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Flat transversals to flats and convex sets of a fixed dimension

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Abstract

Helly and Hadwiger type theorems for transversal m-flats to families of flats and, respectively, convex sets of dimension n are proved in the case of general position. The proofs rely on Helly type theorems for "linear partitions" and "convex partitions," so that a general theory of Helly numbers is also developed.

MSC: 52A35; 52A01

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1. Introduction

Hadwiger considered in [11] the possibility of a Helly type theorem for transversal lines to a family of planar convex sets. He observed that an extra hypothesis about the hitting order of transversal lines to subfamilies of size 3 must be assumed to conclude the existence of a transversal line to the whole family; such a theorem is what we understand as a "Hadwiger type theorem." His result was generalized by Goodman and Pollack [9] (and further by Pollack and Wenger [14]) to one of transversal hyperplanes using the notion of order type, which generalizes order for lines. These ideas have ramified to different contexts by restricting the type of convex

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sets considered (see [10,16]); but few results, other than [1] or the recent [5], are known for transversals of dimension and codimension different from 1. This problem, explicitly posed in [7, Problem 7.9], is the one we address.

In [3] we proved a Hadwiger type theorem for transversal lines to a family of convex sets of dimension 1 (detailed bellow). It is closely related to a Helly type theorem for transversal lines to a family of lines in projective space (of any dimension); namely, if each six of them have a transversal line, they all do. Here, those ideas are extended to existence theorems of transversal flats of dimension m to finite families of convex sets and of projective flats of dimension n; an extra assumption is made concerning the "general position" of the families. There is a work of Lovász about transversal flats to families of flats (seemingly, with no extra hypothesis). It is reported in [8]; but the "Helly number" announced is binomial while ours is linear.

First, we state the main theorems and sketch their proofs; then, to conclude the introduction, we see through an example that when considering convex sets of a fixed dimension greater than 1 some extra hypothesis is always needed.

A projective subspace of a projective space is also called a *flat* or an *n-flat* if it has dimension n. A finite collection $X_0, X_1, \dots, X_k$ of flats in projective space is in *general position* if together they span the biggest possible projective subspace; that is, if

$$\dim\left\langle \bigcup_{i=0}^{k} X_i \right\rangle = \sum_{i=0}^{k} \dim X_i + k,$$

where $\langle A \rangle$ denotes the projective (or linear) span of A. We say that a family of flats is *k-generic* if each $k+1$ of them are in general position, or equivalently, if no $k+1$ of them have a transversal $(k-1)$-flat (this condition was used in [9], see also [7]). For example, they are 1-generic if each pair is disjoint. Likewise, we say that a family of convex sets in Euclidean space is in *general position* (or *k-generic*) if their spanned projective flats are; considering, of course, that they lie in the projective closure of Euclidean space.

Observe that the usual notion of "general position" for points in $\mathbb{P}^n$ corresponds to our notion of being n-generic, for at most $n+1$ points in $\mathbb{P}^n$ can be in general position in the sense we have defined above. Thus, beware of the slight difference with usual terminology.

Considering only finite families to avoid routine topological considerations, our main theorems are:

Theorem 1. *An m-generic family of n-flats has a transversal m-flat if every subfamily of cardinality*

$$\left\lfloor \frac{1}{2}(3n + 2m + 7) \right\rfloor$$

has a transversal m-flat.

Theorem 2. *An m-generic family of convex sets of dimension n has a transversal m-flat if they correspond to an order type of dimension m such that every subfamily of cardinality*

$$2n + m + 3$$

has a transversal m-flat compatible with that order type.

The proofs of these theorems follow the same general sketch. First, to establish the structure of the set T of m-flats transversal to $m+2$ elements of the family—in the first case it is (naturally parametrized by) a projective space of dimension n, in the second, considering the order, it will be a convex set of dimension n. Second, to prove that every other member of the family is transversal to a subset of T of a special type: in the first case what we call a *linear partition*; in the second, a *convex partition*. Finally, to prove and use a Helly theorem for such special subsets.

A *linear partition* is a subset of projective space which is a union of flats in general position. Analogously, a *convex partition* is a subset of Euclidean space such that its connected components are convex sets which, as a family, is in general position. The two Helly theorems we have referred to are:

Theorem 3. *A family of linear partitions in $\mathbb{P}^n$ has non-empty intersection if every $\lfloor 3(n+1)/2 \rfloor$ of them have non-empty intersection. Furthermore, this is the least possible such number.*

Theorem 4. *A family of convex partitions in $\mathbb{R}^n$ has non-empty intersection if every $2n+1$ of them have nonempty intersection. Furthermore, this is the least possible such number.*

1.1. An example

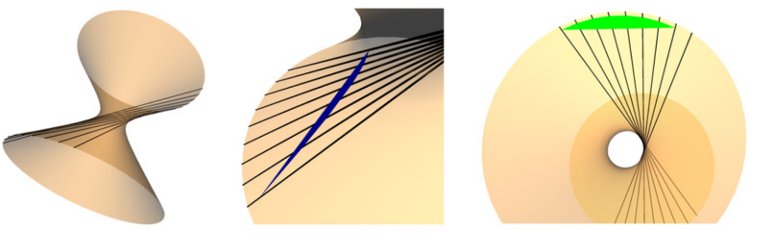
In [3] we proved that if a numbered family of intervals in an affine space has the property that any six of them have a transversal line compatible with the numbering, then all the intervals have a transversal line. The "magic" number six is easily reduced to three if instead of intervals we consider points. However, if we increase the dimension of the convex sets, there is no such "magic" number for transversal lines; some other assumption has to be made.

To see this, consider a family of lines close together which belong to one of the two rulings of the symmetric hyperboloid $x^2 + y^2 = z^2 + 1$ (Fig. 1(a)).

Consider a vertical plane (containing the z axis). A planar convex polygon is spanned by the intersection points of our lines with that plane. For a wide range of such planes, this polygon is contained outside the hyperboloid, as shown in Fig. 1(b). On the other hand, if we do the same with a horizontal plane (parallel to the xy plane), then the polygon obtained lies inside the hyperboloid (Fig. 1(c)).

Since a line cannot cross the hyperboloid in more than two points, we can construct several such polygons, alternating inside and out, to force the fact that any transversal line to them must be one of the a priori chosen family.

Moreover, to generate the polygons we are not obliged to take the convex hull of all the intersection points; we can leave out one or more of them. It is clear then that our freedom is



(a) (b) (c)

Fig. 1. Crossing a ruling with horizontal and vertical planes.

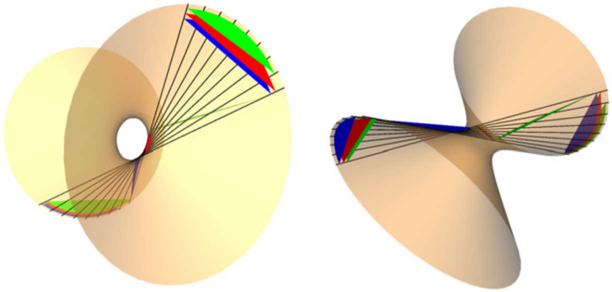


Fig. 2. An example of 9 polygons with the lines transversal to each 8 of them.

enough to force any of the lines to intersect only a subset of the polygons. In particular, we can build up a set of k polygons and k lines in such a way that each polygon avoids exactly one of the lines. In Fig. 2, two perspectives of such an example for $k = 9$ are shown.

