The topology of the space of transversals
through the space of configurations

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Abstract

Let F be a family of convex sets in $\mathbb{R}^n$ and let $T_m(F)$ be the space of m-transversals to F as subspace of the Grassmannian manifold. The purpose of this paper is to study the topology of $T_m(F)$ through the polyhedron of configurations of $(r+1)$ points in $\mathbb{R}^n$. This configuration space has a natural polyhedral structure with faces corresponding to what has been called order types. In particular, if $r = m + 1$ and $T_{m-1}(F)$ is nonempty, we prove that the homotopy type of $T_m(F)$ is ruled by the set of all possible order types achieved by the m-transversals of F. We shall also prove that the set of all m-transversals that intersect F with a prescribed order type is a contractible space. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $F = \{A^0, A^1, \ldots, A^r\}$ be a family of convex sets in $\mathbb{R}^n$. The *space of m-transversals of F*, denoted $T_m(F)$, is the subspace of the Grassmannian $G^*(n, m)$ of (free) m-planes in $\mathbb{R}^n$ that intersect all the members of F.

The purpose of this paper is to study the topology of $T_m(F)$ through C_r^m, *the polyhedron of configurations of $(r + 1)$ points in $\mathbb{R}^m$*; where such a configuration is the affine equivalence class of $(r+1)$ ordered points in $\mathbb{R}^m$ that affinely generate it. The configuration space has a natural polyhedral structure with faces corresponding to what has been called

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order types (see Section 2 for details). In particular, if $r = m + 1$ and $T_{m-1}(F) = \emptyset$, we shall prove that the homotopy type of $T_m(F)$ is ruled by the set of all possible order types achieved by the m-transversals of F. We shall also prove that the set of all m-transversals in $T_m(F)$ that intersect F with a prescribed order type is a contractible space.

More precisely, if $x^0, \ldots, x^{m+1}$ are points in $\mathbb{R}^m$, then, by the classic Radon Theorem, there are subsets $P = \{i_0, \ldots, i_p\} \subset I$ and $Q = \{j_0, \ldots, j_q\} \subset I$, where $I = \{0, \ldots, m+1\}$, with $P \cap Q = \emptyset$, $P \neq \emptyset$, $Q \neq \emptyset$, such that the convex hulls of the points corresponding to P and Q intersect. But it is not hard to see from the proof that if we further assume that $\{x^0, \ldots, x^{m+1}\}$ affinely generate $\mathbb{R}^m$, then P and Q can be uniquely chosen so that they further satisfy that $\{x^{i_0}, \ldots, x^{i_p}\}$ generate a p-simplex Σ_p, $\{x^{j_0}, \ldots, x^{j_q}\}$ generate a q-simplex Σ_q, and $\Sigma_p \cap \Sigma_q$ consists of a single point in the relative interior of both simplices (where, recall that the interior of a 0-simplex is itself). If this is so, we say that $(x^0, \ldots, x^{m+1})$, where the round brackets are now to emphasize a fixed ordering, has the *order type* $\{P, Q\}$. It is not difficult to see that, in this case, our definition of order type coincides with the classic one given by Goodman and Pollack [2]. The finite set of all possible order types $\{\{P, Q\} \mid P, Q \subset I, \ P, Q \neq \emptyset \text{ and } P \cap Q = \emptyset\}$ has the structure of a simplicial complex $\mathcal{OT}$ if we declare that a collection of vertices $\{\{P_{\lambda_0}, Q_{\lambda_0}\}, \ldots, \{P_{\lambda_s}, Q_{\lambda_s}\}\}$ is an s-simplex of $\mathcal{OT}$ if and only if $P_{\lambda_0} \subset \cdots \subset P_{\lambda_s}$ and $Q_{\lambda_0} \subset \cdots \subset Q_{\lambda_s}$. We shall see later that $\mathcal{OT}$ is the first barycentric subdivision of the polyhedron of configurations of $(m + 2)$ points in $\mathbb{R}^m$, introduced in Section 2, following the spirit of Gelfand et al. in [1].

We are now in a position to state our main results.

Definition. Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m + 2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, the *order types achieved* by F, $\mathcal{OT}(F)$, is the finite collection of all order types $\{P, Q\}$ for which there exits an m-plane $H \in T_m(F)$ and points $x_j \in H \cap A^j$, $j = 0, \ldots, m + 1$, with the order type of $(x_0, \ldots, x_{m+1})$ equal to $\{P, Q\}$.

