Rotors in triangles and tethrahedra

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1 Introduction

A polytope P is *circumscribed* about a convex body $\phi \subset \mathbb{R}^n$ if $\phi \subset P$ and each facet of P is contained in a support hyperplane of ϕ. We say that a convex body $\phi \subset \mathbb{R}^n$ is a *rotor* of a polytope P if for each rotation ρ of $\mathbb{R}^n$ there exist a translation τ so that P is circumscribed about $\tau\rho\phi$.

If Q^n is the n-dimensional cube then a convex body Φ is a rotor of Q^n if and only if Φ has constant width. However, there are convex polytopes that have rotors which are not of constant width.

A survey of results in this area has been given by Golberg [4]. See also the book Convex Figures of Boltyanskii and Yaglom [3].

It is well known that if Φ is a convex plane figure which is a rotor in the polygon P, then every support line of Φ intersects its boundary in exactly one point, and if Φ intersects each side of P at the points $\{x_1, \dots x_n\}$, then the normals of Φ at these points are concurrent.

In this paper we shall prove that if P is a triangle, then there is a baricentric formula that describes the curvature of bdΦ at the contact points.We prove also that if $\Phi \subset \mathbb{R}^3$ is a convex body which is a rotor in a tetrahedron T then the normal lines of Φ at the contact points with T generically belong to one ruling of a quadric surface.

2 Rotors in the triangle

Consider Φ a smooth rotor in the triangle T and suppose that the three sides of T intersect the boundary of Φ at the points x_1, x_2, x_3, respectively. As in the case of constant width bodies in which the radii of curvature of the boundary at the ends of a binormal sum to h, we are interested in a formula that involves the curvatures of the boundary of Φ at x_1, x_2, x_3.

A C^m framed curve (α, λ) is a curve of class C^m given by a parametrization of the following form: there is a support function $\mathcal{P} : (-\delta, \delta) \to \mathbb{R}$ of class C^m, $m \geq 2$, such that $\alpha(\theta) = \mathcal{P}(\theta)u(\theta_0 + \theta) + \mathcal{P}'(\theta)u'(\theta_0 + \theta)$ and λ is the tangent line through $\alpha(0) = x$, in the direction $x^{\perp}$. Therefore, $\mathcal{P}'(0) = 0$ and $\alpha(0) = \mathcal{P}(0)u(\theta_0)$ is the closest point of the line λ to the origin and the normal line of α at $\alpha(0)$ passes through the origin. Where $u(\theta) = (\cos\theta, \sin\theta)$ and $u'(\theta) = (-\sin\theta, \cos\theta)$, for every $\theta \in \mathbb{R}$.

A *sliding* along two given C^n framed curves (α_1, λ_1) and (α_2, λ_2) is a one parameter family of Euclidean isometries L_θ, $\theta \in (-\epsilon, \epsilon), \epsilon > 0$, satisfying

- L_0 is the identity map,
- L_θ rotates the plane by an angle of θ,
- $L_\theta(\lambda_i)$ is a tangent line of the curve α_i, for each $\theta \in (-\epsilon, \epsilon)$ and $i = 1, 2$.

Lemma 1. *Let (α_1, λ_1) and (α_2, λ_2) be two C^n framed curves. Suppose that their normal lines at $\alpha_1(0) = x_1$ and $\alpha_2(0) = x_2$ are not parallel and are concurrent at the origin. Then*

1. *there is a unique sliding L_θ, $\theta \in (-\epsilon, \epsilon), \epsilon > 0$, along them,*
2. *there is a C^n map $f : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that $L_\theta(x) = R_\theta(x) + f(\theta)$, for every $x \in \mathbb{R}^2$, $f(0) = f'(0) = 0$, where R_θ is the rotation of the plane about the origin by an angle of θ.*
3. *If the origin does not lie in the line λ_3, then the envelope of $\{L_\theta(\lambda_3)\}_{\theta\in(-\epsilon,\epsilon)}$ is a C^n framed curve (α_3, λ_3), such that the tangent line at $\alpha_3(0)$ is λ_3 and the normal line at $\alpha_3(0)$ passes through the origin.*

Proof. Let $\mathcal{E}$ be the Lie Group of orientation-preserving isometries of the Euclidean space $\mathbb{R}^2$. Let R_θ denote the rotation about the origin by an angle of θ. Since every $g \in \mathcal{E}$ takes the form $g(x) = R_\theta(x) + f$ for some θ and a fixed $f \in \mathbb{R}^2$, we will identify a neighborhood of the identity in $\mathcal{E}$ with $(-\gamma, \gamma) \times \mathbb{R}^2 \subset \mathbb{R}^3$, via the mapping $(\theta, f) \to R_\theta + f$. Observe that the identity in $\mathcal{E}$ is identified with the origin in $\mathbb{R}^3$.

