Chapter 4

Floodlight illumination problems

In the previous chapters, we have assumed that the light sources emit light in all directions, or that the guards can patrol around themselves in all directions. We now present an illumination problem in which the light sources have a restricted angle of illumination. We call such light sources *floodlights*. Thus for the rest of this paper, a floodlight f_i is a source of light located at a point p of the plane, called its apex; f_i illuminates only within a positive angle of illumination α_i, and can be rotated around its apex. We start by study the following problem due to J. Urrutia:

Problem 4.0.1 The 3-floodlight illumination problem: *Let $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ and consider any convex polygon P. Can we place three floodlights of sizes at most $\alpha_1, \alpha_2, \alpha_3$, no more than one per vertex, in such a way that P is completely illuminated?*

We now show that the Three Floodlight Illumination Problem always has a positive solution. Clearly our result is true if P has 3 vertices. Consider any convex polygon P with at least four vertices and suppose that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Notice first that $\alpha_2 < \frac{\pi}{2}$ and that since P has at least four vertices, the interior angle at one vertex v of P is at least $\frac{\pi}{2}$. Find a triangle T with internal angles α_1, α_2, and α_3 such that the vertex of T with angle α_2 lies on v, and the other vertices of T lie on two points x and y on the boundary of P. Suppose that x and y lie on different edges, say e_1 and e_3 of P. (The case when they lie on the same edge will be left to the reader.) See Figure 4.1.

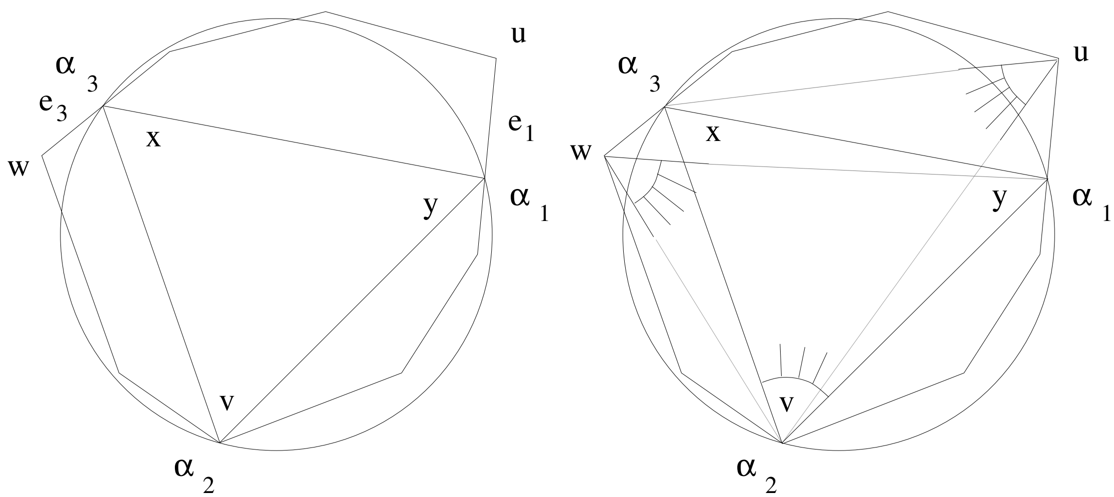

Figure 4.1: Illustration of proof of the Three Floodlight Problem.

Place a floodlight f_2 with angle of illumination α_2 at v, illuminating T. Consider the circle C passing through the vertices of T. It is easy to see that at least one vertex of each of e_1 and e_3 is not contained in the interior of C. Let u and w be these end points. Two cases arise:

1. $u \neq w$. Place floodlights f_1 and f_3 at u and w illuminating the angular region determined by v, u, x and v, w, y. Since u and w are not contained in the interior of C, the angles of illumination of f_1 and f_3 are at most α_1 and α_3 respectively. Since f_1, f_2, and f_3 illuminate P, our result follows.

2. $u = w$. By considering the tangents to C at x and y it is easy to verify that the angle α of P at u is at most $\pi - 2\alpha_2$ which is less than or equal to $\alpha_3 = \pi - (\alpha_1 + \alpha_2)$. Place a floodlight of size α at u. This illuminates P. ∎

4.1 Illuminating the plane

We now study the following floodlight illumination problem of the plane studied in [14]. Suppose we have four points $p_1, \ldots, p_4$ on the plane and four

$\frac{\pi}{2}$ floodlights, one at each p_i. Can we orient them in such a way that all of the plane is illuminated? (We assume here that our lights cast no shadows.) To see that under these conditions the plane can always be illuminated, consider a line that leaves two points on one side of it. Assume without loss of generality that the line is parallel to the x-axis. It is easy to see now that using the floodlights on top of l, we can illuminate the region below it, and using the lights below l, we can illuminate the region above it, as in Figure 4.2. It follows that the number of solutions to our plane illumination problem is actually infinite!