Theorem 1. *Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m + 2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, such that $T_{m-1}(F) = \emptyset$. Then $T_m(F)$ has the homotopy type of $|\mathcal{OT}(F)|$, the subcomplex of $\mathcal{OT}$ induced by the vertices of $\mathcal{OT}(F)$.*

Theorem 2. *Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m + 2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, such that $T_{m-1}(F) = \emptyset$ and let $\{P, Q\}$ be a fixed order type in $\mathcal{OT}(F)$. If $T_{\{P,Q\}}(F)$ is the space of all m-planes $H \in T_m(F)$ with the property that there are $x_j \in H \cap A^j$, $j = 0, \ldots, m + 1$, with the order type of $(x_0, \ldots, x_{m+1})$ equal to $\{P, Q\}$, then $T_{\{P,Q\}}(F)$ is a contractible space.*

Of course, these theorems are false when $T_{m-1}(F) \neq \emptyset$ or if we consider m-transversals of a family of $(r + 1)$ convex sets with $r > m + 1$. For more about Geometric Transversal Theory see [3] and for topological aspects of this theory see [4].

To fix ideas, we end the introduction with a brief discussion of the simplest non trivial example. Let A^0, A^1, A^2 be the three sides of a triangle in $\mathbb{R}^2$, and let $F = \{A^0, A^1, A^2\}$. It

is easy to see that $T_1(F)$ is topologically a circle. On the other hand, we have six order types $\{P, Q\}$: three corresponding to singleton pairs ($P = \{i\}$, $Q = \{j\}$ with $i \neq j$) achieved by lines passing through a vertex ($A^i \cap A^j$) and a point on the interior of the opposite side, and there are three $1, 2$-partitions (of the sort $P = \{0\}$, $Q = \{1, 2\}$, say) achieved by the supporting lines of the sides and taking an interior point (in A^0, say) and the two extremes (in A^1 and A^2). So $\mathcal{OT}(F) = \mathcal{OT}$ which is clearly an hexagon and also realizes a circle. Observe also that $T_{\{P,Q\}}(F)$ is a closed interval for $\sharp P = \sharp Q = 1$ and a single point otherwise.

2. The space of configurations of $(r + 1)$-points in $\mathbb{R}^m$

We follow the basic ideas of what Gelfand et al. did in [1] for vector spaces, but now in the context of affine geometry.

Given points $x^0, x^1, \ldots, x^r$ in $\mathbb{R}^m$, let $\langle x^0, x^1, \ldots, x^r \rangle$ denote the affine subspace spanned (generated) by $x^0, x^1, \ldots, x^r$.

Let

$$C_r^m = \left\{ \left(x^0, x^1, \ldots, x^r \right) \mid x^j \in \mathbb{R}^m, \ \langle x^0, x^1, \ldots, x^r \rangle = \mathbb{R}^m \right\} / \sim,$$

where $(x^0, x^1, \ldots, x^r) \sim (y^0, y^1, \ldots, y^r)$ if and only if there is an affine map $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ such that $\Omega(x^j) = y^j$, $j = 0, 1, \ldots, r$.

If $(x^0, x^1, \ldots, x^r)$ is such that $\langle x^0, x^1, \ldots, x^r \rangle = \mathbb{R}^m$, we will denote by $[x^0, x^1, \ldots, x^r]$ the corresponding element in C_r^m. The elements of C_r^m will be called *the configurations of* $(r + 1)$ *points in* $\mathbb{R}^m$.

If V is a m-plane of $\mathbb{R}^n$ and $(x^0, x^1, \ldots, x^r)$ is such that $\langle x^0, x^1, \ldots, x^r \rangle = V$, we will denote by $[x^0, x^1, \ldots, x^r] \in C_r^m$, the element $[\Omega(x^0), \Omega(x^1), \ldots, \Omega(x^r)] \in C_r^m$, where $\Omega : V \to \mathbb{R}^m$ is any affine isomorphism. Of course our notation is independent of the chosen affine isomorphism Ω.

Now we see that C_r^m is naturally homeomorphic to the Grassmannian manifold $G(r, r - m)$ of $(r - m)$-dimensional linear subspaces of $\mathbb{R}^r$. Let Δ^r be the standard r-simplex of $\mathbb{R}^r$, whose vertices are $e_0, e_1, \ldots, e_r$, where $e_i = (0, \ldots, 1, \ldots, 0)$ is the standard unit vector of $\mathbb{R}^r$ and $e_0 = 0$.

Define

$$\Psi : C_r^m \to G(r, r - m)$$

as follows. If, without loss of generality, $[0, x^1, \ldots, x^r] \in C_r^m$, let $\Gamma : \mathbb{R}^r \to \mathbb{R}^m$ be the linear map defined by $\Gamma(e_i) = x^i$, and then let $\Psi([0, x^1, \ldots, x^r]) = \ker(\Gamma)$. It is not difficult to check that Ψ is indeed a homeomorphism. Compare with [1]. In particular, C_{m+1}^m, the space of configurations of $(m + 2)$ points in $\mathbb{R}^m$, is homeomorphic to the projective space $\mathbb{RP}^m$.

