence of figures, different from the circle, that float in equilibrium in every direction with perimetral densities 1/3, 1/4, 1/5 and 2/5.

2. The carousel

Definition. A *Carousel* (with n chairs) is a system which consists of n smooth curves $\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ in $\mathbb{R}^2$ satisfying the following properties, for every $t \in \mathbb{R}$ and for all $i = 1, \ldots, n$, where $\beta_{i+n}(t) = \beta_i(t)$:

(1) The length of the interval with end points $\beta_i(t)$ and $\beta_{i+1}(t)$, $|\beta_{i+1}(t) - \beta_i(t)|$, is the same non-zero constant for all i.

(2) The curve of midpoints, $m_i(t) = \frac{\beta_i(t)+\beta_{i+1}(t)}{2}$, of the segments from $\beta_i(t)$ to $\beta_{i+1}(t)$, has tangent vector, $m_i'(t)$, parallel to $\beta_{i+1}(t) - \beta_i(t)$.

On this paper, a smooth curve $\phi : \mathbb{R} \to \mathbb{R}^2$ is a C^1-differentiable function with $\phi'(t) \neq 0$, for every $t \in \mathbb{R}$. Furthermore, ϕ is a smooth closed curve if there is $t_0 \in \mathbb{R}$ such that $\phi(t + t_0) = \phi(t)$, for every $t \in \mathbb{R}$. In addition, $\phi(t) = \phi(s)$ if and only if $t = s + kt_0$ for some integer $k \in \mathbb{Z}$, then ϕ is a simple closed curve. If this is the case, the region of the plane bounded by $\phi(\mathbb{R})$ will by denoted by $\Phi(\phi)$.

Lemma 1 (First Carousel Law). *Let $\beta_1(t)$ and $\beta_2(t)$ be smooth curves. Then the system $\{\beta_1(t), \beta_2(t)\}$ is a carousel if and only if $\beta_2'(t)$ is a reflection of $\beta_1'(t)$ across the line generated by $\beta_2(t) - \beta_1(t)$, for every $t \in \mathbb{R}$.*

Proof. Observe that the condition (1) for a carousel is equivalent to: $\langle \beta_2(t) - \beta_1(t), \beta_2(t) - \beta_1(t)\rangle$ is constant, where $\langle\ ,\ \rangle$ is the interior product. Then, taking derivatives, we obtain the equivalent equation

$$\langle \beta_2'(t), \beta_2(t) - \beta_1(t)\rangle = \langle \beta_1'(t), \beta_2(t) - \beta_1(t)\rangle.$$

The condition (2) for a carousel is that $\beta_2'(t) + \beta_1'(t)$ is parallel to $\beta_2(t) - \beta_1(t)$. From here, the lemma follows easily. $\square$

Observe from the previous lemma that $|\beta_2'(t)| = |\beta_1'(t)|$. Then, from now on, if $\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ is a carousel, we may assume that $|\beta_i'(t)| = 1$ so that all the curves $\beta_i(t)$ are parameterized by arc length. Furthermore, we may also assume that $|\beta_{i+1}(t) - \beta_i(t)| = 2$.

Let $\alpha_i(t)$ denote the angle between the vectors $\beta_i'(t)$ and $\beta_{i+1}(t) - \beta_i(t)$ and let $\theta_i(t)$ be the angle between the x-axes and the vector $\beta_{i+1}(t) - \beta_i(t)$. We shall use angles either between $[0, 2\pi)$ or between $[-\pi, \pi)$ as the context requires. In any case, the derivative is the same as we shall see next.

LEMMA 2 (SECOND CAROUSEL LAW). *Let $\{\beta_1(t), \ldots, \beta_n(t)\}$ be a system of smooth curves with the property that $|\beta_{i+1}(t) - \beta_i(t)| = 2$. Then $\{\beta_1(t), \ldots, \beta_n(t)\}$ is a carousel if and only if, for every t, $\theta_i'(t) = \sin(\alpha_i(t))$.*

PROOF. If $m_i(t)$ is the midpoint of $\beta_{i+1}(t) - \beta_i(t)$, we can write $m_i(t) = \beta_i(t) + u(\theta_i(t))$, where

$$u(\theta) = (\cos(\theta), \sin(\theta)).$$

Thus, $m_i'(t) = \beta_i'(t) + u'(\theta_i(t))\theta_i'(t)$. Finally, observe that the condition (2), for a carousel, is equivalent to $\langle m_i'(t), u'(\theta_i(t))\rangle = 0$, which is equivalent to

$$\theta_i'(t) = -\langle \beta_i'(t), u'(\theta_i(t))\rangle = \sin(\alpha_i(t)). \qquad \square$$

DEFINITION. A carousel with n chairs $\{\beta_1(t), \ldots, \beta_n(t)\}$ is a *Zindler carousel* if all the curves $\beta_i(t)$ are parametrizations of the same smooth closed curve γ. If this is so, the smooth closed curve γ will be called a *Zindler curve*. We shall see that Zindler curves studied in [8] are essentially given by Zindler carousels with two chairs which, according to [6], are in one to one correspondence with curves of constant width.

EXAMPLE. Consider $\beta_1(t) = (\cos(t), \sin(t))$, for $t \in \mathbb{R}$, and for $i = 2, 3, \ldots, n$, let $\beta_i(t) = \beta_1(t + \frac{2\pi}{n}(i-1))$. Then $\{\beta_1(t), \ldots, \beta_n(t)\}$ yields a Zindler carousel with n chairs whose Zindler curve is a circle.

Figures 1 to 3 show examples of carousels with 5 chairs. All of them are determined by their differential equations, as we will see in Section 5. Figure 3 shows the unique Zindler carousel with no auto-intersections with 5 chairs, different from the circular carousel.

