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THE SCORPIONS: EXAMPLES IN STABLE NON COLLAPSIBILITY AND IN GEOMETRIC CATEGORY THEORY

JAVIER BRACHO and LUIS MONTEJANO

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1. INTRODUCTION

THE *Scorpion* A is obtained from a pentagon $abcc'b'$ by identifying first two sides $bc = b'c'$ and then the remaining three sides $ab = ab' = cc'$.

The *dunce hat* D is obtained from A by shrinking its essential circle to a point, $D = A/bc$ (compare with Fig. 1 of [9]). The *scorpion* A is the simplest example of a homotopy circle which does not collapse to a circle. However, $A \times I$ does collapse to S^1.

The *scorpion* A can be generalized to higher dimensions as follows: Let $\pi : S^n \to B^n$ be the restriction of the standard projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$; let $\partial : B^n \to S^n$ be the quotient map that identifies the boundary of B^n onto a single point, and let $f^n : S^n \to S^n$ be the null homotopic map given by the composition $\partial\pi$. It is easy to see that A is the mapping torus of f^1.

Define the $(n+1)$-*dimensional scorpion* A^{n+1} as the mapping torus of the map $f^n : S^n \to S^n$. That is:

$$A^{n+1} = (S^n \times [0, 1])/(x, 0) \sim (f^n(x), 1) \quad \text{for every } x \in S^n.$$

Since f^n is nullhomotopic, A^{n+1} is a homotopy circle.

In the next theorem we will not only prove that the scorpions are highly non collapsible but we will completely determine their range of q-collapsibility.

THEOREM 0. *$A^{n+1} \times I^q \searrow S^1$ if and only if $q \geq n$.*

It is well known (see for example Corollary 5.1A of [4]) that the classic notion of collapsibility coincides with the following one:

Let X and Y be polyhedra. If $Y = X \cup_f I^{n+1}$, where $f : I^n \times \{0\} \to X$ is a PL-map, then Y *PL-collapses elementarily* to X. We say that Y *PL-collapses* to X and write $Y \searrow X$ if there is a finite sequence of elementary PL-collapses from Y to X.

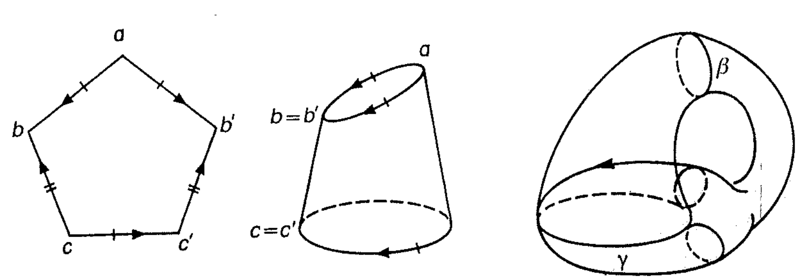


Fig. 1. The Scorpion.

If in the above definition we let X and Y be topological spaces and f any map, we obtain, verbatim, the notion of *topological collapsibility*.

For polyhedra the two notions are different, as the following examples of Berstein, Cohen and Connelly show:

Suppose that Σ^n is a non-simply connected PL homology n-sphere. Let S^p denote the standard PL p-sphere, and let B be a PL $(n+p+1)$-ball in $(S^p * \Sigma^n) - S^p$. Let $\mathscr{B}^{n+p+1} = (S^p * \Sigma^n) - \text{Int}(B)$. The Main Proposition of [1] asserts that $\mathscr{B}^{n+p+1} \times I^{n-2}$ is not PL-collapsible. Nevertheless, if $p \geq 1$, the Double Suspension Theorem [3] and the Schoenflies Theorem [2] imply that $\mathscr{B}^{n+p+1}$ is a topological ball and therefore topologically collapses to a point.

The notion of piecewise linear q-collapsibility has mainly been studied for the one point space. Until now, the examples of Berstein, Cohen and Connelly are essentially the unique known family of contractible non PL q-collapsible polyhedra, but as we already show, they are (in the worst case) *topologically* 2-*collapsible*. Their non collapsibility properties come from their exotic combinatorial PL-structures. However, the non collapsibility of our examples is proved in this stronger sense. It is topological in nature. We restate Theorem 0.

THEOREM 1. (a) *$A^{n+1} \times I^q$ PL-collapses to S^1, if $q \geq n$ and,* (b) *$A^{n+1} \times I^q$ does not topologically collapse to S^1, if $q < n$.*

In the last section, we will return to the notion of non PL-collapsibility. The scorpions yield a natural generalization of the dunce hat to higher dimensions, creating a new family of contractible polyhedra D_n which are highly non PL-collapsible.

The other notion we treat in this paper is the *geometric category* of a space. For a polyhedron X, its geometric category, $\text{gcat}(X)$, is the smallest integer k such that X can be covered with k contractible subpolyhedra. This concept is closely related to the Lusternik–Schnirelmann category and it is also related to the concept of collapsibility because it is easy to see (Lemma 1 below) that if X PL-collapses to Y then $\text{gcat}(X) \leq \text{gcat}(Y)$.