The space of configurations C_r^m has a natural "polyhedral structure", in which the faces correspond to the different "separation structures" or "order types" of the configurations; they turn out to be intersections of Schubert cells of the Grassmannian space $G(r, r - m)$

for the different flags that arise from the total orders of the index set $I = \{0, 1, \ldots, r\}$. This "polyhedral structure" is finer than the one given by Gelfand et al. in [1], but its analysis follows that one almost verbatim. In this paper, we only need the polyhedral structure of C^m_{m+1} which will be described completely.

3. The polyhedron C^m_{m+1} of $(m+2)$ points in $\mathbb{R}^m$

Let $I = \{0, 1, \ldots, m+1\}$ and let $\Delta^I \subset \mathbb{R}^{m+1}$ be the standard $(m+1)$-simplex whose vertices are given by the origin $e_0 = 0$ and the standard unit vectors $e_1, \ldots, e_{m+1}$. For every two nonempty subsets $P = \{i_0, i_1, \ldots, i_p\}$ and $Q = \{j_0, j_1, \ldots, j_q\}$ of I with $P \cap Q = \emptyset$, let $\Delta^P \subset \Delta^I$ be the p-simplex generated by $\{e_{i_0}, \ldots, e_{i_p}\}$ and $\Delta^Q \subset \Delta^I$ be the q-simplex generated by $\{e_{j_0}, \ldots, e_{j_q}\}$.

We may consider $\Delta^I \times \Delta^I$ as a polyhedron whose faces are products of faces of Δ^I. Let $\widetilde{T}^m$ be the subpolyhedron of $\Delta^I \times \Delta^I$ whose faces are all prisms of the form $\Delta^P \times \Delta^Q$, for every two nonempty subsets P, Q of I with $P \cap Q = \emptyset$. Let now

$$T^m = \widetilde{T}^m / {\sim},$$

where $(x, y) \sim (y, x)$, for every $(x, y) \in \Delta^P \times \Delta^Q$.

The face of the polyhedron T^m induced by the nonempty disjoint subsets $P, Q \subset I$, will be denoted by $\{P, Q\}$. Note that the simplicial complex $\mathcal{OT}$, of all order types of $(m+2)$ points in R^m, defined in the introduction, is the first barycentric subdivision of T^m.

We claim that the polyhedron T^m is naturally homeomorphic to C^m_{m+1}, and the basic idea is that the configurations with Radon Partition of type $\{P, Q\}$ are naturally parametrized by $\Delta^P \times \Delta^Q$. To see this, we define a map

$$\psi : T^m \to C^m_{m+1}.$$

If $z \in \{P, Q\}$ a face of T^m, then it corresponds to a point $(x, y) \in \Delta^P \times \Delta^Q$. For $j = 0, 1, \ldots, m+1$, let $x^j \in \Delta^I \times \Delta^I \times \Delta^I$ defined as follows:

$$x^j = \begin{cases} (e_j, y, 0) & \text{if } j \in P, \\ (x, e_j, 0) & \text{if } j \in Q, \\ (0, 0, e_j) & \text{if } j \notin P \cup Q. \end{cases}$$

Note that the set $\{x^0, \ldots, x^{m+1}\}$ generates a m-plane and in this m-plane it has a Radon partition of the type $\{P, Q\}$. Let

$$\psi(z) = [x^0, \ldots, x^{m+1}].$$

The map $\psi : T^m \to C^m_{m+1}$ is a well defined continuous map. Furthermore, the inverse of ϕ is given by Radon's Theorem. More precisely, if $(x^0, \ldots, x^{m+1})$ is such that $\langle x^0, \ldots, x^{m+1} \rangle = \mathbb{R}^m$, then by Radon's Theorem, there are $P = \{i_0, \ldots, i_p\} \subset I$ and $Q = \{j_0, \ldots, j_q\} \subset I$, with $P \cap Q = \emptyset$, and $P, Q \neq \emptyset$, such that $\{x^{i_0}, \ldots, x^{i_p}\}$ generate a p-simplex Σ^P, $\{x^{j_0}, \ldots, x^{j_q}\}$ generate a q-simplex Σ^Q, and $\Sigma^P \cap \Sigma^Q = \{a\}$ consists of a
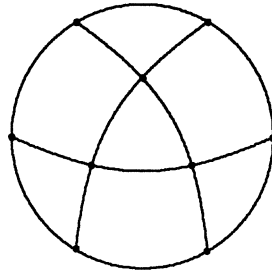
Fig. 1.

single point. Let $\{\gamma_{i_0}, \ldots, \gamma_{i_p}\}$ be the barycentric coordinates of a in Σ^P and $\{\gamma_{j_0}, \ldots, \gamma_{j_p}\}$ be the barycentric coordinates of a in Σ^Q. Define

$$x = \sum_{\lambda=1}^{p} \gamma_{i_\lambda} e_{i_\lambda} \in \Delta^P \quad \text{and} \quad y = \sum_{\lambda=1}^{q} \gamma_{j_\lambda} e_{j_\lambda} \in \Delta^Q,$$

and let $z \in \{P, Q\} \subset T^m$ be the point that corresponds to $(x, y) \in \Delta^P \times \Delta^Q$. Therefore, $\psi^{-1}([x^0, \ldots, x^{m+1}]) = z$. This concludes the proof that $\psi : T^m \to C^m_{m+1}$ is a homeomorphism.

