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Transversals to the convex hulls of all k-sets of discrete subsets of $\mathbb{R}^n$

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ABSTRACT

Let $k, d, \lambda \geqslant 1$ be integers with $d \geqslant \lambda$. What is the maximum positive integer n such that every set of n points in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a transversal $(d-\lambda)$-plane? What is the minimum positive integer n such that every set of n points in general position in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets do not have a transversal $(d-\lambda)$-plane? In this paper, we investigate these two questions. We define a special *Kneser hypergraph* and, by using some topological results and the well-known λ-Helly property, we relate our second question to the chromatic number of such hypergraphs. Moreover, we establish a connection (when $\lambda = 1$) with Kneser's conjecture, first proved by Lovász. Finally, we prove a discrete flat center theorem.

1. Introduction

Let A be a set of eight points in general position in $\mathbb{R}^3$. We claim that there is no transversal line to the convex hulls of all the 4-sets of A. Otherwise, if we let L be such a transversal line and $x_0 \in A$ a point not lying on L, then the plane H through x_0 and L would contain at most three points of A and so there would be at least five points of A not in H. Therefore by the pigeon-hole principle, three of these points would lie on the same side of H. Consequently the line L would not intersect the convex hull of these three points and x_0.

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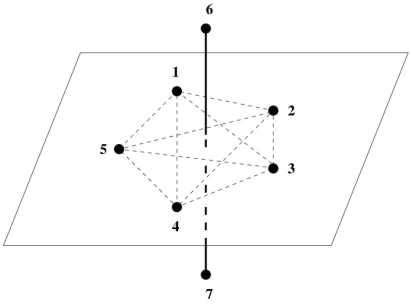

Fig. 1. $\overrightarrow{67}$ is a transversal line of all tetrahedrons.

On the other hand, if A is a set of six points in $\mathbb{R}^3$, then there is always a transversal line to the convex hulls of the 4-sets of A. For this, if $x_0 \in A$, then every 4-set either contains x_0 or is contained in $A - x_0$. Moreover, the family of 4-sets of $A - x_0$ satisfies the 3-Helly property (recall that a family F of convex sets in $\mathbb{R}^d$ has the λ-*Helly property* if every subfamily F' of F with size $\lambda + 1$ is intersecting) and consequently there is a point y_0 in the intersection of the convex hulls of these 4-sets. Therefore the line through x_0 and y_0 is a transversal line to the convex hulls of all the 4-sets of A.

With seven points in $\mathbb{R}^3$ we may have both options. The suspension of a suitable pentagon with two extra points (one above and one below the pentagon) has a transversal line to the convex hulls of the 4-sets, see Fig. 1.

The construction of a set of seven points in general position without a transversal line to the convex hulls of the 4-sets is more difficult. Such construction will be discussed at the end of the paper (see Appendix A).

We define the following two functions: let $k, d, \lambda \geqslant 1$ be integers with $d \geqslant \lambda$.

$m(k,d,\lambda) \overset{\text{def}}{=}$ the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a transversal $(d-\lambda)$-plane,

and

$M(k,d,\lambda) \overset{\text{def}}{=}$ the minimum positive integer n such that for every set of n points in general position in $\mathbb{R}^d$ the convex hulls of the k-sets do not have a transversal $(d-\lambda)$-plane.

The purpose of this paper is to study the above functions. It is clear that $m(k,d,\lambda) < M(k,d,\lambda)$, and from the above we have $m(4,3,2) = 6$ and $M(4,3,2) = 8$. In the next section, we prove the following.

Theorem 1. *Let $k, d, \lambda \geqslant 1$ be integers and $d \geqslant \lambda$. Then*

$$M(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geqslant \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leqslant \lambda. \end{cases}$$

After discussing some topological results in Section 3 and following the spirit of Dol'nikov in [4] and [5], we will introduce a special *Kneser hypergraph* and establish a close connection between its *chromatic number* and both $m(k,d,\lambda)$ and $M(k,d,\lambda)$. We then give an upper bound for the chromatic number of such hypergraphs (Theorem 4) yielding to the lower bound for $m(k,d,\lambda)$ (Corollary 1).

The well-known Rado's central point theorem [13] states that if X is a bounded measurable set in $\mathbb{R}^d$ then there exists a point $x \in \mathbb{R}^d$ such that $measure(P \cap X) \geqslant measure(X/(d+1))$ for each half-space P that contains x (see also [12] for the case $d=2$).

Corollary 1 led us to the following generalization of the discrete version of Rado's theorem.

Theorem 2. *Let X be a finite set of n points in $\mathbb{R}^d$. Then there is a $(d-\lambda)$-plane L such that any closed half-space H through L contains at least $\lfloor \frac{n-d+2\lambda}{\lambda+1} \rfloor + (d-\lambda)$ points.*

In order to show the above theorem, we shall consider the following two functions.

$k(n,d,\lambda) \stackrel{\text{def}}{=}$ the minimum positive integer k such that for any collection X of n points in d-dimensional Euclidean space, there is a $(d-\lambda)$-plane transversal to the convex hulls of all k-sets of X.