REMARK. Let $\beta_2(t)$ be a smooth curve and let $p \neq \beta_2(0)$ (in particular, we may assume as usual that $|p - \beta_2(0)| = 2$). Then, using arguments of ordinary differential equations, it is possible to proof that there is a unique smooth curve $\beta_1(t)$ such that $\beta_1(0) = p$ and $\{\beta_1(t), \beta_2(t)\}$ is a carousel.

3. The floating body problem

DEFINITION. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a smooth simple closed curve. A chord of the figure $\Phi(\phi)$ bounded by $\phi(\mathbb{R})$ is a line segment whose extreme points lie in $\phi(\mathbb{R})$. A *chord system* $\{C(t)\}$ *for* $\Phi(\phi)$ is a continuous selection of an oriented chord $C(t)$ of $\Phi(\phi)$ starting at the point $\phi(t)$. If, except for its extreme points, the chords $C(t)$ lie in the interior of $\Phi(\phi)$, we say that $\{C(t)\}$ is a *system of interior chords for* $\Phi(\phi)$.

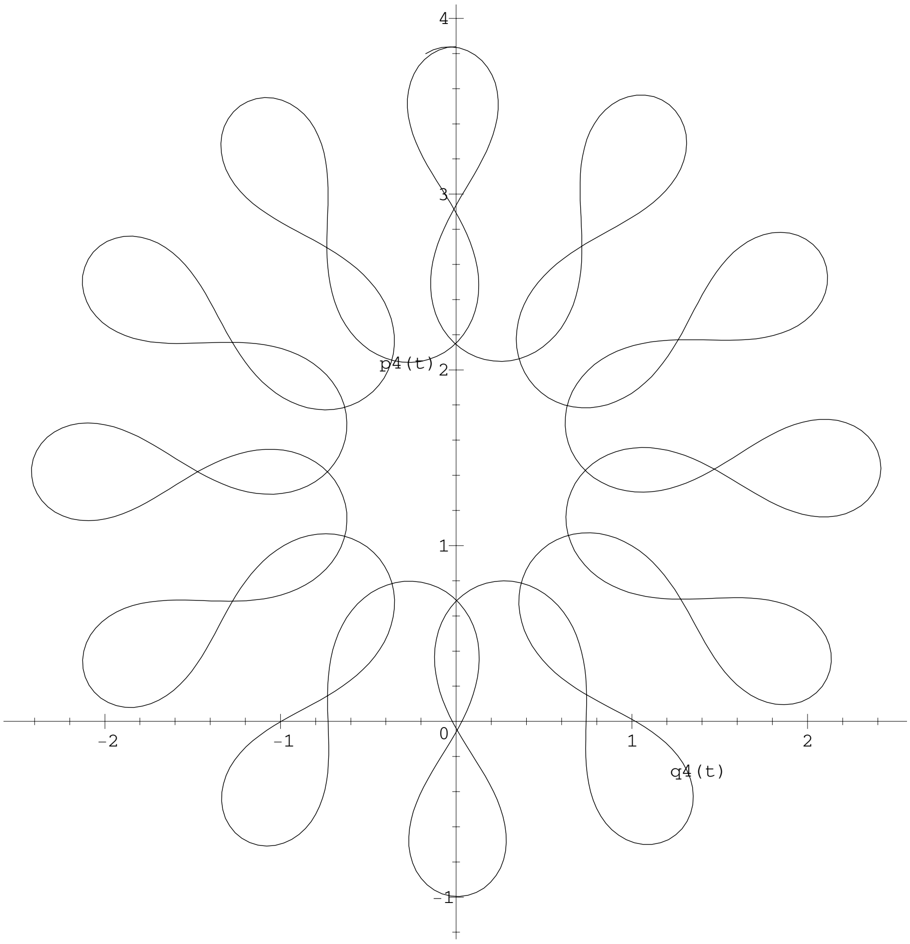

FIGURE 1

There are three natural kinds of chord systems for a figure $\Phi(\phi)$:

1) The system $\{C_{\mathcal{A}}(t)\}$ of interior chords which divide the area of the figure $\Phi(\phi)$ in a fixed ratio ϱ.

2) The system $\{C_{\mathcal{P}}(t)\}$ of chords whose extreme points divide the perimeter of the boundary of $\Phi(\phi)$ in a fixed ratio α.

3) The system $\{C_{\mathcal{L}}(t)\}$ of chords whose length is a positive real number τ.

Note that in general these chord systems do not necessarily exist.

Recall the floating body problem stated in the introduction. Now we make its meaning precise.

DEFINITION. Let $\Phi(\phi)$ be a figure of area $\mathcal{A}$ and suppose that the system of

