A COLORFUL THEOREM ON TRANSVERSAL LINES TO PLANE CONVEX SETS

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Abstract. We prove a colorful version of the Hadwiger's transversal line theorem: if a family of colored and numbered convex sets in the plane has the property that any three differently colored members have a transversal line compatible with the numbering, then there exists a color such that all the convex sets of that color have a transversal line.

1. Introduction

In 1982 (see [1]) Imre Bárány observed that some of the classical theorems in convexity admit interesting and mysterious generalizations which he called "colorful theorems". For example, the Helly Colorful Theorem says that *if a family (repetitions of the same sets are allowed) of compact convex sets in $\mathbb{R}^k$ is colored (properly) with $k+1$ colors and it has the property that any choice of $k+1$ differently colored sets have non void intersection, then there exists a color such that all the convex sets of that color have non void intersection.* In the case that any convex set of the family is repeated $k+1$ times and they are colored with the $k+1$ colors, we obtain the classical Helly's Theorem. So, the colorful version is indeed a generalization. Bárány attributed this theorem to László Lovász (see [2] for his elegant proof) and he proved a colorful version of the Carathéodory Theorem. Since then, several papers have been published on this matter. See, for example [3], [7] and [5].

However, in the study of colorful theorems there was a missing piece. Does the Hadwiger Theorem on transversal lines to plane convex sets admits a colorful version? We will see that the answer is yes. More precisely, the purpose of this note is to proof the following theorem.

Theorem 1 (Colorful Hadwiger)**.** *Let $A_1, ...A_n$ be a finite, ordered family of compact convex sets in the plane colored red, green and blue. If any choice of differently colored A_i , A_j and A_k with $i<j<k$ have a transversal line consistent with the order $i<j<k$, then there is a color such that there is a line transversal to all the convex sets of this color.*

Let us clarify some details. First, the phrase "*consistent with the order $i<j<k$*" means exactly that A_j intersects the convex closure of $A_i \cup A_k$. Second, the coloring must be surjective: any color is present. Finally, it is not supposed that $A_i \neq A_j$, i.e. repetitions of the same convex sets are allowed. With the

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same argument as above, we see that it is indeed a generalization of the Hadwiger theorem (whose first proof in its full generality is due to Wenger [8]).

The proof of the Colorful Hadwiger Theorem has three key parts: a geometric, a combinatorial and a topological one. We will explain them in that order. In Section 2 we introduce balanced colored sign vectors and expose they key properties to the proof of the theorem. In Section 3 we prove a property of colored sign vectors which can be stated and proved without any geometrical interpretation. In the last section we will explain why the fact that the sphere of dimension 1 is a connected topological space is the final element for the proof of the theorem.

2. Balanced colored sign vectors

A vector whose coordinates are elements of the set $\{-,0,+\}$ is called a **sign vector**. Given an ordered family $A_1,...A_n$ of convex sets in the plane and an oriented line ℓ it is natural to construct the **separating sign vector** $\mathbf{x}=(x_1,\dots,x_n)$ whose coordinate x_i is zero if ℓ intersects A_i, is positive if A_i lies in the right open semiplane of ℓ and is negative if A_i lies in the left open semiplane of ℓ.

Let d be a direction in the plane and $d^{\perp}$ its orthogonal direction (i.e. $\left(d,d^{\perp}\right)$ is an orthonormal positive ordered basis). Chose any oriented line $\ell^{\perp}$ in the direction of $d^{\perp}$. When we orthogonally project any convex set A_i to the line $\ell^{\perp}$ we obtain an interval $\left[p^i,q^i\right]$ and we can think that p_i and q_i are real numbers. For the ordered family $\mathcal{A}=(A_1,...A_n)$ define $p=\operatorname{Sup}\left\{p^i\right\}$ and $q=\operatorname{Inf}\left\{q^i\right\}$. Let ℓ be the oriented line in the direction d which meets $\ell^{\perp}$ in the point $(p+q)/2$. We will call ℓ the **middle line of the family $\mathcal{A}$ in the direction** d. The separating sign vector of the line ℓ will be called the **middle separating sign vector of the direction** d.

It is easy to see that the family has no transversal line in the direction d if and only if $q<p$. This is equivalent to the property that the middle separating sign vector has plus and minus coordinates. So, we will say that a sign vector is **balanced** when it has plus and minus coordinates.

The **opposite** of a sign vector $\mathbf{x}$ is the vector $-\mathbf{x}=(-x_1,\dots,-x_m)$ where the usual rules ($-0=0$, $-+=-$ and $--=+$) are used. If $\mathbf{x}$ is the separating sign vector of the oriented line ℓ, then $-\mathbf{x}$ is the separating sign vector of the same line with the reversed orientation. If a sign vector is balanced, then his opposite also is.

A **colored sign vector** is a sign vector together with a coloring $\{1,\dots,n\}\to$ $\{$red, green, blue$\}$ of the indices of its coordinates. Of course, all the separating sign vectors of a colored family of convex sets are naturally colored with the same coloring. We will need to extend the concepts of middle separating sign vectors and balanced sign vectors to the colored case.