$\tau(n,d,\lambda) \stackrel{\text{def}}{=}$ the maximum positive integer τ such that for any collection X of n points in d-dimensional Euclidean space, there is a $(d-\lambda)$-plane L_X such that any closed half-space H through L_X contains at least τ points.

It is clear that $n - \tau(n,d,\lambda) + 1 = k(n,d,\lambda)$. We shall see that Corollary 1 implies that $k(n,d,\lambda) \leqslant \lfloor \frac{\lambda(n-d+\lambda)}{\lambda+1} \rfloor + 1$ and therefore $\tau(n,d,\lambda) \geqslant \lfloor \frac{n-d+2\lambda}{\lambda+1} \rfloor + (d-\lambda)$ from which our generalization follows (see the proof of Theorem 2).

We will use Theorem 2 to give a result (Corollary 3) that can be considered as a discrete version of the following result due to R. Živaljević and S. Vrećica [19, Theorem 1].

Theorem 3. *(See [19].) Let $1 \leqslant \lambda \leqslant d$, and let $\mu_0, \ldots, \mu_{d-\lambda,S,ST}$ be σ-additive probability measures on $\mathbb{R}^d$. Then there is a $(d-\lambda)$-flat L with the property that every closed half-space containing L has μ_i-measure at least $\frac{1}{\lambda+1}$, for all $0 \leqslant i \leqslant d-\lambda$.*

Theorem 3 reduces to Rado's central point theorem in the case $\lambda = d$ and to the *ham sandwich theorem*[2] in the case $\lambda = 1$. As remarked in [16], Rado's central point theorem can also be obtained by using the well-known Tverberg's generalization of Radon's theorem [17]. Tverberg-type results on flat transversal are natural strengthenings of the central (flat) transversal theorem and thus closely related to our work. In particular, Tverberg's flat-type result due to S.A. Bogatyi [2] yields to an alternative proof of Theorem 2 from which Corollary 1 can be achieved; see also [8,18,20].

We shall consider the following.

Conjecture 1. $m(k,d,\lambda) = (d-\lambda) + k + \lceil \frac{k}{\lambda} \rceil - 1$.

We will see that Theorem 2 is sharp if Conjecture 1 is true. We finally show that Conjecture 1 is true when $d = \lambda$ (Theorem 6) and if either $\lambda = 1$ or $k \leqslant \lambda$ or $\lambda = k-1$ or $k = 2, 3$ (Theorem 7).

2. Formula for $M(k,d,\lambda)$

Let $conv(x_1, \ldots, x_n)$ denote the convex hull of the points $x_1, \ldots, x_n$. We prove Theorem 1.

Proof of Theorem 1. We first prove that

$$M(k,d,\lambda) \leqslant \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geqslant \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leqslant \lambda. \end{cases}$$

[2] For every collection of n measurable sets $\mathbb{R}^d$ there exists a hyperplane which bisects all of them, see [11,14,15].

Case (1) If $k \geqslant \lambda$ then we can use essentially the same idea as in the introduction. Let A be a collection of $d + 2(k - \lambda) + 1$ points in general position in $\mathbb{R}^d$ and assume $X^{d-\lambda}$ is a transversal $(d-\lambda)$-plane for the convex hulls of all the k-sets of A. Sine $X^{d-\lambda}$ contains at most $(d-\lambda+1)$ points of A then there are at least $d + 2(k - \lambda) + 1 - (d - \lambda + 1) = (2k - \lambda) > 0$ points of A not lying on $X^{d-\lambda}$. Let $x_1 \in A$, not belonging to $X^{d-\lambda}$, and let $X^{d-\lambda+1}$ be the $(d - \lambda + 1)$-plane generated by $X^{d-\lambda}$ and x_1. Again, we have that $X^{d-\lambda+1}$ contains at most $(d-\lambda+2)$ points of A, and so there are at least $2k - \lambda - 1 > 0$ points of A not lying on $X^{d-\lambda+1}$. Let $x_2 \in A$, not belonging to $X^{d-\lambda+1}$ (and therefore x_2 is neither in $X^{d-\lambda}$), and let $X^{d-\lambda+2}$ be the $(d - \lambda + 2)$-plane generated by $X^{d-\lambda+1}$ and x_2. Note that $conv(x_1, x_2) \cap X^{d-\lambda} = \emptyset$. By carrying on this procedure, we can construct a set $\{x_1, \dots, x_{\lambda-1}\} \subset A$ and a $(d - 1)$-plane X^{d-1} containing $X^{d-\lambda}$ such that $\{x_1, \dots, x_{\lambda-1}\} \subset X^{d-1}$, but where $conv(x_1, \dots, x_{\lambda-1})$ does not intersect our original transversal $(d-\lambda)$-plane $X^{d-\lambda}$. Therefore, since X^{d-1} can have at most d points of A then there still are at least $2(k - \lambda) + 1$ points of A not lying on X^{d-1}, and so there are at least $(k - \lambda) + 1 > 0$ points of A in one of the open half-spaces determined by X^{d-1}. These $(k - \lambda) + 1$ points of A together with $\{x_1, \dots, x_{\lambda-1}\} \subset A$ give rise to a k-set of A whose convex hull does not intersect $X^{d-\lambda}$.