These examples can be easily generalized to convex sets of higher dimension as follows. Suppose we have a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$. In $\mathbb{R}^{d+1}$, multiply each of the sets in $\mathcal{F}$ by the interval $0 \leqslant w \leqslant 1$, where w is the new coordinate, to obtain a family $\mathcal{F}'$ of convex sets one dimension higher. It is not difficult to see that a subfamily of $\mathcal{F}$ has a transversal line if and only if the corresponding subfamily of $\mathcal{F}'$ has one.

Thus, we showed that such general transversal line theorems are possible only when the convex sets have dimension 0 (trivial) or 1 (the one in [3]).

2. Rulings

We will work in a high-dimensional projective space as a universal ambient space. It will be denoted $\mathbb{P}^N$ but may also be thought of as $\mathbb{P}^\infty$ with only finite-dimensional subspaces considered. Some subspaces of a specific dimension k will be relevant as ambient spaces and then are denoted $\mathbb{P}^k$. By $X \lhd \mathbb{P}^N$ we mean that X is a flat of $\mathbb{P}^N$; and by $X^n \lhd \mathbb{P}^N$ we mean, furthermore, that X has dimension n. Superscripts will always mean dimension and so are used to establish it, but later on they may be dropped to ease reading.

Given a family $\mathcal{F}$ of subsets of $\mathbb{P}^N$, an *m-transversal* to $\mathcal{F}$ is an m-flat that intersects every member of $\mathcal{F}$. Let us denote by $T_m(\mathcal{F})$ the set of all m-transversals to $\mathcal{F}$, that is,

$$T_m(\mathcal{F}) := \left\{ Y^m \lhd \mathbb{P}^N \;\middle|\; Y \cap A \neq \emptyset \text{ for all } A \in \mathcal{F} \right\}.$$

Let $X_1^n, X_2^n, \ldots, X_{m+1}^n$ be projective flats of dimension n in general position. They span a projective space of dimension $n(m+1) + m$. Let

$$\mathbb{P}^{n*m} = \langle X_1, \ldots, X_{m+1} \rangle,$$
$$\text{where } n * m := (n+1)(m+1) - 1.$$

Observe that any m-transversal Y^m to $X_1, \ldots, X_{m+1}$ is contained in $\mathbb{P}^{n*m}$ because Y contains a point in each of the X_i and, by general position, these $m+1$ points span Y. So that $T_m(X_1, \ldots, X_{m+1})$ is naturally parametrized by the Cartesian product $X_1 \times \cdots \times X_{m+1}$.

Let $p \in \mathbb{P}^{n*m}$ be a *generic* point, which means that the family $p, X_1, \ldots, X_{m+1}$ is m-generic. Then p lies in a unique m-flat transversal to $X_1, \ldots, X_{m+1}$, which we denote Y_p. Indeed, if we let

$$p_i = X_i \cap \langle p, X_1, \ldots, \widehat{X}_i, \ldots, X_{m+1} \rangle$$

(where $\widehat{X}_i$ means "omit X_i"), which is a well-defined point by a simple-dimensional argument, then $Y_p = \langle p_1, \ldots, p_{m+1} \rangle$. Therefore, if $Z_0^k \lhd \mathbb{P}^{n*m}$ is such that $\{Z_0, X_1, \ldots, X_{m+1}\}$ is m-generic (and thus, $0 \leqslant k \leqslant n$), then $T_m(Z_0, X_1, \ldots, X_{m+1})$ is naturally parametrized by Z_0 ($p \leftrightarrow Y_p$), because Z_0 consists of generic points. Moreover, for each $i \in \{1, \ldots, m+1\}$, the map $p \mapsto X_i \cap Y_p$ from Z_0 to the k-dimensional subspace of X_i,

$$Z_i = X_i \cap \langle Z_0, X_1, \ldots, \widehat{X}_i, \ldots, X_{m+1} \rangle,$$

is a projective isomorphism. Observe that $T_m(Z_0, X_1, \ldots, X_{m+1})$ coincides with $T_m(Z_0, Z_1, \ldots, Z_{m+1})$. The union of these m-flats, which lies in the projective space $\langle Z_1, \ldots, Z_{m+1} \rangle$ of dimension $k * m$, is what we call a (k, m)-*ruling* and we denote it by

$$R(k, m) = \bigcup_{p \in Z_0} Y_p = \bigcup_{Y \in T_m(Z_0, X_1, \ldots, X_{m+1})} Y.$$

Each $Y^m \in T_m(Z_0, Z_1, \ldots, Z_{m+1})$ is called an m-*rule* of $R(k, m)$; the set of m-rules is denoted $R(k, \underline{m})$, that is, $R(k, \underline{m}) = T_m(Z_0, Z_1, \ldots, Z_{m+1})$.

It happens that the (k, m)-ruling $R(k, m)$ can also be expressed as a union of k-*rules*: if we define $R(\underline{k}, m) = T_k(R(k, \underline{m}))$, we have

$$R(k, m) = \bigcup_{Z \in R(\underline{k}, m)} Z.$$

So that $Z_0, Z_1, \ldots, Z_{m+1}$, which are, by definition, k-rules of $R(k, m)$, extend to a family of k-flats naturally parametrized by $\mathbb{P}^m$ via intersection with any of the m-rules $Y^m \in R(k, \underline{m})$. In fact, $R(k, m)$ is the algebraic variety $\mathbb{P}^k \times \mathbb{P}^m$ together with a fixed embedding which is linear in each factor.

To sketch a proof of the assertions we have made, let us ease the notation. Assume that the space Z_0 we started with has dimension n. Let $Z_0 = X_0, X_1, \ldots, X_{m+1}$ be m-generic n-flats in $\mathbb{P}^{n*m}$, and let $R(n, m)$ be their associated (n, m)-ruling. Consider $\mathbb{P}^{n*m}$ as the projective space associated to the vector space $\mathbb{V}$ of dimension $(n * m) + 1 = (n + 1)(m + 1)$, which we write $\mathbb{P}^{n*m} = \mathbb{P}(\mathbb{V})$. For $j = 0, \ldots, m+1$, let V_j be the $(n+1)$-dimensional linear subspace of $\mathbb{V}$ such that $X_j = \mathbb{P}(V_j)$. Observe that the m-generic hypothesis is that $\mathbb{V}$ is expressed as the direct sum of any $m+1$ of $V_0, \ldots, V_{m+1}$. Consider a basis $v_0, v_1, \ldots, v_n$ of V_{m+1}. Then, for $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$, we have uniquely defined $v_{ij} \in V_j$ such that

$$v_i = v_{i0} + v_{i1} + \cdots + v_{im};$$

and moreover, $v_{0j}, \ldots, v_{nj}$ is a basis of V_j. The bilinear map

$$\varphi : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \to \mathbb{V}$$
$$\left((x_i)_{i=0}^n, (y_j)_{j=0}^m\right) \mapsto \sum_{i,j} x_i y_j v_{ij}$$

governs the ruling $R(n,m)$ which is the projectivization of its image. For example, we may write

$$R(n,\underline{m}) = \left\{\mathbb{P}\left(\varphi\left(x \times \mathbb{R}^{m+1}\right)\right) \;\middle|\; x \in \mathbb{R}^{n+1} - \{0\}\right\},$$
$$R(\underline{n},m) = \left\{\mathbb{P}\left(\varphi\left(\mathbb{R}^{n+1} \times y\right)\right) \;\middle|\; y \in \mathbb{R}^{m+1} - \{0\}\right\}.$$

Now, the unproved assertions of our previous paragraphs follow easily from here.