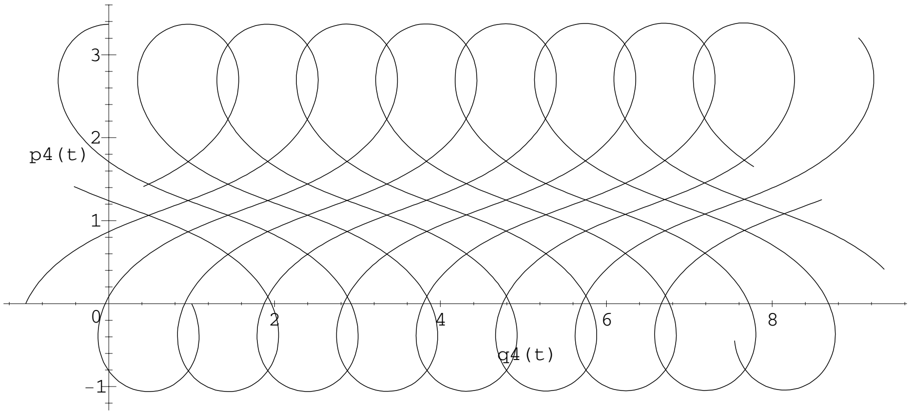

FIGURE 2

interior chords, $\{C_{\mathcal{A}}(t)\}$, which divide the area of Φ in a fixed ratio ϱ, exists. Let G be the mass center of $\Phi(\phi)$ and $G(t)$ the curve of the mass centers of the regions of Φ bounded by $\{C_{\mathcal{A}}(t)\}$, of area $\varrho\mathcal{A}$. Then, according to Archimedes Law, the figure $\Phi(\phi)$ of density ϱ *floats in equilibrium in a given position* t, if the line through G and $G(t)$ is orthogonal to $C_{\mathcal{A}}(t)$.

In 1938 Auerbach proved the following theorem:

THEOREM 1 (cf. [1]). *A figure $\Phi(\phi)$ of density ϱ floats in equilibrium in every position if and only if the system of interior chords $\{C_{\mathcal{A}}(t)\}$ which divides the area of $\Phi(\phi)$ in a fixed ratio ϱ exists and it is also of the type $\{C_{\mathcal{L}}(t)\}$ of constant length.*

In this theorem, $\Phi(\phi)$ is not necessarily convex. For the proof, Auerbach used the following fact of local character that will be used later:

A) Let $\{C(t)\}$ be a system of interior chords for a figure $\Phi(\phi)$. Denote by $\mathcal{A}(t)$, the area of the region of the figure $\Phi(\phi)$ left to the right by the chord $C(t)$, then, $\mathcal{A}'(t) = 0$ if and only if the chord $C(t)$ is tangent to the curve described by the midpoints of $C(t)$.

Let $\Phi(\phi)$ be a figure and suppose that the system of interior chords which divide the area of $\Phi(\phi)$ in a fixed ratio ϱ, $\{C_{\mathcal{A}}(t)\}$, exists. Suppose also that the figure $\Phi(\phi)$ of density ϱ floats in equilibrium in every position. Let us define, for every $t \in \mathbb{R}$, $\gamma(t) = \phi(t_1)$, where $\phi(t_1)$ is the other extreme point of $C_{\mathcal{A}}(t)$. It is known (see [1]), that γ is a smooth curve. By Theorem 1, $|\gamma(t) - \phi(t)|$ is constant. Furthermore, by **A)**, the system $\{\phi(t), \gamma(t)\}$ is a carousel and hence, by Lemma 1, $|\gamma'(t)| = |\phi'(t)|$. Therefore, the chords $\{C_{\mathcal{A}}(t)\}$ divide the perimeter of the figure $\Phi(\phi)$ in a fixed ratio α. In this case, we say that $\Phi(\phi)$ has *perimetral density* α, with respect to $\{C_{\mathcal{A}}(t)\}$.

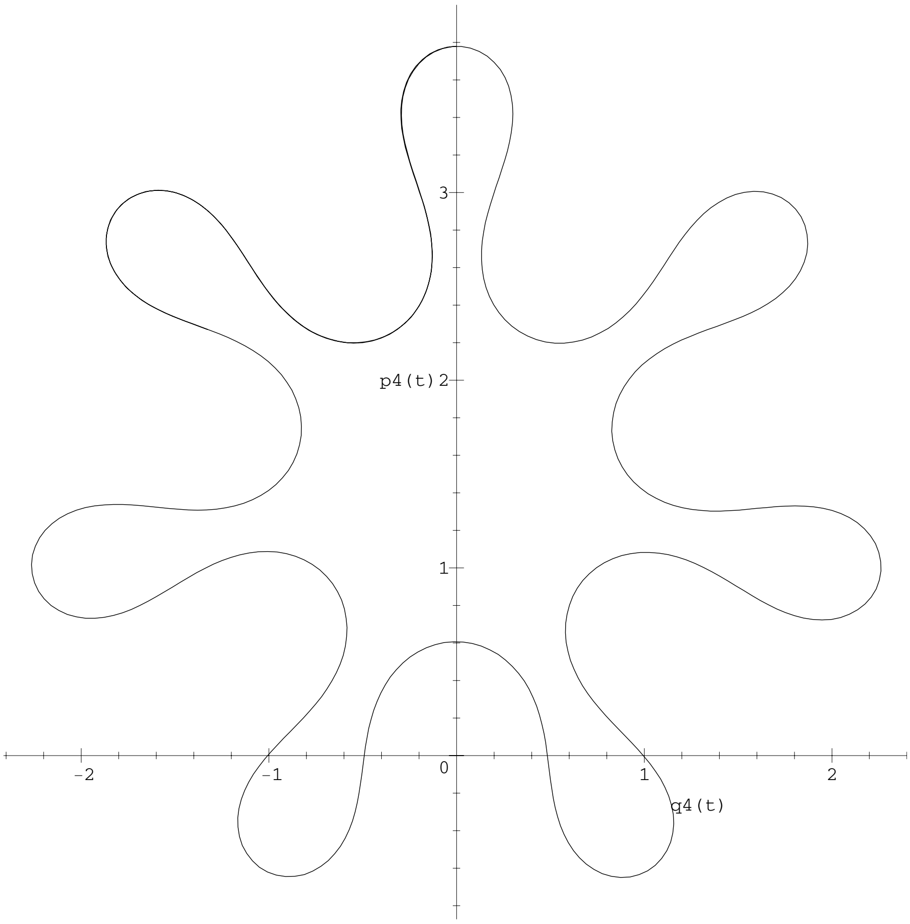

FIGURE 3

In the following, we will classify the figures that float in equilibrium in every position, according to their perimetral density.