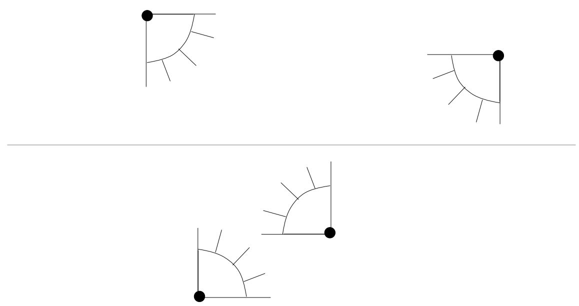

Figure 4.2: Four $\frac{\pi}{2}$ floodlights suffice to illuminate the plane.

We now prove an interesting generalization of the previous result. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of angles such that each of them is at most π and $\alpha_1 + \ldots \alpha_n = 2\pi$. Consider a set of floodlights $f_1, \ldots, f_n$ such that the size of f_i is α_i, $i = 1, \ldots, n$.

Theorem 4.1.1 *Let P_n be a collection of n points on the plane, and $f_1, \ldots, f_n$ a set of floodlights of sizes $\{\alpha_1, \ldots, \alpha_n\}$ such that $\alpha_1 + \ldots + \alpha_n = 2\pi$. Then we can locate all our floodlights, one per element of P_n and point them in such a way that the plane is completely illuminated.*

Some preliminary results will be needed to prove our result. They are of interest in their own right. Suppose that we have three angles $\alpha_1, \alpha_2, \alpha_3$ such that their sum is 2π, and each of them is at most π. Consider any point set P_n with n points, and three integers n_1, n_2, n_3 such that $n_1 + n_2 + n_3 = n$. We now prove:

Theorem 4.1.2 *There is a point p such that the plane can be split into three infinite angular regions by three rays emanating from p such that the angles between the rays are $\alpha_1, \alpha_2, \alpha_3$ and the region with angle α_i contains exactly n_i elements of P_n in its interior.*

Proof: Choose the three directions 0, α_1, and $\alpha_1 + \alpha_2$, and assume without loss of generality that the angle of inclination of any line connecting two elements of P is not 0, α_1, or $\alpha_1 + \alpha_2$. If an element q of P is such that the three rays emanating from it at angles 0, α_1, and $\alpha_1 + \alpha_2$ partition the plane in three regions that contain i, j,and k elements such that two of these values are equal to n_1, n_2, or n_3 in this order, say $i = n_1$, $j = n_2$, then by choosing p slightly above q our result follows. Assume then that this is not the case.

Assign to each point with coordinates (x, y) the vector (i, j, k) in which i is the number of elements q of P_n with the property that the direction d of the line connecting p to (x, y) is $0 \leq d < \alpha_1$. The integer j is the number of elements of P_n with $\alpha_1 \leq d < \alpha_1 + \alpha_2$, and $k = n - (i + j)$. If (x, y) is in P_n, it will be considered to belong to the region bounded by the rays emanating from it with directions $\alpha_1 + \alpha_2$ and 2π.

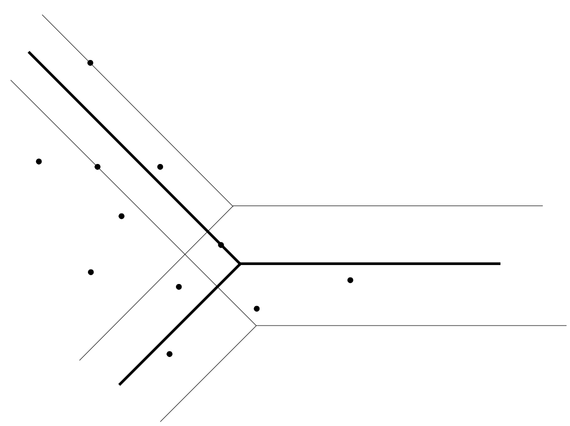

Figure 4.3: Finding a partitioning of the plane for $n = 11$, $n_1 = 2, n_2 = 6$, and $n_3 = 3$, $\alpha_1 = \frac{3\pi}{4}$, and $alpha_2 = \frac{\pi}{2}$ The solution appears in thick lines.