In summary: if $X_0, \dots, X_{m+1}$ are m-generic n-flats, then $T_m(X_0, \dots, X_{m+1})$ is naturally parametrized by a projective space of dimension at most n; namely, by $Z_0 = X_0 \cap \langle X_1, \dots, X_{m+1}\rangle$.

3. Linear partitions

Our interest in linear partitions arises from the following.

Proposition 1. *Let $X_0, \dots, X_{m+1}, X$ be m-generic n-flats. Then*

$$T_m(X_0, \dots, X_{m+1}, X) \subset T_m(X_0, \dots, X_{m+1})$$

is a linear partition, where $T_m(X_0, \dots, X_{m+1})$ is considered naturally as a projective space.

Before we begin the proof, we must give some insight into linear partitions and establish basic facts about them. As defined in the introduction, a linear partition is a union of flats in general position. In terms of the vector space covering, a linear partition corresponds to a union of linear subspaces whose sum is a direct sum.

First, observe that the intersection of linear partitions is again a linear partition. Thus, there is a *linear partition closure* operator. Given any subset S of a projective space, its *$\mathcal{LP}$-closure* $\langle S\rangle_{\mathcal{LP}}$ is the minimal linear partition that contains it, or, equivalently, the intersection of all that do. The subscript is to distinguish it from the linear closure operator, or projective span, that we denote $\langle\ \rangle$. They may clearly be different. For example, if p_1 and p_2 are distinct points then $\langle p_1, p_2\rangle_{\mathcal{LP}} = \{p_1, p_2\}$ while $\langle p_1, p_2\rangle$ is the line through p_1 and p_2; but if we choose a third point $p_0 \in \langle p_1, p_2\rangle - \{p_1, p_2\}$, then $\langle p_0, p_1, p_2\rangle_{\mathcal{LP}} = \langle p_0, p_1, p_2\rangle = \langle p_1, p_2\rangle$. Thus, three points are needed to generate a line as a linear partition.

A set S of $(k+2)$ points is *minimally degenerate* if they are not in general position but every proper subset of S is (they are k-generic and span a k-flat). We will need the following characterization of linear partitions.

Lemma 1. *$A \subset \mathbb{P}^N$ is a linear partition if and only if whenever a set of points $S \subset A$ is minimally degenerate then $\langle S\rangle \subset A$.*

The proof is simple so we leave it to the reader. Formally, it can be traced back to Tutte [15], where he studied "connectivity in matroids." In our case, considering projective space as a matroid in the natural way, our minimally degenerate sets are the *circuits* on which Tutte based the notion of connectivity. This general approach is studied further in [4].

Proof of Proposition 1. Identify $T_m(X_0, \dots, X_{m+1})$ with a subspace of X_0 by intersection. Suppose $Y_0^m, \dots, Y_{k+1}^m \in T_m(X_0, \dots, X_{m+1}, X)$ are such that their intersection points $p_i := Y_i \cap X_0$ are minimally degenerate. Let $Z_0 = \langle p_0, \dots, p_{k+1}\rangle$; it is a subspace of X_0 of dimension k, and also of $\langle X_1, \dots, X_{m+1}\rangle$; therefore, we may use the notation of Section 2. By Lemma 1, it is enough to prove that for every $p \in Z_0$ the corresponding $Y_p \in T_m(X_0, \dots, X_{m+1})$ is also transversal to X.

For $i = 1, \dots, m+1$, let $Z_i := \{Y_p \cap X_i \mid p \in Z_0\} \subset X_i$ be the corresponding k-flat in X_i. Then $Z_0, \dots, Z_{m+1}$ generate a (k,m)-ruling $R(k,m)$ in a projective space of dimension $k * m$, $\mathbb{P}^{k*m}$. Observe that $Y_0, Y_1, \dots, Y_{k+1}$ are dual generators of $R(k,m)$.

By hypothesis, we have points $q_i \in Y_i \cap X$, $i = 0, \dots, k+1$. Let $Z = \langle q_0, \dots, q_{k+1}\rangle \subset X$. If $\dim Z = k$, then Z is a k-rule of $R(k,m)$ ($Z \in R(\underline{k}, m)$) and is therefore transversal to all its m-rules which is what we wanted to prove. Otherwise, Z has dimension $k+1$. But then, $Z_1, \dots, Z_m, Z$ are not in general position because they are all in $\mathbb{P}^{k*m}$ and

$$\sum_{i=1}^{m} \dim Z_i + \dim Z + m = mk + k + 1 + m = (k+1)(m+1) > k * m.$$

This contradicts the hypothesis that $X_0, \dots, X_{m+1}, X$ are m-generic, because then any collection of their subspaces should also be m-generic. □

Observe in the proof that the subspace Z of X is also a k-flat within one of the n-rules of $R(n,m)$; where, to ease notation, we assume that $X_0 \subset \mathbb{P}^{n*m}$ and identify $T_m(X_0, \dots, X_{m+1})$ with $R(n, \underline{m})$. This way we may obtain a little more. For each component of the linear partition $T_m(X_0, \dots, X_{m+1}, X)$, there exists an n-rule $X_y \in R(\underline{n}, m)$ such that $X_y \cap X$ is a flat parameterizing that component. From this observation, one can construct examples of X that produce any linear partition on X_0; simply, lift the components to different "heights" of one m-rule and then take their linear span.

4. Helly theory

We have observed and used that the intersection of linear partitions is a linear partition. We are now interested in proving that they have a *Helly number* like convex sets do. However, we will need a similar result for convex partitions and since many of the arguments are essentially the same, it will be more efficient to work abstractly in a general setting where Helly numbers make sense. Our point of view is different from the classic one introduced by Levi [13] and further developed in [12] (see also [6]) for "abstract convexity spaces," merely in the sense that we look at Helly numbers from within a lattice, not only for the maximal element. However, the main concepts are the same.

Let Λ be a *meet semilattice*; that is, it is a partial order such that for any two elements a and b, there is a well-defined *intersection*, or *meet, $a \wedge b$* satisfying that

$$c \leqslant a \quad \text{and} \quad c \leqslant b \quad \Rightarrow \quad c \leqslant a \wedge b.$$

We will always assume that our partial orders have a minimum element, denoted by $\emptyset$ which we call the *empty set*. This is strictly the case for our relevant examples, which are: convex sets, $\mathcal{C}$; linear partitions, $\mathcal{LP}$; convex partitions, $\mathcal{CP}$; projective subspaces, $\mathcal{L}$; and finite sets. All of them are with set inclusion as order and intersection as meet.

For any $a \in \Lambda$, $a \neq \emptyset$, define its *Helly number* $h(a)$, or $h_\Lambda(a)$ to be more explicit, as the minimum k such that if $a_0, \dots, a_k \leqslant a$ satisfy that

$$a_0 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k \neq \emptyset \quad \text{for every } i = 0, \dots, k,$$

then $a_0 \wedge \cdots \wedge a_k \neq \emptyset$. When $h(a) > 1$, it clearly coincides with the maximum k for which there exist $a_1, \dots, a_k \leqslant a$ with

$$a_1 \wedge \cdots \wedge a_k = \emptyset, \quad \text{and}$$
$$a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k \neq \emptyset \quad \text{for every } i = 1, \dots, k;$$

in this case, we call $a_1, \dots, a_k \leqslant a$ a *Helly family* for a. If $h(a) < \infty$, then a satisfies a Helly theorem for finite families; namely, a (finite) family of elements of $a_{\leqslant} := \{b \in \Lambda \mid b \leqslant a\}$ *intersects* (i.e., has non-empty intersection) if every $h(a)$ of them have non-empty intersection. If $h_\Lambda(a) < \infty$ for every $a \in \Lambda$, we say that h_Λ is the *Helly function* of Λ, and that Λ is a *Helly semilattice*.

