Chapter 3

Guarding traditional art galleries

In the classical Art Gallery Theorem, an art gallery is a simple polygon on the plane. In a more realistic setting, a *traditional art gallery* is housed in a rectangular building subdivided into rectangular rooms. Assume that any two adjacent rooms have a door connecting them. (See Figure 3.1.)

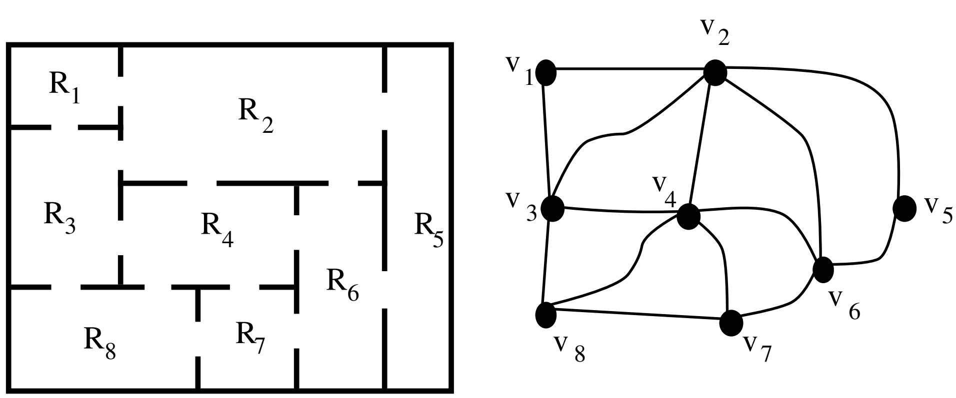

Figure 3.1: Traditional art gallery and its dual graph.

How many guards need to be stationed in the gallery so as to guard all its rooms? Notice that a guard who is stationed at a door connecting two rooms will be able to guard both rooms at once, and since no guard can guard three rooms, it follows that if the art gallery has n rooms we need at least $\lceil \frac{n}{2} \rceil$ guards. One of our objectives in this chapter is to prove:

Theorem 3.0.1 *[41] Any rectangular art gallery with n rooms can be guarded with exactly $\lceil \frac{n}{2} \rceil$ guards.*

3.1 A taste of matching theory

Some basic results in matching theory have been widely used in several results on illumination problems. To help the reader better understand the results about to be proved, a brief revisiting of these results is in order.

Given a graph G, a *matching* M of G is a subset of edges of G such that no two edges of M have a common vertex. A matching M of G is called *perfect* if each vertex of G is incident to an edge in M; see Figure 3.2.

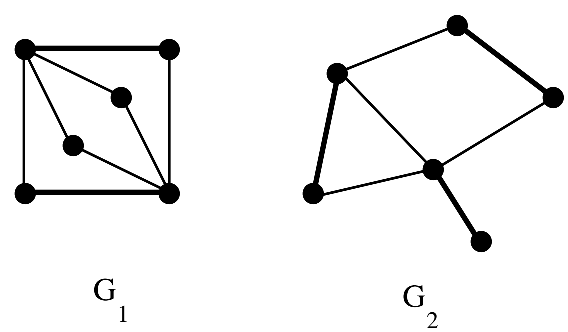

Figure 3.2: Graph G_1 has a matching with two edges, but no perfect matching. G_2 has a perfect matching.

Given a graph G and a subset S of the vertices of G, we define $Odd(S)$ to be the number of components of $G - S$ with an odd number of vertices. We now prove the following result due to W. T. Tutte which provides necessary and sufficient conditions for the existence of perfect matchings:

Theorem 3.1.1 *(Tutte [128]): A graph G has a perfect matching iff for every subset S of $V(G)$, $Odd(G - S) \leq |S|$.*

We first observe that if G has a subset of vertices S such that $Odd(G - S) > |S|$, then G cannot have a perfect matching. To see this we simply observe that in any perfect matching of G at least one element of an odd component of $G - S$ must be matched to a vertex in S. Thus if we have more odd components in $G - S$ than elements in S, a perfect matching cannot exist; see Figure 3.3.

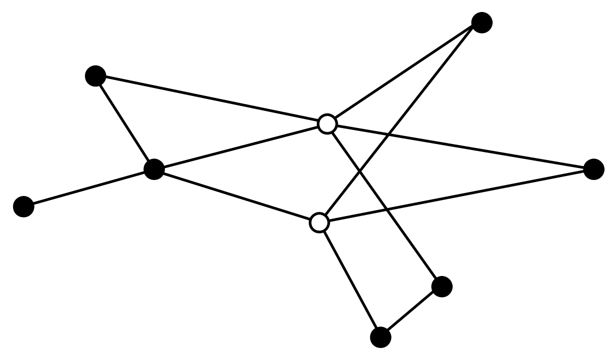

Figure 3.3: A graph with no perfect matching. The vertices of S are represented by small emty disks.

A graph G is called *saturated* if G has no perfect matching, but adding any other edge to G produces a graph that has a perfect matching.

Let S be the set of vertices of a saturated graph G adjacent to all the vertices of G, and let $T = V(G) - S$. We now prove:

Lemma 3.1.1 *Let u, v, w be elements of T such that u and w are adjacent to v. Then u and w are also adjacent.*

Proof: Assume that u and w are not adjacent. Since v is in T, there is a vertex x of G not adjacent to v. Let G' and G'' be the graphs obtained by adding the edges $u - w$ and $v - x$ to G respectively. Since G is saturated, G' and G'' have perfect matchings. Let M' and M'' be these matchings respectively. Since G has no perfect matchings, $u - w$ and $v - x$ are in M' and M'' respectively.

In the graph obtained by joining M' and M'' each vertex has degree two, and this graph consists of a set of even cycles with edges alternating between M' and M''. Two cases arise:

1. $u - w$ and $v - x$ belong to different cycles of $M' \cup M''$.

2. $u - w$ and $v - x$ belong to the same cycle.