Case (2) If $k \leqslant \lambda$, then $k + (d - \lambda) \leqslant d$. Let A be a collection of $k + d - \lambda + 1$ points in general position in $\mathbb{R}^d$ and assume $X^{d-\lambda}$ is a transversal $(d-\lambda)$-plane for the convex hulls of all the k-sets of A. We have $X^{d-\lambda}$ contains at most $(d - \lambda + 1)$ points of A, and so there are at least $k + d - \lambda + 1 - (d - \lambda + 1) = k$ points of A not lying on $X^{d-\lambda}$. The convex hull of these k points does not intersect $X^{d-\lambda}$.

We can now prove that

$$M(k,d,\lambda) \geqslant \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geqslant \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leqslant \lambda. \end{cases}$$

Case (1) If $k - \lambda \geqslant 0$, then we construct a collection of $d + 2(k - \lambda) = (d - \lambda + 1) + (2k - \lambda - 1)$ points in general position in $\mathbb{R}^d = \mathbb{R}^{d-\lambda} \oplus \mathbb{R}^\lambda$ with the property that the convex hulls of its k-sets have a transversal $(d - \lambda)$-plane.

A classic result of Gale [7] states that there is a set of $2k' + d'$ points in general position in $S^{d'}$ such that every open half-space contains at least k' points. In particular, this implies that the origin lies in the interior of the convex hulls of every $(k' + d' + 1)$-set, otherwise there would be an open half-space with less than k' points. Therefore if we put $k' = k - \lambda$ and $d' = \lambda - 1$, we obtain a finite set A of $2(k - \lambda) + (\lambda - 1) = 2k - \lambda - 1$ points in general position in $\mathbb{R}^\lambda - \{0\}$ with the property that the origin lies in the interior of the convex hulls of all k-sets of A. Now let B be a set of $(d - \lambda + 1)$ points in general position in $\mathbb{R}^{d-\lambda}$. By suitably moving the points of A, we can obtain a set of points A' such that $A' \cup B$ is a set of $(d - \lambda + 1) + (2k - \lambda) - 1$ points in general position in $\mathbb{R}^d = \mathbb{R}^{d-\lambda} \oplus \mathbb{R}^\lambda$. Furthermore, A' has the property that $\mathbb{R}^{d-\lambda} \oplus \{0\}$ is a transversal $(d - \lambda)$-plane for the convex hulls of all its k-sets, and hence $\mathbb{R}^{d-\lambda} \oplus \{0\}$ is a transversal $(d - \lambda)$-plane for the convex hulls of all k-sets of $A \cup B$.

Case (2) If $k \leqslant \lambda$, then $k+(d-\lambda) \leqslant d$. Hence a collection $A = \{a_1, \dots, a_{k+d-\lambda}\}$ of $k+(d-\lambda)$ points in general position in $\mathbb{R}^d$ is a simplex, and so the $(d-\lambda)$-plane generated by $\{\sum_{i=1}^{k} \frac{1}{k} a_i, a_{k+1}, \dots, a_{k+d-\lambda}\}$ is transversal to all k-sets of A. $\square$

3. Topological results and Kneser hypergraphs

Let $G(d,\lambda)$ be the Grassmannian $\lambda(d - \lambda)$-manifold of all λ-planes through the origin in Euclidean space $\mathbb{R}^d$ and let $M(d,\lambda)$ be the set of all λ-planes in $\mathbb{R}^d$. Thus $G(d,\lambda) \subset M(d,\lambda)$. We shall regard $M(d,\lambda)$ as an open subset of $G(d + 1, \lambda + 1)$, making the following identifications:

Let $z_0 \in \mathbb{R}^{d+1} - \mathbb{R}^d$ be a fixed point and, without loss of generality, let $G(d+1, \lambda+1)$ be the space of all $(\lambda + 1)$-planes in R^{d+1} through z_0. Let us identify $H \in M(d,\lambda)$ with the unique $(\lambda + 1)$-plane $H' \in G(d + 1, \lambda + 1)$ which contains H and passes through z_0. Thus

$$G(d,\lambda) \subset M(d,\lambda) \subset G(d + 1, \lambda + 1),$$

where $M(d,\lambda)$ is an open subset of $G(d + 1, \lambda + 1)$ and $G(d,\lambda)$ is a retract of $M(d,\lambda)$.

A *system* Ω of λ*-planes* in $\mathbb{R}^d$ is a continuous selection of a unique λ-plane in every direction of $\mathbb{R}^d$. More precisely, it is a continuous function $\Omega : G(d,\lambda) \to M(d,\lambda)$ with the property that $\Omega(H)$ is parallel to H, for every $H \in G(d,\lambda)$.

If $\gamma^{d,\lambda} : E^{d,\lambda} \to G(d,\lambda)$ is the standard vector bundle of all λ-planes through the origin in $\mathbb{R}^d$, then a system of λ-planes is just a section $s : G(d, d-\lambda) \to E^{d,d-\lambda}$ for the vector bundle $\gamma^{d,d-\lambda}$. That is, $\Omega(H) = H + s(H^{\perp})$.