Given a C^m framed curve (α, λ) with support function $\mathcal{P}(\theta)$, consider the set

$$S = \{g \in \mathcal{E} \mid g(\lambda) \quad \text{is a tangent line to} \quad \alpha\}$$

defined in the neighborhood of the identity in $\mathcal{E}$ (or of the origin in $\mathbb{R}^3$). We shall prove that S is a surface of class C^m. Indeed, we have the following explicit parametrization: consider the map $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(\theta, t) = (\theta, h(\theta, t))$, where $h(\theta, t) = (\mathcal{P}(\theta) - \mathcal{P}(0))u(\theta_0 + \theta) + tu'(\theta_0 + \theta)$, It is not difficult to verify that the for every $-\delta \le \theta \le \delta$ and $t \in R$, the isometry $L_\theta + h(\theta, t)$ sends the line λ to a tangent line of α. Furthermore,

$$\frac{d\psi}{d\theta}(0) = (1, \mathcal{P}'(0)u(\theta_0)) = (1, 0, 0)$$

and

$$\frac{d\psi}{dt}(0) = (0, u'(\theta_0)).$$

Moreover, it follows that the normal vector to S at the origin is $(0, -u(\theta_0))$.

Now, given two C^m framed curves, (α_1, λ_1) and (α_2, λ_2), Let S_1 and S_2 be their corresponding surfaces. If $\alpha_i(0) = \mathcal{P}_i(0)u(\theta_i)$, then the normal vector to S_i at the origin is $(0, -u(\theta_i))$, $i = 1, 2$, and since $\theta_1 \neq \theta_2$, we have that in a neighborhood of the origin S_1 and S_2 intersect transversally in a curve of the form $(\theta, f(\theta))$ and hence the sliding can be written as

$$L_\theta = R_\theta + f(\theta)$$

where $f : (-\epsilon, \epsilon) \to \mathbb{R}^2$ is of class C^m.

Thus, for $i = 1, 2$ the support function of α_i is given by

$$\mathcal{P}_i(\theta) = \mathcal{P}_i(0) + \langle f(\theta), u(\theta_i + \theta)\rangle.$$

where $\langle \cdot, \cdot \rangle$ denotes the interior product.

This implies that $f(0) = 0$ and furthermore, $0 = \mathcal{P}_i'(0) = \langle f'(0), u(\theta_i)\rangle$. Since $\theta_1 \neq \theta_2$, then $f'(0) = 0$.

Finally, let θ_3 be such that $u(\theta_3)$ is orthogonal to the line λ_3 and let r_3 be the distance from λ_3 to the origin. Then the support function of α_3 is given by $\mathcal{P}_3(\theta) = r_3 + \langle f(\theta), u(\theta_3 + \theta)\rangle$ and $\mathcal{P}_3'(0) = 0$ as we wished. □

For curves of constant width h, the sum of the radii of curvature at extreme points of every diameter is h. For rotors in a triangle, the analogous result is the following baricentric formula.

Theorem 1. *Let Φ be a rotor in the triangle T with vertices $\{A_1, A_2, A_3\}$. Suppose the boundary of Φ is twice continuous differentiable and let $x_3 = \Phi \cap A_1A_2$, $x_1 = \Phi \cap A_2A_3$ and $x_2 = \Phi \cap A_3A_1$. Let $\{a_1, a_2, a_3\}$ be the baricentric coordinates of the point O with respect to the triangle T, where O is the point at which the normal lines to T at the points x_1, x_2 and x_3 concur. If r_i is the distance from O to x_i and κ_i the curvature of the boundary of Φ at x_i, $i = 1, 2, 3$, then*

$$\frac{a_1}{\kappa_1 r_1} + \frac{a_2}{\kappa_2 r_2} + \frac{a_3}{\kappa_3 r_3} = 1.$$

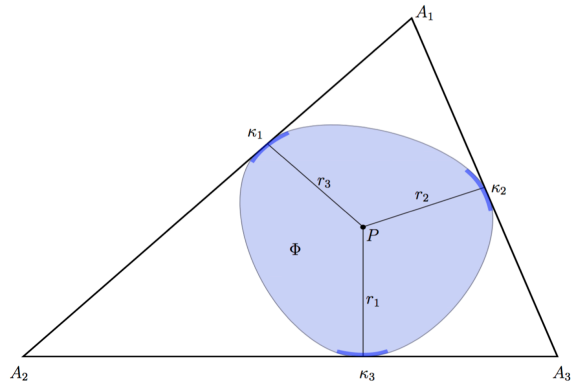

Figure 1

Proof. Let $\alpha_i : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a C^2-parametrization of a neighborhood of the boundary of Φ around x_i, with $\alpha_i(0) = x_i$ and let λ_i be the line through $A_{i+1}A_{i+2}$, mod 3, so that (α_i, λ_i) are C^2 framed curves, whose corresponding normal lines at x_i are concurrent at O. Suppose without loss of generality that O is the origin. By Lemma 1, there is a sliding along the three framed curves. That is, there is a one parameter family of Euclidean isometries L_θ, $\theta \in (-\epsilon, \epsilon), \epsilon > 0$, satisfying

- L_0 is the identity map,
- L_θ rotates the plane by an angle of θ,
- $L_\theta(\lambda_i)$ is a tangent line of the curve α_i, for each $\theta \in (-\epsilon, \epsilon)$ and $i = 1, 2, 3$.

