Two geometric representation theorems for separoids

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Abstract

Separoids capture the combinatorial structure which arises from the separations by hyperplanes of a family of convex sets in some Euclidian space. Furthermore, as we prove in this note, every abstract separoid S can be represented by a family of convex sets in the $(|S|-1)$-dimensional Euclidian space. The *geometric dimension* of the separoid is the minimum dimension where it can be represented and the upper bound given here is tight. Separoids have also the notions of *combinatorial dimension* and *general position* which are purely combinatorial in nature. In this note we also prove that: *a separoid in general position can be represented by a family of points if and only if its geometric and combinatorial dimensions coincide.*

1 Introduction

Consider a family $\mathcal{S}$ of convex sets in $\mathbb{E}^d$. Two subfamilies A and B are said to be *separated*, and writen $A \mid B$, if there exists a hyperplane H leaving all elements of A strictly in one side of H and all of B in the other. The separation relation on the subsets of $\mathcal{S}$ clearly satisfies

$$\begin{array}{rl} \circ & A \mid B \Longrightarrow B \mid A, \\ \circ\circ & A \mid B \Longrightarrow A \cap B = \emptyset, \\ \circ\circ\circ & A \mid B \text{ and } A' \subset A \Longrightarrow A' \mid B. \end{array}$$

A *separoid* is a set $\mathcal{S}$ endowed with a separation relation $\mid\ \subseteq 2^{\mathcal{S}} \times 2^{\mathcal{S}}$ satisfying these simple properties [7]. The separoid is said to be *acyclic* if the empty set is separated from the total one (i.e., if $\emptyset \mid \mathcal{S}$).

Conversely, any abstract finite separoid can be represented by a family of convex sets [1]. Therefore, separoids have a notion of *geometric dimension*; namely, the minimum dimension of a Euclidian space where it can be represented. This notion was introduced in [1] to study the topology of hyperplanes transversal to families of convex sets. In this note we prove that

Theorem 1 *The geometric dimension of a separoid $\mathcal{S}$ is at most $|\mathcal{S}|-1$.*

We also use the geometric dimension to characterise those separoids which can be represented by a family of points in general position. For, let the *d-dimensional simploid* $\sigma = \sigma^d$ be the separoid of order $d+1$ such that every subset is separated from its complement (which yields $A \mid B \iff A \cap B = \phi$). The simploid can be represented with the vertex set of the simplex, hence its name. The *(combinatorial) dimension* of a separoid, denoted by $\mathrm{d}(\mathcal{S})$, is the maximum dimension of its induced simploids. A separoid is said to be in *general position* if every subset of $\mathcal{S}$ of cardinality $\mathrm{d}(\mathcal{S})+1$ is an induced simploid.

Theorem 2 *Let $\mathcal{S}$ be a separoid in general position. $\mathcal{S}$ is a separoid of some points in an Euclidian space if and only if its combinatorial and geometric dimensions coincide.*

Since an oriented matroid can be thought of as a separoid satisfying some extra properties (see e.g. [5, 6]), thus Theorem 2 addresses the classic problem of stretchability of oriented matroids (see [2] Chapter 8). In particular, Theorem 2 implies that *a uniform oriented matroid is linear (stretchable) if and only if its geometric dimension equals its combinatorial one*; that is, a uniform oriented matroid of rank r is not linear if and only if its representations by convex sets cannot be done in dimension $r-1$.

Observe that, without the hypothesis of general position the conclusion of the theorem need not be true. For example, consider the 3-element separoid consisting of a line segment and its two end points; its combinatorial and geometric dimension coincide (they are equal to 1) but it cannot be represented with points. Furthermore, there are non-linear non-uniform oriented matroids of rank 3 (dimension 2) which can be easily represented by convex sets in the plane.

We conclude this introduction with some definitions.

A pair of disjoint subsets A and B which are not separated are called a *Radon partition* and denoted $A \dagger B$. Each part, A and B, is called a *(Radon) component* and the union $A \cup B$ will be called the *support* of the partition. A *minimal Radon partition* is a partition $A \dagger B$ where each component is minimal by contention, i.e. if $A' \subset A \Longrightarrow A' \mid B$ and $B' \subset B \Longrightarrow A \mid B'$. Clearly, the set of all minimal Radon partitions encodes the separoid and it will be denoted by MRP; i.e., $A \dagger B \in MRP$ means that $A \dagger B$ is a minimal Radon partition.

Finally, let $\mathcal{S}$ and $\mathcal{T}$ be two separoids. A *separoid morphism* $\varphi\colon \mathcal{S} \longrightarrow \mathcal{T}$ is a function with the property that for all $A, B \subset \mathcal{T}$, $A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B)$. Two separoids are *isomorphic* if there exists a bijective morphism from one onto the other whose inverse function is also a morphism. Therefore, $\mathcal{S}$ is isomorphic to $\mathcal{T}$ if and only if there exists a bijective morphism $\varphi\colon \mathcal{S} \longrightarrow \mathcal{T}$ such that $A \dagger B \Longrightarrow \varphi^{-1}(A) \dagger \varphi^{-1}(B)$, for each minimal Radon partition $A \dagger B$.