For example, the affine diameters of a strictly convex body $K \subset \mathbb{R}^d$ are a system of 1-planes or a system of lines in $\mathbb{R}^d$, although the standard system of lines in $\mathbb{R}^d$ is the collection of lines through a fixed point p_0 in $\mathbb{R}^d$. It is not difficult to verify that two systems of lines in $\mathbb{R}^d$ agree in some direction. In particular, this is the reason why there is an affine diameter of K through any point p_0 of $\mathbb{R}^d$. In the plane the lines that divide the area or the perimeter of K in half are a system of lines; therefore there is always a line that divides the area and the perimeter of K in half and through every point there is a line that divides the perimeter of K in half. In 3-space the planes that divide the volume or the surface of K in half are a system of 2-planes or a system of planes. This time it is a little more difficult to verify that three systems of planes (independently of the dimension of $\mathbb{R}^d$) agree in some direction. So, for example, through every point of $\mathbb{R}^3$ there is a plane that divides the volume and the surface of K in half or through every line of $\mathbb{R}^3$ there is a plane that divides the volume K in half (recall the ham sandwich theorem [11]).

For completeness, we review the basics from Grassmannian geometry. Let $\lambda_1,\dots,\lambda_m$ be a sequence of integers such that $0 \leqslant \lambda_1 \leqslant \dots \leqslant \lambda_m \leqslant d-m$. Let us denote $\{\lambda_1,\dots,\lambda_m\} = \{H \in G(d,m) \mid dim(H \cap \mathbb{R}^{\lambda_j + j}) \geqslant j,\ \ j = 1,\dots,m\}$. It is known that $\{\lambda_1,\dots,\lambda_m\}$ is a compact subset of $G(d,m)$ of dimension $\lambda = \lambda_1 + \dots + \lambda_m$, which is a closed connected λ-manifold, except possibly for a closed connected subset of codimension three. Thus, $H^{\lambda}(\{\lambda_1,\dots,\lambda_m\},\mathbb{Z}_2) = \mathbb{Z}_2 = H_{\lambda}(\{\lambda_1,\dots,\lambda_m\},\mathbb{Z}_2)$. Let $(\lambda_1,\dots,\lambda_m) \in H_{\lambda}(G(m,d),\mathbb{Z}_2)$ be the λ-cycle which is induced by the inclusion $\{\lambda_1,\dots,\lambda_m\} \subset G(m,d)$. These cycles are called *Schubert-cycles*. A canonical basis for $H_{\lambda}(G(m,d),\mathbb{Z}_2)$ consists of all Schubert-cycles $(\xi_1,\dots,\xi_m)$ such that $0 \leqslant \xi_1 \leqslant \dots \leqslant \xi_n \leqslant d-m$ and $\lambda = \xi_1 + \dots + \xi_n$. Let us denote by $[\lambda_1,\dots,\lambda_m] \in H^{\lambda}(G(d,m),\mathbb{Z}_2)$ the λ-cocycle whose value is one for $(\lambda_1,\dots,\lambda_m)$ and zero for any other Schubert-cycle of dimension λ. Thus, a canonical basis for $H^{\lambda}(G(d,m),\mathbb{Z}_2)$ consists of all *Schubert-cocycles* $[\xi_1,\dots,\xi_m]$ such that $0 \leqslant \xi_1 \leqslant \dots \leqslant \xi_n \leqslant d-m$ and $\lambda = \xi_1 + \dots + \xi_n$. The cohomology classes $[0,\dots,0,1,\dots,1]$, where the last symbol consists of $m-i$ zeros and i ones, $i = 1,\dots,m$, are the classical *Stiefel–Whitney characteristic classes* of the standard vector bundle over $G(m,d)$. The isomorphism $D : H_{\lambda}(G(m,d),\mathbb{Z}_2) \to H^{d-\lambda}(G(d,m),\mathbb{Z}_2)$ given by $D((\lambda_1,\dots,\lambda_m)) = [d-m-\lambda_1,\dots,d-m-\lambda_m]$ is the classical *Poincaré Duality Isomorphism* (we refer the reader to [3] for further details).

So, a *system* Ω of λ*-planes* in $\mathbb{R}^d$ determines the Schubert-cycle $(0,d-\lambda,\dots,d-\lambda) \in H_{\lambda(d-\lambda)}(G(d+1,\lambda+1),\mathbb{Z}_2)$. Its dual under the Poincaré Duality Isomorphism $D : H_{\lambda(d-\lambda)}(G(d+1,\lambda+1),\mathbb{Z}_2) \to H^{d-\lambda}(G(d+1,\lambda+1),\mathbb{Z}_2)$ is the Schubert-cocycle $[0,\dots,0,d-\lambda] \in H^{d-\lambda}(G(d+1,\lambda+1),\mathbb{Z}_2)$. The fact that using the Cap-product in $H^*(G(d+1,\lambda+1),\mathbb{Z}_2)$ we obtain $[0,\dots,0,d-\lambda]^{\lambda+1}$ as the fundamental class $[d-\lambda,\dots,d-\lambda] \in H^{(\lambda+1)(d-\lambda)}(G(d+1,\lambda+1),\mathbb{Z}_2)$ of $G(d+1,\lambda+1)$, implying the following result (which can be considered as a restatement of Dol'nikov's lemma [6, Section 1]).

