TRANSVERSALS TO THE CONVEX HULL OF ALL k-SET OF DISCRETE SUBSETS OF $\mathbb{R}^n$

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Abstract. What is the maximum positive integer n such that every set of n points in $\mathbb{R}^d$ has the property that the convex hull of all k-set have a transversal $(d-\lambda)$-plane? In this paper, we investigate this and closely related questions. We establish a connection with a special *Kneser hypergraph* by using some topological results and the well-known λ-*Helly property*.

1. Introduction

Let A be a set of eight points in general position in $\mathbb{R}^3$. We claim that there is not a transversal line to the convex hull of the 4-sets of A. Otherwise, if we let L be such a transversal line and $x_0 \in A$ a point not lying on L, then the plane H through x_0 and L would contain at most three points of A and so, there would be at least five points of A not in H. Therefore, by the pigeon-hole principle, three of these points would lie on the same side of H. Consequently, the line L would not intersect the convex hull of these three points and x_0.

On the other hand, if A is a set of six points in $\mathbb{R}^3$ then there is always a transversal line to the convex hull of the 4-sets of A. For this, if $x_0 \in A$ then every 4-set either contains x_0 or is contained in $A - x_0$. Moreover, the family of 4-sets of $A - x_0$ satisfies the 4-Helly property (recall that a family F of convex sets in $\mathbb{R}^d$ has the λ-*Helly property* if every subfamily F' of F, with size $\lambda + 1$, is intersecting) and consequently there is a point y_0 in the intersection of the convex hull these 4-sets. Therefore, the line through x_0 and y_0 is a transversal line to the convex hull of the 4-sets of A.

With seven points in $\mathbb{R}^3$ we may have both options. The suspension of a suitable pentagon with two extra points (one above and one below the pentagon) has a transversal line to the convex hull of the 4-sets, see Figure 1.

The construction of a set of seven points in general position without a transversal line to the convex hull of the 4-sets is more difficult (and tricky). We first notice that if there exists a transversal line L of convex hull of the 4-sets in a set of seven points A in $\mathbb{R}^3$ then L must contains two points of A. Otherwise, if we let L be such a transversal line and $x_0 \in A$ a point not lying on L, then the plane H through

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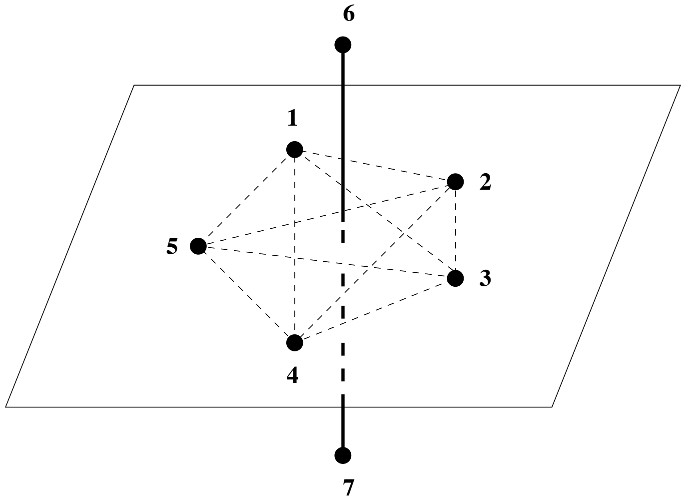

FIGURE 1. $\overrightarrow{67}$ is a transversal line of all tetrahedrons.

x_0 and L would contain at most three points of A (none lying on L). If there were three points in H then there would be four points of A not in H, and so L would not intersect the convex hull of the three points in H and a point out H. If there were two points in H then there would be five points of A not in H and, by the pigeon-hole principle, there would be at least two on the same side of H. The line L would not intersect the convex hull of these two points and the two points in H. Finally, if there was just one point (x_0) in H then there would be six points of A not in H and, by the pigeon-hole principle, there would be at least three points on the same side of H. The line L would not intersect the convex hull of these three points and x_0.