In 1967 T. Ganea [6] introduced the notion of *strong category* of a compact polyhedron $\text{Cat}(X) = \text{Min}\{\text{gcat}(Y) \mid Y \text{ is a compact polyhedron with the homotopy type of } X\}$.

The strong category differs from the classic Lusternik–Schnirelmann category in at most one unit and there is essentially only one known example, due to I. Berstein, for which both notions do not agree. Furthermore, the difference between the geometric category and the strong category may be arbitrarily large. For more relations about these concepts see [7] and [8].

In [8] it was proved that $\text{Cat}(X) = \text{Min}\{\text{gcat}(X \times I^q) \mid q \geq 0\}$. In most of the cases $\text{gcat}(X \times I) = \text{Cat}(X)$, hence it is natural to ask how big must the integer q be in order to get $\text{gcat}(X \times I^q) = \text{Cat}(X)$. Since $\text{Cat}(A^n) = \text{Cat}(S^1) = 2$, the following theorem shows that this integer may be arbitrarily large, thus solving Question 5 of [7].

THEOREM 2.

$$gcat(A^{n+1} \times I^q) = \begin{cases} 3 & \text{if } \; 0 \leq q < n \\ 2 & \text{if } \; q \geq n \end{cases}$$

The essence of Theorem 2 is that $A^{n+1} \times I^q$ cannot be covered with two contractible sets when q is smaller than n. The following intuitive proof of this fact, for the case $n = 1$, is illustrative of the complete proof. Suppose $A = K \cup L$ where K and L are contractible sets. The argument has two steps:

(1) Since L is contractible, any essential circle of A is not contained in L, which implies by "duality" that K must contain a null homotopic circle "isotopic" to β. (See Fig. 1).

(2) There is "essentially" only one way to deform β in A into a single point, which is by passing through γ. Therefore, if $\beta \subset K$, K must contain the circle γ. Repeating the same idea, γ cannot be deformed in A into a single point without using all points of A, thus $K = A$ which is a contradiction.

The precise definitions and the formalization of these ideas will take the following two sections.

This paper is dedicated to Dan Kan, in whose seminars we were introduced to Zeeman's "On the Dunce Hat".

2. THE COLLAPSING

This section is devoted to exhibit a collapse $A^{n+1} \times I^n \searrow S^1$.

First, we describe the $(n+1)$-*scorpion* A^{n+1} in a suitable way. Identify $B^n_{1/2}$ with $B^n_{1/2} \times 0 \subset B^{n+1}$. Observe, from the definition of A^{n+1}, that

$$A^{n+1} = B^{n+1}/\sim$$

where $\sim$ is the identification; $x \sim \partial(2x)$ for every $x \in B^n_{1/2}$.

Remember that $\partial : B^n \to S^n$ is the quotient map that identifies the boundary of B^n onto a single point, and $B^n_{1/2}$ denotes the 1/2-radius n-ball.

Now, consider a PL-map $g : B^{n+1} \to I^q$, and let Γ_g be its graph in $B^{n+1} \times I^q$. We claim that

$$A^{n+1} \times I^q \searrow (((S^n \cup B^n_{1/2}) \times I^q) \cup \Gamma_g)/\sim$$

where $\sim$ now denotes the restriction of the product extension of $\sim$ to $B^{n+1} \times I^q$.

The above follows from Lemma 2 of [9] or Lemma 38 of [10], since $\sim$ only identifies points in $S^n \cup B^n_{1/2}$.

Observe now that $((S^n \cup B^n_{1/2}) \times I^q)/\sim \; = S^n \times I^q$, so that

$$A^{n+1} \times I^q \searrow (S^n \times I^q) \cup_{\tilde{g}} B^{n+1}$$

where $\tilde{g} : \partial B^{n+1} \cup B^n_{1/2} \to S^n \times I^q$ is given by

$$\tilde{g}(x) = \begin{cases} (x, g(x)) & \text{if } x \in S^n \text{, and} \\ (\partial(2x), g(x)) & \text{if } x \in B^n_{1/2}. \end{cases}$$

Suppose that we may choose g in such a way that:

(a) $S^n \times I^q \searrow \tilde{g}(\partial B^{n+1} \cup B^n_{1/2})$,
(b) $\tilde{g}(\partial B^{n+1}) \cap \tilde{g}(B^n_{1/2})$ is a single point and
(c) $\tilde{g}|B^n_{1/2}$ is an embedding.

Then

$$A^{n+1} \times I^q \searrow (S^n \times I^q) \cup_{\tilde{g}} B^{n+1} \searrow B^{n+1}/\{x, y\} \searrow S^1$$

where $x \in \partial B^{n+1}$ and $y \in B^n_{1/2}$.