Suppose then that $u - v$ and $v - x$ belong to different cycles in $M' \cup M''$. Let C be the cycle containing $u - w$. Then by deleting from M' all the edges of C in M' and replacing them by those edges of C in M'', we obtain a perfect matching of G' that does not contain $v - w$, which is a contradiction.

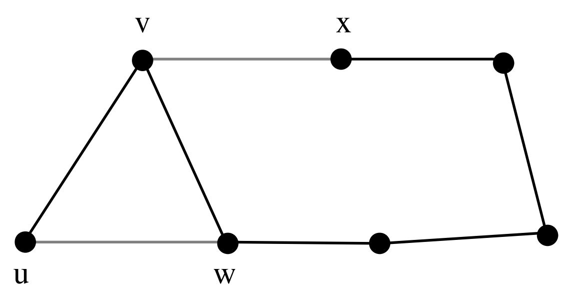

Figure 3.4: Graph G_1 has a matching with two edges, but no perfect matching. G_2 has a perfect matching.

If both edges belong to the same cycle C of $M \cup M''$, relabel the vertices of C $v_0 = v, v_1 = x, \ldots,$ according to the order in which they appear in C, and suppose without loss of generality that w is relabelled with a smaller label than v. Then since the edges of C alternate between M' and M'', $w = v_{2k+1}$ for some $k \geq 0$. However since v and w are adjacent, the cycle with vertices $v_0 = v, v_1 = x, \ldots, v_{2k+1}, v_0 = v$ is an even cycle C' such that every second edge of it belongs to M'; see Figure 3.4. By removingthe edges of C' in M' and replacing them by the edges of C' not in M' we obtain again a matching of G' that does not contain $v - x$, a contradiction. Therefore u and w are adjacent in G. ∎

We are now ready to prove Tutte's Theorem:

Proof: As mentioned before, if $Odd(G - S) > |S|$ then G cannot have a perfect matching. Suppose then that G has no perfect matching, and that G has an even number of vertices, otherwise if G has an odd number of vertices, take S to be the empty set, and our result follows.

We now show that G must contain a subset S of vertices such that $Odd(G - S) > |S|$. Let us add to G as many edges as possible to obtain a saturated graph G'. Since G is not a complete graph, G' is not complete either. As before, let S be the set of vertices of G' adjacent to all the vertices of G'. Let H_0 be the subgraph of G' induced by S. We now prove that G' can be decomposed into a set of complete subgraphs H_0, $H_1, \ldots, H_k$ such that $|S| = k - 2$. Moreover by Lemma 3.1.1, all the components og $G' - S$ are complete graphs. Let us label them $H_1, \ldots, H_k$.

If at most S components of $G' - S$ are odd, then a perfect matching of G' can be easily found contradicting that G' is saturated. Thus $G' - S$ has at least $S + 1$ components, and by a parity argument, it must have at least $S + 2$ odd components. If $G' - S$ contains more than $S + 2$ odd components, we can check that we can add to G' an extra edge connecting two odd components and the resulting graph still has no perfect matching, a contradiction.

Similarly, if $G' - S$ contains an even component, then any edge connecting it to another component of $G' - S$ can be added without creating a perfect matching, which contradicts that G' is saturated.

To finish our proof, we delete from G' the edges we added to G, and observe that in doing so, each odd component of $G' - S$ generates at least an odd component of $G - S$. Thus $Odd(G - S) > |S|$ and our result follows. ∎

3.2 The proof of Theorem 3.0.1

We now prove Theorem 3.0.1 using Tutte's Theorem.

Proof: Given a rectangular art gallery T with n rooms $R_1, \ldots, R_n$, we can associate to it a dual graph $G(T)$ by representing each room R_i of T by a vertex v_i in $G(T)$, two vertices being adjacent if their corresponding rectangles share a line segment in their common boundary. See Figure 3.1.

Notice that the boundary of the union of the rectangles corresponding to the vertices of any connected subgraph of $G(T)$ forms an orthogonal polygon, possibly with some orthogonal holes.

We now show that if $G(T)$ has an even number of vertices, $G(T)$ has a perfect matching M. This suffices to prove our result since our $\lceil \frac{n}{2} \rceil$ points can now be chosen using M as follows: for every edge $\{v_i, v_j\}$ of $G(T)$ in M station a guard at the door connecting R_i and R_j. Clearly these guards will cover all the subrectangles of T. The case when $G(T)$ has an odd number of vertices follows by subdividing any room of T into two.

Suppose then that $G(T)$ has an even number of vertices. We now show that $G(T)$ satisfies Tutte's Theorem. Let S be any subset of vertices of $G(T)$, and k be the number of connected components of $G(T)-S$. Each component C_i of $G(T)-S$ is represented by an orthogonal subpolygon P_i of T. Each such polygon has at least four corner points and thus the total number of corner points generated by the k components in $G(T)-S$ is at least $4k$; see Figure 3.5.

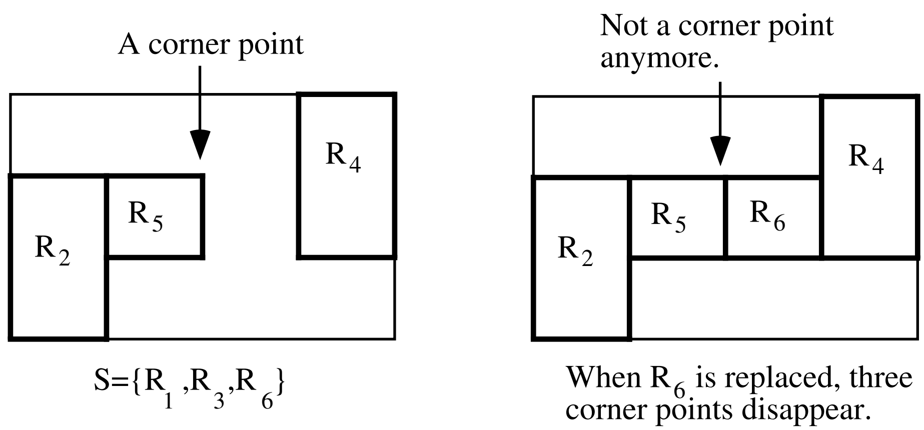

Figure 3.5: The components of $G-S$ are themselves orthogonal polygons.