Lemma 1. *Given $\lambda+1$ systems of λ-planes in $\mathbb{R}^d$; $\Omega_0,\dots,\Omega_\lambda : G(d,\lambda) \to M(d,\lambda)$, they all agree in at least one direction. In other words, there is $H \in G(d,\lambda)$ such that $\Omega_0(H) = \dots = \Omega_\lambda(H)$.*

We say that a system Ω of λ-planes is *transversal* to a given family F of convex sets in $\mathbb{R}^d$ if every λ-plane of Ω is a transversal λ-plane for the family F. Notice that if $\lambda \leqslant d$ and the family F has the λ-Helly property, then F has a transversal system Ω_F of $(d-\lambda)$-planes. Indeed, for a given $(d-\lambda)$-plane $H \in G_0(d,d-\lambda)$, we may project the family F orthogonally onto the λ-plane $H^{\perp}$. By Helly's theorem, there is a $(d-\lambda)$-plane $\Omega_F(H)$ parallel to H and transversal to F. Furthermore, it is easy to see that we can choose $\Omega_F(H)$ continuously. See [19] for proof.

Given a family F of convex sets in $\mathbb{R}^d$, we say that a coloration of F is λ-admissible if every subfamily of F consisting of all convex sets of F with the same color has the λ-Helly property, that is, if every monochromatic subfamily of F of size $\lambda+1$ is intersecting. We denote by $\chi^{\lambda}(F)$ the minimum positive integer r such that there is a λ-admissible coloration of the convex sets of F with r colors.

Proposition 1. *Let F be a family of convex set in $\mathbb{R}^d$ and suppose that F has a λ-admissible coloration with $d-\lambda+1$ colors, $\lambda \leqslant d$. Then F admits a transversal $(d-\lambda)$-plane. In other words, if $\chi^\lambda(F) \leqslant d-\lambda+1$, then there is a transversal $(d-\lambda)$-plane to all convex sets of F.*

Proof. For every color $i \in \{0, 1, \dots, d-\lambda\}$, there is a system Ω_i of $(d-\lambda)$-planes for the subfamily of convex sets of color i. By Lemma 1, there is a $(d-\lambda)$-plane transversal to subfamily of convex sets of every color. □

Proposition 1 was first announced by Dol'nikov in [4] and published with proof in [5].

3.1. Kneser hypergraphs

Let $n \geqslant k \geqslant 1$ be integers. Let $[n]$ denote the set $\{1, \dots, n\}$ and $\binom{[n]}{k}$ the collection of k-subsets of $[n]$. The well-known *Kneser graph* has vertex set $\binom{[n]}{k}$, and two k-subsets are connected by an edge if they are disjointed. We shall consider a generalization of this graph in terms of hypergraphs. A *hypergraph* is a family $\mathcal{S} \subseteq 2^N$ where the set N is its *ground set*. Let $\lambda \geqslant 1$ be an integer. We define the *Kneser hypergraph* $KG^{\lambda+1}(n,k)$ as the hypergraph whose vertices are $\binom{[n]}{k}$ and a collection of vertices $\{S_1, \dots, S_\rho\}$ is a hyperedge of $KG^{\lambda+1}(n,k)$ if and only if $2 \leqslant \rho \leqslant \lambda+1$ and $S_1 \cap \dots \cap S_\rho = \phi$. We remark that $KG^{\lambda+1}(n,k)$ is the Kneser graph when $\lambda = 1$. Let $s \geqslant 1$ be an integer. In [1], a Kneser hypergraph $KG(n,k,r,s)$ is defined in which the vertices are all the k-subsets of $[n]$ and a collection of r vertices forms a hyperedge if each pair of the corresponding k-sets have an intersection of cardinality smaller than an integer $s \geqslant 1$. Notice that $KG^{\lambda+1}(n,k)$ is different from $KG(n,k,\lambda+1,1)$.

A *coloring* of a hypergraph $\mathcal{S} \subseteq 2^N$ with m colors is a function $c : N \to [m]$ that assigns colors to the ground set so that no hyperedge $S \in \mathcal{S}$ is *monochromatic*, that is, at least two elements in S have different colors. The *chromatic number* $\chi(\mathcal{S})$ of a hypergraph is the smallest number m such that a coloring of $\mathcal{S}$ with m colors exists.

We notice that the collection of vertices $\{S_1, \dots, S_\xi\}$ of $KG^{\lambda+1}(n,k)$ is independent if and only if either $\xi \leqslant \lambda+1$ and $S_1 \cap \dots \cap S_\xi \neq \phi$ or $\xi > \lambda+1$ and any $(\lambda+1)$-subfamily $\{S_{i_1}, \dots, S_{i_{\lambda+1}}\}$ of $\{S_1, \dots, S_\xi\}$ is such that $S_{1_1} \cap \dots \cap S_{i_{\lambda+1}} \neq \phi$ (satisfies the λ-Helly property). Therefore if A is any finite set with n points in $\mathbb{R}^d$ and F is the family of convex hulls of k-sets of A, then $\chi(KG^{\lambda+1}(n,k)) \geqslant \chi^\lambda(F)$.