If $(x_0^0, x_0^1, \ldots, x_0^{m+1})$ and $(x_1^0, x_1^1, \ldots, x_1^{m+1})$ are such that $\langle x_\theta^0, x_\theta^1, \ldots, x_\theta^{m+1} \rangle = \mathbb{R}^m$ for $\theta = 0, 1$, then we say that $(x_0^0, x_0^1, \ldots, x_0^{m+1})$ and $(x_1^0, x_1^1, \ldots, x_1^{m+1})$ give rise to the same order type, oriented matroid or separoid (see [2,4]) if and only if the corresponding configurations $[x_0^0, x_0^1, \ldots, x_0^{m+1}]$ and $[x_1^0, x_1^1, \ldots, x_1^{m+1}]$ belong to the interior of the same face $\{P, Q\}$ of the polyhedron $T^m = C^m_{m+1}$. Consequently, the faces of C^m_{m+1} are precisely the order types of $(m+2)$ points in $\mathbb{R}^m$.

For example, let us consider C^2_3, the space of configurations of 4 points in $\mathbb{R}^2$. It gives a polyhedral structure to the projective plane $\mathbb{RP}^2$ (see Fig. 1). Its 2-dimensional cells are four triangles (corresponding to configurations where one point lies in the interior of the convex hull of the other three, with order type $\{P, Q\}$ where $\sharp P = 1$ and $\sharp Q = 3$), and three quadrilaterals (corresponding to configurations where the 4 points are in the boundary of its convex hull and the order type is the partition in diagonals). The 1-dimensional cells correspond to order types $\{P, Q\}$ where $\sharp P = 1$ and $\sharp Q = 2$, and the six vertices to configurations where two of the points coincide ($\sharp P = 1$ and $\sharp Q = 1$). The 1-dimensional cells group by triples to form 4 projective lines corresponding to configurations with three colinear points and, in the Grassmannian, to lines that are parallel to one of the planes of the standard simplex.

4. The space of transversals via the space of configurations

Now we turn our attention to the general case, and prove that the first dimension where there are transversals to a family of convex sets can be studied topologically by the configurations that arise.

Let $F = \{A^0, A^1, \ldots, A^r\}$ be a family of convex sets in $\mathbb{R}^n$. The purpose of this section is the study of $T_m(F)$, the space of all m-planes transversal to F through the space of all possible configurations of $(r + 1)$ points that are achieved within m-transversals to F.

Let

$$C_m(F) = \{[x^0, x^1, \ldots, x^r] \in C_r^m \mid x^j \in A^j, \dim\langle x^0, x^1, \ldots, x^r\rangle = m\}.$$

Theorem 3. *Let $F = \{A^0, A^1, \ldots, A^r\}$ be a family of convex sets in $\mathbb{R}^n$ such that $T_{m-1}(F) = \emptyset$. Then, $T_m(F)$ has the homotopy type of $C_m(F)$.*

Proof. Let $\widetilde{F} \subset A^0 \times A^1 \times \cdots \times A^r$ be defined as follows:

$$\widetilde{F} = \{(x^0, x^1, \ldots, x^r) \in A^0 \times A^1 \times \cdots \times A^r \mid \dim\langle x^0, x^1, \ldots, x^r\rangle = m\},$$

and let $\Phi : \widetilde{F} \to T_m(F)$ be

$$\Phi(x^0, x^1, \ldots, x^r) = \langle x^0, x^1, \ldots, x^r\rangle.$$

If $H \in T_m(F)$, and $(x^0, \ldots, x^r) \in (A^0 \cap H) \times \cdots \times (A^r \cap H)$ then $\langle x^0, \ldots, x^r\rangle = H$ because $T_{m-1}(F) = \emptyset$. Therefore Φ is surjective and we clearly have that

$$\Phi^{-1}(H) = (A^0 \cap H) \times \cdots \times (A^r \cap H).$$

This implies that $\Phi : \widetilde{F} \to T_m(F)$ is a homotopy equivalence because it is surjective and the fibers $\Phi^{-1}(H)$ are convex and hence contractible.