The next observation is essential to our proof: When a rectangle represented by a point in S is now replaced, at most four corner points will disappear.

Once all rectangles in S are replaced, all the corner points generated by the components of $G(T)-S$ will disappear, except for the four corner points of T. It follows that $k \leq \mid S \mid +1$. The reader may verify that if $k = \mid S \mid +1$, then at least one of the components of $G(T)-S$ is even. Our result now follows. ∎

A different proof of Theorem 3.0.1 can be obtained by using another well known result on planar graphs. It is common practice in most museums to indicate to visitors, using arrow signs, a sequence in which the rooms of the museum may be visited without either missing or repeating a room. Our second proof of Theorem 3.0.1 shows that in fact, any traditional Art Gallery always has such a path. A graph G is called *hamiltonean* if there is a cycle that contains all the vertices of G. A graph has a *hamiltonean path* if there are two vertices u and v and a path starting at u and ending in v that contains all the vertices of the graph. We now prove:

Theorem 3.2.1 *Every rectangular art gallery has a hamiltonean path. Moreover, this path starts and ends at external rooms.*

To prove our result, we will use the following result due again to Tutte [129]:

Theorem 3.2.2 *Every planar* 4*-connected planar graph is hamiltonean.*

A graph is called k-connected if we need to delete at least k vertices of it to obtain a disconnected graph. We now prove:

Theorem 3.2.3 *Any rectangular art gallery with n rooms has a hamiltonean path.*

To prove our result, we will show that $G(T)$ has a hamiltonean path. We notice first that $G(T)$ is planar, but not necessarily connected. For example the graph arising from Figure 3.1 is 3-connected. To sove this problem we first *frame* our rectangular art gallery T using four rectangles as shown in Figure 3.6.

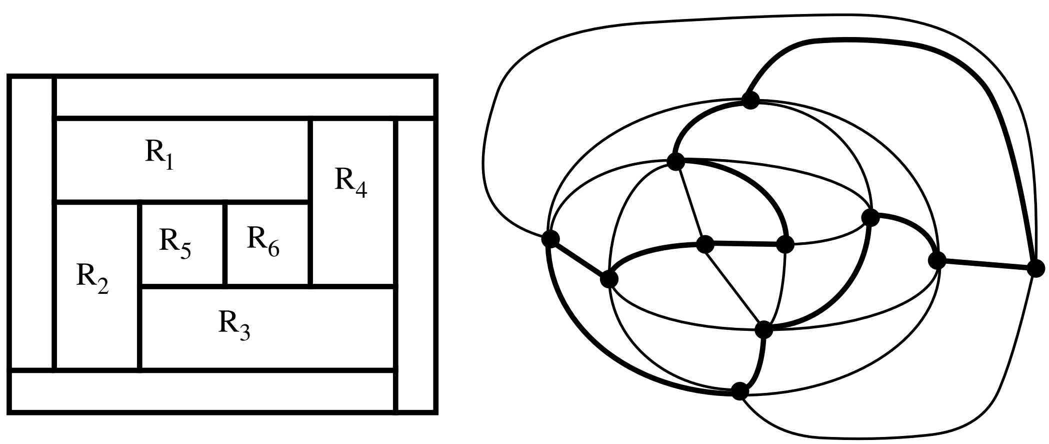

Figure 3.6: Framing a rectangular art gallery.

We now notice that the dual $G(T^*)$ of the resulting rectangular art gallery T^* in which we add a vertex to the unbounded external face is indeed 4-connected. Thus by Theorem 3.2.3 it is hamiltonean. This however is not enough, as the hamiltonean path of $G(T^*)$ guaranteed by Theorem 3.2.3 may induce two paths in $G(T)$. Referring again to Figure 3.6 the hamiltonean cycle of $G(T^*)$ shown in thick lines first visits R_2, R_5, R_6 and R_1, exits T and reenters it to visit R_4 and R_3, thus failing to produce a hamiltonean path for T. This can however be fixed as follows: Take three copies of T, and frame them as in Figure 3.7. The dual graph $G(T^{**})$ of the resulting configuration is again hamiltonean. Hovever, it is now easy to verify that the hamiltonean cycle of $G(T^{**})$ induces a hamiltonean path in at least one of the three copies of T used in our construction. That this path begins and ends at external rooms of T is obvious. Our result follows. ∎

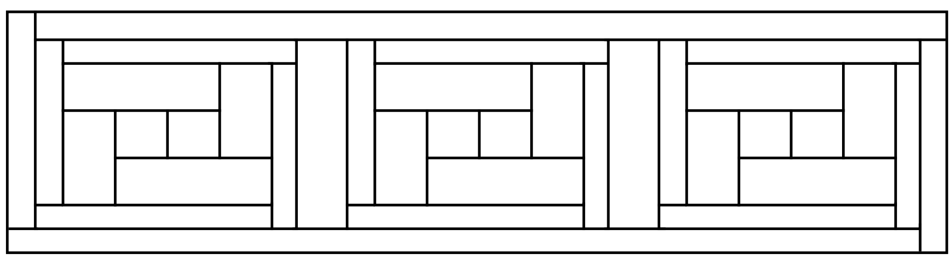

Figure 3.7: Framing three copies of a rectangular art gallery.

To show that Theorem 3.2.1 implies Theorem 3.0.1, we simply follow our hamiltonean path for T, and station a guard at the door connecting the first room we visit to the second, the door connecting the third to the fourth, etc. our result follows.