2 Proof of Theorem 1

Let $S = \{1, \ldots, n\}$ be an acyclic separoid (i.e., $A \dagger B \Longrightarrow |A||B| > 0$). To each element $i \in S$ and each (minimal Radon) partition $A \dagger B \in MRP$ such that $i \in A$, we assign a point of $\mathbb{R}^n$

$$(*) \qquad \rho^i_{A\dagger B} := \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

(where $\{\mathbf{e}_j\}$ denotes the canonical basis) and represent each element $i \in S$ as the convex hull of all such points

$$i \mapsto \mathcal{K}_i := \langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B \in MRP \rangle.$$

Observe that these convex sets are in the $(n-1)$-dimensional affine subspace spanned by the basis, because $(*)$ is, in fact, an affine combination (see Figure 1).

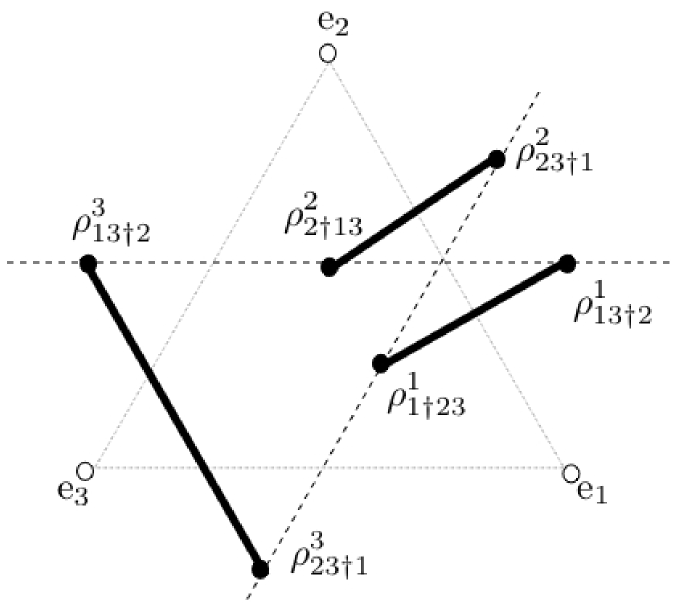

Figure 1. *The separoid with Radon partitions* $1 \dagger 23$ *and* $2 \dagger 13$.

To see that this family of convex polytopes represent the separoid we first prove that, for a given partition $A \dagger B$ it follows that $\langle \mathcal{K}_a : a \in A\rangle \cap \langle \mathcal{K}_b : b \in B\rangle \neq \phi$.

Clearly, it is enough to prove that $\langle \rho^a_{A\dagger B} : a \in A\rangle \cap \langle \rho^b_{B\dagger A} : b \in B\rangle \neq \phi$ because $\rho^a_{A\dagger B} \in \mathcal{K}_a$ and therefore $\langle \rho^a_{A\dagger B} : a \in A\rangle \subset \langle \mathcal{K}_a : a \in A\rangle$ (analogously with B).

For, let $\rho : \mathbb{R}^n \to \mathbb{R}^n$ be the translation

$$\rho(\mathbf{x}) = \mathbf{x} + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right].$$

Observe that $\rho^a_{A\dagger B} = \rho(\mathbf{e}_a)$. It follows that the baricenter of $\langle \rho^a_{A\dagger B} : a \in A\rangle$ is

$$\frac{1}{|A|}\sum_{a\in A}\rho^a_{A\dagger B} = \frac{1}{|A|}\sum_{a\in A}\rho(\mathbf{e}_a) = \rho\left(\frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right) = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

Analogously, using that $\rho^b_{B\dagger A} = -\rho(-\mathbf{e}_b)$, we have that

$$\frac{1}{|B|}\sum_{b\in B}\rho^b_{B\dagger A} = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

and therefore $\langle \rho^a_{A\dagger B} : a \in A\rangle \cap \langle \rho^b_{B\dagger A} : b \in B\rangle \neq \phi$.