Define now $\phi : \widetilde{F} \to C_m(F)$ as follows:

$$\phi(x^0, x^1, \ldots, x^r) = [x^0, x^1, \ldots, x^r].$$

Again, ϕ is a continuous surjective map. We shall prove that inverse images of ϕ are convex in $A^0 \times \cdots \times A^r$. Suppose that $(x^0, x^1, \ldots, x^r)$ and $(y^0, y^1, \ldots, y^r) \in \widetilde{F}$ are such that $[x^0, x^1, \ldots, x^r] = [y^0, y^1, \ldots, y^r]$. We must prove that the segment, in $\mathbb{R}^{nr}$, from one point to the other is in the same fibre; that is, for every $t \in [0, 1]$, we have to verify that

$$(tx^0 + (1-t)y^0, \ldots, tx^r + (1-t)y^r) \in \widetilde{F},$$

and

$$[tx^0 + (1-t)y^0, \ldots, tx^r + (1-t)y^r] = [x^0, x^1, \ldots, x^r].$$

If $[x^0, x^1, \ldots, x^r] = [y^0, y^1, \ldots, y^r]$, then there is a set of $(r + 1)$ points $z^0, z^1, \ldots, z^r$ that affinely generate $\mathbb{R}^m$ and affine embeddings $f, g : \mathbb{R}^m \to \mathbb{R}^n$, such that $f(z^j) = x^j$ and $g(z^j) = y^j$, $j = 0, \ldots, r$.

For every $t \in [0, 1]$, $tf + (1-t)g : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map. Its image, $(tf + (1-t)g)(\mathbb{R}^m)$ is transversal to F because it contains

$$tf(z^j) + (1-t)g(z^j) = tx^j + (1-t)y^j \in A^j, \quad j = 0, \ldots, r.$$

Since $T_{m-1}(F) = \emptyset$, then $\dim\langle tx^0 + (1-t)y^0, \ldots, tx^r + (1-t)y^r\rangle \geqslant m$ so that $tf + (1-t)g$ is an affine embedding and equality holds. This clearly implies that $(tx^0 + (1-t)y^0, \ldots, tx^r + (1-t)y^r) \in \widetilde{F}$ and

$$\left[tx^0 + (1-t)y^0, \ldots, tx^r + (1-t)y^r\right]$$
$$= \left[z^0, \ldots, z^r\right] = \left[x^0, \ldots, x^r\right] = \left[y^0, \ldots, y^r\right].$$

The above proves that the inverse images of ϕ are contractible, which implies that ϕ is a homotopy equivalence. This, together with the fact that Φ is also a homotopy equivalence, concludes the proof of the theorem. □

5. The technical lemmas

The purpose of this section is to prove three lemmas.

Lemma 1. *Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m+2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, such that $T_{m-1}(F) = \emptyset$. Let H_0, $H_1 \in T_m(F)$ be two transversal m-planes and for $\theta = 0, 1$ and $j = 0, \ldots, m+1$, let $a_\theta^j \in A^j \cap H_\theta$ be such that a_θ^0 lies in the m-simplex generated by $\{a_\theta^1, \ldots, a_\theta^{m+1}\}$. Then, there are continuous maps*

$$H : [0, 1] \to T_m(F),$$
$$a^j : [0, 1] \to A^j, \quad j = 0, 1, \ldots, m+1,$$

such that:

(a) *for $\theta = 0, 1$ and $j = 0, 1, \ldots, m+1$,*

$$H(\theta) = H_\theta, \quad \text{and} \quad a^j(\theta) = a_\theta^j,$$

(b) *for every $t \in [0, 1]$, $\langle a^1(t), a^2(t), \ldots, a^{m+1}(t) \rangle = H(t)$, and $a^0(t) = ta_0^0 + (1-t)a_1^0$ lies in the m-simplex generated by $\{a^1(t), a^2(t), \ldots, a^{m+1}(t)\}$.*

Moreover, if $\{\gamma_1(t), \ldots, \gamma_{m+1}(t)\}$ are the barycentric coordinates of $a^0(t)$ in the m-simplex generated by $\{a^1(t), a^2(t), \ldots, a^{m+1}(t)\}$, then for every $j = 1, \ldots, m+1$,

$$\gamma_j(t) = t\gamma_j(0) + (1-t)\gamma_j(1).$$

Proof. First observe that the simultaneous linear movement of a_0^j to a_1^j does not necessarily work (see Fig. 2 for a simple example); so we have to be much more cautious.

Let Δ be the $(2m+1)$-simplex generated by $\{e_1, \ldots, e_{m+1}, e_{m+2}, \ldots, e_{2m+2}\}$. Then Δ can be thought of as the join of $(m+1)$ closed intervals. That is:

$$\Delta = [e_1, e_{m+2}] * [e_2, e_{m+3}] * \cdots * [e_{m+1}, e_{2m+2}].$$

Therefore, for every $z \in (\Delta - \bigcup_{j=1}^{m+1}[e_j, e_{j+m+1}])$, there is a unique m-simplex generated by $\{y_1(z), \ldots, y_{m+1}(z)\}$ with $y_j(z) \in [e_j, e_{j+m+1}]$, $j = 1, \ldots, m+1$, and $z \in \Delta(z)$. Furthermore, this is a continuous association.
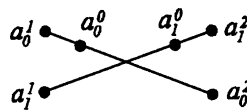

Fig. 2.