3.2.1 Algorithms

In this section we show that the placement of the $\lfloor \frac{n}{2} \rfloor$ guards to protect a rectangular art gallery can be done in linear time. Our result follows fron the following result due to N. Chiba and T. Nishizeki [27]

Theorem 3.2.4 *Finding hamiltonean cycles in 4-connected planar graphs can be done in linear time.*

By using Chiva and Nishizeki's algorithm on $G(T^{**})$, which is 4-connected, we have:

Theorem 3.2.5 *Guarding a rectangular art gallery with* $\lfloor \frac{n}{2} \rfloor$ *can be done in linear time.*

3.3 Protecting Families of Line Segments

We now study the following problem for families of sets. Given a collection $F = \{S_1, \ldots, S_n\}$ we say that a set of points $Q = \{p_1, \ldots, p_k\}$ protects F if every element S_i of F contains a point visible from some point p_j in Q.

A natural motivation for this concept is that a guard is protecting an art work S if it can see at least a point of it. In this case, a robber cannot remove S without our guard noticing it! In Figure 3.8 we show a family of 7 line segments that need two points to protect them. Notice that the smaller line segments cannot be protected by the same guard.

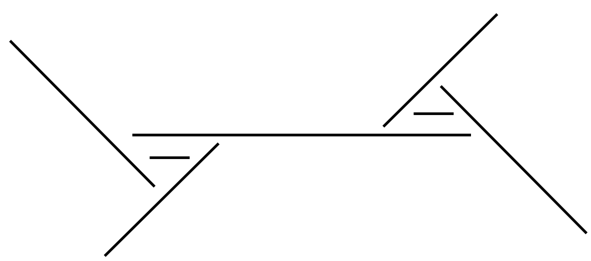

Figure 3.8: Two guards suffice.

Our main objective in this section is to prove the following result.

Theorem 3.3.1 *Any collection of* n *line segments can always be guarded using at most* $\lfloor \frac{n}{2} \rfloor$ *points;* $\lfloor \frac{2n-3}{5} \rfloor$ *points are occasionally necessary.*

Some preliminary results and definitions will be needed before we can prove our result.

Consider a collection $F = \{S_1, \ldots, S_n\}$ of n disjoint line segments on the plane. Construct a graph $G(F)$ with n vertices $v_1, \ldots, v_n$ such that v_i is adjacent to v_j if and only if there is a point x on the plane that sees at least a point in the boundary of each of $S_i$and S_j, i.e. x can protect both line

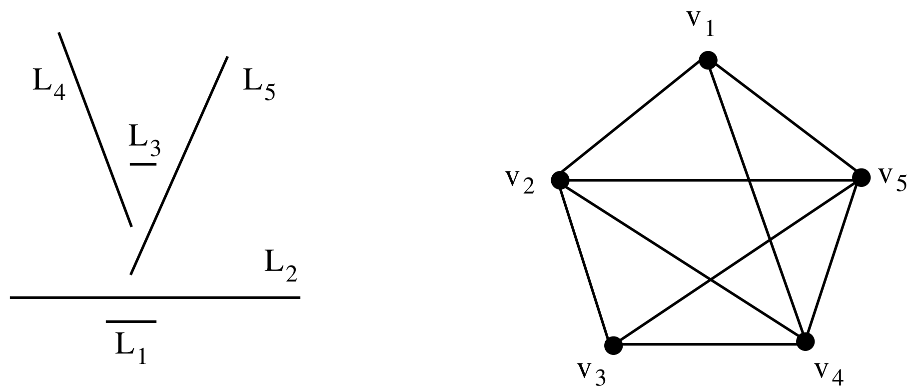

Figure 3.9: A family F of five line segments, and its corresponding graph $G(F)$.

segments. In Figure 3.9 a collection F of five segments and its corresponding graph $G(F)$ is shown.

The main idea in our proof of Theorem 3.3.1 is to show that for any family F with an even number n of disjoint segments, the graph $G(F)$ has a perfect matching.

The following lemma, given without proof, will be used to prove our main result.

Lemma 3.3.1 *Let Q be any convex polygon, $F = \{S_1, \ldots, S_n\}$ a family of n disjoint segments and let H be the subset of elements of F which intersect Q. The subgraph of $G(F)$ induced by the vertices of $G(F)$ representing elements in H is connected.*

We proceed now to prove Theorem 3.3.1.

Proof: Let $F = \{L_1, \ldots, L_n\}$ be a collection of n disjoint segments and let $G(F)$ be its associated graph. Assume that n is even, otherwise add another segment to F. We show that $G(F)$ satisfies Tutte's condition and thus has a perfect matching .

Consider any subset H of F and let S be the set of vertices of $G(F)$ representing the elements of H. We show that the number of connected components of $G(F) - S$ is at most $\mid S \mid = \mid H \mid$.

To start, delete from the plane all the segments *not* in H. One at a time, extend the elements of H until they meet another element of F, meet a previously extended element of F or become lines or semilines. Let π be the plane partition induced by the extended elements of H. It is easy to verify that π contains exactly $\mid H \mid +1$ polygonal faces. Replace the elements of F not in H; see Figure 3.10.

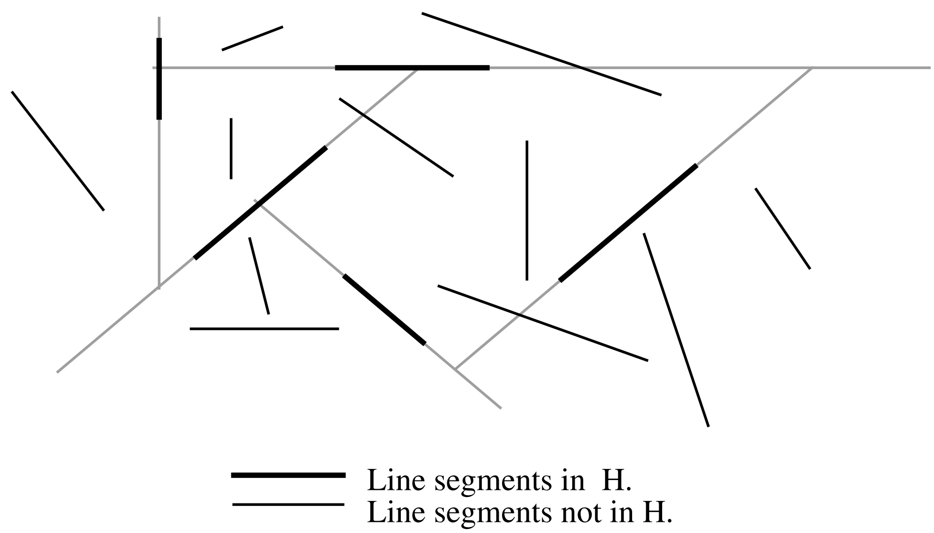

Figure 3.10: The partitioning generated by the elements in H.