In order to achieve (a), (b) and (c), choose $q = n$ and $g : B^{n+1} \to I^n$ in such a way that $g(\partial B^{n+1})$ is a single point and g restricted to $B^n_{1/2}$ is a homeomorphism. Clearly, with this choice, (b) and (c) follows immediately. In order to prove (a), let $x_1, \ldots, x_{n+1}$ be the standard orthonormal basis for $\mathbb{R}^{n+1}$ and regard $\mathbb{R}^n$ as the subspace generated by $x_1, \ldots, x_n$. For every $x \in \mathbb{R}^n - \{0\}$, let $\mathbb{R}^2_x$ be the subspace generated by x_{n+1} and $x/|x|$. Let us define $\phi : B^n \to SO(n+1)$ as follows: $\phi(0) = \text{Id}$ and if $x \neq 0$, let $\phi(x)$ be the linear isometry which fixes the subspace orthogonal to $\mathbb{R}^2_x$ and, in $\mathbb{R}^2_x$, is a rotation of $\pi|x|$ radians in the direction of x. Note that if $\varepsilon : SO(n+1) \to S^n$ is the evaluation map on x_{n+1}, then

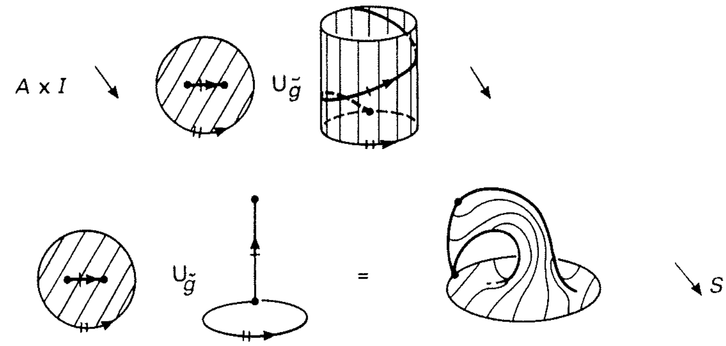


Fig. 2. The collapsing.

$\varepsilon\phi = \partial : B^n \to S^n$. Furthermore, ϕ give rise to a fiber preserving homeomorphism $\bar{\phi}$: $S^n \times B^n \to S^n \times B^n$ given by $\bar{\phi}(y, x) = (\phi(x)(y), x)$. Note that hence the pair $(S^n \times B^n, \{x_{n+1}\} \times B^n)$ is homeomorphic to the pair $(S^n \times B^n, \partial \times 1(B^n))$. Consequently, in our case, the pair $(S^n \times I^n, \tilde{g}(\partial B^{n+1} \cup B^n_{1/2}))$ is homeomorphic to the pair $(S^n \times I^n, (\{x_{n+1}\} \times I^n) \cup (S^n \times \{d\}))$, where $\{d\} = g(\partial B^{n+1})$. Now, it is easy to see that (a) follows from our choice of $q = n$ and $g: B^{n+1} \to I^n$. This concludes the proof.

Remark. There is a technical problem in our proof of the collapsing. The problem is that the map $\partial: B^n \to S^n$ is not *PL*. Nevertheless, this technical problem causes no difficulties mainly because the topological mapping cylinder of a simplicial map is homeomorphic to its simplicial mapping cylinder.

Note that $S^n \times I^q$ does not collapse to $\tilde{g}(\partial B^{n+1} \cup B^n_{1/2})$ unless $q \geq n$, because $\tilde{g}$ embeds the interior of $B^n_{1/2}$ and, if $q < n$, closes $\partial B^n_{1/2}$ over I^q as a "zipper", creating n-homotopy other than $\tilde{g}(\partial B^{n+1})$. Figure 2 shows, for $n = 1 = q$, the choice of g such that $g(\partial B^{n+1}) = 0$ and is the "identity" on $B^n_{1/2}$.

3. THE PROOF OF THE THEOREMS

In this work we are interested in topological collapsibility, so that we need a slight modification of the notion of geometric category:

Definition. Given a topological space X, gcat(X) denotes the smallest integer k such that there is an open cover $\{U_1, \ldots, U_k\}$ of X, where each U_i is contractible and is contained in some closed contractible subset of X.

We will prove Theorem 2 in this setting, although it will be clear that the proof also works when we use the classic notion of geometric category.

LEMMA 1. *If Y topologically collapses to X then* $gcat(Y) \leq gcat(X)$.

Proof. It is enough to prove the lemma for the case of an elementary collapse. So, we may assume $Y = X \cup_f I^{n+1}$ where $f: I^n \times \{0\} \to X$. Let $U \subset C$ be two contractible subsets of X with U open and C closed. Then

$$U \bigcup_{f|f^{-1}(U)} (f^{-1}(U) \times I) \subset C \bigcup_{f|f^{-1}(C)} (f^{-1}(C) \times I)$$

are two contractible subsets of $X \cup_f I^{n+1} = Y$, with the first open and the second closed. Thus any suitable cover of X can be extended to Y and the lemma follows. □