We now consider the points of a tetrahedron and those of a suitable triangle placed under the tetrahedron, see Figure 2. We claim that any line containing two of these points has empty intersection with the convex hull of a 4-set. By the symmetry of the configuration, there are just five cases to be checked, see Figure 3.

We define the following two functions. Let $k, d \geq 1$ and $d \geq \lambda \geq 1$ be integers.

$M(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hull of all k-set have a transversal $(d-\lambda)$-plane.

and

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ the minimum positive integer n such that for every set of n points in general position in $\mathbb{R}^d$ the convex hull of the k-sets does not have a transversal $(d-\lambda)$-plane.

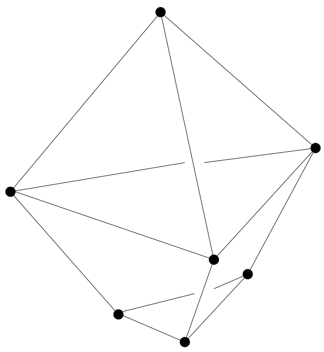

FIGURE 2. Configuration of 7 points without transversal line to the convex hull of the 4-sets.

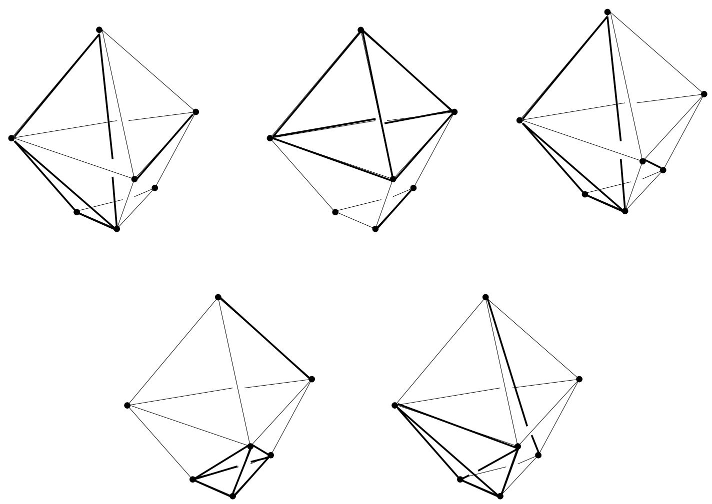

FIGURE 3. Transversals missing a tetrahedron.

The purpose of this paper is to study the above functions. It is clear that $M(k,d,\lambda) < m(k,d,\lambda)$ and, from the above, we have $M(4,3,2) = 6$ and $m(4,3,2) = 8$. In the next section, we prove the following.

Theorem 1. *Let $k, d \geq 1$ and $d \geq \lambda \geq 1$ be integers. Then,*

$$m(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

After discussing some topological results, we will introduce a special *Kneser hypergraph* and establish a close connection between its *chromatic number* and both $M(k,d,\lambda)$ and $m(k,d,\lambda)$, in Section 3. We then give an upper bound for the chromatic number of such hypergraphs (Theorem 4) yielding to a lower bound for $M(k,d,\lambda)$. We will end this section by showing that the following conjecture is true when $d = \lambda$ (Theorem 5) and if either $\lambda = 1$ or $k \leq \lambda$ or $\lambda = k-1$ or $k = 2, 3$ (Theorem 6).

Conjecture 1. $M(k,d,\lambda) = (d-\lambda) + k + \lceil \frac{k}{\lambda} \rceil - 1$.

2. Formula for $m(k,d,\lambda)$

Let $conv(x_1,\ldots,x_n)$ denote the convex hull of the points $x_1,\ldots,x_n$. We may prove Theorem 1.