By Lemma 3.3.1, the number of components of $G(F) - S$ is at most the number of faces of π, which is $\mid H \mid +1$. It can be easily verified that there are at least two adjacent faces in π such that the segments that intersect them are in the same component in $G(F) - S$. Thus the number of components in $G(F) - S$ is at most $\mid S \mid$, implying that $G(F)$ has a perfect matching M. It follows that F can be illuminated by a set, consisting of at most $\frac{n}{2}$ points, one for each edge of M.

To get the lower bound, construct an example of a family F with n segments in which $\lfloor \frac{2n-3}{5} \rfloor$ points are required to guard F, as follows.

Let H be a cubic plane graph with triangular outer face in which all the vertices, except the outer ones, are such that the three vectors emanating from the vertex along the edges positively span the plane. Let H have k vertices; it has $\frac{3k}{2}$ edges.

Substitute the edges of H by segments such that at each of the $k-3$ inner vertices we obtain a triangular face in which we insert a small segments; see Figure 3.11.

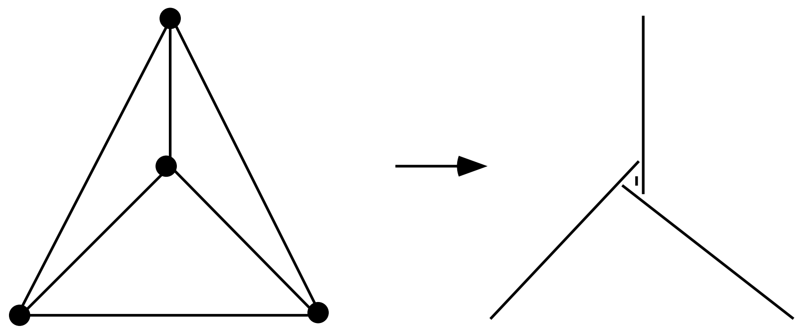

Figure 3.11: Generating a set of n line segments that require $\lfloor \frac{2n-3}{5} \rfloor$ points to protect them.

Discard the three edges of the outer face of H and disconnect each edge in a small neighborhood of its end vertices to form a collection of $n = (\frac{3k}{2}) - 3 + k - 3$ segments. No two of our $k-3$ small segments are visible from a single point, hence $k-3$ points are needed to guard the collection of segments. It is easy to verify that $k-3$ points are also sufficient. $k-3 = \lfloor \frac{2n-3}{5} \rfloor$, completing the proof of Theorem 3.3.1. ■

3.4 More on families of line segments

An attractive variation to guarding problems involving families of line segments was studied by J. O'Rourke [104].

Suppose we have a set of disjoint line segments $F = \{L_1, \ldots, L_n\}$ representing walls built anywhere on the plane, and we want to guard the whole plane. How many guards are always sufficient to achieve our goal?

We must first clarify the conditions set by O'Rourke in his result. A guard at a point p sees a point q on the plane if the line segment joining p to q does not *cross* any element of F. For this purpose, it is better to consider the elements of F as open line segments. Moreover, O'Rourke allows for the line segment joining p to q to be collinear with any element of F.

Under these restrictions, O'Rourke proved:

Theorem 3.4.1 $\lfloor\frac{2n}{3}\rfloor$ *guards are always sufficient and occasionally necessary to guard the plane in the presence of n line segment obstacles.*

We will assume that our line segments are in *general position*, i.e. we assume that no two of them are collinear, and that the lines generated by the elements of F are such that no three meet at a point. These conditions can easily be taken care of.

To prove our result, we use the following result due to Nishizeki [96] (see also [97]).

Theorem 3.4.2 *Let G be a 2-connected planar graph such that the degree of every vertex in G is at least 3. Then for all $n \geq 14$, G has a matching of size at least $\lceil\frac{n+4}{3}\rceil$. For $n < 14$, G has a matching of size $\lceil\frac{n}{2}\rceil$.*

We will not prove Nishizeki's result here. The interested reader may find the proof of this result in [97].

We can now prove Theorem 3.4.1.

Proof: Consider a family $F = \{L_1, \ldots, L_n\}$ of n line segments in general position. As we did in the proof of Theorem 3.3.1, one by one extend each element of F until it hits another element of F, or an extension of it, or becomes a semiline or a whole line.

This creates a partitioning of the plane into exactly $n+1$ convex polygons. Consider the dual of the obtained partitioning. If $n \geq 3$ this graph is 2-connected, however it may have vertices of degree 2; see Figure 3.12. It is easy to see that all vertices of degree 2 lie on infinite regions of our partitioning, and thus by adding an *artificial* vertex adjacent to all these vertices, we obtain a planar graph that satisfies Nishizeki's Theorem, and thus has a matching of size at least $\lceil\frac{((n+1)+1)+4}{3}\rceil = \lceil\frac{n+6}{2}\rceil$.

