A Helly type theorem for abstract projective geometries

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Abstract

We prove that the lattice of linear partitions in a projective geometry of rank n has Helly number $\lfloor 3n/2 \rfloor$.

1 Introduction

Consider a set P whose elements will be called *points* and a family L of subsets of P whose elements will be called *lines*. For $A \subseteq P$, let $L(A)$ be the set of all lines which have at least two points in A. Denote $\langle A \rangle$ the union of all lines in $L(A)$. The function $A \mapsto \langle A \rangle$ is a closure operator whose closed sets are called *flats*. The rank of a flat f is the minimum cardinality of a set of points A such that $f = \langle A \rangle$. A n-flat is a flat of rank n. The pair (P, L) is a *projective geometry of rank* n if the following axioms are satisfied:

- Two distinct points belong to exactly one line.
- If a line intersects two sides of a triangle (not at their intersection), then it also intersects the third line.
- Every line has at least three points.
- P is a n-flat.

The main example of a projective geometry is obtained considering a vector space k^n over a division ring k and taking the set of all its subspaces moduli the action of the multiplicative group k^*. Such geometry will be called *projective geometry over* k. A remarkable fact (see [2]) is that any projective geometry of finite rank at least 4 is a projective geometry over some division ring.

If f, g are two flats, then $f + g$ denotes the flat $\langle f \cup g \rangle$. It is well known (see e.g. [2]) that the collection of all flats of a projective geometry with the operations $+, \cap$ is a modular lattice i.e. for any two flats f, g the equality $\operatorname{rk} f + \operatorname{rk} g = \operatorname{rk}(f + g) + \operatorname{rk}(f \cap g)$ holds. Observe also that every point is a flat.

A flat f is *transversal* to a family of flats $\mathcal{F}$ if f intersects every member of $\mathcal{F}$. A finite family of flats $\mathcal{F}$ is *independent* if together they span the biggest possible flat; that is, if

$$\operatorname{rk} \sum_{f \in \mathcal{F}} f = \sum_{f \in \mathcal{F}} \operatorname{rk} f$$

We say that a family of flats is *m-generic* if each m of them are independent.

In [1] its is proved the following: *a m-generic finite family of n-flats of a projective geometry over the field of real numbers has a transversal m-flat if every subfamily of cardinality $\left\lfloor \frac{3n}{2} \right\rfloor + m + 1$ has a transversal m-flat.* However, it was used only that the ring of real numbers is commutative. Therefore, this fact is also true for any field.

The proof of this result has two key elements: a Helly type theorem for *linear partitions* (see below) and the use of double ruled linear manifolds. Unfortunately, the existence of double ruled linear manifolds in a projective geometry over a division ring k can be shown to be equivalent to the commutativity of multiplication in k. So, the result above seems to be false for projective geometries over non-commutative division rings.

A *linear partition* is the union of an independent family of flats. This concept is natural and can be easily introduced in a wide class of lattices. The Helly type theorem mentioned above is the following:

Theorem 1 *A family of linear partitions in a projective geometry of rank n has non-empty intersection if every*

$$\left\lfloor \frac{3n}{2} \right\rfloor$$

of them have non-empty intersection. Furthermore, this is the least possible such number.

Therorem 1 was proved in [1] for projective spaces over the field of real numbers. In this note we will show that modularity is enough to prove such result, i.e. commutativity is not essential. Beware of the differences between formulas and concepts in [1] and here. They arise because the numbers in [1] are calculated based on the *dimension* and here on the rank. It seems that in the abstract case the use of the rank function is more appropriate.

2 The Helly number of a lattice

Let L be a lattice. We will suppose that L is complete and atomic i.e. it has a minimum element, every element of L is a join of a set of points and every set of points has a join. So, L is defined by a closure operator (denoted by $\langle\rangle$) on the set of points. The elements of L will be called flats. Any flat is defined by the set of points which it contains. The join, is the closure of the union and the meet is equal to the intersection.

We will say that L is k**-Helly** if for any finite family $T \subseteq L$ the condition that every subfamily of T of cardinality k has non-empty intersection is sufficient to conclude that the whole family has non-empty intersection. If L is k-Helly, then it is k'-Helly for any $k' \geq k$. The **Helly number** of L is the minimum number h such that the lattice is h-Helly. Of course, L may have or have not Helly number. If every chain of L has bounded length, then it has (see [1]). The classic Helly theorem states that the lattice of all convex sets (ordered by inclusion) of an affine space of dimension d has Helly number $d+1$. From now on, the existence of the Helly number will be obvious and our task will be to find its value. Therefore, we will not make any provisions for the case that the Helly number does not exist. We will denote by $h(L)$ the Helly number of L.