Proof of Theorem 1. We first prove that

$$m(k,d,\lambda) \leq \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

We might use essentially the same idea as in the introduction. Let A be a collection of $d+2(k-\lambda)+1$ points in general position in $\mathbb{R}^d$ and assume $X^{d-\lambda}$ is a transversal $(d-\lambda)$-plane for the convex hull of the k-sets of A. We have that $X^{d-\lambda}$ contains at most $(d-\lambda+1)$ points of A, and so there are at least $(2k-\lambda) > 0$ points of A not lying on $X^{d-\lambda}$. Let $x_1 \in A$ not belonging to $X^{d-\lambda}$ and let $X^{d-\lambda+1}$ be the $(d-\lambda+1)$-plane generated by $X^{d-\lambda}$ and x_1. Again, we have that $X^{d-\lambda+1}$ contains at most $(d-\lambda+2)$ points of A, and so there are at least $2k-\lambda-1 > 0$ points of A not lying on $X^{d-\lambda+1}$. Let $x_2 \in A$ not belonging to $X^{d-\lambda+1}$ and let $X^{d-\lambda+2}$ be the $(d-\lambda+2)$-plane generated by $X^{d-\lambda+1}$ and x_2. Note that $conv(x_0, x_1) \cap X^{d-\lambda} = \emptyset$.

Case 1) If $k \geq \lambda$ then we can construct inductively $\{x_0,\ldots,x_{\lambda-1}\} \subset A$ and a $(d-1)$-plane X^{d-1} containing $X^{d-\lambda}$ such that $\{x_0,\ldots,x_{\lambda-1}\} \subset X^{d-1}$, but where $conv(x_1,\ldots,x_{\lambda-1})$ does not intersect our original transversal $(d-\lambda)$-plane $X^{d-\lambda}$. Therefore, since X^{d-1} can have at most d points of A then there still are at least $2(k-\lambda)+1$ points of A not lying on X^{d-1}, and so there are at least $(k-\lambda)+1$ points of A in one the open half-spaces determined by X^{d-1}. These $(k-\lambda)+1$ points of A together with $\{x_0,\ldots,x_{\lambda-1}\} \subset A$ give rise to a k-set of A whose convex hull does not intersect $X^{d-\lambda}$.

Case 2) If $k \leq \lambda$ then $k + (d-\lambda) \leq d$ and hence we can construct inductively $\{x_0,\ldots,x_k\} \subset A$ such that $conv(x_1,\ldots,x_k)$ does not intersect our original transversal $(d-\lambda)$-plane $X^{d-\lambda}$.

We may now prove that

$$m(k,d,\lambda) \geq \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

Case 1) If $k-\lambda \geq 0$ then we shall construct a collection of $d+2(k-\lambda) = (d-\lambda+1) + (2k-\lambda-1)$ points in $\mathbb{R}^d = \mathbb{R}^{d-\lambda} \oplus \mathbb{R}^{\lambda}$ with the property that the convex hull of its k-sets have a transversal $(d-\lambda)$-plane.

A classic result of Gale [4] states that there are a set of $2k' + d'$ points in general position in $S^{d'}$, such that every open half-space has at least k points. In particular, this implies that the origin lies in the interior of the convex hull of every $(k'+d'+1)$-set, otherwise there would be an open half-space with less than k points. Therefore, if we put $k' = k - \lambda$ and $d' = \lambda - 1$, we obtain a finite set A of $2(k-\lambda)+(\lambda-1) = 2k-\lambda-1$ points in general position in $\mathbb{R}^\lambda - \{0\}$ with the property that the origin lies in the interior of the convex hull of all k-sets of A. Now, let B be a set of $(d-\lambda+1)$ points in general position in $\mathbb{R}^{d-\lambda}$. So, by suitably moving the points of A we may obtain a set of points A' such that $A' \cup B$ is a set of $(d-\lambda+1)+(2k-\lambda)-1$ points in general position in $\mathbb{R}^d = \mathbb{R}^{d-\lambda} \oplus \mathbb{R}^\lambda$. Furthermore, A' has the property that $\mathbb{R}^{d-\lambda} \oplus \{0\}$ is a transversal $(d-\lambda)$-plane for the convex hull of all its k-sets, and hence $\mathbb{R}^{d-\lambda} \oplus \{0\}$ is a transversal $(d-\lambda)$-plane for the convex hull of all k-sets of $A \cup B$.