Next, we will prove Theorem 2. To prove that $\mathrm{gcat}(A^{n+1} \times I^q) \leq 3$, assume $\partial((0, \ldots, 0)) = (0, \ldots, 0, -1)$. Let D^{n+1} be a small ball contained in $\{(x_1, \ldots, x_{n+1}) \in \mathrm{Int}(B^{n+1}) | x_{n+1} > 0\}$. It is easy to see that

$$B^{n+1} - \mathrm{Int}(D^{n+1}) \searrow S^n \cup B^n_{1/2} \cup \{(0, \ldots, 0, x_{n+1}) | -1 \leq x_{n+1} \leq 0\}$$

By Lemma 2 of [9], $A^{n+1} - \mathrm{Int}(D^{n+1}) \searrow S^n \vee S^1$. Since $\mathrm{gcat}(S^n \vee S^1) = 2$, then $\mathrm{gcat}(A^{n+1}) \leq 3$, and therefore $\mathrm{gcat}(A^{n+1} \times I^q) \leq 3$ for all $q \geq 0$.

For $q \geq n$, $\mathrm{gcat}(A^{n+1} \times I^q) = 2$ follows from Lemma 1 and the fact that $A^{n+1} \times I^q \searrow S^1$, proved in Section 2.

The hard part of the proof is that $\mathrm{gcat}(A^{n+1} \times I^q) > 2$ for $q < n$. For that purpose we need to describe the universal cover of A^{n+1}.

Let M_{f^n} be the mapping cylinder of the null homotopic map $f^n: S^n \to S^n$ described in the introduction. That is

$$M_{f^n} = ((S^n \times [0, 1]) \amalg S^n)/(x, 1) \sim f^n(x) \quad \text{for every } x \in S^n.$$

and let $S^n \times \{0\}$ and S^n be the two standard copies of the n-sphere in M_{f^n}.

Note that A^{n+1} is obtained from M_{f^n} by identifying x with $f(x)$, for every $x \in S^n$.

For every $i \in \mathbb{Z}$, let S^n_i be a copy of S^n and let $f^n_i: S^n_i \to S^n_{i+1}$ be a copy of the map $f^n: S^n \to S^n$. Let $M_{f^n_i}$ be the mapping cylinder of each f^n_i. Since A^{n+1} is the mapping torus of the null homotopic map f^n, hence A^{n+1} has the homotopy type of a circle and its universal cover $\tilde{A}^{n+1}$ can be described as follows:

$$\tilde{A}^{n+1} = \left(\bigcup_{i=-\infty}^{\infty} M_{f^n_i} \right) \Big/ \sim,$$

where $f^n_i(x) \sim (x, 0) \in S^n_{i+1} \times \{0\} \subset M_{f^n_{i+1}}$, for every $x \in S^n_i$ and $i \in \mathbb{Z}$.

Clearly $\tilde{A}^{n+1}$ is the universal covering space of A^{n+1} and its group of covering transformations $\mathbb{Z}$ is generated by a homeomorphism $\tau: \tilde{A}^{n+1} \to \tilde{A}^{n+1}$ such that $\tau(M_{f^n_i}, S^n_i) = (M_{f^n_{i+1}}, S^n_{i+1})$.

For every $i \in \mathbb{Z}$, let $S_i = S^n_i \times I^q \subset \tilde{A}^{n+1} \times I^q$.

The following two propositions correspond respectively to the two steps of the intuitive proof given in the introduction. All our homomorphisms, unless otherwise stated, are induced by inclusions and homology is with $\mathbb{Z}_2$-coefficients.

Proposition 1. *Suppose $\mathrm{gcat}(A^{n+1} \times I^q) = 2$. Then there exists an open subset U of $\tilde{A}^{n+1} \times I^q$ with compact closure, such that $\tilde{H}_*(U) = 0$ and $S_0 \subset U$.*

Proposition 2. *Let U be an open subset of $\tilde{A}^{n+1} \times I^{n-1}$ with $H_n(U) = 0$. If $H_n(U \cap S_i) \to H_n(S_i)$ is non-trivial then so is $H_n(U \cap S_{i+1}) \to H_n(S_{i+1})$.*

We can now conclude the proof of Theorem 2. It will be enough to prove that $\mathrm{gcat}(A^{n+1} \times I^{n-1}) > 2$. Suppose $\mathrm{gcat}(A^{n+1} \times I^{n-1}) = 2$. By Proposition 1, there is a homologically trivial open subset U of $\tilde{A}^{n+1} \times I^{n-1}$ with compact closure which contains S_0. Hence $H_n(U \cap S_0) \to H_n(S_0)$ is non-trivial, which implies, by Proposition 2, that $H_n(U \cap S_1) \to H_n(S_1)$ is non-trivial. Then $H_n(U \cap S_i) \to H_n(S_i)$ is non-trivial for every $i \geq 0$, which is a contradiction to the compactness of the closure of U. This finishes the proof of Theorem 2. Furthermore, the second part of Theorem 1 follows immediately from Theorem 2, Lemma 1 and the fact that $\mathrm{gcat}(S^1) = 2$.