For a finite set A of points we will call $Z(A) = \bigcap_{a \in A} \langle A - a \rangle$ the **Radon center** of A. The Radon center is a monotone function, i.e. $A \subseteq B \Rightarrow Z(A) \subseteq Z(B)$. Denote by $h^*(L)$ the maximum cardinality of a finite set of points of L with empty Radon center. The following proposition generalizes to an abstract setting the proof of the classic Helly theorem using the Radon theorem.

Proposition 2 *$h(L) = h^*(L)$.*

Proof. Let A be a set of points with empty Radon center. The family $T = \{\langle A - a \rangle\}_{a \in A}$ has the property that any subfamily of cardinal $\#A - 1$ has non-empty intersection but the whole family has empty intersection. Therefore L is not $(\#A - 1)$-Helly and so $h(L) \geq \#A$. This shows that $h(L) \geq h^*(L)$.

Now, let T be any finite family with empty intersection and such that any of its subfamilies of cardinality $h(L) - 1$ has non-empty intersection. By the definition of the Helly number, there must exist $T' \subseteq T$ with empty intersection and such that $\#T' = h(L)$. Any subfamily of T' of cardinality $h(L) - 1$ has non-empty intersection and there are $h(L)$ such subfamilies. For each such subfamily we choose a point in the intersection, thus obtaining a set of points A of cardinality $h(L)$ (no two of the chosen points can be equal). It is clear that the Radon center of A is contained in the intersection of T' and therefore is empty. So, $h(L) \leq h^*(L)$. ■

We will use a property of the Radon center function defined in the direct product of lattices. Let $L = L_1 \times L_2$ be the direct product of two atomic complete lattices. Then L is also atomic and complete. The points in L are of the form $(a, \varnothing)$ or $(\varnothing, b)$ with $a \in L_1$ and $b \in L_2$. Therefore, we can think that any set of points of L is the (disjoint) union of a set of points of L_1 with a set of points of L_2. Moreover, for any sets of points $A \subseteq L_1$ and $B \subseteq L_2$ we have that $\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle$.

Lemma 3 *If $L = L_1 \times L_2$, then for any sets of points $A \subseteq L_1$ and $B \subseteq L_2$ the equality $Z(A \cup B) = Z(A) \cup Z(B)$ holds.*

Proof. Using the distributive properties between union and intersection we obtain:

$$\bigcap_{a \in A} \langle A - a \cup B \rangle \cap \bigcap_{b \in B} \langle A \cup B - b \rangle = (Z(A) \cup \langle B \rangle) \cap (Z(B) \cup \langle A \rangle) = Z(A) \cup Z(B)$$

■

Lemma 3 will mainly be used when considering ideals in a lattice which are isomorphic to a direct product of two lattices. Suppose that the set of points $A \cup B$ is such that $A \cap B = \varnothing$ and $\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle$. Then the ideal generated by $\langle A \cup B \rangle$ is isomorphic to the direct product of the ideals generated by $\langle A \rangle$ and $\langle B \rangle$ and by lemma 3 we have that $Z(A \cup B) = Z(A) \cup Z(B)$.

A f flat is called *irreducible* if its ideal $f_{\leq} = \{g \in L \mid g \leq f\}$ can not be decomposed into the direct product of nontrivial lattices. A set of points A of an atomic complete lattice is called *block* (a word taken from graph theory) if it minimally spans the irreducible flat $\langle A \rangle$. Singletons are always blocks and have empty Radon center.

Observe that if in a set of points A there is a point a such that $\langle A\rangle = \langle A-a\rangle$, then $a \in Z(A)$. Therefore, by the preceding remarks, to calculate the Helly number of a lattice it is enough to find out which of its blocks have non-empty Radon center.

3 The Radon center of linear partitions.

A combinatorial geometry (see e.g. [4],[5]) is a bounded length atomic lattice with submodular $(\operatorname{rk}(f+g) + \operatorname{rk}(f\cap g) \leq \operatorname{rk} f + \operatorname{rk} g)$ rank function. Combinatorial geometries have been intensively studied often under different names (e.g. "geometric lattices" [3] or "matroids" [6]).

Let L be a combinatorial geometry. *Independent* families of flats are defined as in the introduction. A family of flats $\mathcal{F}$ is *dependent* if it is not independent. Minimal dependent families of flats are called *circuits* (another word taken from graph theory). When the flats are points these three concept coincide with the usual ones for sets of points.

We will need the following lemma which is valid in any combinatorial geometry, it is straightforward and difficult to find in the literature. So, we state it without a proof.