The next Theorem establishes the main relation between carousels and the floating body problem

THEOREM 2. *Let $\Phi(\phi)$ be a figure. Suppose that the system of interior chords which divide the area of $\Phi(\phi)$ in a fixed ratio ϱ, $\{C_{\mathcal{A}}(t)\}$, exists and that the figure $\Phi(\phi)$ of density ϱ floats in equilibrium in every position with perimetral density α, where $\alpha = q/n$ (irreducible fraction). Then there exist a Zindler carousel with n-chairs $\{\beta_1(t), \ldots, \beta_n(t)\}$, such that $\phi = \beta_1$. Conversely, if $\{\beta_1(t), \ldots, \beta_n(t)\}$ is a Zindler carousel, with the property that β_i is a simple closed curve and the chords* *with extreme points $\beta_i(t)$ and $\beta_{i+1}(t)$ are interior chords of $\Phi(\beta_i)$, then $\Phi(\beta_i)$ is a figure that floats in equilibrium in every position.*

Proof. The interior chords $C_{\mathcal{A}}(t)$ divide the area and the perimeter of $\Phi(\phi)$ in a fixed ratio, and by Auerbach's Theorem, all of them have the same length. Let $\beta_1 = \phi$ and, for every $t \in \mathbb{R}$, let us define $\beta_2(t) = \beta_1(t_1)$, where $\beta_1(t_1)$ is the other extreme of the chord $C_{\mathcal{A}}(t)$. Clearly β_2 is a smooth simple closed curve. Similarly, for $i > 1$, let $\beta_i(t) = \beta_{i-1}(t_{i-1})$, where $\beta_{i-1}(t_{i-1})$ is the other extreme of the chord $C_{\mathcal{A}}(t_{i-2})$. So, since the perimetral density $\alpha = q/n$, we finally obtain that $\beta_{n+1} = \beta_1$, where the curves β_i are all parametrizations of ϕ. By construction and by A), the two conditions asked from the system $\{\beta_1(t), \dots, \beta_n(t)\}$ to obtain a carousel are satisfied.

Conversely if $\{\beta_1, \dots, \beta_n\}$ is a Zindler Carousel, then every curve β_i determines the same figure. So the chords of $\Phi(\beta_1)$ with extreme points $\beta_1(t)$ and $\beta_2(t)$ give rise to a system of interior chords. By A) and condition (2) for carousels, we have that these interior chords divide the area of $\Phi(\beta_1)$ in a fixed ratio and by condition (1) for carousels and Auerbach's Theorem, the figure $\Phi(\beta_1)$ floats in equilibrium in every position. $\square$

4. Some intrinsic properties of carousels

Let P be a polygon with cyclically oriented vertices $\{v_1, v_2, \dots, v_n\}$. One can define its (signed) *area* as

$$A\,(P) = \frac{1}{2}\sum_{i=1}^{n} |v_i, v_{i+1}| \tag{1}$$

where $|a, b|$ denotes the determinant of the vectors a, b – and is therefore twice the area of the triangle they form with the origin. It is easy to see that the area does not depend on where the origin is and that it coincides with the classic area for the convex or non self-intersecting polygons, so that it is a natural extension to general polygons such as carrousels.

Similarly, for polygons P with $A(P) \neq 0$ one can naturally generalize the notion of *center of mass* by the formula

$$g\,(P) = \frac{1}{6A\,(P)}\sum_{i=1}^{n} |v_i, v_{i+1}|\,(v_i + v_{i+1})\,. \tag{2}$$

For the convex case with the origin inside, it is clearly the classical definition of mass center by means of the natural triangulation. It is not too hard to see that in general it does not depend on where the origin is.

Theorem 3. *The area of a carrousel is constant and if it is not zero then its center of mass is a fixed point.*

PROOF. Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a carrousel and denote by $A(t)$ the area of the polygon $X(t) = \{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$. By the properties of the determinant, and omitting the variable t, we have

$$A' = \frac{1}{2}\sum_{i=1}^{n} \frac{d}{dt}\left(|\beta_i, \beta_{i+1}|\right) = \frac{1}{2}\sum_{i=1}^{n} |\beta_i', \beta_{i+1}| - \left|\beta_{i+1}', \beta_i\right|. \tag{3}$$

The first carrousel law can be written $\left|\beta_i' + \beta_{i+1}', \beta_{i+1} - \beta_i\right| = 0$, which can clearly be restated as

$$|\beta_i', \beta_{i+1}| - \left|\beta_{i+1}', \beta_i\right| = |\beta_i', \beta_i| - \left|\beta_{i+1}', \beta_{i+1}\right|.$$

Substituting in (3) we get that $A' = 0$ and thus that the area is constant.