Proof of Proposition 1. Let us suppose $A^{n+1} \times I^q = W \cup V$, where $W \subset \mathscr{W}$ and $V \subset \mathscr{V}$ are contractible sets with W, V open and $\mathscr{W}$, $\mathscr{V}$ closed and contractible. By the Mayer–Vietoris exact sequence, $W \cap V$ has exactly two components and these are homologically trivial. Let $\Pi : \tilde{A}^{n+1} \to A^{n+1}$ be the universal covering space map. Then, $(\Pi \times 1)^{-1}(W) = \coprod_{i=-\infty}^{\infty} W_i$, and $(\Pi \times 1)^{-1}(V) = \coprod_{i=-\infty}^{\infty} V_i$, where W_i and V_i are copies of W and V respectively, and $\tau(W_i) = W_{i+1}$, $\tau(V_i) = V_{i+1}$, $i \in \mathbb{Z}$.

Simple applications of Mayer–Vietoris imply that with the appropriate choice of indices:

(a) $W_i \cap V_j \neq \phi$ if and only if $i = j, j-1$, and

(b) $U_m = \bigcup_{i=-m}^{m} (W_i \cup V_i)$ is homologically trivial.

Now, since $\mathscr{W}$ and $\mathscr{V}$ are compact and contractible, it is easy to see that U_m has compact closure. Hence, since S_0 is compact, $S_0 \subset U_{m_0}$ for sufficiently large m_0. □

In order to prove Proposition 2, we need another proposition whose proof requires the following version of Alexander Duality.

Alexander Duality Lemma. *Let X be a subpolyhedron of $I^n \times I^m$ such that $I^n \times \partial I^m \subset X$ and $\partial I^n \times \mathring{I}^m \subset (I^n \times I^m) - X$. If $H_{m-1}(I^n \times \partial I^m) \to H_{m-1}(X)$ is non-trivial (respectively, trivial) then $H_{n-1}(\partial I^n \times \mathring{I}^m) \to H_{n-1}((I^n \times I^m) - X)$ is trivial (non-trivial).*

Proof. Let us identify S^{n+m} with the one point compactification of $\mathbb{R}^{n+m}$ Then $I^n \times I^m \subset \mathbb{R}^{n+m} \subset S^{n+m}$. Consider the following commutative diagram, where D is the Alexander Duality Isomorphism.

$$\begin{array}{ccc} H_{m-1}(I^n \times \partial I^m) & \xrightarrow[\cong]{D} & H^n(S^{n+m} - (I^n \times \partial I^m)) \\ \downarrow & & \downarrow \\ H_{m-1}(X) & \xrightarrow[\cong]{D} & H^n(S^{n+m} - X) \end{array}$$

Since $H_{m-1}(I^n \times \partial I^m) \to H_{m-1}(X)$ is non-trivial (trivial), then $H^n(S^{n+m} - (I^n \times \partial I^m)) \to H^n(S^{n+m} - X)$ is non-trivial (trivial), and hence, since we are working with $\mathbb{Z}_2$-coefficients, $H_n(S^{n+m} - X) \to H_n(S^{n+m} - (I^n \times \partial I^m))$ is non-trivial (trivial).

Let $U = (S^{n+m} - (I^n \times I^m)) \cup (\partial I^n \times \mathring{I}^m)$. Note that U is an AR. Consider the following commutative diagram, where the columns are Mayer–Vietoris exact sequences induced by $(S^{n+m} - X) = ((I^n \times I^m) - X) \cup U$ and $(S^{n+m} - (I^n \times \partial I^m)) = (I^n \times \mathring{I}^m) \cup U$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ H_n((I^n \times I^m) - X) \oplus H_n(U) & \longrightarrow & 0 = H_n(I^n \times \mathring{I}^m) \oplus H_n(U) \\ \downarrow & & \downarrow \\ H_n(S^{n+m} - X) & \longrightarrow & H_n(S^{n+m} - (I^n \times \partial I^m)) \\ \downarrow \partial & & \downarrow \cong \\ H_{n-1}(\partial I^n \times \mathring{I}^m) & \xrightarrow{\cong} & H_{n-1}(\partial I^n \times \mathring{I}^m) \\ \downarrow & & \downarrow \\ H_{n-1}((I^n \times I^m) - X) \oplus H_{n-1}(U) & \longrightarrow & 0 = H_{n-1}(I^n \times \mathring{I}^m) \oplus H_{n-1}(U) \\ \downarrow & & \downarrow \end{array}$$

Remember that $H_n(S^{n+m} - X) \to H_n(S^{n+m} - (I^n \times \partial I^m))$ is non-trivial (trivial), hence we have that ∂: $H_n(S^{n+m} - X) \to H_{n-1}(\partial I^n \times \mathring{I}^m)$ is non-trivial (trivial) and consequently, by exactness, since $\tilde{H}_*(U) = 0$ and $H_{n-1}(\partial I^n \times \mathring{I}^m) = \mathbb{Z}_2$, we have that $H_{n-1}(\partial I^n \times \mathring{I}^m) \to H_{n-1}((I^n \times I^m) - X)$ is trivial (non-trivial). This concludes the proof of the Lemma. □

In order to state Proposition 3 we need to simplify notation.