Lemma 4 *Let $C_1 \cup p$ and $C_2 \cup p$ be two circuits of points such that $\langle C_1 \cup p\rangle \cap \langle C_2 \cup p\rangle = p$. Then $C_1 \cup C_2$ is also a circuit of points spanning p.*

A *linear partition* in L is the union of an independent family of flats. In particular, independent sets of points are linear partitions. The following characterization of linear partitions in combinatorial geometries can be traced back to Tutte [7], where he studied "connectivity in matroids".

Lemma 5 *A set of points $A \subseteq L$ is a linear partition if and only if whenever a set of points $S \subseteq A$ is a circuit, then $\langle S\rangle \subseteq A$.*

For a set of points $A \subseteq L$ denote $\|A\| = \{p \in \langle S\rangle \mid S \subseteq A \text{ is a circuit}\}$. By Lemma 5 the set $\|A\|$ is a linear partition and the function $A \mapsto \|A\|$ is a closure operator. Therefore, the set of all linear partitions in L ordered by inclusion is an atomic complete lattice and Proposition 2 can be applied to compute its Helly number.

Another way to define the operator $\|\|$ is the following. Let A be a set of points. For $a, b \in A$ define $a \sim b$ if $a = b$ or there is a circuit $S \subseteq A$ such that $\{a, b\} \subseteq S$. It is well known (and can be easily proved from Lemma 5) that $\sim$ is an equivalence relation. So, the set A splits into the disjoin union of the equivalence classes $A_1, \ldots, A_r$ called the *components* of A and $\|A\| = \langle A_1\rangle \cup \cdots \cup \langle A_r\rangle$. Therefore, if a linear partition $\|A\|$ is irreducible, then $r = 1$ and $\|A\| = \langle A\rangle$.

Let A be a set of points. Denote by $Z^*(A)$ its Radon center in the lattice of linear partitions, i.e. $Z^*(A) = \bigcap_{a\in A} \|A-a\|$. Circuits of points are blocks in the lattice of linear partitions. Since for a point a in a circuit A we have that $\|A-a\| = A-a$ it follows that circuits have empty Radon center in the lattice of linear partitions.

Proposition 6 *Suppose that L is modular. If A is a block in the lattice of linear partitions which is not a circuit nor a singleton, then $Z^*(A) \neq \varnothing$.*

Proof. Let a be a point in A. Observe that $a \in \langle A - a \rangle$ because if not, then no circuit in A contains a and this contradicts that $\|A\|$ is irreducible. The set $A - a$ has more than one component because if not, then $\langle A \rangle = \|A\| \supsetneq \|A - a\| = \langle A - a \rangle$.

Let $X, Y, \ldots, Z$ be the components of $A - a$. If all components of $A - a$ are singletons, then A is a circuit. So, we can suppose that X is not a point and therefore for any x in X there is a circuit in X which contains x. Denote $B = Y \cup \cdots \cup Z$. Since the flats $\langle X \rangle, \langle Y \rangle, \ldots, \langle Z \rangle$ are independent, then $\langle X \rangle$ and $\langle B \rangle$ are disjoint. Observe that $a \notin \langle B \rangle$ because this would imply that X is a component of A.

Therefore, $\operatorname{rk} \langle B \cup a \rangle = \operatorname{rk} \langle B \rangle + 1$ and $\operatorname{rk} \langle X \rangle + \operatorname{rk} \langle B \rangle = \operatorname{rk} \langle A - a \rangle = \operatorname{rk} \langle A \rangle$. By modularity, we have that the rank of $\langle X \rangle \cap \langle B \cup a \rangle$ is equal to

$$\operatorname{rk} \langle X \rangle + \operatorname{rk} \langle B \cup a \rangle - \operatorname{rk} (\langle X \rangle + \langle B \cup a \rangle) = \operatorname{rk} \langle X \rangle + \operatorname{rk} \langle B \rangle + 1 - \operatorname{rk} \langle A \rangle = 1$$

Hence $\langle X \rangle \cap \langle B \cup a \rangle$ is a point which we denote by p.