Let $\Gamma : \mathbb{R}^{2m+2} \to \mathbb{R}^n$ be the linear map such that, for $j = 1, \ldots, m+1$, $\Gamma(e_j) = a_0^j$ and $\Gamma(e_{j+m+1}) = a_1^j$. Let, $\tilde{a}_0^0$ in the m-simplex generated by $\{e_1, \ldots, e_{m+1}\}$ and $\tilde{a}_1^0$ in the m-simplex generated by $\{e_{m+2}, \ldots, e_{2m+2}\}$ be such that $\Gamma(\tilde{a}_0^0) = a_0^0$ and $\Gamma(\tilde{a}_1^0) = a_1^0$.

For $j = 1, \ldots, m+1$, let $a^j : [0, 1] \to A^j$ be defined as

$$a^j(t) = \Gamma\big(y_j\big(t\tilde{a}_0^0 + (1-t)\tilde{a}_1^0\big)\big),$$

for every $t \in [0, 1]$, and let

$$H(t) = \big\langle a^1(t), \ldots, a^{m+1}(t)\big\rangle.$$

Since $T_{m-1}(F) = \emptyset$ we have that $\dim H(t) = m$, and thus $H : [0, 1] \to T_m(F)$ is well defined. By construction, $a^0(t) = ta_0^0 + (1-t)a_1^0$ lies in the m-simplex generated by $\{a^1(t), \ldots, a^{m+1}(t)\}$, $t \in [0, 1]$, hence proving (a) and the first part of (b).

If $\{\gamma_1(t), \ldots, \gamma_{m+1}(t)\}$ are the barycentric coordinates of $a^0(t) = ta_0^0 + (1-t)a_1^0$ in the m-simplex generated by $\{a^1(t), a^2(t), \ldots, a^{m+1}(t)\}$, then, by linearity of Γ, $\{\gamma_1(t), \ldots, \gamma_{m+1}(t)\}$ are the barycentric coordinates of $t\tilde{a}_0^0 + (1-t)\tilde{a}_1^0$ in the m-simplex generated by $\{y_1(t\tilde{a}_0^0 + (1-t)\tilde{a}_1^0), \ldots, y_{m+1}(t\tilde{a}_0^0 + (1-t)\tilde{a}_1^0)\}$.

Let $\Gamma_1 : \mathbb{R}^{2m+1} \to \mathbb{R}^{m+1}$ be the linear map such that, for $j = 1, \ldots, m+1$, $\Gamma_1(e_j) = \Gamma_1(e_{j+m+1}) = e_j$. Then, for $t \in [0, 1]$,

$$\Gamma_1\big(t\tilde{a}_0^0 + (1-t)\tilde{a}_1^0\big) = \big(\gamma_1(t), \ldots, \gamma_{m+1}(t)\big).$$

By the linearity of Γ_1, $\Gamma_1(t\tilde{a}_0^0 + (1-t)\tilde{a}_1^0) = t\Gamma_1(\tilde{a}_0^0) + (1-t)\Gamma_1(\tilde{a}_1^0)$, thus obtaining, coordinate by coordinate, our desired conclusion:

$$\gamma_j(t) = t\gamma_j(0) + (1-t)\gamma_j(1).$$

This finishes the proof of Lemma 1. □

Lemma 2. *Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m+2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, such that $T_{m-1}(F) = \emptyset$. Then, for every face σ of the polyhedron C_{m+1}^m,*

$$\sigma \cap C_m(F) \text{ is convex.}$$

Proof. As in Section 3, let $\sigma = \{P, Q\}$, where $I = \{0, 1, \ldots, m+1\}$, $P = \{i_0, i_1, \ldots, i_p\} \subset I$, $Q = \{j_0, j_1, \ldots, j_q\} \subset I$, $P \cap Q = \emptyset$, $P, Q \neq \emptyset$. Let $[x_0^0, x_0^1, \ldots, x_0^{m+1}]$ and $[x_1^0, x_1^1, \ldots, x_1^{m+1}]$ be two points of $\sigma \cap C_m(F)$. Therefore, $\{x_\theta^{i_0}, x_\theta^{i_1}, \ldots, x_\theta^{i_p}\}$ generate a p-simplex Σ_θ^P and $\{x_\theta^{j_0}, x_\theta^{j_1}, \ldots, x_\theta^{j_q}\}$ generate a q-simplex Σ_θ^Q where $\Sigma_\theta^P \cap \Sigma_\theta^Q = \{a_\theta\}$ consists of a single point. Let $\{\gamma_\theta^{i_0}, \ldots, \gamma_\theta^{i_p}\}$ be the barycentric coordinates of a_θ in Σ_θ^P and $\{\gamma_\theta^{j_0}, \ldots, \gamma_\theta^{j_q}\}$ be the barycentric coordinates of a_θ in Σ_θ^Q. Furthermore, let H_θ^P be the p-plane that contains Σ_θ^P and H_θ^Q be the q-plane that contains Σ_θ^Q, $\theta = 0, 1$.