The *upper-rank,* $r(a)$, may be defined as the maximum k such that there exist $a_0 < a_1 < \cdots < a_k = a$. And if $r(a) < \infty$ for every $a \in \Lambda$, one says that Λ has an *upper-rank function*. It is the *rank function* if the length of maximal chains depends only on the extremes.

Convex sets $\mathcal{C}$ do not have an upper-rank function because they have infinite strict chains; however, they are a Helly lattice with $h_{\mathcal{C}} = \dim + 1$ by Helly's theorem. It is not difficult to see that projective subspaces $\mathcal{L}$ and finite sets have both a rank function and a Helly function and that they coincide.

Lemma 2. *If Λ is a meet semilattice with upper-rank function r, then it is a Helly semilattice and*

$$h_\Lambda \leqslant r.$$

Proof. Atoms (which have no smaller element other than $\emptyset$) have upper-rank and Helly functions well-defined as 1. So we may inductively assume for $k > 1$ that if $r(a) < k$ then $h(a) \leqslant r(a)$. Suppose that $a \in \Lambda$ has $r(a) = k$, and let $a_0, a_1, \dots, a_k$ be elements of $a_{\leqslant}$ such that for every $j = 0, \dots, k$, we have that $\bigwedge_{i \neq j} a_i \neq \emptyset$. If we prove that then $\bigwedge_{i=0}^k a_i \neq \emptyset$, we may conclude that $h(a) \leqslant k$ and we are done.

We may assume that one of the a_i is different from a, otherwise their intersection is $a \neq \emptyset$. Suppose $a_0 < a$, so that $r(a_0) < r(a) = k$ and, by induction, we have that $h(a_0) < k$.

For $i = 1, \dots, k$, let $b_i := a_i \wedge a_0$, so that $b_1, \dots, b_k \in (a_0)_{\leqslant}$. For each $j = 1, \dots, k$ and i in $\{1, \dots, k\}$, we have that

$$\bigwedge_{i \neq j} b_i = \bigwedge_{i \neq j} (a_i \wedge a_0) = a_0 \wedge \left(\bigwedge_{i \neq j} a_i \right) \neq \emptyset,$$

because we are only missing an index. Therefore, since $h(a_0) < k$, we have that $\bigwedge_{i=1}^k b_i \neq \emptyset$. But $\bigwedge_{i=1}^k b_i = \bigwedge_{i=0}^k a_i$ and thus the proof is complete. $\square$

Observe that we essentially proved that

$$h_\Lambda(a) \leqslant \max\{h_\Lambda(b) \mid b < a\} + 1, \tag{1}$$

so that if strict chains are not infinite (an upper-rank function exists) there is a Helly number bounded by the upper-rank.

The intersection lattice of linear partitions $\mathcal{LP}$ has a rank function and thus, by the lemma, a Helly function. To see that $r_{\mathcal{LP}}(\mathbb{P}^n) = 2n+1$, consider points in general position $p_0, p_1, \ldots, p_n \in \mathbb{P}^n$. Then the chain of linear partitions

$$\begin{aligned}&\emptyset < \{p_0\} < \{p_0, p_1\} < \cdots < \{p_0, \ldots, p_n\} < \langle p_0, p_1\rangle \cup \{p_2, \ldots, p_n\}\\ &\quad < \langle p_0, p_1, p_2\rangle \cup \{p_3, \ldots, p_n\} < \cdots < \langle p_0, \ldots, p_{n-1}\rangle \cup \{p_n\} < \langle p_0, \ldots, p_n\rangle = \mathbb{P}^n\end{aligned}$$

gives that $r_{\mathcal{LP}}(\mathbb{P}^n) \geqslant 2n+1$. Moreover, it is easy to see that this chain is maximal. However, as we will shortly see, the Helly function of $\mathcal{LP}$ is, in general, strictly smaller.

Suppose furthermore that Λ is a *lattice*, that is, it also has *join*: well-defined minima $a \vee b$ such that $a, b \leqslant a \vee b$. This is the case for our lattices of interest because they all have an associated *closure operator* defined on any set.

For any pair of elements $a, b \in \Lambda$, there is a natural morphism

$$\begin{aligned} a_{\leqslant} \times b_{\leqslant} &\to (a \vee b)_{\leqslant}\\ (a', b') &\mapsto a' \vee b', \end{aligned}$$

where $\times$ denotes the standard Cartesian product. When this morphism is a lattice isomorphism, we say $a \vee b$ is a *direct join* and denote it by $a \veebar b$. It is easy to see that $a \vee b$ is a direct join if and only if for every $c \leqslant a \vee b$ one has that $c = (a \wedge c) \vee (b \wedge c)$.

Observe that if A is a linear partition whose connected components are $A_0, A_1, \ldots, A_k$ then in $\mathcal{LP}$: $A = A_0 \veebar A_1 \veebar \cdots \veebar A_k$.

Lemma 3. *If Λ is a Helly lattice, then*

$$h(a \veebar b) = h(a) + h(b).$$

Proof. Let $k_1 = h(a)$, $k_2 = h(b)$ and $k = k_1 + k_2$. To show that $h(a \veebar b) \leqslant k$, we must prove that if $c_0, c_1, \ldots, c_k \leqslant a \vee b$ are such that every k of them have non-empty intersection then they all have non-empty intersection.

Let $a_i = a \wedge c_i$ for $i = 0, \ldots, k$. If $\emptyset \neq \bigwedge_{i=0}^k a_i \leqslant \bigwedge_{i=0}^k c_i$ then we are done. So suppose $\bigwedge_{i=0}^k a_i = \emptyset$. Then, because the Helly number of a is k_1, there exist k_1 indices, say $I \subset \{0, \ldots, k\}$ with $\sharp I = k_1$, such that $\bigwedge_{i \in I} a_i = \emptyset$.

Let $b_i = b \wedge c_i$ for $i = 0, \ldots, k$. Because $h(b) = k_2$, to see that $\emptyset \neq \bigwedge_{i=0}^k b_i \leqslant \bigwedge_{i=0}^k c_i$, it is enough to prove that for any given $J \subset \{0, \ldots, k\}$ with $\sharp J = k_2$, we have $\bigwedge_{i \in J} b_i \neq \emptyset$. We claim that

$$\bigwedge_{i \in J} b_i \geqslant \bigwedge_{i \in I \cup J} b_i \neq \emptyset.$$

Let $c = \bigwedge_{i \in I \cup J} c_i$. Because $\sharp(I \cup J) \leqslant k_1 + k_2$, we know that $c \neq \emptyset$ by hypothesis. We also have that $a \wedge c = \bigwedge_{i \in I \cup J}(a \wedge c_i) = \bigwedge_{i \in I \cup J} a_i \leqslant \bigwedge_{i \in I} a_i = \emptyset$. So that, because $a \vee b$ is a direct join, $b \wedge c = \bigwedge_{i \in I \cup J} b_i = \emptyset$ implies that $c = (a \wedge c) \vee (b \wedge c) = \emptyset$, which is a contradiction. Therefore, $\bigwedge_{i \in I \cup J} b_i \neq \emptyset$ proving that $h(a \veebar b) \leqslant k$.