For the mass center, let $f : R \to R^2$ be

$$f(t) = \sum_{i=1}^{n} |\beta_i(t), \beta_{i+1}(t)|\left(\beta_i(t) + \beta_{i+1}(t)\right).$$

Another way to write the first carrousel law is that there exist real valued functions h_i for which

$$\beta_i' + \beta_{i+1}' = h_i\left(\beta_{i+1} - \beta_i\right).$$

Then we have

$$\begin{aligned}
f' &= \sum_{i=1}^{n}\left[\left(|\beta_i', \beta_i| - \left|\beta_{i+1}', \beta_{i+1}\right|\right)(\beta_i + \beta_{i+1}) + h_i\,|\beta_i, \beta_{i+1}|\,(\beta_{i+1} - \beta_i)\right] \\
&= \sum [2\,|\beta_i', \beta_i|\,\beta_i - 2\left|\beta_{i+1}', \beta_{i+1}\right|\beta_{i+1} \\
&\qquad + \left(|\beta_i', \beta_i| + \left|\beta_{i+1}', \beta_{i+1}\right| + h_i\,|\beta_i, \beta_{i+1}|\right)(\beta_{i+1} - \beta_i)] \\
&= \sum\left(|\beta_i', \beta_i| + \left|\beta_{i+1}' + h_i\beta_i, \beta_{i+1}\right|\right)(\beta_{i+1} - \beta_i) \\
&= \sum\left(|\beta_i', \beta_i| - |\beta_i', \beta_{i+1}|\right)(\beta_{i+1} - \beta_i) \\
&= -\sum |\beta_i', \beta_{i+1} - \beta_i|\,(\beta_{i+1} - \beta_i).
\end{aligned}$$

With our previous notation and conventions about the length of the chairs and the velocity of the points in a carrousel, the second carrousel law can be written $\theta_i' = \frac{1}{2}\left|\beta_i', \beta_{i+1} - \beta_i\right|$, because $2\sin\alpha_i = |\beta_i', \beta_{i+1} - \beta_i|$. Therefore, recalling that $(\beta_{i+1} - \beta_i) = 2u\,(\theta_i)$, we obtain from our last equation that

$$\begin{aligned}
f' &= -4\sum \theta_i' u\,(\theta_i) \\
&= 4\sum \frac{d}{dt}\left(u\left(\theta_i + \frac{\pi}{2}\right)\right) \\
&= 4\frac{d}{dt}\left(\sum u\left(\theta_i + \frac{\pi}{2}\right)\right).
\end{aligned}$$

Since $\sum u\left(\theta_i + \frac{\pi}{2}\right)$ is the constant function 0 because the carrousel is a closed polygon, we conclude that $f' = 0$ and thus that $f(t)$ is a fixed point. Finally, when $A(t) \neq 0$, from the equation $6A(t)g(t) = f(t)$, having the area constant we obtain that the center of mass $g(t)$ is also a fixed point. □

We must remark that the preceding proof should also have a geometrical meaning for some carrousels of area zero. We have proved that $f(t)$ is constant, if it is also non-zero for an area zero carrousel then it determines a fixed direction (point at infinity) which is the limit point of the mass centers of carrousels with small area tending to it. Figure 2 is the unique such case for carrousels with 5 chairs, it moves in the perpendicular direction to its "center of mass" as the limit of circles with a large radius.

THEOREM 4. *Let $\{\beta_1(t), \dots, \beta_n(t)\}$ be a carousel with n chairs, and let $x_i(t) \in [0, 2\pi)$ be the angle between the sides $\beta_{i+1}(t) - \beta_i(t)$ and $\beta_{i-1}(t) - \beta_i(t)$. Then, if n is even, there exists an integer k such that*

$$\sum_{i=1}^{n} (-1)^i x_i(t) = 2k\pi.$$

PROOF. Take a fixed time $t \in \mathbb{R}$. By Lemma 1, we know that β'_{i+1} is a reflection Ω_{θ_i} of β'_i on the line (through the origin) with angle θ_i, between $\beta_{i+1} - \beta_i$ and the x-axis. Then, if n is even we have that

$$\begin{aligned}\beta'_1 &= \Omega_{\theta_n} \circ \Omega_{\theta_{n-1}} \circ \cdots \circ \Omega_{\theta_1}(\beta'_1) \\ &= R_{2((\theta_1+\theta_3+\dots\theta_{n-1})-(\theta_2+\theta_4+\dots\theta_n))}(\beta'_1),\end{aligned}$$

where R_σ is the rotation of an angle σ.

But this equality happens only if for some integer $k_o(t)$,

$$2\big((\theta_1(t) + \theta_3(t) + \cdots + \theta_{n-1}(t)) - (\theta_2(t) + \theta_4(t) + \cdots + \theta_n(t))\big) = 2k_o(t)\pi.$$

Using now the fact that for an n-gon there exists an integer $k_i(t)$ such that $\theta_{i+1}(t) - \theta_i(t) - \pi + x_{i+1}(t) = 2k_i(t)\pi$, for $i = 1, \dots, n$, we conclude that $\sum_{i=1}^{n}(-1)^i x_i(t) = 2k(t)\pi$.

Assume that $x_i(t_0) = 0$ then $\beta_{i-1}(t_0) = \beta_{i+1}(t_0)$. Consider $\{\beta_{i-1}(t), \beta_i(t)\}$ and $\{\beta_{i+1}(t), \beta_i(t)\}$ as carousels of two chairs, then by the remark at the end of Section 2, we have that $\beta_{i-1}(t) = \beta_{i+1}(t)$ for every t. Therefore, $x_i(t) = 0$ for every t.

Hence, $x_i : \mathbb{R} \to [0, 2\pi)$ is continuous (never passes discontinuously between 0 and 2π) therefore $\sum_{i=1}^{n}(-1)^i x_i(t)$ is a continuous map and hence $k(t)$ must be constant. The proof of the theorem follows from this. □

REMARK. If n is odd, the composition of the n reflections is again a reflection, then there exists a unique vector w, up to scalar factor, such that $\Omega_{\theta_n} \circ \Omega_{\theta_{n-1}} \circ \cdots \circ \Omega_{\theta_1}(w) = \Omega_\Psi(w)$, for some suitable angle Ψ. Consequently, w should be the velocity vector β_1'.

In contrast to the fact that there are non circular figures that float in equilibrium in every position with perimetral density 1/2 ([1]), we have the following theorem.