Proposition 2. *If $\chi(KG^{\lambda+1}(n,k)) \leqslant d-\lambda+1$, then $n \leqslant m(k,d,\lambda)$.*

Proof. If $\chi^\lambda(F) \leqslant \chi(KG^{\lambda+1}(n,k)) \leqslant d-\lambda+1$, then by Proposition 1, there is a transversal $(d-\lambda)$-plane to the convex hulls of all k-sets of A where A is any subset of n points in $\mathbb{R}^d$, and therefore $n \leqslant m(k,d,\lambda)$. □

Theorem 4. *Let $n \geqslant k + \lceil \frac{k}{\lambda} \rceil$ and $\lambda \geqslant 1$. Then $\chi(KG^{\lambda+1}(n,k)) \leqslant n-k-\lceil \frac{k}{\lambda} \rceil + 2$.*

Proof. Let $\alpha \geqslant 1$ be an integer. We first claim that if $A_1 \cup \dots \cup A_\alpha \subset X$, where $|X| = m$ and $|A_j| = k$, then $|\bigcap_{j=1}^{\alpha} A_j| \geqslant \alpha k - (\alpha-1)m$. We prove it by induction on α. It is clearly true for $\alpha = 1$. We suppose that it is true for $\alpha - 1$ and prove it for α. Consider the subsets A_α and $A' = \bigcap_{j=1}^{\alpha-1} A_j$ of X. Note that $|A_\alpha| = k$ and $|A'| \geqslant (\alpha-1)k - (\alpha-2)m$. So $|\bigcap_{j=1}^{\alpha} A_j| = |A' \cap A_\alpha| \geqslant (\alpha-1)k - (\alpha-2)m + k - m = \alpha k - (\alpha-1)m$.

Thus, by setting $\alpha = \lambda+1$, we have that the family of k-sets of a set X with cardinality m has the λ-Helly property if and only if $(\lambda+1)k - \lambda m > 0$ or equivalently if and only if $k + \frac{k}{\lambda} > m$. Therefore, by taking $m = k + \lceil \frac{k}{\lambda} \rceil - 1$, we have that the family of k-sets of $B = \{1, \dots, k + \lceil \frac{k}{\lambda} \rceil - 1\}$ has the λ-Helly property. Let $C_j = \{S \in \binom{[n]}{k} \mid k + \lceil \frac{k}{\lambda} \rceil + j \in S\}$ for each $j = 0, \dots, n - (k + \lceil \frac{k}{\lambda} \rceil)$. Notice that each

C_j also has the λ-Helly property. So the family of k-sets (corresponding to vertices of $KG^{\lambda+1}(n,k)$) of B and the families of k-sets (also corresponding to vertices of $KG^{\lambda+1}(n,k)$) of each C_i with $j = 0, \ldots, n-(k+\lceil \frac{k}{\lambda} \rceil)$ are independent. These sets of independent vertices give rise to an admissible coloration for $KG^{\lambda+1}(n,k)$ with $n-k-\lceil \frac{k}{\lambda} \rceil + 2$ colors. □

We have the following corollaries:

Corollary 1. $d-\lambda+k+\lceil \frac{k}{\lambda} \rceil - 1 \leqslant m(k,d,\lambda)$.

Proof. By combining Theorem 4 and Proposition 2. □

Corollary 2.

$$\chi\big(KG^{\lambda+1}(n,k)\big) > \begin{cases} n-2k+\lambda & \text{if } k \geqslant \lambda, \\ n-k & \text{if } k \leqslant \lambda. \end{cases}$$

Proof. By Proposition 2, we have that if $m(k,d,\lambda) < n$, then $d-\lambda+1 < \chi(KG^{\lambda+1}(n,k))$. The result follows by setting $n = M(k,d,\lambda)$ and by using the values of $M(k,d,\lambda)$ given in Theorem 1. □

As an immediate consequence of Corollary 2 and Theorem 4 (with $\lambda = 1$) we obtain the following theorem conjectured by Kneser [9] and first proved by Lovász [10].

Theorem 5. *(See [10].) Let $n \geqslant 2k \geqslant 4$. Then $\chi(KG^2(n,k)) = n-2k+2$.*

We may now prove Theorem 2.

Proof of Theorem 2. It is clear that

$$n - \tau(n,d,\lambda) + 1 = k(n,d,\lambda). \tag{1}$$

We claim that $k(n,d,\lambda) \leqslant \lfloor \frac{\lambda(n-d+\lambda)}{\lambda+1} \rfloor + 1$. Indeed, this follows since, by Corollary 1, we have that if $k = \lfloor \frac{\lambda(n-d+\lambda)}{\lambda+1} \rfloor + 1$, then

$$(d-\lambda) + \left\lceil \frac{(\lambda+1)\lfloor \frac{\lambda(n-d+\lambda)}{\lambda+1} + 1 \rfloor}{\lambda} \right\rceil - 1 \geqslant n,$$

due to the fact that

$$\frac{(\lambda+1)\lfloor \frac{\lambda(n-d+\lambda)}{\lambda+1} + 1 \rfloor}{\lambda} > n-d+\lambda.$$