To see that $h(a \veebar b) = k$, let $a_1, \ldots, a_{k_1} \leqslant a$ be a Helly family for a, and let $b_1, \ldots, b_{k_2} \leqslant b$ be a Helly family for b. Define $c_i = a_i \vee b$ for $i = 1, \ldots, k_1$ and $c_i = a \vee b_{i-k_1}$ for $i = k_1 + 1, \ldots, k$. It is easily seen that $c_1, \ldots, c_k$ is a Helly family for $a \veebar b$. $\square$

4.1. *Helly for linear partitions*

We turn our attention to the lattice of Linear Partitions $\mathcal{LP}$. Observe that by Lemma 3, once we know the Helly number of the flats, $h_{\mathcal{LP}}(\mathbb{P}^n)$, we know the Helly function because any linear partition is the direct join of its connected components. Thus, the Helly function of $\mathcal{LP}$ is determined by the following theorem which is a reformulation of Theorem 3.

Theorem 5. $h_{\mathcal{LP}}(\mathbb{P}^n) = \lfloor 3(n+1)/2 \rfloor$.

Proof. By definition, we have that $h_{\mathcal{LP}}(\mathbb{P}^0) = 1$. Suppose inductively that for $m < n > 0$, $h_{\mathcal{LP}}(\mathbb{P}^m) = \lfloor 3(m+1)/2 \rfloor$, and let $k = \lfloor 3(n+1)/2 \rfloor$. To see that $h_{\mathcal{LP}}(\mathbb{P}^n) \geqslant k$, consider $\lfloor (n+1)/2 \rfloor$ lines in general position in $\mathbb{P}^n$ and choose three different points in each of them; for n even, choose yet another point in general position with the lines. Label these points $p_1, p_2, \ldots, p_k$, and let $A_i := \langle p_j \mid j \neq i \rangle_{\mathcal{LP}} = \langle p_1, \ldots, \widehat{p_i}, \ldots, p_k \rangle_{\mathcal{LP}}$. We clearly have that $\emptyset \neq \{p_i\} = \bigcap_{j \neq i} A_j$ and $\bigcap_i A_i = \emptyset$, so $k \leqslant h_{\mathcal{LP}}(\mathbb{P}^n)$.

We are left to prove that $h_{\mathcal{LP}}(\mathbb{P}^n) < k + 1$. Suppose $A_0, A_1, \ldots, A_k$ are linear partitions in $\mathbb{P}^n$ such that every k of them *intersect* (observe that we use this shorthand for "have non-empty intersection"). We must prove that they all intersect.

For every $i = 0, \ldots, k$, choose a point $p_i \in \bigcap_{j \neq i} A_j$, and let $A_i' := \langle p_j \mid j \neq i \rangle_{\mathcal{LP}}$. Observe that $A_i' \subset A_i$ so that if the A_i' intersect, so do the original ones. To ease notation, we may assume that $A_i = A_i'$. We will call $p_0, \ldots, p_k$ the *special points*: $S = \{p_0, \ldots, p_k\}$.

By induction, we may assume that *for every proper flat $X^m \lhd \mathbb{P}^n$ we have that*

$$\sharp(S \cap X) \leqslant h_{\mathcal{LP}}\big(X^m\big) = \big\lfloor 3(m+1)/2 \big\rfloor, \tag{2}$$

because if not, it is easy to find an intersection point of all the A_i in X.

If $A_0 = \langle p_1, \ldots, p_k \rangle_{\mathcal{LP}} = \mathbb{P}^n$, then $p_0 \in A_0$ and so $\bigcap_{i=0}^k A_i \supset \{p_0\} \neq \emptyset$ which completes the proof. Therefore, assume A_0 breaks into flat components. All of them cannot be points because $k > n + 1$ and two special points cannot be equal by (2); this takes care of the case $n = 1$. So, we may consider a flat component X of A_0 which is not a point. Let $I = \{i \mid p_i \in X,\ 1 \leqslant i \leqslant k\}$ so that $X = \langle p_i \mid i \in I \rangle_{\mathcal{LP}}$. Let Y be the linear span of the other components of A_0, so that $A_0 \subset X \cup Y$. Using (2) one easily sees that X and Y span $\mathbb{P}^n$ and that $p_0 \notin X \cup Y$; so that we have a well-defined point $q = X \cap \langle Y, p_0 \rangle$. We will prove that $q \in \bigcap_{i=0}^k A_i$ to conclude the proof of the theorem.

For $j \notin I$ we have that $q \in X = \langle p_i \mid i \in I \rangle_{\mathcal{LP}} \subset \langle p_0, \ldots, \widehat{p_j}, \ldots, p_k \rangle_{\mathcal{LP}} = A_j$. Fix $i \in I$. We have that $\langle p_j \mid j \in I - \{i\} \rangle = X$; indeed, because X is not a point, if $\langle p_j \mid j \in I - \{i\} \rangle$ was a proper subspace of X, then p_i would be a component of $\langle p_i \mid i \in I \rangle_{\mathcal{LP}}$. Let $I_i \subset I - \{i\}$ be a minimal set that generates $q \in X$, that is, such that $q \in \langle p_j \mid j \in I_i \rangle$. On the other hand, let

J be a minimal set that generates q in $\langle Y, p_0\rangle$, that is, $J \cap I = \emptyset$ and $q \in \langle p_j \mid j \in J\rangle$. Then $\{p_j \mid j \in I_i \cup J\}$ is minimally degenerate. Therefore, by Lemma 1, we have that

$$q \in \langle p_j \mid j \in I_i \cup J\rangle = \langle p_j \mid j \in I_i \cup J\rangle_{\mathcal{LP}} \subset \langle p_0, \dots, \widehat{p_i}, \dots, p_k\rangle_{\mathcal{LP}} = A_i,$$

and the proof is complete. □

We now have all the ingredients to prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{F}$ be an m-generic family of n-flats such that every $\lfloor (3n+2m+7)/2\rfloor$ of them have a transversal m-flat. We will prove that they all have a transversal m-flat.

Consider $X_0, \dots, X_{m+1} \in \mathcal{F}$ and the ruling they define. Identify the m-rules (their transversal m-flats) naturally with a subspace of $\mathbb{P}^n \cong X_0$. For every $X \in \mathcal{F}$, let $T_X = T_m(X_0, \dots, X_{m+1}, X) \subset \mathbb{P}^n$ which is a linear partition by Proposition 1. Observe that

$$\left\lfloor \frac{3(n+1)}{2} \right\rfloor + m + 2 = \left\lfloor \frac{3n+2m+7}{2} \right\rfloor$$

so that, by hypothesis, every subfamily with $\lfloor 3(n+1)/2\rfloor$ elements of $\mathcal{F}_{\mathcal{LP}} = \{T_X \mid X \in \mathcal{F}\}$ intersects. Since $h_{\mathcal{LP}}(\mathbb{P}^n) = \lfloor 3(n+1)/2\rfloor$ by Theorem 5, all the members of $\mathcal{F}_{\mathcal{LP}}$ have nonempty intersection. Each point there corresponds to a transversal m-flat to $\mathcal{F}$. □

5. The convex case

We first establish the Helly number for convex partitions including Theorem 4. Then we prove Theorem 2.