THEOREM 5. *Except from the circle, there are no figures that float in equilibrium in every position with perimetral density* $\frac{1}{3}$ *and* $\frac{1}{4}$.

PROOF. Recall that if a figure floats in equilibrium in every position with perimetral density ρ, then the system of chords that divide the area in the corresponding ratio are all interior. Let us start with the case of perimetral density $\frac{1}{3}$. The equilateral triangle is the only possible n-gon with 3 sides of equal length. By Theorem 3, the mass center of the equilateral triangles determined by a Zindler carousel with 3 chairs is a fixed point. So, the circular carousel is the only non trivial Zindler carousel with 3 chairs and therefore, by Theorem 2, except from the circle, there are no figures that float in equilibrium in every position with perimetral density $\frac{1}{3}$.

Let $\Phi(\phi)$ be a figure that floats in equilibrium in every position with perimetral density $\frac{1}{4}$. Clearly, the 4-gons that arise from $\Phi(\phi)$ can not have area zero otherwise density $\frac{1}{4}$ never achieve. Thus, these 4-gons are convex parallelograms with positive area and by Theorem 4, they should be squares. By Theorem 3, their mass centers are a single point. Then, except from the circle, there are no figures that float in equilibrium in every position with perimetral density $\frac{1}{4}$. □

When n is even the existence carousels in which $\beta_i = \beta_{n-i+2}$ gives rise to a type of carousels that can be studied in their own right and will not be considered in this paper. For this reason, from now on, we shall consider only carousels with an odd number of chairs.

5. The differential equations

The purpose of this section is to exhibit the differential equations of carousels.

Recall, that given a carousel $\{\beta_1(t), \dots, \beta_n(t)\}$, we assume, that $|\beta_i'(t)| = 1$ and $|\beta_{i+1}(t) - \beta_i(t)| = 2$ and remember that $\alpha_i(t)$ denotes the angle between the vectors $\beta_i'(t)$ and $\beta_{i+1}(t) - \beta_i(t)$ and $\theta_i(t)$ denotes the angle between the x-axes and the vector $\beta_{i+1}(t) - \beta_i(t)$. Let $x_i(t) \in [0, 2\pi)$ be the angle between the vectors $\beta_{i+1}(t) - \beta_i(t)$ and $\beta_{i-1}(t) - \beta_i(t)$.

THEOREM 6. *Let $\{\beta_1(t), \ldots, \beta_n(t)\}$ be a carousel with n-chairs. Then, the interior angles $x_i(t)$, $i = 1, \ldots, n$, satisfy the following system of differential equations*

$$x_i'(t) = \sin(\alpha_{i-1}(t)) - \sin(\alpha_i(t)), \tag{1}$$

If n is odd then

$$\alpha_i(t) = x_{i+2}(t) + x_{i+4}(t) + \cdots + x_{i+(n-1)}(t) - \left(\frac{k-1}{2}\right)\pi, \tag{2}$$

where k is the integer number such that $\sum_{i=1}^{n} x_i(0) = k\pi$.

Conversely, if n is odd and we have functions $x_i(t)$, $i = 1, \ldots, n$, satisfying the system of differential equations (1), (2); and such that the initial conditions $(x_1(0), \ldots, x_n(0))$ are the interior angles of an equilateral n-gon with sides of length 2. Then, there exists a carousel of n chairs $\{\beta_1(t), \ldots, \beta_n(t)\}$, with the property that $x_i(t)$ is the angle between $\beta_{i+1}(t) - \beta_i(t)$ and $\beta_{i-1}(t) - \beta_i(t)$.

PROOF. As we know, $x_i(t) = k_i(t)\pi + \theta_{i-1}(t) - \theta_i(t)$, where $k_i(t)$ is a step function. By Lemma 2, we have that $\theta_i'(t) = \sin(\alpha_i(t))$, so

$$x_i'(t) = \sin(\alpha_{i-1}(t)) - \sin(\alpha_i(t)).$$

On the other hand, by Lemma 1, $\alpha_{i-1}(t) + x_i(t) + \alpha_i(t) = \pi$. Taking the alternating sum of these equations, if n is odd, it is possible to conclude that

$$\alpha_i(t) = x_{i+2}(t) + x_{i+4}(t) + \cdots + x_{i+(n-1)}(t) - \left(\frac{k(t)-1}{2}\right)\pi,$$

where $k(t)$ is the integer number such that $\sum_{i=1}^{n} x_i(t) = k(t)\pi$. As we already noted in the proof of Theorem 4, the functions $x_i : \mathbb{R} \to [0, 2\pi)$ are continuous (never pass discontinuously between 0 and 2π). Therefore $k(t) = k(0) = k$.

Suppose that we have functions $x_i(t)$, $i = 1, \ldots, n$, n odd, satisfying the system of differential equations (1), (2); and such that the initial conditions $(x_1(0), \ldots, x_n(0))$ are the interior angles of an equilateral n-gon with sides of length 2. That is, $\sum_{i=1}^{n} x_i(0) = k\pi$ and

$$u(0) + u(\pi - x_2(0)) + \cdots + u\big((n-1)\pi - (x_2(0) + \cdots + x_n(0))\big) = 0.$$

For $i = 1, \ldots, n$, define $\theta_i(t)$ and $\beta_i(t)$ so that

$$\theta_i'(t) = \sin(\alpha_i(t)),$$

where

$$\alpha_i(t) = x_{i+2}(t) + x_{i+4}(t) + \cdots + x_{i+(n-1)}(t) - \left(\frac{k-1}{2}\right)\pi,$$

and

$$\beta_i'(t) = (\cos(\theta_i(t) - \alpha_i(t)), \sin(\theta_i(t) - \alpha_i(t))),$$

with initial conditions, $(\beta_1(0), \ldots, \beta_n(0))$, the vertices of an equilateral n-gon with sides of length 2, and $\theta_i(0)$ the angle between the corresponding side $\beta_{i+1}(0) - \beta_i(0)$

and the x-axes. That is, $\theta_i(0) = \theta_1(0)+(i-1)\pi-(x_2(0)+\cdots+x_i(0))$ for $i = 1, \ldots, n$, and $|\beta_{i+1}(0) - \beta_i(0)| = 2$.