By Lemma 1, there are continuous functions:

$$H^P : [0, 1] \to G(n, p),$$
$$H^Q : [0, 1] \to G(n, q),$$
$$x^j : [0, 1] \to A^j, \quad j = 1, \ldots, m+1,$$
$$a : [0, 1] \to [a^0, a^1],$$

such that for $\theta = 0, 1$ and $j = 0, 1, \ldots, m + 1$,

$$H^P(\theta) = H^P_\theta, \qquad H^Q(\theta) = H^Q_\theta, \qquad x^j(\theta) = x^j_\theta,$$

and for $i \notin P \cup Q$ and $t \in [0, 1]$, $x^i(t) = tx^i_0 + (1 - t)x^i_1$ and $a(t) = ta_0 + (1 - t)a_1$.

Moreover, for every $t \in [0, 1]$, we have that $\langle x^{i_0}(t), \ldots, x^{i_p}(t) \rangle = H^P(t)$, $\langle x^{j_0}(t), \ldots, x^{j_q}(t) \rangle = H^Q(t)$, $H^P(t) \cap H^Q(t) = \{a(t)\}$ and that $a(t)$ lies in the p-simplex generated by $\{x^{i_0}(t), \ldots, x^{i_p}(t)\}$ and in the q-simplex generated by $\{x^{j_0}(t), \ldots, x^{j_q}(t)\}$.

Furthermore, if $\{\gamma_{i_0}(t), \ldots, \gamma_{i_p}(t)\}$ are the barycentric coordinates of $a(t)$ in the p-simplex generated by $\{x^{i_0}(t), \ldots, x^{i_p}(t)\}$ and $\{\gamma_{j_0}(t), \ldots, \gamma_{j_q}(t)\}$ are the barycentric coordinates of $a(t)$ in the q-simplex generated by $\{x^{j_0}(t), \ldots, x^{j_q}(t)\}$ then, for every $i \in P \cup Q$,

$$\gamma_i(t) = t\gamma_i(0) + (1 - t)\gamma_i(1).$$

Define the continuous map

$$H : [0, 1] \to T_m(F)$$

as follows: for every $t \in [0, 1]$, let

$$H(t) = \langle x^0(t), \ldots, x^{m+1}(t) \rangle.$$

Then,

$$\left\{ \left[x^0(t), \ldots, x^{m+1}(t) \right] \in \sigma \cap C_m(F) \mid t \in [0, 1] \right\}$$

is the closed interval in $\sigma = \{P, Q\}$ with extreme points $[x^0_0, x^1_0, \ldots, x^{m+1}_0]$ and $[x^0_1, x^1_1, \ldots, x^{m+1}_1]$. This concludes the proof of Lemma 2. $\quad\square$

Lemma 3. *Let K be a polyhedron, K' its barycentric subdivision and let X be a closed subset of K with the property that $\sigma \cap X$ is convex for every face σ of K. Let $|X|$ be the subpolyhedron of K' induced by the set of vertices $\{\sigma' \in K' \mid \sigma \text{ is a face of } K \text{ and } \overset{\circ}{\sigma} \cap X \neq \emptyset\}$. Then, $|X|$ has the same homotopy type of X.*

Proof. Let L be the set of all faces σ of K such that $\overset{\circ}{\sigma} \cap X = \emptyset$ and let $|L|$ be the subpolyhedron of K' induced by the vertices of L. Then, there is a strong deformation retraction $r : K' - |L| \to |X|$ which takes place through the linear structure of every simplex of K', because, for every simplex τ of K', τ is the join of $\{\sigma' \in \tau \mid \overset{\circ}{\sigma} \cap X \neq \emptyset\}$ and $\{\sigma' \in \tau \mid \overset{\circ}{\sigma} \cap X = \emptyset\}$. By convexity, the restriction of r to $X \subset K' - |L|$ is also a strong deformation retraction. This concludes the proof of Lemma 3. $\quad\square$

6. The main results

The purpose of this section is to prove our main results.