Now, for any two positive integers x and y, we have that $\lfloor \frac{xy}{x+1} \rfloor = y - \lfloor \frac{x+y}{x+1} \rfloor$. So, by taking $x = \lambda$ and $y = n-d+\lambda$ we have

$$k(n,d,\lambda) \leqslant \left\lfloor \frac{\lambda(n-d+\lambda)}{\lambda+1} \right\rfloor + 1 = n-d-\lambda - \left\lfloor \frac{2\lambda+n-d}{\lambda+1} \right\rfloor + 1. \tag{2}$$

By combining (1) and (2), we obtain

$$\tau(n,d,\lambda) \geqslant \left\lfloor \frac{n-d+2\lambda}{\lambda+1} \right\rfloor + (d-\lambda)$$

and the result follows. □

Notice that if $d-\lambda+\lceil\frac{(\lambda+1)k}{\lambda}\rceil-1=m(k,d,\lambda)$, then $\tau(n,d,\lambda)=\lfloor\frac{n-d+2\lambda}{\lambda+1}\rfloor+(d-\lambda)$. The bound $\lfloor\frac{n-d+2\lambda}{\lambda+1}\rfloor+(d-\lambda)=\lfloor\frac{n+\lambda(d-\lambda+1)}{\lambda+1}\rfloor$ yields to the following result that can be considered as a discrete version of Theorem 3.

Corollary 3. *For every $i=1,\ldots,d-\lambda+1$ let $A_i\subset\mathbb{R}^d$. Then, there is a $(d-\lambda)$-plane L such that any closed half-space H through L contains at least $\lfloor\frac{|A_i|+\lambda}{\lambda+1}\rfloor$ points of A_i.*

Proof. The result follows immediately from Lemma 1, when we orthogonally project A_i over every λ-dimensional linear subspace of $\mathbb{R}^d$ and by the discrete central theorem (Theorem 2 with $d=\lambda$). The continuity can be achieved by the fact that given a finite set $A\subset\mathbb{R}^\lambda$, the set of points x, with the property that every closed half-space H through x contains at least $\lfloor\frac{|A|+\lambda}{\lambda+1}\rfloor$ points of A, is a convex set whose barycentric varies continuously with A. □

3.2. Results on $m(k,d,\lambda)$

Let us first notice that Conjecture 1 is equivalent to the following conjecture (by setting $d=\alpha+\lambda$).

Conjecture 2. *There is a set A with $\alpha+k+\lceil\frac{k}{\lambda}\rceil$ points in $\mathbb{R}^{\alpha+\lambda}$ such that the convex hulls of the k-sets do not admit a transversal α-plane.*

Theorem 6. $m(k,\lambda,\lambda)=k+\lceil\frac{k}{\lambda}\rceil-1$.

Proof. We shall show that $m(k,\lambda,\lambda)<k+\lceil\frac{k}{\lambda}\rceil$. The result follows since by Corollary 1 (with $\lambda=d$), we have that $k+\lceil\frac{k}{\lambda}\rceil-1\leqslant m(k,\lambda,\lambda)$. So by Conjecture 2, it is enough to prove that there is a set A with $k+\lceil\frac{k}{\lambda}\rceil$ points in $\mathbb{R}^\lambda$ such that the family of convex hulls of the k-sets of A does not have a common point in the intersection. We have two cases.

Case (1) If $k>\lambda$, then $k=p\lambda+j-1$ for some integers $p\geqslant 1$ and $2\leqslant j\leqslant\lambda+1$, and so

$$k+\left\lceil\frac{k}{\lambda}\right\rceil=p\lambda+j-1+\left\lceil\frac{p\lambda+j-1}{\lambda}\right\rceil=p(\lambda+1)+j-1+\left\lceil\frac{j-1}{\lambda}\right\rceil=p(\lambda+1)+j.$$

We shall next prove that there is an embedding of $p(\lambda+1)+j$ points with the property that the convex hulls of the $(p\lambda+j-1)$-sets have no common point. To this end, we take a simplex in $\mathbb{R}^\lambda$ with $\lambda+1$ vertices. We split the vertices of the simplex into j red vertices and $\lambda+1-j$ blue vertices. At every red vertex we put $p+1$ points and at every blue vertex we put p points. So in each facet we have at least $p(\lambda+1-j)+(p+1)(\lambda-(\lambda+1-j))=p\lambda+j-1=k$ points. Therefore for each facet, we can form a k-set, and clearly the intersection of the convex hulls of all such k-sets has no common point.

Case (2) If $k\leqslant\lambda$, then $k+\lceil\frac{k}{\lambda}\rceil=k+1$. In this case, we consider a simplex with $k+1$ vertices embedded in $\mathbb{R}^\lambda$. It is clear that the family of $(k-1)$-faces of the simplex has an empty intersection. □

Theorem 7. *Conjecture 1 is true if either* (a) $\lambda=1$ *or* (b) $k\leqslant\lambda$ *or* (c) $\lambda=k-1$ *or* (d) $k=2,3$.