We observe now that for each y we can choose x large enough that $(i, j, k) = (0, n, 0)$. As x decreases in value and y remains constant, the second coordinate of (i, j, k) decreases one by one until it becomes n_2. Let

$f(y)$ be the value such that if $x < f(y)$, then $j < n_2$. It is easy to verify that the function defined by $(f(y), y)$ is a piecewise continuous linear function. Moreover if we choose y_0 small enough, then for the point $(f(y_0), y_0)$ we have that $(i, j, k) = (n - n_2, n_2, 0)$ and that for y_1 large enough, we have $(i, j, k) = (0, n_2, n - n_2)$. It now follows that by increasing y_0 continuously, the value of the first coordinate of (i, j, k) decreases one by one, until it reaches the value n_3; see Figure 4.3. Our result follows. ■

Let $\alpha \leq \pi$. An *α-wedge* of the plane is a region W bounded by two rays emanating from a point p, called the apex of W, such that the angle between the rays generating W is α; α is called the size of W. We define W^{-1} to be the wedge with apex at p but generated by rays emanating from p in opposite directions from those that generated W; see Figure 4.4.

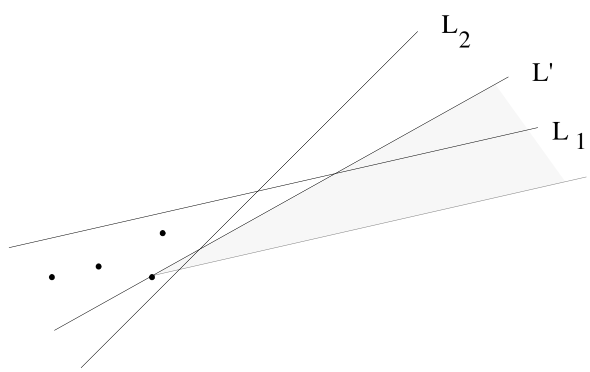

Figure 4.4: Illuminating a wedge.

We now prove:

Lemma 4.1.1 *Consider an α wedge W, and set of floodlights with sizes $\{\alpha_1, \ldots, \alpha_m\}$ such that $\alpha_1 + \ldots + \alpha_m = \alpha$, and a set of points P_m contained in W^{-1}. Then we can always illuminate W by placing a floodlight at each element of P_m.*

Proof: Our result is clearly true if $m = 1$. Suppose then that $m > 1$. Suppose that W and W^{-1} are opposite regions bounded by two lines L_1 and L_2 that intersect at a point p, and that W is above L_1 as shown in Figure 4.4. Consider a line L' that intersects L_1 at an angle α_1, contains exactly one point

q of P_m, and all the remaining elements of P_m lie on or above l'. Notice that L' and L_1 define a wedge W_1 of size α_1 such that q belongs to W_1'. W_1 can now be illuminated from q using a floodlight of size α_1. Notice now that L' also defines with l_2, a wedge W" of size $\alpha - \alpha_1$, and all the elements of $P_m - q$ lie on W''^{-1}. By induction, we can now illuminate W''^{-1} with the floodlights of sizes $\alpha_2, \ldots, \alpha_m$, and our result follows. ∎

We are ready to prove Theorem 4.1.1. Partition our set of floodlights into three sets $F_1 = \{f_1, \ldots, f_i\}$, $F_2 = \{f_{i+1}, \ldots, f_j\}$ and $F_3 = \{f_{j+1}, \ldots, f_n\}$ such that $\beta_1 = \alpha_1 + \ldots + \alpha_i$, $\beta_2 = \alpha_{i+1} + \ldots + \alpha_j$ and $\beta_3 = \alpha_{j+1} + \ldots + \alpha_n$ are all less than π. Let $n_1 = i$, $n_2 = j - i$, and $n_3 = n - j$. By Theorem 4.1.2, we can partition the plane into 3 wedges W_1, W_2 and W_3 of sizes β_1, β_2 and β_3 such that each of them contains exactly n_1, n_2 and n_3 points. By Lemma 4.1.1, we can now illuminate W_1^{-1}, W_2^{-1} and W_3^{-1} by using the points in W_i, and using the floodlights at F_i, we can illuminate W_i^{-1}, $i = 1, \ldots, 3$. However $W_1^{-1} \cup W_2^{-1} \cup W_3^{-1}$ covers the plane! ∎

4.2 Illumination of stages

We now study the following problem introduced by J. Urrutia [14] in 1990. Suppose that we are testing the set of stage lights of a new theater, with a number of stage lights. The *Stage illumination problem*, consists in determining if the set of stage lights of the theater are enough to illuminate all of the stage at once. In mathematical terms we cast this problem as follows:

Suppose we have a line segment, representing the stage of a theater, and a set of stage lights, modeled by floodlights. Is it possible to point our floodlights in such a way that the line segment is completely illuminated?