Proof of Theorem 1. Remember that the simplicial complex $\mathcal{OT}$ of all order types of $(m + 1)$ points in $\mathbb{R}^m$ is the first barycentric subdivision of the polyhedron C^m_{m+1}. Then,

by Lemmas 2 and 3, $C_m(F)$ has the homotopy type of $|\mathcal{OT}(F)|$. To finish the proof just remember that by Theorem 3, $C_m(F)$ has the homotopy type of $T_m(F)$.

Theorem 4. *Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m+2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, such that $T_{m-1}(F) = \emptyset$ and let σ be a face of the polyhedron C_{m+1}^m. Then,*

$$\left\{ \left(a^0, a^1, \ldots, a^{m+1}\right) \in A^0 \times A^1 \times \cdots \times A^{m+1} \mid \left[a^0, a^1, \ldots, a^{m+1}\right] \in \overset{\circ}{\sigma} \right\}$$

is contractible.

Proof. As in the proof of Theorem 3, let

$$\widetilde{F} = \left\{ \left(x^0, \ldots, x^{m+1}\right) \in A^0 \times \cdots \times A^{m+1} \mid x^j \in A^j \,, \dim\!\left\langle x^0, \ldots, x^{m+1}\right\rangle = m \right\}$$

and $\phi : \widetilde{F} \to C_m(F) \subset C_{m+1}^m$ be defined as $\phi((x^0, \ldots, x^{m+1})) = [x^0, \ldots, x^{m+1}]$. Then,

$$\phi^{-1}\!\left(\overset{\circ}{\sigma} \cap C_m(F)\right) = \left\{ \left(a^0, \ldots, a^{m+1}\right) \in A^0 \times \cdots \times A^{m+1} \mid \left[a^0, \ldots, a^{m+1}\right] \in \overset{\circ}{\sigma} \right\}$$

and furthermore, the inverse images of ϕ are convex in $A^0 \times \cdots \times A^{m+1}$. Since, by Lemma 2, $\overset{\circ}{\sigma} \cap C_m(F)$ is convex, then $\{(a^0, \ldots, a^{m+1}) \in A^0 \times \cdots \times A^{m+1} \mid [a^0, \ldots, a^{m+1}] \in \overset{\circ}{\sigma}\}$ is contractible. $\square$

Theorem 2 can be restated as follows:

Theorem 5. *Let $F = \{A^0, A^1, \ldots, A^{m+1}\}$ be a family of $(m+2)$ convex sets in $\mathbb{R}^n$, $n \geqslant m$, such that $T_{m-1}(F) = \emptyset$. Let σ be a face of C_{m+1}^m and let $T_\sigma(F) \subset T_m(F)$ be the set of all m-transversals that intersect the members of F consistently with the order type σ. Then $T_\sigma(F)$ is contractible.*

Proof. Let us consider, by Theorem 4, the contractible space $\{(a^0, \ldots, a^{m+1}) \in A^0 \times \cdots \times A^{m+1} \mid [a^0, \ldots, a^{m+1}] \in \overset{\circ}{\sigma}\}$. And let

$$\Phi : \left\{ \left(a^0, \ldots, a^{m+1}\right) \in A^0 \times \cdots \times A^{m+1} \mid \left[a^0, \ldots, a^{m+1}\right] \in \overset{\circ}{\sigma} \right\} \to T_\sigma(F)$$

be $\Phi(a^0, \ldots, a^{m+1}) = \langle a^0, \ldots, a^{m+1} \rangle$. Clearly Φ is a continuous surjective map. Moreover, if $H \in T_m(F)$, we have that $\Phi^{-1}(H)$ is precisely the set

$$\left\{ \left(a^0, \ldots, a^{m+1}\right) \in \left(A^0 \cap H\right) \times \cdots \times (A^{m+1} \cap H) \mid \left[a^0, \ldots, a^{m+1}\right] \in \overset{\circ}{\sigma} \right\}.$$

Therefore, by Theorem 4, when $n = m$, $\Phi^{-1}(H)$ is contractible for every $H \in T_\sigma(F)$ and hence Φ is a homotopy equivalence. This implies that $T_\sigma(F)$ is contractible. With this we conclude the proofs of Theorems 2 and 5. $\square$

References

[1] I.M. Gelfand, R.M. Goresky, R.D. Macpherson, V. Serganova, Combinatorial geometries, convex polyhedra and Schubert cells, Adv. in Math. 63 (1987) 301–317.

[2] J.E. Goodman, R. Pollack, Multidimensional sorting, SIAM J. Comput. 12 (1983) 484–507.

[3] J.E. Goodman, R. Pollack, R. Wenger, Geometric transversal theory, in: J. Pach (Ed.), New Trends in Discrete and Computational Geometry, Algorithms Combin., Vol. 1, Springer, Berlin, 1995, pp. 163–198.

[4] J. Arocha, J. Bracho, L. Montejano, D. Oliveros, R. Strausz, Separoids, their categories and a Borsuk–Ulam theorem for transversals, Preprint.