5.1. *Helly for convex partitions*

Observe that there is a *convex partition closure* operator which we denote $\langle\ \rangle_{\mathcal{CP}}$. It can be thought of as first applying the linear partition closure (in the projective compactification) and then, back in Euclidean space, taking the usual convex closure within each flat component.

Theorem 6. *If C is a convex partition whose connected components are $K_1, \dots, K_r$ of dimensions $n_1, \dots, n_r$, respectively, then*

$$h_{\mathcal{CP}}(C) = \sum_{i=1}^{r} (2n_i + 1).$$

Proof. Observe that Theorem 4, which simply states that $h_{\mathcal{CP}}(\mathbb{R}^n) = 2n + 1$, is a special case of Theorem 6.

First, let us give an example proving that $h_{\mathcal{CP}}(\mathbb{R}^n) \geqslant 2n + 1$; it will turn out to be a Helly family. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{R}^n$. Let p_0 be the origin, and for $i = 1, \dots, n$, let $p_{2i-1} := e_i$ and $p_{2i} := 2e_i$. For $j = 0, \dots, 2n$, let $C_j := \langle p_0, \dots, \widehat{p}_j, \dots, p_{2n}\rangle_{\mathcal{CP}}$. Observe that, for $n > 1$, C_0 is the convex hull of the basis and their “doubles” (a simplex of dimension n truncated at a vertex), and that every other C_j consists of a point component in a coordinate axis and a simplex (of dimension $n - 1$) in its orthogonal complement. Each $2n$ of them intersect in the corresponding common point p_j, but all of them have empty intersection, thus $h_{\mathcal{CP}}(\mathbb{R}^n) \geqslant 2n+1$. Since this example can essentially be built within any convex set K of dimension n, we also have that $h_{\mathcal{CP}}(K) \geqslant 2n+1$.

Assume inductively that if K is a convex set of dimension $m < n$, then $h_{\mathcal{CP}}(K) = 2m+1$; which is true for a point to start the induction with.

To see that $h_{\mathcal{CP}}(\mathbb{R}^n) \leqslant 2n+1$, let $C_0, \ldots, C_{2n+1}$ be convex partitions in $\mathbb{R}^n$ such that each $2n+1$ of them intersect; we must prove that they all intersect. If they are all connected, that is, if they are all convex sets, then by Helly's classic theorem (used with ample margin) they do intersect. Assume that one of them, say C_0, is not connected. Suppose it has connected components $K_1, \ldots, K_r$ of dimensions $n_1, \ldots, n_r$ respectively. We have that $\sum_{i=1}^r n_i + (r-1) \leqslant n$ and $r \geqslant 2$.

Since $C_0 = K_1 \cup \cdots \cup K_r$ is a direct join in the lattice of convex partitions, by Lemma 3 and induction ($n_i < n$), we have that

$$h_{\mathcal{CP}}(C_0) = \sum_{i=1}^r h_{\mathcal{CP}}(K_i) = \sum_{i=1}^r (2n_i + 1) = \sum_{i=1}^r 2n_i + r \leqslant 2n - (r-2) \leqslant 2n.$$

Then the convex partitions $(C_0 \cap C_1), (C_0 \cap C_2), \ldots, (C_0 \cap C_{2n+1})$ in $(C_0)_{\leqslant}$ intersect because each $2n$ of them do by hypothesis (such an intersection is an intersection of $2n+1$ of the originals). This proves that $h_{\mathcal{CP}}(\mathbb{R}^n) = 2n+1$.

To complete the induction process, consider any convex set K of dimension n. $K \subset \mathbb{R}^n$ implies $h_{\mathcal{CP}}(K) \leqslant h_{\mathcal{CP}}(\mathbb{R}^n)$ and we have seen that $h_{\mathcal{CP}}(K) \geqslant 2n+1$, therefore $h_{\mathcal{CP}}(K) = 2n+1$.

The theorem now follows from Lemma 3. □

5.2. *Hadwiger for convex sets*

For the rest of this section we turn our attention to Theorem 2. We first establish the setting we will be working in. From it, we construct a family of convex partitions in an affine flat by first analyzing the order types of the m-rules in an appropriate ruling, and then conclude the proof of the theorem.

5.2.1. *Setting*

Let $\mathcal{F} = \{K_i\}_{i \in \Gamma}$ be a family of convex sets in $\mathbb{R}^N$ such that $\{X_i\}_{i \in \Gamma}$ is a corresponding family of m-generic n-flats in $\mathbb{P}^N$, with $K_i \subset X_i$ for every $i \in \Gamma$. We think of $\mathbb{P}^N$ as the projective closure of $\mathbb{R}^N$. Denote by H_∞ the hyperplane at infinity, so that $\mathbb{R}^N = \mathbb{P}^N - H_\infty$. Then, for every flat $F \lhd \mathbb{P}^N$ we may refer to its affine flat $F - H_\infty = F - (F \cap H_\infty)$, provided that $F \not\subset H_\infty$. Observe that convexity and order type make implicit reference to these affine flats.

We also have a given *order type* of dimension m for the same index set, that is, a family of points $Q = \{q_i\}_{i \in \Gamma} \subset \mathbb{R}^m$ which we will call the *abstract points*. Our hypothesis is that for a certain k (for the moment we only need to know that $k > m+2$), we have that for any $I \subset \Gamma$ with $\sharp I \leqslant k$, there exists an affine m-flat Y in $\mathbb{R}^N$ for which there are points $y_i \in Y \cap K_i$ ($i \in I$), such that $\{q_i\}_{i \in I} \subset \mathbb{R}^m$ and $\{y_i\}_{i \in I} \subset Y$ define the same oriented matroid, order type or separoid. More precisely, and using the simple terminology of separoids [2]: for every *index partition* $\alpha \cup \beta = I$ with $\alpha \cap \beta = \emptyset$, there exists a hyperplane $L \lhd \mathbb{R}^m$ that separates $\{q_i\}_{i \in \alpha}$ from $\{q_i\}_{i \in \beta}$ if and only if there exists a hyperplane $L' \lhd Y$ that separates $\{y_i\}_{i \in \alpha}$ from $\{y_i\}_{i \in \beta}$. (For further reference, the combinatorial information of all the index partitions that do separate is called a *separoid*; it defines the order type.)

Our aim is to prove that $\mathcal{F}$ has a transversal m-flat.

First, observe that *the abstract points Q are m-generic*. If not so, there exist $q_0, \dots, q_m \in Q$ contained in a hyperplane of $\mathbb{R}^m$. By hypothesis, their corresponding convex sets $K_0, \dots, K_m$ have a transversal m-flat Y that intersects them in $y_0, \dots, y_m$ respectively, such that they define the same separoid as $q_0, \dots, q_m$. Therefore, $y_0, \dots, y_m$ lie in a hyperplane of Y, so that $K_0, \dots, K_m$ have a transversal flat of dimension $m-1$. But then they are not m-generic contradicting the hypothesis, and proving the claim.

5.2.2. The ruling and order types

Consider $(m+2)$ n-flats $X_0, \dots, X_{m+1}$ in the family and their generated (n', m)-ruling with $n' \leqslant n$, where, recall, $n' = \dim(X_0 \cap \langle X_1 \cap \dots \cap X_{m+1} \rangle)$. The case $n' < n$ is basically the same as the one when $n' = n$, with minor adjustments that, however, complicate notation needlessly (we should replace n by n' and X_i for $X'_i = X_i \cap \langle X_0 \cap \cdots \widehat{X_i} \cdots \cap X_{m+1} \rangle$ below). So let us assume that $X_0, \dots, X_{m+1}$ generate an (n,m)-ruling $R(n,m)$. Identify the m-rules by their intersection with X_0, so that for $x \in X_0$, let $Y_x \in R(n, \underline{m})$ be such that $x = Y_x \cap X_0$.