By uniqueness of solutions, $\theta_i(t) = \theta_1(t)+(i-1)\pi-(x_2(t)+\cdots+x_i(t))$, because for $t = 0$ the condition is true and $\theta_1'(t) - (x_2'(t) + \cdots + x_i'(t)) = \sin(\alpha_i(t)) = \theta_i'(t)$.

Define $\gamma_2(t) := \beta_1(t) + 2u(\theta_1(t))$. Then, by Lemma 2, $\{\beta_1(t), \gamma_2(t)\}$ is a carousel, because $|\gamma_2(t) - \beta_1(t)| = 2$ and $\theta_1'(t) = \sin(\alpha_1(t))$, where by construction, $\alpha_1(t)$ is the angle between $\gamma_2(t) - \beta_1(t)$ and $\beta_1'(t)$.

Using trigonometric identities and after some simplifications

$$\gamma_2'(t) = (\cos(\theta_1(t) + \alpha_1(t)), \sin(\theta_1(t) + \alpha_1(t))).$$

But $\theta_{i+1}(t) = \theta_i(t) + \pi - x_i(t)$ and $\pi - x_2(t) - \alpha_1(t) = \alpha_2(t)$, thus $\theta_1(t) + \alpha_1(t) = \theta_2(t) - \alpha_2(t)$. Then $\beta_2'(t) = \gamma_2'(t)$ and, by hypothesis, $\beta_2(0) = \gamma_2(0)$. By uniqueness of solutions, $\beta_2(t) = \gamma_2(t)$. Finally, if we define inductively $\gamma_{i+1}(t) := \beta_i(t) + 2u(\theta_i(t))$, we conclude the proof of the theorem. $\square$

The following two corollaries which follow from our previous results, will be used in the next section.

COROLLARY 1. *Let $X(0)$ be a regular n-gon with interior angles $(x_1(0), \ldots, x_n(0))$, n odd. Then, there exists a unique carousel $\{\beta_1(t), \ldots, \beta_n(t)\}$ up to orientation, with initial condition $X(0)$.*

COROLLARY 2. *Let $\{\beta_1(t), \ldots, \beta_n(t)\}$ be a carousel with n-chairs, n odd. If there exists $t_0 \in \mathbb{R}$ such that $x_i(t_0) = x_{i+1}(0)$ $(i = 1, \ldots, n)$, then the curves $\beta_1(t), \ldots, \beta_n(t)$ are congruent.*

PROOF. Since $x_i(t_0) = x_{i+1}(0)$, there is a congruence Δ such that $\Delta(\beta_i(t_0)) = \beta_{i+1}(0)$. Let $\gamma_{i+1}(t) = \Delta(\beta_i(t + t_0))$. Clearly the system $\{\gamma_1(t), \ldots, \gamma_n(t)\}$ is a carousel with initial conditions $X(0) = \{x_1(0), \ldots, x_n(0)\}$, because $\gamma_i(0) = \beta_i(0)$. By Corollary 1, $\{\gamma_1(t), \ldots, \gamma_n(t)\} = \{\beta_1(t), \ldots, \beta_n(t)\}$. Then, $\Delta(\beta_i(t + t_0)) = \beta_{i+1}(t)$. This concludes the proof of the corollary. Note that if the area of the n-gon $\{\beta_1(0), \ldots, \beta_n(0)\}$ is not zero, by Theorem 3, Δ is a rotation. $\square$

6. An outline towards the classification of carousels with five chairs

In this section we describe the main ideas leading to the classification of carousels with five chairs. Such a complete understanding yields, again, the non existence of figures, different from the circle, that float in equilibrium in every direction with perimetral densities 1/5 and 2/5. The detailed proofs, which become quite technical, can be seen in [2] and [6].

Carousels are systems of curves, but they can also be thought of as a linkage with rigid rods moving in time. If we consider the space of regular plane polygons, then, by Corollary 1, carousels are simply a flow there. But there is another interesting, and simpler, associated flow in the compact space of regular polygons modulo congruence. This flow simply describes how the "shape" of the polygon moves as the carousel goes.

More precisely, and restricting ourselves from now on to the case $n = 5$, let $\mathcal{P}^5$ be the space of all equilateral pentagons in the plane, one of whose sides is the distinguished interval $[(-1,0),(1,0)]$, and all the other sides have length two. We will call it the *phase space of equilateral pentagons.* Corollary 1 implies that given $P(0) \in \mathcal{P}^5$ we have a carousel $\{\beta_1(t),\ldots,\beta_5(t)\}$ with initial conditions $X(0) = P(0)$ and therefore we can define $P(t)$ to be the image of the regular pentagon $X(t) = (\beta_1(t),\ldots,\beta_5(t))$ under the unique orientation preserving isometry that sends its edge $[\beta_1(t),\beta_2(t)]$ to the distinguished interval $[(-1,0),(1,0)]$. This defines a flow in $\mathcal{P}^5$, which may be rightly called the *carousel flow* on the phase space of equilateral pentagons, and its orbits correspond to carousels (modulo orientation preserving isometries, which are the natural thing to consider).