First, we prove that $p \notin A$. Since $\langle X \rangle$ and B are disjoint, then $p \notin B$. If $p = a$, then $a \in \langle X \rangle \subseteq \|A - a\|$ and this contradicts that A minimally spans $\|A\|$. Since $p \in \langle B \cup a \rangle$ but $p \notin B \cup a$, then there is a circuit $C_1 \cup p$ with $C_1 \subseteq B \cup a$. If $p \in X$, then there is a circuit $C_2 \cup p$ with $C_2 \subseteq X$. By Lemma 4 the circuit $C_1 \cup C_2 \subseteq A - p$ spans p and therefore $p \in \|A - p\|$. From this, A does not minimally span $\|A\|$

Finally, we prove that $p \in Z^* (A)$, i.e. for any $q \in A$ we have that $p \in \|A - q\|$. Suppose $q \notin X$ and let $X', Y', \ldots, Z'$ be the components of $A - q$. Since $X \subseteq A - q$, then any circuit in X is a circuit in $A - q$. Therefore, all the elements of X are equivalent in $A - q$. So, we can suppose that $X \subseteq X'$ and hence $p \in \langle X \rangle \subseteq \langle X' \rangle \subseteq \langle X' \rangle \cup \langle Y' \rangle \cup \cdots \cup \langle Z' \rangle = \|A - q\|$.

Suppose $q \in X$. Since X is not a point, then q is in some circuit of X and therefore $\langle X - q \rangle = \langle X \rangle \ni p$. Let $C_1 \subseteq X - q$ be such that $C_1 \cup p$ is a circuit. Since $p \in \langle B \cup a \rangle - (B \cup a)$, then there exist a circuit $C_2 \cup p$ with $C_2 \subseteq B \cup a$. By Lemma 4 the circuit $C_1 \cup C_2 \subseteq A - q$ spans p and therefore $p \in \|A - q\|$. ■

4 The proof of Theorem 1

Proof. By Proposition 6 the blocks with empty Radon center in the linear partition lattice of a modular combinatorial geometry are precisely the singletons and circuits. Therefore, by the results in Section 2, all we have to do is to solve a maximization problem: how big can the number of points be in a set which is the union of circuits or points provided that they span an independent family of flats.

Let $N = N_1 \cup \cdots \cup N_r$ be such a set. Let s be the number of N_i which are points and t the number of N_i which are circuits. For $j \geq 3$ let t_j the number of N_i which are circuits with j points. Since

$$n \geq \operatorname{rk} (N) = s + \sum_j (j - 1) t_j = \#N - \sum_j t_j = \#N - t,$$

then the inequality $\#N \leq n + t$ must hold. From this, we conclude that we must make t as big as possible.

If n is even, then this is achieved when all the N_i are 3-point circuits and $\#N = 3t_3 = 3n/2$. If n is odd, then there are two solutions. The first is when one of the N_i is a point and all the other N_i are 3-point circuits. The second is when one of the N_i is a 4-point circuit and all the other N_i are 3-point circuits. For both solutions, $\#N = (3n-1)/2$.

By Proposition 2, we have that the Helly number of the lattice of linear partitions in a modular combinatorial geometry of rank n is bounded from above by the number $\lfloor 3n/2 \rfloor$. To show the equality we must find a concrete set of points $N = N_1 \cup \cdots \cup N_r$ with the properties described above. This is not difficult to do for projective geometries using their representation as projective geometries over a division ring. ■

5 Conclusion

Denote by $\pi(L)$ the lattice of linear partitions in L. A theorem due to Garret Birkhoff (see [3]) states that modular combinatorial geometries of finite rank are isomorphic to the direct product of projective geometries. Since $\pi(L_1 \times L_2) = \pi(L_1) \times \pi(L_2)$ and $h(L_1 \times L_2) = h(L_1) + h(L_2)$, then Theorem 1 allows to calculate the Helly number of $\pi(L)$ for all L which are modular combinatorial geometries of finite rank.

The case when L is a non-modular combinatorial geometry can be tricky. To see this, first observe that if L has rank 3, then any block in $\pi(L)$ is a circuit or a singleton and therefore $h(\pi(L)) = 4$, i.e. Theorem 1 also holds for such combinatorial geometries.

On the other hand deleting a point from a combinatorial geometry gives another combinatorial geometry. Consider the real projective space $\mathbb{P}^n$ as $\mathbb{R}^n$ augmented with the hyperplane at infinity. The set A of vertices of the cross-polytope $\operatorname{conv}\{\pm e_i \mid \{e_1, \cdots, e_n\}$ is a basis of $\mathbb{R}^n\}$ is a block in $\pi(\mathbb{P}^n)$ and $Z^*(A)$ is the origin of coordinates O. So, $h(\pi(\mathbb{P}^n)) = \lfloor 3(n+1)/2 \rfloor$ but by Proposition 2 we have that $h(\pi(\mathbb{P}^n - O)) \geq \#A = 2n$. Therefore, Theorem 1 does not hold for $\mathbb{P}^n - O$ for $n \geq 4$.

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