The existence of an efficient algorithm to solve the stage illumination problem remains open. We now proceed to solve a problem that at first appears harder:

Given a set $F = \{f_1, \ldots, f_n\}$ of floodlights each with angle $\alpha_i, i = 1, \ldots, n$ we can associate to F an angular cost equal to $\alpha_1 + \ldots + \alpha_n$. We now solve the following problem:

Optimal floodlight illumination of stages Consider a stage, represented by a line segment S and a set $P = \{p_1, \ldots, p_n\}$ of n points. Determine a set of

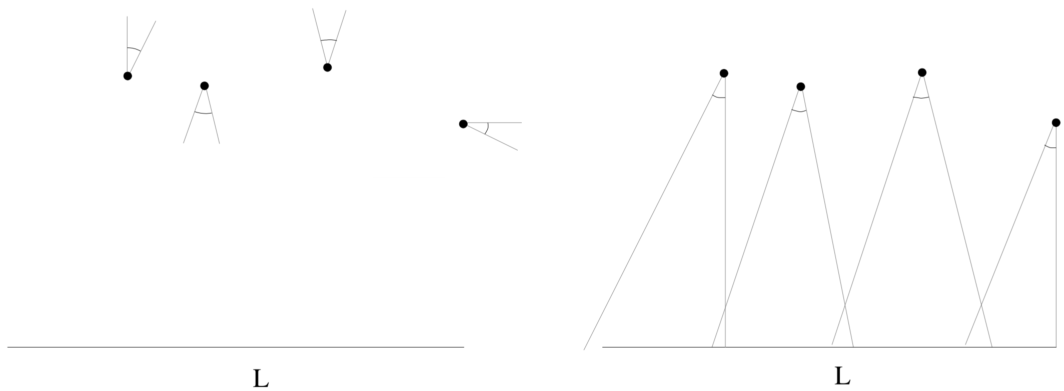

Figure 4.5: The stage illumination problem.

floodlights F that illuminates S such that the angular cost of F is minimized and the apex of each floodlight $f_i \in F$ is located at some point $p_j \in P$.

We point out here that we will allow for more than one floodlight to be located at each point of P. Moreover, we assume that each floodlight has size strictly greater than 0. Interestingly enough, our solution is such that only at one point in P we may place two lights; at the other points we place one or no light at all.

4.2.1 Illuminating the real line

Our first step consists in solving the problem of illuminating the real line using floodlights placed on the elements of a point set $P = \{p_1, \ldots, p_n\}$. To make our presentation easier, we will assume that all of our points have different y-coordinates. This restriction can be easily taken care of.

We study now the problem of illuminating the real line using only two points p_i and p_j. Assume that p_i is lower than p_j. Consider the two circles tangent to the real line and containing p_i and p_j, and let their intersection points with the real line be labelled $x_{i,j}$ and $y_{i,j}$ in such a way that $x_{i,j} < yi, j$; see Figure 4.6(a).

We now prove:

Lemma 4.2.1 *In the optimal floodlight illumination of the real line L from $\{p_i, p_j\}$ all points in the interval $[x_{i,j}, y_{i,j}]$ are illuminated from p_j, and all points in the intervals $(\infty, x_{i,j}]$ and $[y_{i,j}, \infty)$ are illuminated from p_i.*

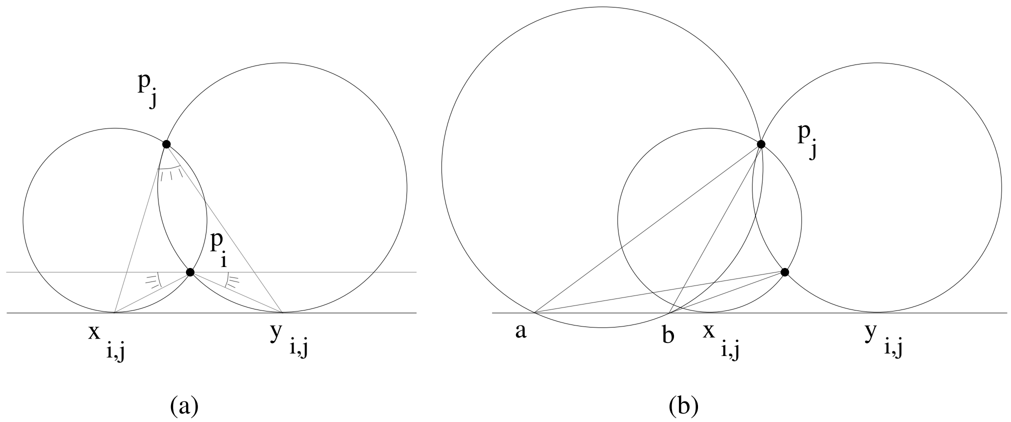

Figure 4.6: Illuminating the real line from two points.