There is a homeomorphism $B^{n+1}/S^{n-1}_{1/2} \to M_{f^n}$ which sends ∂B^{n+1} homeomorphically onto $S^n \times \{0\}$ and also $B^n_{1/2}/S^{n-1}_{1/2}$ homeomorphically onto S^n, where $S^{n-1}_{1/2}$ is the boundary of $B^n_{1/2}$.

Let ρ: $B^{n+1} \to M_{f^n}$ be the corresponding quotient map and let us identify M_{f^n} with $M_{f_0^n}$, so that $S_0 = \rho(\partial B^{n+1}) \times I^{n-1}$ and $S_1 = \rho(B^n_{1/2}) \times I^{n-1}$. Let $T = \rho(S^{n-1}_{1/2}) \times I^{n-1}$, thus T is homeomorphic to I^{n-1}.

Let $Y \subset M_{f^n} \times I^{n-1}$ and consider the following diagrams:

$$H_n(Y \cap S_0) \begin{array}{l} \xrightarrow{s_0} H_n(S_0) \\ \xrightarrow{j_0} H_n(Y) \end{array}$$

$$H_n(Y \cap (S_0 \cup S_1)) \begin{array}{l} \xrightarrow{s} H_n(S_0 \cup S_1) \\ \xrightarrow{j} H_n(Y) \end{array}$$

Proposition 3. *Let Y be open and assume there is $\gamma \in H_n(Y \cap S_0)$ such that $s_0(\gamma) \neq 0$ but $j_0(\gamma) \in j(\mathrm{kern}\, s)$. Then $H_n(Y) \to H_n(M_{f^n} \times I^{n-1})$ is non-trivial.*

Proof. We first observe that it is enough to prove the proposition in the case Y is a compact polyhedron instead of open, by restricting to the carrier of the homology between the appropriate cycles.

The main idea is to lift the situation to a ball and use Alexander-Duality.

Let $X = (\rho \times 1)^{-1}(Y) \cup (S^{n-1}_{1/2} \times I^{n-1})$. We may assume that

$$X \cap (B^{n+1} \times \partial I^{n-1}) = (S^{n-1}_{1/2} \times \partial I^{n-1}).$$

Claim. $H_{n-1}(S^{n-1}_{1/2} \times I^{n-1}) \to H_{n-1}(X)$ is trivial.

Suppose it is not, then $H_{n-1}(S^{n-1}_{1/2} \times I^{n-1}) \to H_{n-1}(X')$ is non-trivial, where $X' = X \cap (B^n_{1/2} \times I^{n-1})$. By the Alexander-Duality Lemma in $B^n_{1/2} \times I^{n-1}$,

$$H_{n-2}(\{0\} \times \partial I^{n-1}) \to H_{n-2}((B^n_{1/2} \times I^{n-1}) - X') \quad \text{is trivial.}$$

Then, there exists a compact subpolyhedron K of

$$(B^n_{1/2} \times I^{n-1}) - X' \subset (B^{n+1} \times I^{n-1}) - X$$

such that $K \cap \partial(B^n_{1/2} \times I^{n-1}) = \{0\} \times \partial I^{n-1}$ and such that $H_{n-2}(\{0\} \times \partial I^{n-1}) \to H_{n-2}(K)$ is trivial. Again, by the Alexander-Duality Lemma in $B^{n+1} \times I^{n-1}$,

$$H_n(\partial B^{n+1} \times I^{n-1}) \to H_n((B^{n+1} \times I^{n-1}) - K)$$

is non-trivial and hence it is a monomorphism.