By intersection with the m-rules we have naturally given projective isomorphisms between the n-rules that we refer to as *projections*—dually, the m-rules are projectively identified by intersection with the n-rules. Our main problem to be solved is that, for $i = 1, \dots, m+1$, the convex sets $K_i \subset X_i$ project to the affine part of X_0 to sets that may have two connected components. Indeed, the image of a convex set under a projectivity is either convex or the union of two unbounded convex components: we will have to choose the appropriate one.

For $i = 0, \dots, m+1$, let $H_i := \{x \in X_0 \mid Y_x \cap X_i \in H_\infty\} \subset X_0$, which is the hyperplane at infinity of X_i ($X_i \cap H_\infty$) projected to X_0 by the ruling; observe that H_0 is the hyperplane at infinity of X_0. Each H_i is a hyperplane because the X_i come from affine n-flats (they have non-empty convex sets defined on them). The H_i cut the affine $X_0 - H_0$ into convex regions.

Claim 1. *m-rules in different regions of $X_0 - (\cup_{i=0}^{m+1} H_i)$ intersect the n-rules $X_0, \dots, X_{m+1}$ in points with a different order type.*

Let x and x' in $X_0 - (\bigcup_{i=0}^{m+1} H_i)$ be such that the affine segment σ from x to x' intersects some of the H_i. More precisely, let

$$\alpha = \{i \mid \sigma \cap H_i \neq \emptyset\}.$$

We assume that $\alpha \neq \emptyset$; and know that $0 \notin \alpha$ because affine segments do not cross infinity.

Let $Y = Y_x$ and $Y' = Y_{x'}$ be the m-rules at x and x', and let $y_i = Y \cap X_i$ (respectively, $y'_i = Y' \cap X_i$) so that $x = y_0$ (respectively, $x' = y'_0$). We must prove that the order types $\{y_0, \dots, y_{m+1}\}$ in $Y - H_\infty$ and $\{y'_0, \dots, y'_{m+1}\}$ in $Y' - H_\infty$ are different.

Denote by $\varphi : Y' \to Y$ the projective isomorphism defined by the ruling $R(n,m)$ so that $\varphi(y'_i) = y_i$. Let $L = \varphi(Y' \cap H_\infty) \subset Y$, which is a hyperplane because $x' \in Y' \not\subset H_\infty$.

First, we prove that *L separates α from its complement* $\bar{\alpha}$, which shall be written as $\alpha \mid_L \bar{\alpha}$. Since $0 \in \bar{\alpha}$, it is enough to prove that the segment from y_0 to y_i (in Y) intersects L if and only if σ intersects H_i.

The lines $\ell := \langle x, x' \rangle \subset X_0$ and $\eta := \langle y_0, y_i \rangle \subset Y$ define a $(1,1)$-ruling $\mathcal{H}$ contained in $R(n,m)$. Indeed, $R(n,m)$ may be naturally identified with $X_0 \times Y$; then $\mathcal{H}$ is the $(1,1)$-ruling identified with the inclusion $\ell \times \eta \subset X_0 \times Y$. The standard hyperboloid $\mathcal{H}$ lies in a projective space $\mathbb{P}^3$, and we have two cases to consider: when it is transversal or tangent to the plane at infinity.

When $\mathcal{H}$ is transversal to H_∞ there is a natural projective identification of ℓ and η given by

$$\ell \ni s \leftrightarrow t \in \eta \quad \Leftrightarrow \quad \eta_s \cap \ell_t \in H_\infty$$

where, for $s \in \ell$, η_s is the other rule passing through s, and analogously, ℓ_t is the rule *parallel* to ℓ passing through $t \in \eta$. Observe that $x \in \ell$ corresponds to the point at infinity of η, $\infty_\eta := \eta \cap H_\infty$, because $\eta_x = \eta$; that $x' \in \ell$ corresponds to $L \cap \eta$; that $\infty_\ell := \ell \cap H_\infty$ corresponds to $y_0 \in \eta$, and that $H_i \cap \ell$ corresponds to $y_i \in \eta$:

$$\begin{array}{ccccc} x & x' & \infty_\ell & H_i \cap \ell & (\in \ell) \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \infty_\eta & L \cap \eta & y_0 & y_i & (\in \eta). \end{array}$$

From this information it follows that $H_i \cap \ell$ is in the affine segment σ from x to x' ($i \in \alpha$) if and only if $L \cap \eta$ is in the affine segment from y_0 to y_i (L separates y_0 from y_i). See Fig. 3 for a rough sketch.

If H_∞ is tangent to $\mathcal{H}$, then $\mathcal{H}$ has two rules at infinity, one in each ruling. So that, with notation as above, $\infty_\ell = H_i \cap \ell$ and $\infty_\eta = L \cap \eta$. Then $i \in \bar{\alpha}$ and L leaves y_0 and y_i on the same side. Completing the proof that L separates α from its complement.

Now, Claim 1 follows from:

Lemma 4. *Let $y_0, \ldots, y_{m+1}$ be m-generic points in $\mathbb{R}^m \subset \mathbb{P}^m$, and let L be a hyperplane in $\mathbb{R}^m$ that separates $\{y_i\}_{i \in \alpha}$ from $\{y_i\}_{i \in \bar{\alpha}}$, where $\emptyset \neq \alpha \subsetneq \{0, \ldots, m+1\}$. If $f : \mathbb{P}^m \to \mathbb{P}^m$ is a projectivity that sends L to the hyperplane at infinity then $\{y_0, \ldots, y_{m+1}\}$ and $\{f(y_0), \ldots, f(y_{m+1})\}$ define different order types in $\mathbb{R}^m$.*

Proof. From Radon's classic theorem, and the general position hypothesis, it follows that there exists a unique $\beta \subsetneq \{0, \ldots, m+1\}$, $\beta \neq \emptyset$, such that the simplexes $\Delta_\beta := \langle y_i \mid i \in \beta \rangle_{\mathcal{C}}$ and $\langle y_i \mid i \in \bar{\beta} \rangle_{\mathcal{C}}$ (where $\langle\ \rangle_{\mathcal{C}}$ denotes the convex hull) intersect in a point, say y, in their interior. Furthermore, $\beta, \bar{\beta}$ is the only such *Radon partition* in the order type of $\{y_0, \ldots, y_{m+1}\}$. Then, we have that $\{\alpha, \bar{\alpha}\} \neq \{\beta, \bar{\beta}\}$ because $\alpha \mid_L \bar{\alpha}$. We may assume that $\alpha \cap \beta \neq \emptyset \neq \bar{\alpha} \cap \beta$. This implies that L intersects the simplex Δ_β. Therefore, we have that $f(\Delta_\beta) \neq \langle f(y_i) \mid i \in \beta \rangle_{\mathcal{C}}$ because

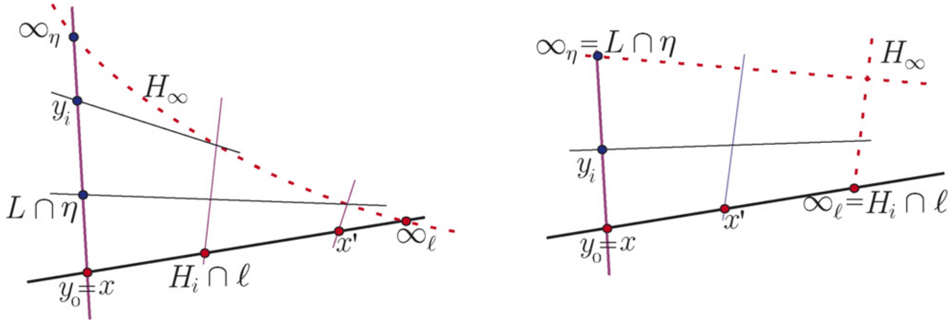


Fig. 3. The two cases of hyperboloids.