Proof: Suppose that in an optimal illumination of the real line with a set of floodlights F, an interval with endpoints $a < b$ contained to the left of $x_{i,j}$ is illuminated by a small floodlight f at p_j. Consider the circle passing through a, b and p_j. It is easy to see that this circle leaves p_i outside, and thus the angle a, p_i, b is smaller than the angle a, p_j, b. Thus if we substitute f by a floodlight f' at p_i we obtain a set of lights that illuminates the real line with smaller weight than that of F, which is a contradiction. In now follows that all the points to the left of $x_{i,j}$ are illuminated from p_j. Similarly we can conclude that the interval $[x_{i,j}, y_{i,j}]$ is illuminated from p_j and $[y_{i,j}, \infty)$ is illuminated from p_i, see Figure 4.6. ∎

Next we prove:

Lemma 4.2.2 *Let $P = \{p_1, \ldots, p_n\}$ be a collection of points, and p_i a point in the interior of the convex hull of P. Then in any optimal illumination of the real line with floodlights at points of P, there is no floodlight located at p_i.*

Proof: Suppose that p_i is an interior point of the convex hull of P, and that there is an optimal illumination of the real line in which a floodlight f_i placed at p_i illuminates an interval, say $[x, y]$. Consider the smallest disk D containing x, y and all of the elements of P, see Figure 4.7.

Let p_j be a point of P located in t he boundary of D. Since p_i is in the interior of the convex hull of P, $p_i \neq p_j$; moreover p_i belongs to the interior of

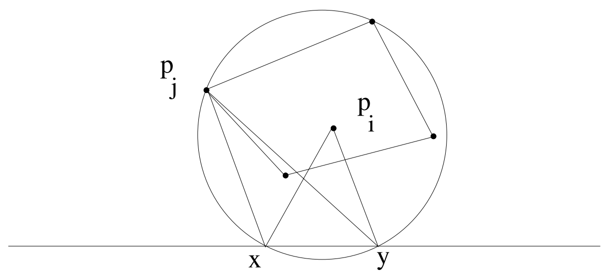

Figure 4.7: Only points in the convex hull are useful.

D. Therefore the angle x, p_i, y is greater than angle x, pj, y. Thus we could substitute the floodlight at p_i that illuminates the interval $[x, y]$ by a smaller one placed at p_j that illuminates the same interval . This contradicts our assumption on the optimality of F. ∎

The following corollary now follows:

Corollary 4.2.1 *Consider an optimal floodlight illumination of the real line wit a set of floodlights F on a set of points P. Then if a point x is illuminated by a floodlight of F located at a point p_i of P, the disk circle C tangent to the real line and containing p_i contains all of the elements of P.*

It now follows that the optimal solution to the floodlight illumination problem of the real line can be obtained as follows: Consider the leftmost point $y_{i,j}$ generated by all pairs of points in the convex hull of P, and obtain the circle tangent to the real line and passing throuh p_i and p_j. Slide to the left a point p initially located at $y_{i,j}$. While we move p to the left, maintain a circle C tangent to the real line at p, and tangent to the convex hull of P. For any position of p, the vertex p_k on the convex hull of P and C, is the point were the floodlight that will illuminate p must be located, see Figure 4.8. Notice that at a finite set of positions of p, C will contain two points in the convex hull of P. These points correspond to the beginning and ending of the intervals to be illuminated by the elements of P, this situation is illustrated in Figure 4.8.

we now proceed to develop an algorithm to solve our floodlight illumination problem:

Algorithm 4.2.1 *FLIP*

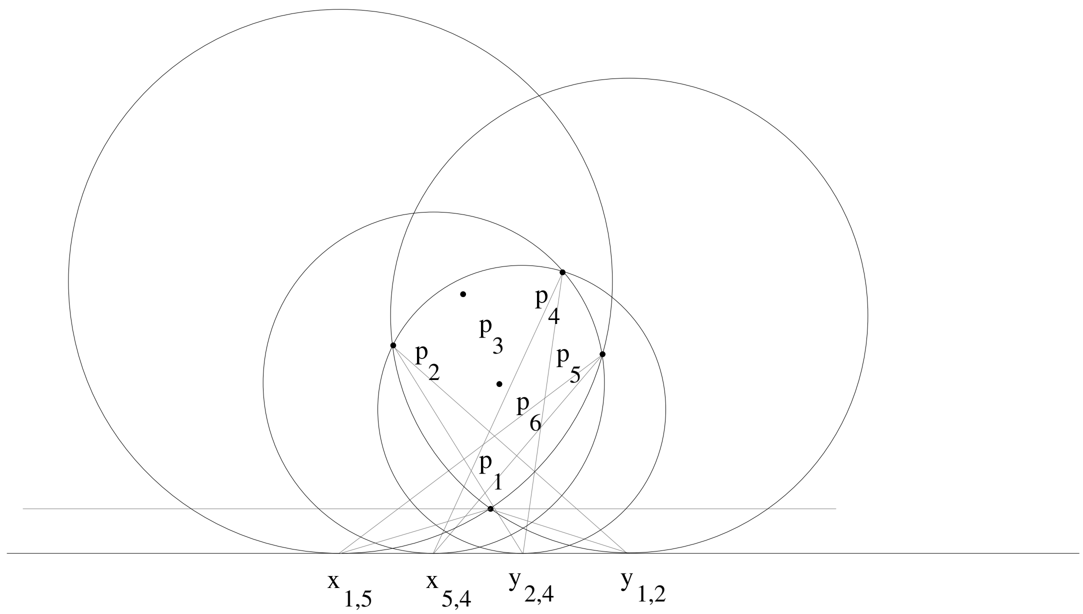

Figure 4.8: Optimal illumination of the real line.

Input : *A set $P = \{p_1, \ldots, p_n\}$ of n points on the plane with positive y-coordinates, and the real line L.*

Output : *A partitioning of the lines into a sequence of at most $n+1$ intervals, each assigned to a point of P from where that interval is to be illuminated.*

1. *Calculate the convex hull of P. Relabel the vertices of the convex hull of P in the clockwise direction by $\{p_1, \ldots, p_k\}$ where p_1 is the point of P closest to L and k is the number of vertices in the convex hull of P.*
2. *Determine the point $y = y_{1,i}$ which is the rightmost point in $\{y_{1,j}; j = 2, ..., n\}$. Illuminate all of the points in the interval $[y, \infty)$ from p_1.*

   While $i1$ do:

   (a) *Find the smallest index $j > i$ (or take $j = 1$ if no such j exists) such that the disk D defined by the circle tangent to L containing p_i and p_j contains all the elements of P. Let x be the point in which C is tangent to L.*

(b) *Illuminate the interval* $[x, y]$ *from* p_i.

(c) $i \leftarrow j$, $y \leftarrow x$

EndWhile

3. : *Illuminate the interval* $(-\infty, y]$ *from* p_1. **Stop**

For example in Figure 4, y initially takes the value $y_{1,2}$, and i the value 2. In the next iteration, y changes to $y_{2,4}$ and i to 4. Notice that even though p_3 is a vertex in the convex hull of $P = \{p_1, \ldots, p_6\}$, no interval is illuminated by p_3. This happens because the circle tangent to L through p_2 and p_3 does not contain p_4, and in the execution of our While loop for $i = 2$, j skips the value 3. The subsequent values for y are $x_{5,4}$ and $x_{1,5}$ and the values for i are 5 and 1 respectively. Thus all the points in the intervals $(-\infty, x_{1,5}]$ and $[y_{1,2}, \infty)$ are illuminated from p_1, and $[y_{2,4}, y_{1,2}]$, $[x_{5,4}, y_{2,4}]$ and $[x_{1,5}, x_{5,4}]$ are illuminated from p_2, p_4 and p_5 respectively.

4.3 Floodlight illumination of polygons

Let us consider a polygon P with n vertices. Observe that if we are allowed to place $n-2$ floodlights of size $\frac{\pi}{3}$ at the vertices of P, we can always do so in such a way that P is completely illuminated. This follows from the observation that any triangulation of P has $n-2$ triangles, each of which in turn has an internal angle of size at most $\frac{\pi}{3}$. Observe, however, that using this rule to illuminate P, we may place more than one floodlight at some vertices of P. If we allow a floodlight of size at most π at each vertex, we can indeed illuminate all of P. If P is a triangle, we can illuminate it by placing a floodlight at any of its vertices of size at most π. Suppose then that P has more than 3 vertices. Take any triangulation T of P, choose an ear e of T. Let v be the vertex of e of degree 2 in T. Place a floodlight of size equal to the internal angle of P at v. Delete v and by an inductive argument we can illuminate $P - v$. Our result follows.