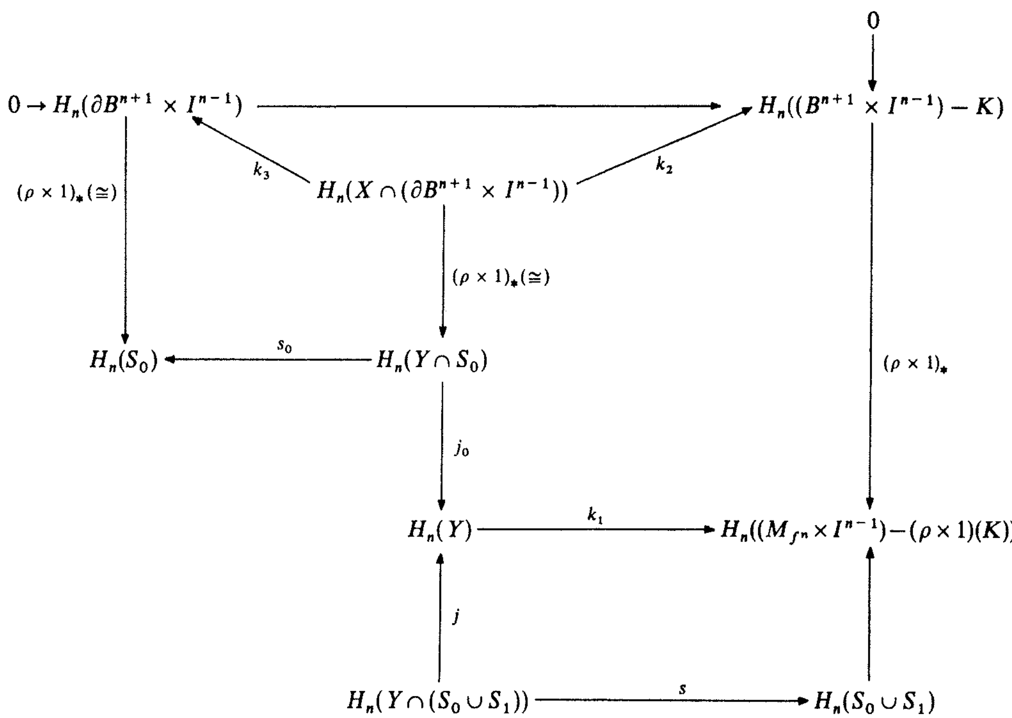
Observe that also

$$H_n((B^{n+1} \times I^{n-1}) - K) \xrightarrow{(\rho \times 1)_*} H_n((M_{f^n} \times I^{n-1}) - (\rho \times 1)(K))$$

is a monomorphism because the following diagram commutes

$$\begin{array}{ccc} 0 \to H_n((B^{n+1} \times I^{n-1}) - K) & \to & H_n((B^{n+1} \times I^{n-1}) - K, S^{n-1}_{1/2} \times I^{n-1}) \\ \downarrow {\scriptstyle (\rho \times 1)_*} & & \downarrow {\scriptstyle (\rho \times 1)_* (\cong)} \\ H_n((M_{f^n} \times I^{n-1}) - (\rho \times 1)(K)) & \xrightarrow{\cong} & H_n((M_{f^n} \times I^{n-1}) - (\rho \times 1)(K), T) \end{array}$$

Let us now consider the following commutative diagram, where the top and upper right maps are monomorphisms.



By hypothesis there is $\gamma \in H_n(Y \cap S_0)$ such that $s_0(\gamma) \neq 0$ but $j_0(\gamma) = j(\alpha)$, where $\alpha \in H_n(Y \cap (S_0 \cup S_1))$ and $s(\alpha) = 0$. Consequently, $k_1 j_0(\gamma) = 0$. Let $\beta \in H_n(X \cap (\partial B^{n+1} \times I^{n-1}))$ be such that $(\rho \times 1)_*(\beta) = \gamma$, hence $k_2(\beta) = 0$ because $(\rho \times 1)_* k_2 = k_1 j_0 (\rho \times 1)_*$. Then $k_3(\beta) = 0$ which is a contradiction because $s_0(\gamma) \neq 0$. This proves the claim.

Hence, $H_n(X, S^{n-1}_{1/2} \times I^{n-1}) \xrightarrow{\partial} H_{n-1}(S^{n-1}_{1/2} \times I^{n-1})$ is an epimorphism and therefore non-trivial. The following commutative diagram completes the proof of Proposition 3.

$$
\begin{array}{ccc}
 & \overset{\partial}{\nearrow} & H_{n-1}(S^{n-1}_{1/2} \times I^{n-1}) \\
 & & \uparrow \partial(\cong) \\
H_n(X, S^{n-1}_{1/2} \times I^{n-1}) & \longrightarrow & H_n(B^{n+1} \times I^{n-1}, S^{n-1}_{1/2} \times I^{n-1}) \\
\downarrow (\rho \times 1)_*(\cong) & & \downarrow (\rho \times 1)_*(\cong) \\
H_n(Y \cup T, T) & \longrightarrow & H_n(M_{f^n} \times I^{n-1}, T) \\
\uparrow \cong & & \uparrow \cong \\
H_n(Y) & \longrightarrow & H_n(M_{f^n} \times I^{n-1})
\end{array}
$$

□

Define, for every $i \in \mathbb{Z}$

$$R_i^- = \left(\bigcup_{j=-\infty}^{i-1} M_{f_j^n} \right) \times I^{n-1} \subset \tilde{A}^{n+1} \times I^{n-1}, \quad \text{and}$$

$$R_i^+ = \left(\bigcup_{j=i}^{\infty} M_{f_j^n} \right) \times I^{n-1} \subset \tilde{A}^{n+1} \times I^{n-1}.$$

It is not difficult to see, since f^n is null homotopic, that R_i^+ is contractible and the inclusions $S_{i+1} \subsetneq M_{f_i^n} \times I^{n-1} \subsetneq R_{i+1}^-$ are homotopy equivalences.

Proof of Proposition 2. Let U be an open subset of $\tilde{A}^{n+1} \times I^{n-1}$ with $H_n(U) = 0$. Suppose that s_0: $H_n(U \cap S_0) \to H_n(S_0)$ is non-trivial. We will prove that $s_1: H_n(U \cap S_1) \to H_n(S_1)$ is non-trivial.