Notice that since two adjacent regions can be guarded with a single point, located possibly on a line segment of F, using $\lceil\frac{n+6}{2}\rceil$ points we can guard at least $2\lceil\frac{n+6}{2}\rceil - 1$ regions. The last -1 in the equation is to take into consideration the case when our artificial vertex is the endvertex of an edge in our matching. The remaining ones can be guarded with a point each. This gives a total of:

$$(n+1) - 2\lceil\frac{n+6}{2}\rceil - 1 + \lceil\frac{n+6}{2}\rceil = \lfloor\frac{2n}{3}\rfloor$$

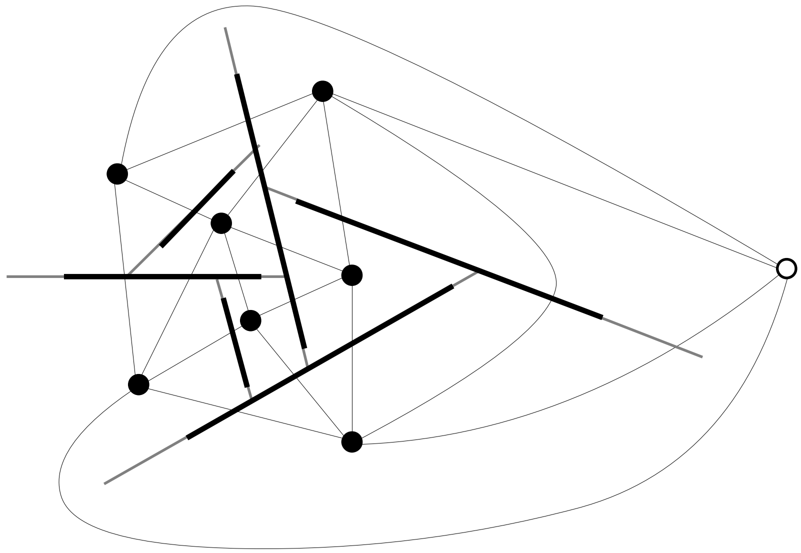

Figure 3.12: A convex partitioning of the plane generated by a set of line segments.

guards. We now show that there are families of line segments for which $\lfloor \frac{2n}{3} \rfloor$ points are required. We start again with a cubic graph G with m vertices drawn on the plane such that its external face is a triangle, and for all internal vertices, the edges incident to them span the plane. We again discard the edges in the outer face and modify the remaining edges of G in such a way that at each of the internal vertices we obtain a *triangular* face. Our configuration has exactly three line segments, say a, b and c that can be extended to infinity.

We now shorten each of the resulting line segments by a sufficiently small ammount, and add three more lines l_a, l_b and l_c close to the end points of a, b, and c respectively as shown in Figure 3.13.

Since G has exactly $\frac{m}{2}$ edges, our construction yields a total of $\frac{3m}{2}$ edges. If we place a point in the middle of each triangle in our previous construction, plus a point between each of a and l_a, b and l_b, and c and l_c, we obtain a set of m points such that no two of them can be guarded by a single guard. Our result now follows. ∎

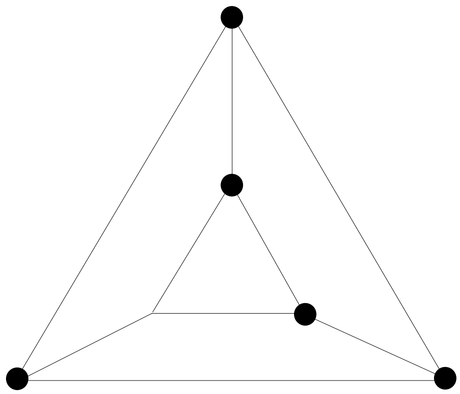
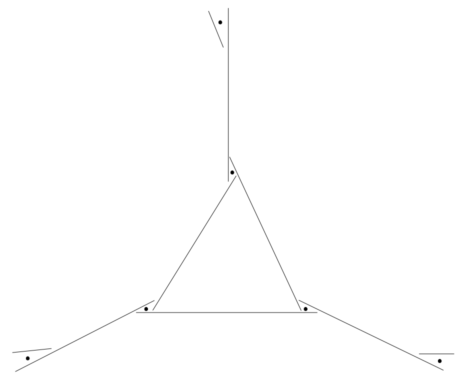

Figure 3.13: A family of n line segments for which $\lfloor \frac{2n}{3} \rfloor$ guards are necessary.

3.5 Illuminating line segments

We now turn our attention to the following problem studied by Czyzowicz, Rivera-Campo, Urrutia, and Zaks [40]: Instead of viewing our line segments as obstacles, let us try to illuminate them. Notice however that a point on a line can be illuminated by a ray hitting the line from either side of it. How many lights are needed to illuminate all the line segments? We prove:

Theorem 3.5.1 *Any collection F of n disjoint line segments can be illuminated with at most $\lceil \frac{2n}{3} \rceil - 3$ light sources.*

Proof: Let $F = \{L_1, \ldots, L_n\}$ be a family of n disjoint line segments. Choose a triangle T containing all of the elements of F in its interior and let F' be the family containing all of the elements of F together with three line segments L_{n+1}, L_{n+2} L_{n+3} obtained from T by shortening the three sides of T by an $\epsilon > 0$, ϵ sufficiently small; see Figure 3.14.