Suppose our ordered family $\mathcal{A}=(A_1,...A_n)$ is now colored. Let d be a direction and let the line $\ell^{\perp}$ and the intervals $\left[p^i,q^i\right]$ be defined as above. Denote $p_{\mathbf{R}}=\operatorname{Sup}\left\{p^i\mid A_i\text{ is red}\right\}$ and $q_{\mathbf{R}}=\operatorname{Inf}\left\{q^i\mid A_i\text{ is red}\right\}$. In the same way we define the real numbers $p_{\mathbf{G}},q_{\mathbf{G}}$ and $p_{\mathbf{B}},q_{\mathbf{B}}$ for the green and blue convex sets.

The numbers $p_{\mathbf{R}}$, $p_{\mathbf{G}}$ and $p_{\mathbf{B}}$ can be in any order. They can even coincide. Denote by p_1 one of the smallest of them. Denote by p_3 one of the greatest of the remaining two. Finally denote by p_2 the remaining number. So, we have $p_1\le p_2\le p_3$ and $\{p_1,p_2,p_3\}=\{p_{\mathbf{R}},p_{\mathbf{G}},p_{\mathbf{B}}\}$. In the same way we can define q_1,q_2 and q_3 such that $q_1\ge q_2\ge q_3$ and $\{q_1,q_2,q_3\}=\{q_{\mathbf{R}},q_{\mathbf{G}},q_{\mathbf{B}}\}$. Let ℓ be the oriented line in the direction d which meets $\ell^{\perp}$ in the point $\theta=(p_2+q_2)/2$. We will call ℓ the **middle line of the colored family $\mathcal{A}$ in the direction** d. The colored separating sign vector of the line ℓ will be called the **middle colored separating sign vector of the direction** d.

The following two results are easy to check and we will omit their proofs.

Lemma 1. *The map that to any direction of the plane assigns the middle line of the colored family in that direction is continuos.*

Lemma 2. *If two directions are opposite, then their middle colored separating sign vector are opposite.*

A colored sign vector will be called **balanced** if *for any two colors, there is a plus of some of the two colors and a minus of the other color.*

Lemma 3. *If the colored family $\mathcal{A}$ has the property that any monochromatic subfamily has no transversal line in the direction d, then the middle colored separating sign vector of that direction is balanced.*

Proof. Let $p_{\mathbf{R}}$, $p_{\mathbf{G}}$, $p_{\mathbf{B}}$, p_1, p_2, p_3, $q_{\mathbf{R}}$, $q_{\mathbf{G}}$, $q_{\mathbf{B}}$, q_1, q_2 ,q_3 and θ be as above. Since there is no transversal line to the red convex sets in the direction d, then we have $p_{\mathbf{R}} > q_{\mathbf{R}}$. Analogous arguments give $p_{\mathbf{G}} > q_{\mathbf{G}}$ and $p_{\mathbf{B}} > q_{\mathbf{B}}$. We shall see that $p_2 > q_2$. Indeed, if $q_2 \geq p_2$, then $q_1 \geq q_2 \geq p_2 \geq p_1$ and this contradicts the fact that there is a bijection $\varphi : \{1, 2, 3\} \to \{1, 2, 3\}$ such that $p_i > q_{\varphi(i)}$ for $i \in \{1, 2, 3\}$.

Denote by $\mathbf{x}$ the middle colored separating sign vector of the direction d. If $\mathbf{x}$ is not balanced, then there are two colors say red and green such that all non zero red and green coordinates are (say) pluses. This implies that $q_{\mathbf{R}} \geq \theta$ and $q_{\mathbf{G}} \geq \theta$. Moreover, since $p_2 > q_2$, then $\theta > q_2$. But $q_{\mathbf{R}} > q_2$ and $q_{\mathbf{G}} > q_2$ contradict the definition of q_2. $\square$

3. The sign of colored sign vectors

Among all n-dimensional sign vectors there is a natural partial order relation $\preceq$ defined by $\mathbf{x} = (x_1, \dots, x_n) \preceq \mathbf{x}' = (x'_1, \dots, x'_n)$ if and only if $(x_i \neq 0) \Rightarrow (x_i = x'_i)$. The minimum element of this order relation is the zero vector. For the case that $\mathbf{x}$ and $\mathbf{x}'$ are the separating sign vectors corresponding to the oriented lines ℓ and ℓ' the relation $\mathbf{x} \preceq \mathbf{x}'$ means that ℓ separates less sets than ℓ'. This relation is important because *if the convex sets are compact and ℓ' is in a sufficiently small neighborhood of ℓ, then ℓ separates less sets than ℓ' i.e. the relation $\mathbf{x} \preceq \mathbf{x}'$ always hold.* We will use the relation $\mathbf{x} \preceq \mathbf{y}$ among colored sign vectors only if their colorings are the same.