Proof. Part (a) follows by Theorem 5. For parts (b) and (c), we remark that by Theorem 1 and Corollary 1,

$$d-\lambda+k+\left\lceil\frac{k}{\lambda}\right\rceil-1\leqslant m(k,d,\lambda)<M(k,d,\lambda)=\begin{cases}d+2(k-\lambda)+1 & \text{if } k\geqslant\lambda,\\ k+d-\lambda+1 & \text{if } k\leqslant\lambda.\end{cases}\tag{3}$$

So if $\lambda\geqslant k$, then $d-\lambda+k\leqslant m(k,d,\lambda)<k+d-\lambda+1$, and therefore $m(k,d,\lambda)=k+d-\lambda$, giving Conjecture 1. If $\lambda=k-1$, then $d+2\leqslant m(k,d,\lambda)<d+3$, and therefore $m(k,d,\lambda)=d+2$, also giving Conjecture 1. We may then suppose that $\lambda<k$. Finally for part (d), if $k=2$, then $\lambda=1$, and it follows by part (a); and if $k=3$, then either $\lambda=1$ or 2, and it follows by parts (a) and (c). □

We notice that if (3) is used Conjecture 1 is also true if $k = 4$ when $\lambda = 1$ or 3, but $\lambda = 2$ does not yield the validity of the conjecture. This case is more complicated and we leave it for future work. In fact, we are investigating a general improved upper bound for $m(k, d, \lambda)$ giving the conjectured value for $k = 4$ and 5 (work in progress).

Note: One of the referees informed us that in [4] Dol'nikov announced that $\chi(KG^{\lambda}(n,k)) = n - k - \lfloor \frac{k}{\lambda-1} \rfloor + 2$ and proved the result for the case $\lambda = 1$. Also, we were informed that proof of the inequality $\chi(KG^{\lambda}(n,k)) \leqslant n - k - \lfloor \frac{k}{\lambda-1} \rfloor + 2$ was given in a recent MSc thesis by A.A. Belova (unpublished), presenting a particular coloring similar to the above.

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Appendix A

We shall discuss the construction of a set of seven points in general position without a transversal line to the convex hulls of the 4-sets. For, we need the following result:

Lemma 2. *Let A be a set of seven points in general position in $\mathbb{R}^3$ and let L be a transversal line to the convex hulls of the 4-sets in A. Then either L contains two points of A, or L contains one point of A and it intersects three intervals whose ends are among the other six points of A.*

Proof. Since the points are in general position, L contains at most two points of A. Let us first show that L contains at least one point of A. We proceed by contradiction, let us then suppose that L does not contain any points of A. Let $x_0 \in A$ and let H be the plane through x_0 and L. The plane H contains at most three points of A (none lying on L). If H contains exactly three points, then there would be four points of A not in H, and by the pigeon-hole principle, there would be at least two points $\{a, b\} \subset A$ on the same side of H. By the same reasoning, there would be at least two points $\{c, d\} \subset A \cap H$ on the same side of L. This implies that L does not intersect the convex hull of $\{a, b, c, d\}$, which is a contradiction. If H contains at most two points, then there would be at least five points of A not in H and by the pigeon-hole principle, there would be at least three points on the same side of H. The line L would not intersect the convex hull of these three points and x_0, which is a contradiction. Therefore L must contain either one or two points of A. Let us suppose that L contains one point, say x_0, and let H be the plane generated by L and a point $u \in A \setminus \{x_0\}$. We shall show that there exists a unique point $v \in A \setminus \{u, x_0\}$ such that the interval $[u, v]$ intersects L. We know that H contains at most three points; if it contained at most two, then by using arguments as above, we can show that there would be at least three points of A on the same side of H. These three points and u would form a tetrahedron having empty intersection with L, which is not possible. Then we suppose that H contains exactly three points and thus there are two points, say $p, q \in A \setminus \{x_0\}$, on the same side of H. Moreover, among the three points in H (say u, v and $x_0 \in L$), we cannot have that u and v lie on the same side of L, otherwise the tetrahedron formed by u, v, p and q would have an empty intersection with L, which is not possible. Therefore u and v lie on opposite sides of L, and thus $[u, v]$ intersects L, since $u, v \in H$. □

We now consider the points of a tetrahedron and those of a suitable triangle placed under the tetrahedron, see Fig. 2.

We claim that any line containing two of these points has empty intersection with the convex hull of a 4-set. By the symmetry of the configuration, there are just five cases to be checked, see Fig. 3.

Moreover it can be verified that any line passing through one of the vertices does not intersect three intervals having ends on the other six points (a little perturbation of the vertices may be needed). Therefore by Lemma 2, this configuration does not have a transversal line to the convex hulls of the 4-sets.

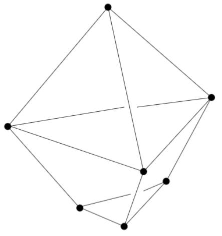

Fig. 2. Configuration of 7 points without transversal line to the convex hull of the 4-sets.

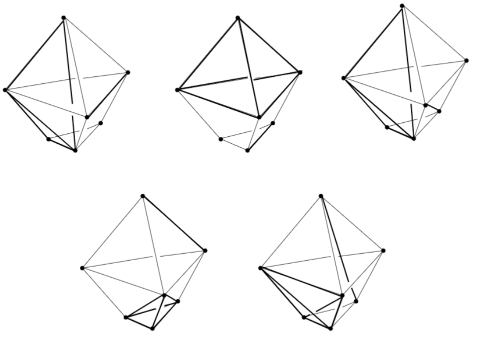

Fig. 3. Transversals missing a tetrahedron.

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