Let $Y = U \cap (M_{f^n} \times I^{n-1})$. Hence $Y \cap S_i = U \cap S_i$, $i = 0, 1$. By hypothesis, let $\gamma \in H_n(Y \cap S_0)$ be such that $s_0(\gamma) \neq 0$. Let s, j, s_0 and j_0 as in the proof of Proposition 3. As you would expect, the idea is now to use Proposition 3.

The Mayer–Vietoris exact sequence of the decomposition

$$U = Y \cup ((U \cap R_0^-) \cup (U \cap R_1^+))$$

yields

$$H_n(U \cap S_0) \oplus H_n(U \cap S_1) \to H_n(Y) \oplus H_n(U \cap R_0^-) \oplus H_n(U \cap R_1^+) \to H_n(U) = 0$$

Thus there are $a \in H_n(U \cap S_0)$, $b \in H_n(U \cap S_1)$ such that $j(a, b) = j_0(\gamma)$ and $i_0(a) = 0$, where i_0: $H_n(U \cap S_0) \to H_n(U \cap R_0^-)$.

Therefore $s_0(a) = 0$, because the following diagram commutes

$$\begin{array}{ccc} H_n(U \cap S_0) & \xrightarrow{s_0} & H_n(S_0) \\ \downarrow i_0 & & \downarrow \cong \\ H_n(U \cap R_0^-) & \to & H_n(R_0^-) \end{array}$$

If $s_1(b) \neq 0$, there is nothing to prove. Then suppose $s_1(b) = 0$. Hence $s(a, b) = (s_0(a), s_1(b)) = 0$, and therefore $j_0(\gamma) \in j$ (kern s).

Consequently, by Proposition 3,

$$k: H_n(U \cap (M_{f^n} \times I^{n-1})) \to H_n(M_{f^n} \times I^{n-1}) \quad \text{is non-trivial.}$$

By Mayer–Vietoris, we have that

$$H_n(U \cap S_1) \to H_n(U \cap R_1^-) \oplus H_n(U \cap R_1^+) \to H_n(U) = 0$$

which implies that $i_1: H_n(U \cap S_1) \to H_n(U \cap R_1^-)$ is an epimorphism.

Finally, consider the following commutative diagram

$$\begin{array}{ccccc} 0 & & H_n(U \cap (M_{f^n} \times I^{n-1})) & \xrightarrow{k} & H_n(M_{f^n} \times I^{n-1}) \\ \uparrow & \swarrow & & & \downarrow \cong \\ H_n(U \cap R_1^-) & & \xrightarrow{k^-} & & H_n(R_1^-) \\ \uparrow i_1 & & & & \uparrow \cong \\ H_n(U \cap S_1) & & \xrightarrow{s_1} & & H_n(S_1) \end{array}$$

Since k is non-trivial so is k^-, and hence so is s_1: $H_n(U \cap S_1) \to H_n(S_1)$. □

4. CONTRACTIBLE, NON q-COLLAPSIBLE POLYHEDRA

As we know, we can describe the $(n + 1)$-*scorpion* A^{n+1} as:

$$A^{n+1} = B^{n+1}/\sim$$

where $\sim$ is the identification; $x \sim \partial(2x)$ for every $x \in B^n_{1/2}$.

Let C' be a straight line segment from p to $\partial(2p)$, for some $p \in S^{n-1}_{1/2}$, and let C be the circle $C'/\sim$. Then, the inclusion $C \subsetneq A^{n+1}$ is a homotopy equivalence.

Define the $(n+1)$-*dimensional dunce hat* D_{n+1} as

$$D_{n+1} = A^{n+1}/C$$

Clearly D_n is contractible and D_2 is the classical dunce hat. Let η: $A^{n+1} \to D_{n+1}$ be the quotient map and let $v = \eta(C)$.

THEOREM 3. *D_{n+1} is not PL q-collapsible for $n > 3q + 4$.*

Proof. Suppose $D_{n+1} \times I^q$ PL-collapses. Start to simplicially collapse $D_{n+1} \times I^q$ by collapses of decreasing dimension, but stop as soon as all $(q+2)$-simplices and some $(q+1)$-simplices are gone. Then $D_{n+1} \times I^q$ PL-collapses to T^{q+1}, where T^{q+1} is a $(q+1)$-dimensional polyhedron which contains $\{v\} \times I^q$. Then $Z^{q+1} = (\eta \times 1)^{-1}(T^{q+1})$ is a $(q+1)$-dimensional subpolyhedron of $A^{n+1} \times I^q$ and $(\eta \times 1)$ sends homeomorphically $(A^{n+1} \times I^q) - Z^{q+1}$ onto $(D_{n+1} \times I^q) - T^{q+1}$. Therefore, by Lemma 2 of [9], $A^{n+1} \times I^q$ PL-collapses to Z^{q+1}. By Corollary 5 of [5], $Z^{q+1} \times I^{2q+4}$ PL-collapses to S^1. Consequently $A^{n+1} \times I^{3q+4}$ PL-collapses to S^1 and therefore, by Theorem 1, it follows that $n \leq 3q + 4$. This concludes the proof of Theorem 3. □

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Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, Circuito Exterior
México D.F., 04510
México