Case 2) If $k \leq \lambda$ then $k + (d-\lambda) \leq d$. Hence, a collection $A = \{a_1, \ldots, a_{k+d-\lambda}\}$ of $k + (d-\lambda)$ points in general position in $\mathbb{R}^d$ is a simplex, and so the $(d-\lambda)$-plane generated by $\{\sum_{i=1}^{k} \frac{1}{k} a_i, a_{k+1}, \ldots, a_{k+d-\lambda}\}$ is transversal to all k-set of A. □

3. Topological's results and Kneser hypergraphs

Let $G(d,\lambda)$ be the Grassmanian space of all λ-planes in $\mathbb{R}^d$ and let $G_0(d,\lambda)$ be the Grassmanian space of all λ-planes through the origin in $\mathbb{R}^d$. A *system* Ω of λ-*planes* in $\mathbb{R}^d$ is a continuous selection of a unique λ-plane in every direction of $\mathbb{R}^d$. More precisely, it is a continuos function $\Omega : G_0(d,\lambda) \to G(d,\lambda)$ with the property that $\Omega(H)$ is parallel to H, for every $H \in G_0(d,\lambda)$.

If $\gamma^{d,\lambda} : E^{d,\lambda} \to G_0(d,\lambda)$ is the standard vector bundle of all λ-planes through the origin in $\mathbb{R}^d$, then a system of λ-planes is just a section $s : G_0(d, d-\lambda) \to E^{d,d-\lambda}$, for the vector bundle $\gamma^{d,d-\lambda}$. That is: $G(H) = H + s(H^\perp)$.

For example, the diametral lines of a strictly convex body $K \subset \mathbb{R}^d$ is a system of 1-planes or a system of lines in $\mathbb{R}^d$, although the standard system of lines in $\mathbb{R}^d$ is the collection of lines through a fixed point p_0 in $\mathbb{R}^d$. It is not difficult to verify that two systems of lines in $\mathbb{R}^d$ agree in some direction. In particular this is the reason why there is a diametral line of K through any point p_0 of $\mathbb{R}^d$. In the plane, the lines that divide the area or the perimeter of K in half are system of lines, therefore there is always a line that divide the area and the perimeter of K in half and through every point there is a line that divide the perimeter of K in half. In 3-space the planes that divide the volume or the surface of K in half are system of 2-planes or system of planes. This time it is a little more difficult to verify that three systems of planes (independently of the dimension of $\mathbb{R}^d$) agree in some direction. So, for example, through every point of $\mathbb{R}^3$ there is a plane that divide de volume and the surface of K in half or through every line of $\mathbb{R}^3$ there is a plane that divide de volume K in half. In general, we have the following.

Theorem 2. [2, 3] *Given* $\lambda+1$ *systems of* λ*-planes in* $\mathbb{R}^d$*;* $\Omega_0, \dots, \Omega_\lambda : G_0(d,\lambda) \to G(d,\lambda)$*, they all agree in at least on direction. In other words, there is* $H \in G_0(d,\lambda)$ *such that* $\Omega_0(H) = \cdots = \Omega_\lambda(H)$*.*

We say that a system Ω of λ-planes is *transversal* to a given family F of convex sets in $\mathbb{R}^d$ if every λ-plane of Ω is a transversal λ-plane for the family F. Notice that if $\lambda \le d$ and the family F has λ-Helly property, then F has a transversal system Ω_F of $(d-\lambda)$-planes. Indeed, for a given $(d-\lambda)$-plane $H \in G_0(d, d-\lambda)$, we may project orthogonally the family F into the λ-plane $H^\perp$. By Helly's Theorem, there is a $(d-\lambda)$-plane $\Omega_F(H)$, parallel to H, and transversal to F. Furthermore, it is easy to see that we can chose continuously $\Omega_F(H)$.

Given a family F of convex sets in $\mathbb{R}^d$, we say that a coloration of F is λ-admissible if the every subfamily of F, consisting of all convex sets of F with the same color, has the λ-Helly property. We denote by $\chi^\lambda(F)$ the minimum positive integer r such that there is a λ-admissible coloration of the convex sets of F with r colors.