Construct a family $H = \{S_1, \ldots, S_n, S_{n+1}, S_{n+2}, S_{n+3}\}$ of $n + 3$ *strictly convex* compact sets with mutually disjoint interiors (our sets are allowed to touch each other at a single tangency point located in their boundaries) satisfying the following properties:

1. L_i is contained in S_i, $i = 1, \ldots, n + 3$.

2. The number of points at which pairs of elements of H are tangent is maximized.

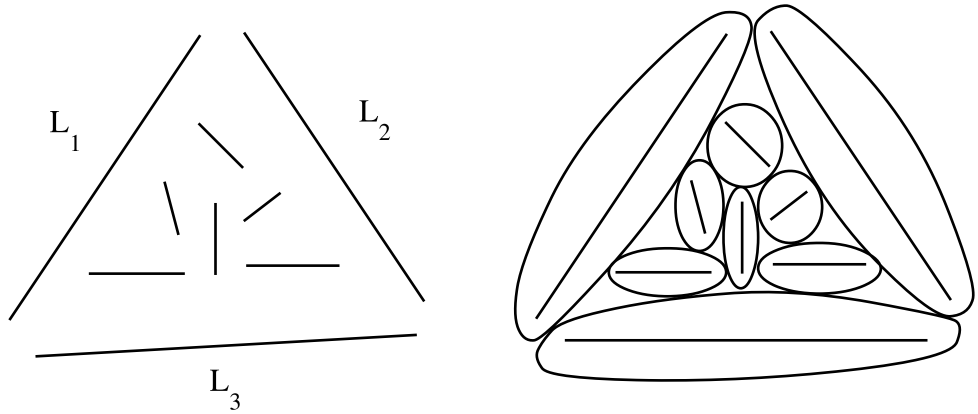

Figure 3.14: Illuminating line segments.

We can now see easily that every element in H is tangent to at least three elements in H. Construct a graph G as follows: For each element of H insert a vertex in G. Two vertices are adjacent if their corresponding sets in G are tangent; see Figure 3.15.

It is now easy to see that G is planar and 2-connected. In addition, since each set in H is tangent to at least three other elements in H the degree of each vertex in G is at least three. Then by Theorem 3.4.2, G has a matching M of size at least $\lceil \frac{n+3+4}{3} \rceil = \lceil \frac{n+1}{3} \rceil + 2$. For each pair of elements S_i, S_j matched in M by an edge of G, place a light source at the point in which they intersect. This light source will completely illuminate the line segments L_i and L_j contained in S_i and S_j respectively. Since M has at least $\lceil \frac{n+1}{3} \rceil + 2$ elements, $2(\lceil \frac{n+1}{3} \rceil + 2)$ elements of F will be illuminated using $\lceil \frac{n+1}{3} \rceil + 2$ lights. For the remaining elements of F, an extra light source per element is needed. Then the total number of lights required with our technique is:

$$(\lceil \tfrac{n+1}{3} \rceil + 2) + ((n+3) - 2(\lceil \tfrac{n+1}{3} \rceil)) = n + 1 - \lceil \tfrac{n+1}{3} \rceil \leq \lceil \tfrac{2n}{3} \rceil - 3. \qquad \blacksquare$$

In terms of lower bounds, we notice that the set of n lines obtained in Theorem 3.3.1 also requires $\lfloor \frac{2n-3}{5} \rfloor$ to illuminate them. If we also consider illumination of sets of line segments and semilines, we can obtain sets that require $\lceil \frac{n}{2} \rceil$ points to illuminate them as follows: Consider a regular polygon P_n with n vertices. In the clockwise direction, extend all the sides of P_n to

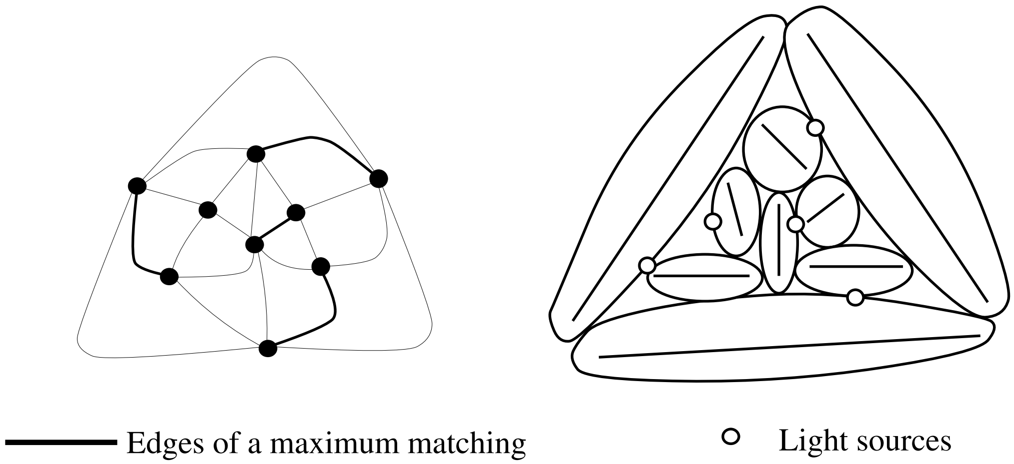

Figure 3.15: Placement of lights.

obtain a set of n semilines. Shorten them a bit at their finite endpoint to make them disjoint.

The gap between the lower and upper bounds for the illumination of line segments (or semilines for that matter) is large. A challenging problem is that of closing those gaps. For line segments, I believe that the correct bound is somewhere around $\lfloor \frac{2n-3}{5} \rfloor$. However even a proof that any family of n line segments can always be illuminated with $\lceil \frac{n}{2} \rceil + c$ lights has proved elusive, c a constant.

The case when our lines are *orthogonal*, i.e. parallel to the x- or y-axis is also open; see Exercise 4. We can easily prove that any set of n orthogonal line segments can always be illuminated with $\lceil \frac{n}{2} \rceil$ lights. For the lower bound, we can only produce families that require $\lfloor \frac{n}{3} \rfloor$ lights; see Figure 3.16.

3.6 Hiding behind walls

In this section, we study the following problem introduced by Hurtado, Serra and Urrutia [78]: Consider a set of disjoint line segments F. A collection of points P is called *hidden* with respect to F if any line segment joining two points in P intersects an element of F.

We now prove:

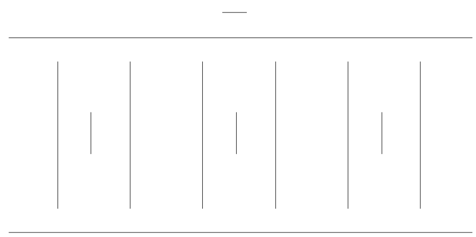

Figure 3.16: A set of orthogonal lines that requires $\lfloor \frac{n}{3} \rfloor$ lights.