It can be seen in [7], that $\mathcal{P}^5$ is a genus 4 oriented surface which is most conveniently described as an analytic smooth sub variety of the 5 dimensional torus $(\mathbb{S}^1)^5$ when we take as parameters the 5 internal angles $(x_1,\ldots,x_5)$ of the pentagons. By Theorem 3 we know that carousels preserve area, therefore the carousel flow turns out to be an integral flow of the area function $f : \mathcal{P}^5 \to \mathbb{R}$, that is, the orbits stay on the same area pentagons of a fixed area.

The Morse Theory of the area function $f : \mathcal{P}^5 \to R$ reads as follows. It has six critical values $-a < -b < -c < c < b < a$ and 14 critical points: 2 maxima (which are the positively oriented regular convex pentagon, of area a, and the negatively oriented regular pentagram); 2 minima (which are the negatively oriented regular convex pentagon and the positively oriented regular pentagram, of area c), and 10 saddle points (corresponding to pentagons which look like regular triangles of area b, or $-b$, but where one side is used three times). Of course, the complete topological description of $\mathcal{P}^5$ can be carried out in detail, and the analysis of the carousel flow should be made on each of the regular pieces.

Let us concentrate on the "top" piece $\mathcal{P}^5_0 := f^{-1}((b,a])$ which corresponds to pentagons whose sides do not intersect and are positively oriented. Topologically this is an open disc where the regular equilateral pentagon lies in the center and all of the remaining orbits are closed; they are $f^{-1}(A)$ for some area $A \in (b,a)$. Then, we have a *period* η_A, which is the minimum number for which $P(0) = P(\eta_A)$ for any $P(0) \in f^{-1}(A)$.

Observe that if a pentagon $P \in \mathcal{P}^5_0$ is parametrized by the angles $(x_1, x_2, \ldots, x_5)$ then the other four pentagons obtained by cyclic permutations of its angles have the same area. Thus, they lie in the same orbit. It happens that the first of these pentagons that $P(0) = P \in f^{-1}(A)$ encounters as it flows is $P(\varepsilon_A) = (x_4, x_5, x_1, x_2, x_3)$ where $5\varepsilon_A = \eta_A$.

Turning our attention from the pentagons (modulo congruence) back to their carousels. If $P \in f^{-1}(A) \subset \mathcal{P}^5_0$ is considered as the initial conditions of a carousel $\{\beta_1(t),\ldots,\beta_5(t)\}$, then there is an angle σ_A, called the *basic angle* of A, for which $\beta_i(t+\varepsilon_A) = R_{\sigma_A}\beta_{i+3}(t)$, where R_{σ_A} is the rotation of σ_A around the center of mass, because the pentagon has come back to itself but in a different cyclic order and rotated. If we ask the curve $\beta_1(t)$ to "match up" at some time with the curve $\beta_2(t)$ (that is, that $\beta_1(t+m\varepsilon_A)=\beta_2(t)$ for some m) to obtain a Zindler carousel, then this imposes some rationality conditions on the basic angle. A complete classification of Zindler carousels may be derived in these lines, but we are mainly interested in those of index 1, that is, those whose closed curve winds once around the center.

THEOREM 7. *Any $\frac{1}{5}$-Zindler carousel of index 1 has basic angle of the form $\frac{4s+2}{5s+2}\pi$, for $s>0$, a natural number and conversely, if a carousel with area $A\in(a,b)$ has basic angle $\sigma_A=\frac{4s+2}{5s+2}\pi$, for $s>0$ a natural number, then it is a $\frac{1}{5}$-Zindler carousel.*

Next, using the fact that $f^{-1}(a)$ is an isolated singular point of the vector field and a non-degenerate center, that is, the linear part of the vector field has eigenvalues $\pm i\omega$, $\omega>0$, we may prove, using the Classical Poincaré–Lyapunov Center Theorem [3], that the limit of the period function $\eta:(b,a)\to R$, when $A\to a$, is $2\pi/\omega$ which, after the corresponding calculations, gives $\eta(a)=2\pi/\omega\sim 2.4002$. Furthermore, there is clear evidence, that the period function $\eta:(b,a)\to R$ is a decreasing function. In fact, $\eta(A)\geq\eta(m)$, for every $A\in(b,a)$. But it can be shown [2] that $\sigma_A<2.9132$ and $\eta_A>2.4002$, for every $A\in(b,a)$.

This, together with the fact that a $\frac{1}{5}$-Zindler carousel of index 1 must have a period smaller than $2\pi/\omega\sim 2.4002$ shows that a $\frac{1}{5}$-Zindler carousel of index 1 never gives rise to a figure that floats in equilibrium. The same ideas can be analogously applied to study $\frac{1}{5}$-Zindler carousels of index -1 and $\frac{2}{5}$-Zindler carousels.

In Section 4, it was proved that there are no figures that float in equilibrium in every position with perimetral density $\frac{1}{3}$ and $\frac{1}{4}$, although we know from [1] that there are with perimetral density $\frac{1}{2}$. The previous discussion gives a complete understanding of carousels with 5 chairs that yields, again, the non existence of figures, different from the circle, that float in equilibrium in every position with perimetral densities $\frac{1}{5}$ and $\frac{2}{5}$.

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J. Bracho
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuíto Exterior C. U. 04510
México D.F. México
E-mail: roli@matem.unam.mx

L. Montejano
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuíto Exterior C. U. 04510
México D.F. México
E-mail: luis@matem.unam.mx

D. Oliveros
Department of Mathematics and Statistics
University of Calgary
2500 University Drive NW.
Calgary, Alberta, T2N 1N4, Canada
E-mail: deborah@math.ucalgary.ca