A natural question now arises: is it possible to illuminate any polygon by placing at each vertex of P at most one floodlight of size at most α for some $\alpha < \pi$? We will now prove that for arbitrary polygons, the answer to this question is, surprisingly, no. We now prove:

Theorem 4.3.1 *For any* $\epsilon > 0$ *there is a polygon* P *that cannot be illuminated by placing at most one floodlight of size* $\pi - \epsilon$ *at each of its vertices.*

Proof: We will prove our result only for orthogonal floodlights. The proof presented here is due to O'Rourke and Xu [105]. The full result follows easily from the ideas presented here, and is left as an exercise. The interested reader can find the complete proof in [49]. Consider the symmetric polygon P shown in Figure 4.9. Divide P into a left and right section by cutting it along a vertical through vertex v. Notice that all the convex internal angles at the vertices of P are slightly bigger than $\frac{\pi}{2}$. Label the vertices of P as shown in Figure 4.9 and notice that the extensions l_2, l_1 of the edges of P connecting v_2 to r_2, and v_1 to r_1, respectively are such that l_2 hits the base of P to the left of the point at which l_1 hits it.

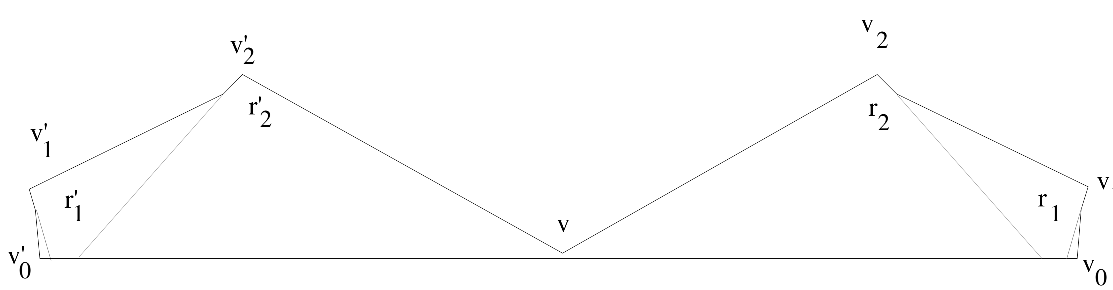

Figure 4.9: A polygon that cannot be illuminated by placing an orthogonal floodlight at each vertex.

We now show that P cannot be illuminated by placing an orthogonal floodlight at each of its vertices. The orthogonal floodlight at v illuminates vertices in L or R, but not both. Assume without loss of generality that this light illuminates vertices in L. Since the internal angle at v_2 is bigger than $\frac{\pi}{2}$ the only way we can illuminate all the points in a small neighborhood of v_2 is by using the floodlights at v_2 and r_2. Similarly to illuminate all the points in a small neighborhood of v_1, we need the lights at v_1 and r_1. Notice, however, that this will prevent the floodlights at r_1 and r_2 from pointing towards v_0. It now follows that all the points in a small neighborhood of v_0 have to be illuminated using only the floodlight at v_0. Notice that this is impossible, since the internal angle of P at v_0 is greater than $\frac{\pi}{2}$. Our result now follows. ∎

The following problem arises naturally from the above result:

> *How many π-floodlights are always sufficient to illuminate any polygon with n vertices?*

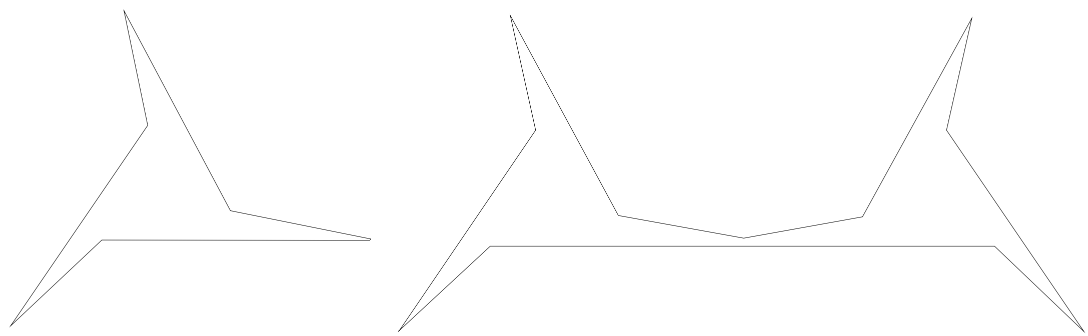

Figure 4.10: A family of polygons with $5n+1$ vertices that require $3n$ π vertex floodlights.