Proposition 1. *Let* F *be a family of convex set in* $\mathbb{R}^d$ *and suppose that* F *has a* λ*-admissible coloration with* $d-\lambda+1$ *colors,* $\lambda \le d$*. Then,* F *admits a transversal* $(d-\lambda)$*-plane. In other words, if* $\chi^\lambda(F) \le d-\lambda+1$ *then there is a transversal* $(d-\lambda)$*-plane to all convex sets of* F*.*

Proof. For every color $i \in \{1, \dots, d-\lambda\}$, there is a system Ω_i of $(d-\lambda)$-planes for the subfamily of convex sets of color i. By Theorem 2, there is a $(d-\lambda)$-plane transversal to subfamily of convex sets of every color. □

3.1. **Kneser Hypergraphs.** Let $n \ge k \ge 1$ be integers. We denote by $[n]$ the set $\{1, \dots, n\}$ and let $\binom{[n]}{k}$ denote the collection of k-subsets of $[n]$. The well known *Kneser graph* has vertex set $\binom{[n]}{k}$ and two k-subsets are connected by an edge if they are disjoint. We shall consider a generalization of this graph in terms of hypergraphs. A *hypergraph* is a family is a set family $\mathcal{S} \subseteq 2^N$ where the set N is its *ground set.* Let $\lambda \ge 1$ be an integer. We define the *Kneser hypergraph* $KG^{\lambda+1}(n,k)$ as the hypergraph whose vertices are $\binom{[n]}{k}$ and a collection of vertices $\{S_1, \dots, S_\rho\}$ is a hyperedge of $KG^{\lambda+1}(n,k)$ if and only if $2 \le \rho \le \lambda+1$ and $S_1 \cap \cdots \cap S_\rho = \phi$. We remark that $KG^{\lambda+1}(n,k)$ is the Kneser graph when $\lambda = 1$. Let $s \ge 1$ be an integer. In [1] is defined a Kneser hypergraph $KG(n,k,r,s)$ in which the vertices are all the k-subsets of $[n]$ and a collection of r vertices forms a hyperedge if each pair of the corresponding k-sets have intersection of cardinality smaller than an integer $s \ge 1$. Notice that our Kneser hypergraph is different from that defined in [1]. For, we notice that $KG(n,k,r,s)$ is a r*-uniform hypergraph*, that is, each hyperedge contains exactly r elements which is not always the case for $KG^{\lambda+1}(n,k)$.

A *coloring* of a hypergraph $\mathcal{S} \subseteq 2^N$ with m colors is a function $c : N \to [m]$ that assigns colors to the ground set so that each hyperedge $S \in \mathcal{S}$ is *heterochromatic*, that is, all elements in S have different colors[1]. The *chromatic number* $\chi(\mathcal{S})$ of a hypergraph is the smallest number m such that a coloring of $\mathcal{S}$ with m colors exists. The so-called Kneser's conjecture [5], first proved by Lovász [6] states.

Theorem 3. [6] *Let $n \geq 2k \geq 4$. Then, $\chi(KG^2(n,k)) = n - 2k + 2$.*

A subset $A \subset N$ is *independent* if $|S \cap A| \leq 1$ for all hyperedge $S \in \mathcal{S}$. We notice that in a m-coloring of a hypergraph the vertices of each color class is independent. Thus, a collection of vertices $\{S_1, \dots, S_\xi\}$ of $KG^{\lambda+1}(n,k)$ is independent if and only if either $\xi \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\xi \neq \phi$ or $\xi > \lambda + 1$ and any $(\lambda+1)$-subfamily $\{S_{i_1}, \dots, S_{i_{\lambda+1}}\}$ of $\{S_1, \dots, S_\xi\}$ is such that $S_{1_1} \cap \cdots \cap S_{i_{\lambda+1}} \neq \phi$ (satisfies the λ-Helly property). Therefore, if A is any finite set with n points in $\mathbb{R}^d$ and F is the family of convex hull of k-sets of A then $\chi(KG^{\lambda+1}(n,k)) = \chi^\lambda(F)$.