$f(\Delta_\beta)$ has points at infinity. Then

$$\langle f(y_i) \mid i \in \beta \rangle_{\mathcal{C}} \cap \langle f(y_i) \mid i \in \bar{\beta} \rangle_{\mathcal{C}} = \emptyset$$

because the flats $\langle f(y_i) \mid i \in \beta \rangle$ and $\langle f(y_i) \mid i \in \bar{\beta} \rangle$ (which contain, respectively, the above simplexes) intersect in the unique point $f(y)$ which is in $f(\mathrm{Interior}(\Delta_\beta))$ and outside of $\langle f(y_i) \mid i \in \beta \rangle_{\mathcal{C}}$. Therefore, β separates from its complement in the order type of $\{f(y_0), \ldots, f(y_{m+1})\}$, completing the proof of the lemma. □

5.2.3. The convex partitions

The abstract points $q_0, \ldots, q_{m+1}$ corresponding to our flats $X_0, \ldots, X_{m+1}$ define a separoid S on $\{0, \ldots, m+1\}$. By Claim 1 and the fact that the convex sets $K_0, \ldots, K_{m+1}$ have transversal m-flats compatible with that order type ($k \geqslant m+2$), there exists a unique open connected component $R \subset X_0 - (\bigcup_{i=0}^{m+1} H_i)$ for which the m-rules Y_x with $x \in R$ intersect the X_i in points that define the separoid S. This region defines a *positive hemispace* $H_i^+ \supset R$ in $X_0 - H_0$ for each hyperplane H_i, $i = 1, \ldots, m+1$.

Let $C_0 := K_0 \subset X_0$. For $i \in \{1, \ldots, m+1\}$, we have that the convex set $K_i \subset X_i$ projects by the ruling to a set K_i' in X_0 which consists of one or two connected convex components in the affine $X_0 - H_0$, according to whether the hyperplane H_0 misses or hits K_i'. In the first case, let $C_i := K_i' \subset X_0 - H_0$. In the second case, observe that the hyperplane H_i (which corresponds to the plane at infinity in X_i) separates the two components of K_i', then let $C_i \subset X_0 - H_0$ be the component of $K_i' - H_0$ lying in the positive side H_i^+. In any case, C_i is a convex set in the affine $X_0 - H_0$.

Now, we define $C_j \subset X_0$ for the general $j \in \Gamma - \{0, 1, \ldots, m+1\}$. Suppose first that the n-flat X_j is a rule in $R(\underline{n}, m)$. Replacing j instead of $m+1$ in the above discussion, we obtain a hyperplane $H_j \lhd X_0$ with a distinguished positive side H_j^+ satisfying the property that if an m-rule $Y \in R(n, \underline{m})$ is such that $\{X_i \cap Y \mid i = 0, \ldots, m, j\} \subset Y - H_\infty$ defines the same order type as $\{q_0, \ldots, q_m, q_j\}$ in $\mathbb{R}^m$, then we have that $Y \cap X_0 \in H_j^+$. Then we define C_j as above (the component of K_j projected to X_0 that lies in H_j^+).

If X_j is not a rule in $R(\underline{n}, m)$, then $T_m(X_0, \ldots, X_{m+1}, X_j)$ viewed in X_0, is a linear partition $Z = Z_1 \cup \cdots \cup Z_r$ where each Z_i is a flat. By the proof of Proposition 1 (see the observation that follows it), we have rules $X_{j,i} \in R(\underline{n}, m)$, $i = 1, \ldots, r$, such that $Z_i' := X_{j,i} \cap X_j$ is a flat that is projected unto Z_i under the ruling identification $X_{j,i} \to X_0$. Let $K_{j,i} := X_{j,i} \cap K_j$, and proceeding as in the case above, define $C_{j,i} \subset X_0 - H_0$, observing that $C_{j,i} \subset Z_i$. Finally, define $C_j := C_{j,1} \cup \cdots \cup C_{j,r}$ and observe that it is a convex partition because each component lies in the corresponding component of the linear partition Z.

We have defined the corresponding family of convex partitions $\{C_i\}_{i \in \Gamma}$ in $X_0 - H_0$ with the property that if $Y \in T_m(K_0, \ldots, K_{m+1}, K_j)$ intersects them compatibly with the order type Q, then Y passes through $\bigcap_{i=0}^{m+1} C_i \cap C_j$.

5.2.4. Completion of the proof

We are left to prove that if the number k, for which there exist compatible m-flat transversals, is

$$k = 2n + m + 3 = (m+2) + (2n+1),$$

then the whole family $\mathcal{F} = \{K_i\}_{i \in \Gamma}$ has a transversal m-flat.

Let $I_0 := \{0, 1, \ldots, m+1\} \subset \Gamma$ be the index subset we have been working with. Given $I \subset \Gamma$ with $\sharp I = 2n+1$, we will prove that $\bigcap_{i \in I} C_i \neq \emptyset$. Since $\sharp(I \cup I_0) \leqslant k$, the family of convex sets $\{K_i \mid i \in I \cup I_0\}$ has a transversal m-flat Y that intersects them compatibly with the order type. But $Y \in R(n, \underline{m}) = T_m(X_0, \ldots, X_{m+1})$, so Y passes through each C_i $(i \in I)$ by the construction of C_i, therefore $\bigcap_{i \in I} C_i \neq \emptyset$. By Helly's theorem for convex partitions (Theorem 4), we have that $\bigcap_{i \in \Gamma} C_i \neq \emptyset$; if we take $x \in \bigcap_{i \in \Gamma} C_i$, Y_x is an m-flat transversal to $\{K_i\}_{i \in \Gamma}$.

Observe that no claim is made about the order type of $\{Y_x \cap K_i\}_{i \in \Gamma}$ in $Y_x - H_\infty$, because the construction only gives information about the subsets of type $\{0, 1, \ldots, m, j\} \subset \Gamma$.

6. Concluding remarks

The fact that there exists a Helly number for flat transversals to flats is due to Lovász, see [8]. Here, we reduced the number for the generic case, and established (again, in the generic case) that there exists a “magic” number for flat transversals to families of convex sets of a fixed dimension. However, in both settings, there seems to be room for improvement in our numbers. In particular, we know that for $m = 1$ and $n = 2$ in which Theorem 2 gives 8 as the “magic” number, it can be reduced to 7 with ad hoc arguments to agree with Theorem 1. Only for $m = 1 = n$ is there a proof, by explicit examples in [3], that the numbers (6 in both settings) are the best possible. To have new critical examples, and thus lower bounds, would be helpful to establish the best possible “magic” numbers: the natural problem to pose.

Finally, observe that the proofs of Theorems 1 and 2 work for the more general case of families of linear partitions and, respectively, convex partitions of a fixed *dimension* (understood as the dimension of the linear span). However, the statements and arguments in such generality would have made the presentation quite awkward.

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