Furthermore, there is a C^2 map $f : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that

$$L_\theta(x) = R_\theta(x) + f(\theta),$$

for every $x \in \mathbb{R}^2$, $f(0) = f'(0) = 0$, where R_θ is the rotation of the plane through the origin by an angle of θ.

Let $\mathcal{P}_i(\theta)$ be the pedal function of the framed curve α_i, with $\mathcal{P}_i(0) = r_i = \mid x_i \mid$, $i = 1, 2, 3$. Hence, $\mathcal{P}_i'(0) = 0$ and the radius of curvature of the boundary of Φ at x_i is

$$\frac{1}{\kappa_i} = \mathcal{P}_i(0) + \mathcal{P}_i''(0).$$

On the other hand, $\mathcal{P}_i(\theta) =\mid L_\theta(x_i) \mid=\mid R_\theta(x_i) + f(\theta) \mid$. Hence,

$$\mathcal{P}_i(\theta)^2 = \langle R_\theta(x_i) + f(\theta), R_\theta(x_i) + f(\theta)\rangle.$$

So,

$$\mathcal{P}_i(\theta)\mathcal{P}_i'(\theta) = \langle R_\theta(x_i) + f(\theta), R_\theta(x_i)^\perp + f'(\theta)\rangle.$$

Let $h_i(\theta) = \langle R_\theta(x_i), f'(\theta)\rangle + \langle R_\theta(x_i)^\perp, f(\theta)\rangle + \langle f'(\theta), f'(\theta)\rangle$ in such a way that

$$\mathcal{P}_i'(\theta) = \frac{h_i(\theta)}{\mathcal{P}_i(\theta)}$$

and

$$\mathcal{P}_i''(\theta) = \frac{h_i'(\theta)\mathcal{P}_i(\theta)^2 - h_i(\theta)^2}{\mathcal{P}_i(\theta)^3}.$$

Note that $h_i(0) = 0$ and $h_i'(0) = \langle f''(0), x_\rangle$.

Since the radius of curvature of $\mathrm{bd}\Phi$ at x_i is given by $\mathcal{P}_i(0)+\mathcal{P}_i''(0)$, we have that for $i = 1, 2, 3$

$$\frac{1}{\kappa_i} = r_i + \frac{\langle f''(0), x_i\rangle}{r_i}.$$

Let $\{b_1, b_2, b_3\}$ be the baricentric coordinates of the origin O with respect the triangle with vertices $\{x_1, x_2, x_3\}$. That is: $b_1x_1 + b_2x_2 + b_3x_3 = 0$, with $b_1 + b_2 + b_3 = 1$. Hence, for $i = 1, 2, 3$,

$$\frac{b_i r_i^2}{\kappa_i r_i} = b_i r_i^2 + \langle f''(0), b_i x_i \rangle,$$

and therefore,

$$\sum \frac{b_i r_i^2}{\kappa_i r_i} = \sum b_i r_i^2 + 0.$$

To conclude the proof of the theorem, it will be enough to prove that

$$a_i = \frac{b_i r_i^2}{b_1 r_1^2 + b_2 r_2^2 + b_3 r_3^2}.$$

The basic property that defines A_i is $\langle A_i, x_j \rangle = \langle x_j, x_j \rangle = r_j^2$ for $i \neq j$. Using it, one easily obtains that

$$\langle b_1 r_1^2 A_1 + b_2 r_2^2 A_2 + b_3 r_3^2 A_3, x_j \rangle = \langle r_j^2 A_j, b_1 x_1 + b_2 x_2 + b_3 x_3 \rangle = 0,$$

for $j = 1, 2, 3$. This implies that $b_1 r_1^2 A_1 + b_2 r_2^2 A_2 + b_3 r_3^2 A_3 = 0$ because the x_j generate $\mathbb{R}^2$, and from here

$$\frac{b_1 r_1^2}{\sum b_i r_i^2} A_1 + \frac{b_2 r_2^2}{\sum b_i r_i^2} A_2 + \frac{b_3 r_3^2}{\sum b_i r_i^2} A_3 = 0.$$

It follows that

$$\frac{a_1}{\kappa_1 r_1} + \frac{a_2}{\kappa_2 r_2} + \frac{a_3}{\kappa_3 r_3} = 1,$$

as we wished.