On the other hand, given a separation $\alpha \mid \beta$, let $\psi_{\alpha|\beta}$ be the affine extension of

$$\psi_{\alpha|\beta}(\mathbf{e}_j) = \begin{cases} -1 & j \in \alpha \\ 1 & j \in \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, n.$$

Now, it is enough to prove that for every $i \in \alpha$ (resp. β), we have that $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) < 0$ (resp. > 0) for all $A \dagger B$ such that $i \in A$. For this, observe that, if $i \in \alpha \cap A$ (and $A \dagger B$) then, since $\psi_{\alpha|\beta}(\mathbf{e}_j) = 0$ for all $j \in (A \cup B) \setminus (\alpha \cup \beta)$,

$$\psi_{\alpha|\beta}\left(\rho^i_{A\dagger B}\right) = \psi_{\alpha|\beta}\left(\mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\Big(\sum_{B\cap\alpha} \mathbf{e}_b + \sum_{B\cap\beta} \mathbf{e}_b\Big) - \frac{1}{|A|}\Big(\sum_{A\cap\alpha} \mathbf{e}_a + \sum_{A\cap\beta} \mathbf{e}_a\Big)\right]\right) =$$

$$= -1 + \frac{(|B\cap\beta| - |B\cap\alpha|)}{2|B|} + \frac{(|A\cap\alpha| - |A\cap\beta|)}{2|A|} \leq -1 + \frac{1}{2} + \frac{1}{2} = 0.$$

Equality holds if and only if $B \subseteq \beta$ and $A \subseteq \alpha$ leading to a contradiction. Since the case $i \in \beta$ is analogous, the proof is completed.

3 Proof of Theorem 2

We will suppose that the reader is familiar with the classical theorems in convexity of Kirchberger, Carathédory and Radon (see e.g. [3, 4]); we will use them without further reference.

Lemma 1 *Let $\mathcal{S} = \{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$ be a separoid of convex sets. Given a minimal Radon partition $A \dagger B$, there exists a point on each convex set of the support, $\mathbf{x}_i \in \mathcal{K}_i$, such that*

$$\langle \mathbf{x}_i : \mathcal{K}_i \in A\rangle \cap \langle \mathbf{x}_j : \mathcal{K}_j \in B\rangle \neq \phi.$$

Proof. Let $\mathbf{x} \in \langle \cup A\rangle \cap \langle \cup B\rangle \neq \phi$. We need at most a finite number of points of A to express $\mathbf{x}$ as a convex combination of them. It is easy to see that, if two (or more) of these are on the same convex set $\mathcal{K}_i \in A$, they can be replaced by a single point $\mathbf{x}_i \in \mathcal{K}_i$ which is a convex combination of them. Therefore we need at most one point in each convex set. By the minimality of the partition $A \dagger B$, it is clear that we need at least one point in each convex set of A. The same argument works for B and we are done. •

A *Radon* separoid is a separoid such that the supports of any two different minimal Radon partitions are incomparable.

Lemma 2 *If a separoid is in general position and its geometric dimension is equal to its dimension, then it is a Radon separoid.*

Proof. Let $\mathcal{S}$ be a d-dimensional separoid in general position. If its geometric dimension is equal to its dimension, it can be represented by a family of convex sets in $\mathbb{R}^d$. Suppose that $\mathcal{S}$ **is not** a Radon separoid. Then there are $A \dagger B$, $C \dagger D \in MRP$ such that $A \cup B \subseteq C \cup D$ and $\{A, B\} \neq \{C, D\}$. Since $\mathcal{S}$ is in general position and both partitions are minimal, we may suppose that $|\mathcal{S}| = |A \cup B| = |C \cup D| = d + 2$.

Using Lemma 1, two configuration of points can be defined, two points on each convex set, in such a way that they represent the two Radon partitions. Consider the line segment that joins each couple —inside each convex set— and move continuously from one end to the other. Since we are moving from one configuration of points to another different one, we have to reach a configuration which is not in general position. We conclude that $\mathcal{S}$ **is not** in general position. •

We are now ready for the proof of Theorem 2. Let $\mathcal{S}$ be a separoid of convex sets in $\mathbb{R}^d$. Since its geometric and combinatorial dimensions coincide, we may suppose that $d = \mathrm{d}(\mathcal{S})$. Choose a point in each convex set, denote by $\mathcal{P}$ the separoid of these points, and let $\varphi: \mathcal{P} \longrightarrow \mathcal{S}$ be the obvious bijection. It is clear that φ is a morphism of separoids. We will show that it is an isomorphism of separoids.

Let $A \dagger B \in MRP$ be a minimal Radon partition of $\mathcal{S}$. Since $\mathcal{S}$ is a separoid in general position, the cardinality of the support is $|A \cup B| \geq d + 2$. Then the preimage of this union consists of $d + 2$ or more points in $\mathbb{R}^d$ and there exists a partition $C \dagger D$ of $\varphi^{-1}(A \cup B)$ in $\mathcal{P}$. Since φ is a bijective morphism, $\varphi(C) \dagger \varphi(D)$ is a Radon partition of $A \cup B$. Finally, due to Lemma 2, $\mathcal{S}$ is a Radon separoid and $\{A, B\} = \{\varphi(C), \varphi(D)\}$. Therefore $\varphi^{-1}(A) \dagger \varphi^{-1}(B)$.

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