Theorem 3.6.1 *Any family of n disjoint line segments has a hidden point set with at least $\sqrt{n}$ elements.*

To prove our result, we will use the following result of Erdös and Szekeres [48]:

Theorem 3.6.2 *Any sequence of n numbers has an increasing or a decreasing subsequence with at least $\sqrt{n}$ elements. The bound is tight.*

For example, consider the following sequence of numbers:

$\pi = \{4, 6, 3, 7, 2, 6, 1, 9, 8\}$

The numbers $4, 6, 7, 9$ form an increasing subsequence of π. Notice that the elements of the subsequence do not have to appear consecutively in π. Similarly $7, 2, 1$ form a decreasing subsequence of π. The proof of this result is left as an exercise to the reader. See exercise 6. Consider a set $F = \{L_1, ..., L_n\}$ of line segments such that their projections on the x axis form a disjoint collection of intervals. Let s_i be the slope of L_i, $i = 1, \ldots, n$. We now prove:

Lemma 3.6.1 *If $a_1 < \ldots, < a_n$ then F admits a hidden set of size n.*

Proof: For each L_i let p_i be its midpoint. Choose a point q_1 below p_i at distance ϵ from L_i, $i = 1, \ldots, n$. We now claim that if we choose ϵ small enough, $q_1, \ldots, q_n$ form a hidden set. Consider two integers $i < j$, and the line segment $L_{i,j}$ joining p_i to p_j. If a_i is smaller than the slope $a_{i,j}$ of $L_{i,j}$, then by choosing ϵ small enough, we can guarantee that the line segment joining q_i to q_j intersects L_i. If a_i is greater than or equal to $a_{i,j}$ the segment joining q_i to q_j will intersect L_j; see Figure 3.17. ■

We observe that Lemma 3.6.1 is also true when $a_1 >, \ldots, > a_n$, placing $q_1, \ldots, q_n$ *above* p_i, $i = 1, \ldots, n$.

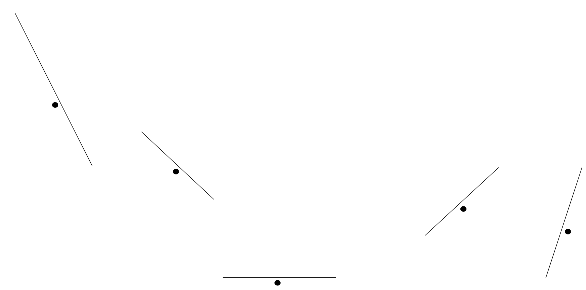

Figure 3.17: A family of 5 line segments with increasing slopes that has a hidden set of size 5.

We now prove Theorem 3.6.1:

Proof: Consider a family of n disjoint line segments. Pick a point p_i in the interior of each L_i in such a way that the x-coordinates $x_1, \ldots, x_n$ of $p_i, \ldots, p_n$ are all different. Assume without loss of generality that $x_1 <, \ldots, < x_n$. For each L_i, choose a segment L'_i contained in it and centered at p_i in such a way that $L'_1, \ldots, L'_n$ have disjoint projections on the x-axis. Let us consider now the sequence of slopes of $L'_1, \ldots, L'_n$. Then by Theorem 3.6.2 this sequence contains an increasing or decreasing subsequence with at least $\sqrt{n}$ elements. Our result now follows from Lemma 3.6.1. ■

3.7 Excercises

Ex. 3.1 Prove that if an orthogonal polygon with k reflex vertices is subdivided into n rectangular rooms as in Theorem 3.0.1 it can be guarded with at most $\lfloor \frac{n+k}{2} \rfloor$ guards. Show that this bound is tight.

Ex. 3.2 Show that Theorem 3.0.1 extends to decompositions of boxes in R^3 into boxes, such that no two of these boxes meet at a single point.

Ex. 3.3 Suppose that we choose to station a set of guards in a rectangular art gallery according to the following greedy algorithm:

> While not all the rectangles of T are guarded, choose, if possible, a pair of adjacent rectangles, guard them with a guard, and delete them them from $G(T)$. Once this can no longer be done, the collection of subrectangles left form an independent set in the dual graph $G(T)$. Guard each of these rectangles with a guard.

Prove that using this algorithm will produce a guarding set with at most $\lfloor \frac{3n}{4} \rfloor$ guards. Show that in some cases, our algorithm will actually choose $\lfloor \frac{3n}{4} \rfloor$ guards.

Ex. 3.4 Prove that any family of n isothetic line segments, i.e. parallel to the x- or y-axis, can be illuminated with at most $\lfloor \frac{n+1}{2} \rfloor$ guards. *Hint: Enclose the segments within a rectangle and then partition it into* $n+1$ *rectangles. Now use Tutte's Theorem. Be careful, in this case we cannot assume that all adjacent faces are connected in the dual graph!*

Ex. 3.5 (Jennings and Lenhart [80] Prove that any family F of n disjoint line segments always has a subset S with at most $\lfloor \frac{n}{2} \rfloor$ elements such that every element in F is in S or is visible from at least one point on an element in S. The bound is tight.

Ex. 3.6 Prove Theorem 3.6.2. *Hint: Let* $\pi = \{p_1, \dots, p_n\}$ *be a sequence of numbers. Associate to each* p_i *a point on the plane with integer coordinates* (a_i, b_i) *such that* a_i *is the length of the longest increasing subsequence of elements of* π *that ends in* p_i*, and* b_i *is the lenght of the longest decreasing subsequence of* π *that* starts *at* p_i*. Show that if* $i \neq j$ *then* $(a_i, b_i) \neq (a_j, b_j)$*. Conclude from this that for some* i*,* a_i *or* b_i *is at least* $\sqrt{n}$.

Ex. 3.7 (Boenke and Shermer [104])Show that if we place n disjoint line segments on the plane, and we restrict our guards to be located only within our line segments, then n guards are always sufficient, and ocassionally necessary to guard the plane.