Proposition 2. *If $\chi(KG^{\lambda+1}(n,k)) \leq d - \lambda + 1$ then $n \leq M(k,d,\lambda)$.*

Proof. If $\chi(KG^{\lambda+1}(n,k)) \leq d - \lambda + 1$ then, by Proposition 1, there is a transversal $(d-\lambda)$-plane to the convex hull of all k-set of A where A is any subset of n points in $\mathbb{R}^d$ and therefore $n \leq M(k,d,\lambda)$. □

Theorem 4. *Let $n \geq k + \lceil \frac{k}{\lambda} \rceil$ and $\lambda \geq 1$. Then, $\chi(KG^\lambda(n,k)) \leq n - k - \lceil \frac{k}{\lambda} \rceil + 2$.*

Proof. Let $\alpha \geq 1$ be an integer. We first claim that if $A_1 \cup \cdots \cup A_\alpha \subset X$ where $\mid X \mid = m$ and $\mid A_j \mid = k$ then $\mid \cap_{j=1}^{\alpha} A_j \mid \geq \alpha k - (\alpha - 1)m$. We prove it by induction on α. It is clearly true for $\alpha = 1$. We suppose that it is true for $\alpha - 1$ and prove it for α. Consider the subsets A_α and $A' = \cap_{j=1}^{\alpha-1} A_j$ of X. Note that $\mid A_\alpha \mid = k$ and $\mid A' \mid \geq (\alpha-1)k - (\alpha-2)m$. So, $\mid \cap_{j=1}^{\alpha} A_j \mid = \mid A' \cap A_\alpha \mid \geq (\alpha-1)k - (\alpha-2)m + k - m = \alpha k - (\alpha-1)m$.

Thus, by setting $\alpha = \lambda + 1$, we have that the family of k-sets of a set X with cardinality m has the λ-Helly property if and only if $(\lambda+1)k - \lambda m > 0$ or equivalently if and only if $k + \frac{k}{\lambda} > m$. Therefore, by taking $m = k + \lceil \frac{k}{\lambda} \rceil - 1$, we have that the family of k-sets of $B = \{1, \dots, k + \lceil \frac{k}{\lambda} \rceil - 1\}$ has the λ-Helly property. Let $C_j = \{S \in \binom{[n]}{k} \mid k + \lceil \frac{k}{\lambda} \rceil + j \in S\}$ for each $j = 0, \dots, n - (k + \lceil \frac{k}{\lambda} \rceil)$. Notice that each C_j has also the λ-Helly property. So, the family of k-sets (corresponding to vertices of $KG^{\lambda+1}(n,k)$) of B and the families of k-sets (also corresponding to vertices of $KG^{\lambda+1}(n,k)$) of each C_i with $j = 0, \dots, n - (k + \lceil \frac{k}{\lambda} \rceil)$ are independent. These sets of independent vertices give rise to an admissible coloration for $G^{\lambda+1}(n,k)$ with $n - k - \lceil \frac{k}{\lambda} \rceil + 2$ colors. □

We have the following corollaries.

Corollary 1. *$d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1 \leq M(k,d,\lambda)$.*

Proof. By combining Theorem 4 and Proposition 2. □

[1]This coloring definition is different from the classic one in which is required that no hyperedge is *monochromatic*, that is, every hyperedge $S \in \mathcal{S}$ contains two elements i, j with $c(i) \neq c(j)$.

Corollary 2.

$$\chi(KG^{\lambda+1}(n,k)) > \begin{cases} n-2k+\lambda & \text{if } k \geq \lambda, \\ n-2k & \text{if } k \leq \lambda. \end{cases}$$

Proof. By Proposition 2, we have that if $M(k,d,\lambda) < n$ then $d-\lambda+1 < \chi(KG^{\lambda+1}(n,k))$. The result follows by setting $n = m(k,d,\lambda)$ and by using Theorem 1. □

As an immediate consequence of Corollary 2 and Theorem 4 (with $\lambda = 1$) we obtain Theorem 3.

3.2. **Results on $M(k,d,\lambda)$.** Let us first notice that Conjecture 1 is equivalent to the following one (by setting $d = \alpha + \lambda$).