□

3 The relation with immobilization problems

Immobilization problems were introduced by Kuperberg [5] and also appeared in [8]. They were motivated by grasping problems in robotics ([6] and [7]).

Let $\Phi \subset \mathbb{R}^n$ be a convex body. A collection of points X on the boundary of Φ is said to immobilize Φ if any small rigid movement of Φ causes one point in X to penetrate the interior of Φ. In the plane, for the case in which three points $X = \{x_1, x_2, x_3\}$ lie in the boundary Φ, there is a baricentric formula involving the curvature of bdΦ at x_i that allows us to know if X immobilizes Φ. See [1].

Theorem 2. *Let Φ be a twice continuous differentiable convex figure and let $X = \{x_1, x_2, x_3\}$ be three points in the boundary of Φ, whose normals are concurrent at the point O. Let $\{a_1, a_2, a_3\}$ be the baricentric coordinates of the point O with respect to the vertices of the triangle formed be the three support lines* *of Φ at x_1, x_2 and x_3. Also, let r_i be the distance from O to x_i, let κ_i be the curvature of the boundary of Φ at x_i, $i = 1, 2, 3$, and let*

$$\omega = a_1\kappa_1 r_1 + a_2\kappa_2 r_2 + a_3\kappa_3 r_3.$$

Then, if $\omega < 1$, $\{x_1, x_2, x_3\}$ immobilize Φ, and if $\omega > 1$, they do not.

There is a duality between Theorem 2 and Theorem 1. While in Theorem2, we have a rigid segment sliding along the boundary of the convex figure Φ, in Theorem 1, we have a rigid angle (formed by two lines) sliding along the boundary of Φ.

In dimension three, immobilization results are much more complicated. See [2]. To characterize when four points in the faces of a tetrahedron T immobilize T we require the following definition.

Let $\{L_1, L_2, L_3, L_4\}$ be four directionally independent lines in $\mathbb{R}^3$. We say that they belong *generically to one ruling of a quadric surface* if

- they are concurrent,
- they belong to one ruling of a quadric surface, or
- they meet in pairs and the planes these pairs generate meet in the line through the intersecting points.

Theorem 3. *A necessary and sufficient condition for four points $\{x_1, x_2, x_3, x_4\}$, in the corresponding four faces of a tetrahedron T, to immobilize it, is that the normal lines to T at x_1, x_2, x_3 and x_4 belong generically to one ruling of a quadratic surface.*

The "duality" mentioned above, gives us the following theorem for rotors in a tetrahedron.

Theorem 4. *Let Φ a twice continuous differentiable rotor in the tetrahedron T, and let $\{x_1, x_2, x_3, x_4\}$ be the points of the boundary of Φ that intersect the four faces of T. Then, the normal lines to T at x_1, x_2, x_3 and x_4 belong generically to one ruling of a quadratic surface.*

Proof. Consider a tetrahedron T that circumscribes Φ. For every $\rho \in SO(3)$, let $T(\rho)$ be the tetrahedron directly homothehtic to ρT circumscribing Φ and let $V_\Phi(\rho)$ be the volume of of $T(\rho)$. It is not difficult to see that $V_\Phi(\rho)$ depends continuously on ρ.

We will prove that if ρ_0 is a local maximum of $V_\Phi(\rho)$, then the four normal lines to the boundary of Φ at the points that touch the four faces of $T(\rho_0)$, belong generically to one ruling of a quadratic surface. If this is so, then the proof the theorem is complete because Φ is a rotor in T if and only if $V_\Phi(\rho)$ is constant. For the proof of the above statement, it will be sufficient to consider the case in which Φ is a tetrahedron. The reason is that if a, b, c and d are the points in which the sides of $T(\rho_0)$ touch the boundary of Φ, then ρ_0 is also a local maximum of $V_K(\rho)$, where K is the tetrahedron with vertices $\{a, b, c, d\}$.

Let H_a, H_b, H_c and H_d be four planes containing the faces of the tetrahedron $T(\rho_0)$, in such a way that $a \in H_a, b \in H_b, c \in H_c$ and $d \in H_d$, respectively. Assume now that a $T(\rho_0)$ is a rigid tetrahedron sliding along a, b, c. That is, $T(\rho_0)$ is sliding rigidly in such a way that the points a, b, c remain fixed but inside the planes H_a, H_b and H_c, and during the rigid sliding movement of $T(\rho_0)$, the fixed point d is always inside $T(\rho_0)$.

The proof of Theorem 4 now follows straightforward from the proof or Theorem 3 in [2], but this time we consider, instead of a rigid triangle sliding along three fixed planes, the dual situation of a 3-dimensional rigid sector (the angle between three planes H_a, H_b and H_c) sliding along three fixed points a, b, c.

$\square$

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