A colored sign vector $\mathbf{x}$ will be called **Hadwiger** if for any choice $i < j < k$ of indices with different colors we have that $(x_i, x_j, x_k) \neq (+ - +)$ and $(x_i, x_j, x_k) \neq (- + -)$. If $\mathbf{x}$ is the separating sign vector of the colored family $A_1, ...A_n$ and the oriented line ℓ, then the property that $\mathbf{x}$ is not Hadwiger means that there exist differently colored $i < j < k$ such that ℓ separates A_j from the convex closure of $A_i \cup A_k$. Therefore, if the hypothesis of the Colorful Hadwiger Theorem holds for the family $A_1, ...A_n$, then *any its colored separating sign vector is Hadwiger.*

The **sign** of a sign vector is zero if all it coordinates are zero and in the other cases is equal to the sign of its leading coordinate, this is, the non zero coordinate with the smallest index.

Lemma 4. *If $\mathbf{x} \preceq \mathbf{y}$ are both balanced and Hadwiger colored sign vectors, then they have the same sign.*

Proof. Assume that $\mathbf{x}$ and $\mathbf{y}$ contradict the lemma. Say the sign of $\mathbf{x}$ is plus and the sign of $\mathbf{y}$ is minus. Denote by a and b the indices of the leading coordinates of $\mathbf{x}$ and $\mathbf{y}$ respectively. Since $\mathbf{x} \preceq \mathbf{y}$, then we have $b < a$. Let us eliminate all coordinates i such that $i < b$ and $b < i < a$ thus obtaining sign vectors $\mathbf{x}'$ and $\mathbf{y}'$. In this process the signs and the order relation are preserved. Both $\mathbf{x}'$ and $\mathbf{y}'$ are Hadwiger and $\mathbf{x}'$ is balanced. Since $\mathbf{x}' \preceq \mathbf{y}'$ the vector $\mathbf{y}'$ is balanced too and therefore $\mathbf{x}'$ and $\mathbf{y}'$ also contradict the lemma. So, without loosing generality we can suppose that $b = 1$ and $a = 2$.

The coordinate 2 is of some color, say blue and since $\mathbf{x} \preceq \mathbf{y}$, then $x_2 = y_2 = +$. We also know that $x_1 = 0$ and $y_1 = -$. We divide the proof into two cases: the coordinate 1 is blue or not.

Suppose the first coordinate is blue. Since $\mathbf{y}$ is balanced there must be a green coordinate (say i) such that $y_i = -$ and a red coordinate (say j) such that $y_j = +$. If $i < j$ then $(y_2, y_i, y_j) = (+-+)$ otherwise $(y_1, y_j, y_i) = (-+-)$ and in both cases we contradict that $\mathbf{y}$ is Hadwiger.

Suppose the first coordinate is red. If there exist a green coordinate (say i) such that $x_i = y_i = -$, then $(y_1, y_2, y_i) = (-+-)$ and this contradicts that $\mathbf{y}$ is Hadwiger. So, there is no such coordinate. Since $\mathbf{x}$ is balanced and $\mathbf{x} \preceq \mathbf{y}$ there must exist a green coordinate (say i) such that $x_i = y_i = +$, a red coordinate (say j) such that $x_j = y_j = -$ and a blue coordinate (say k) such that $x_k = y_k = -$. If $j < i$, then $(y_2, y_j, y_i) = (+-+)$. Hence, $i < j$. If $k < i$, then $(y_k, y_i, y_j) = (-+-)$ otherwise $(y_1, y_i, y_k) = (-+-)$ and in all cases we obtain a contradiction. □

4. The proof

Suppose that the Colorful Hadwiger Theorem is not true and let $A_1, ...A_n$ be an ordered colored family of compact convex sets in the plane which contradict the theorem. Let P (respectively N) be the set of the directions of the plane such that their middle colored separating sign vector has positive sign (respectively negative sign). All middle colored separating sign vectors are Hadwiger and by Lemma 3 are balanced. Since balanced sign vectors have non zero sign, then $P \cup N$ is the set of all directions in the plane, which is homeomorphic to the sphere of dimension 1 denoted by $\mathcal{S}^1$. By Lemma 2 both sets P and N are non empty and by their definition they are disjoint. By Lemma 1, the observation at the beginning of Section 3 and Lemma 4 both sets P and N are open sets. This is a contradiction because $\mathcal{S}^1$ is a connected topological space.

This conclude the proof of the theorem. We shall finish this note with two remarks. First, observe that hidden along the lines of this note there is a very simple proof (it seems to be new) of the classical Hadwiger Theorem. In this case, the sign vectors are not colored; the definition of Hadwiger sign vectors does not require that coordinates are differently colored; the definition of a balanced sign vector is that there exist plus and minus coordinates; the Lemma 4 becomes obvious and we need the sign of middle separating sign vectors (not colored) to construct the open sets P and N.

Second, observe that it is easy to formulate a colorful version of the Goodman-Pollak-Wenger Theorem on transversal hyperplanes (see [4] and [6]). However, the known proof seems to be not adaptable to the colorful version. All the data we have (for example, it is true for families of disjoint convex sets in $\mathbb{R}^3$) point to conjecture that the colorful version of this theorem is true.

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