Conjecture 2. *There is a set A with $\alpha + k + \lceil \frac{k}{\lambda} \rceil$ points in $\mathbb{R}^{\alpha+\lambda}$ such that the convex hull of the k-sets does not admit a transversal α-plane.*

Theorem 5. $M(k,\lambda,\lambda) = k + \lceil \frac{k}{\lambda} \rceil - 1.$

Proof. We shall show that $M(k,\lambda,\lambda) < k + \lceil \frac{k}{\lambda} \rceil$. The result follows since, by Corollary 1 (with $\lambda = d$), we have that $k + \lceil \frac{k}{\lambda} \rceil - 1 \leq M(k,\lambda,\lambda)$. So, by Conjecture 2, it is enough to prove that there is a set A with $k + \lceil \frac{k}{\lambda} \rceil$ points in $\mathbb{R}^{\lambda}$ such that the family of convex hull of the k-sets of A does not have a common point in the intersection. We have two cases.

Case 1) If $k > \lambda$ then $k = p\lambda + j - 1$ for some integers $p \geq 1$ and $2 \leq j \leq \lambda$, and so

$$k + \left\lceil \frac{k}{\lambda} \right\rceil = p\lambda + j - 1 + \left\lceil \frac{p\lambda + j - 1}{\lambda} \right\rceil = p(\lambda+1) + j - 1 + \left\lceil \frac{j-1}{\lambda} \right\rceil = p(\lambda+1) + j.$$

We shall then prove that there is an embedding of $p(\lambda+1)+j$ points with the property that the convex hull of the $(p\lambda+j-1)$-sets has not a common point. To this end, we take in $\mathbb{R}^{\lambda}$ a simplex with $\lambda+1$ vertices. We split the vertices of the simplex into j red vertices and $\lambda+1-j$ blue points. In every red point we put $p+1$ points and in every blue point we put p points. So, in each facet we have at least $p(\lambda+1-j) + (p+1)(\lambda-(\lambda+1-j)) = p(\lambda+1)+j = k$ points. Therefore, for each facet we can form a k-set, and clearly the intersection of the convex hull of each of such k-sets has not a common point.

Case 2) If $k \leq \lambda$ then $k + \lceil \frac{k}{\lambda} \rceil = k+1$. In this case, we consider a simplex with $k+1$ vertices embedded in $\mathbb{R}^{\lambda}$. It is clear that the family of k-faces of the simplex has empty intersection. □

Theorem 6. *Conjecture 1 is true if either (a) $\lambda = 1$ or (b) $k \leq \lambda$ or (c) $\lambda = k-1$ or (d) $k = 2, 3$.*

Proof. Part (a) follows by Theorem 3. For parts (b) and (c), we remark that, by Theorem 1 and Corollary 1,

$$d-\lambda+k+\left\lceil\frac{k}{\lambda}\right\rceil-1 \leq M(k,d,\lambda) < m(k,d,\lambda) = \begin{cases} d+2(k-\lambda)+1 & \text{if } k \geq \lambda, \\ k+d-\lambda+1 & \text{if } k \leq \lambda. \end{cases} \quad (1)$$

So, if $\lambda \geq k$ then $d-\lambda+k \leq M(k,d,\lambda) < k+d-\lambda+1$, and therefore, $M(k,d,\lambda) = k+d-\lambda$, giving Conjecture 1. If $\lambda = k-1$ then $d+2 \leq M(k,d,\lambda) < d+3$, and therefore, $M(k,d,\lambda) = d+2$, giving also the Conjecture 1. We may then suppose that $\lambda < k$. Finally for part (d), if $k=2$ then $\lambda = 1$ and it follows by part (a), and if $k=3$ then either $\lambda = 1$ or 2, and it follows by parts (a) and (c). □

We notice that, by using (1), Conjecture 1 is also true if $k=4$ when $\lambda = 1$ or 3 but it does not yield to the validatity of the conjecture if $\lambda = 2$. This case is more complicated and we leave it for a future work. In fact, we investigate (work in process) a general improved upper bound for $M(k,d,\lambda)$ giving the conjectured value for $k=4$ and 5.

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