F. Santos has produced a family of polygons with $5n+1$ vertices that require $3n$ vertex π floodlights to illuminate them; see Figure 4.10. We now conjecture:

Conjecture 4.3.1 $\lfloor \frac{3n-3}{5} \rfloor$ π *vertex floodlights are always sufficient and occasionally necessary to illuminate any polygon with n vertices.*

4.4 Orthogonal polygons

We now turn our attention to the study of the illumination of orthogonal polygons with vertex floodlights. A floodlight of size $\frac{\pi}{4}$ will be called an orthogonal floodlight. We now prove:

Theorem 4.4.1 *Any orthogonal polygon P with n vertices can be illuminated with at most $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal vertex floodlights.*

Proof: We start by proving that P can be illuminated by choosing a set of orthogonal floodlights according to the following rule, which we call the *top-left* illumination rule:

Top-left illumination rule:

1. At the top vertex of every left edge of P place an orthogonal floodlight illuminating the set of directions between $\frac{3\pi}{4}$ and 2π.
2. At the left vertex of every top edge of P again place an orthogonal floodlight illuminating the set of directions between $\frac{3\pi}{4}$ and 2π.

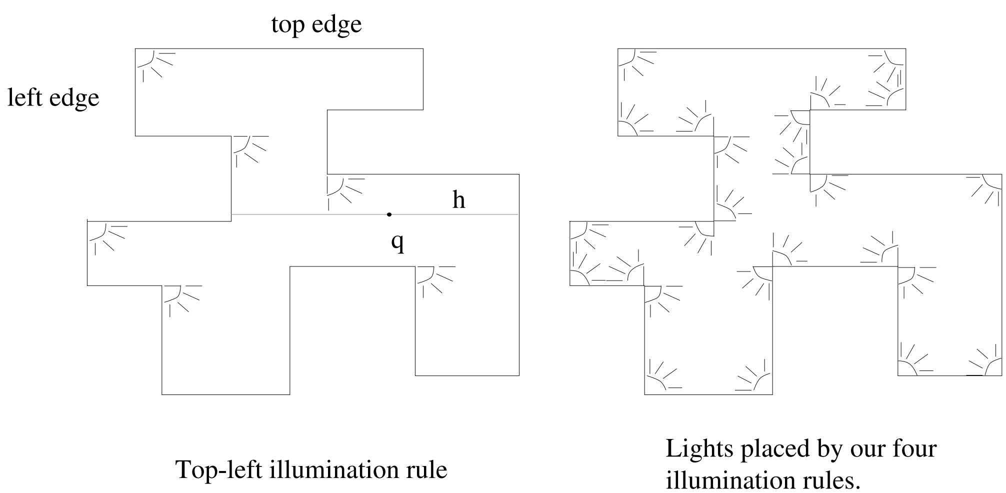

Figure 4.11: Illuminating an orthogonal polygon with orthogonal floodlights.

To see that the set of floodlights chosen by the top-left illumination rule illuminates P, choose any point q in P. Consider the longest horizontal line segment h passing by q, totally contained in P. Slide h upwards until it hits a top edge of P or it reaches the top vertex of the edge of P containing the left end point of h. In the former case, there is a floodlight at this vertex that illuminates q. Suppose then that h hits a top edge e of P. Notice that by our illumination rule, we have a floodlight at the left vertex of e, and this floodlight illuminates q; see Figure 4.11.

In a similar way, we can define the *top-right*, *bottom-right*, and *bottom-left* illumination rules, each of which illuminates P. Moreover, if we use the four illumination rules simultaneously, we will place exactly two floodlights at each reflex vertex, and one at each convex vertex. Thus if P has r reflex vertices, the total number m of floodlights used by our four rules is $2r + (n - r)$. By Lemma 2.3.1 $r = \frac{n-4}{2}$, and thus we have that $m = 2\frac{n-4}{2} + \frac{n+4}{2} = \frac{3n-4}{2}$. It now follows that one of our illumination rules has at most $\lfloor \frac{3n-4}{8} \rfloor$ floodlights.

The polygon P_{12} shown in Figure 4.12(a) has 12 vertices and requires $4 = \frac{3(12)-4}{8}$ orthogonal vertex floodlights. If we now *paste* copies of this polygon as shown in the same figure, we can generate a family of polygons with $12+8k$ vertices, each of which requires $4+3k$ vertex floodlights. Our result now follows. ∎

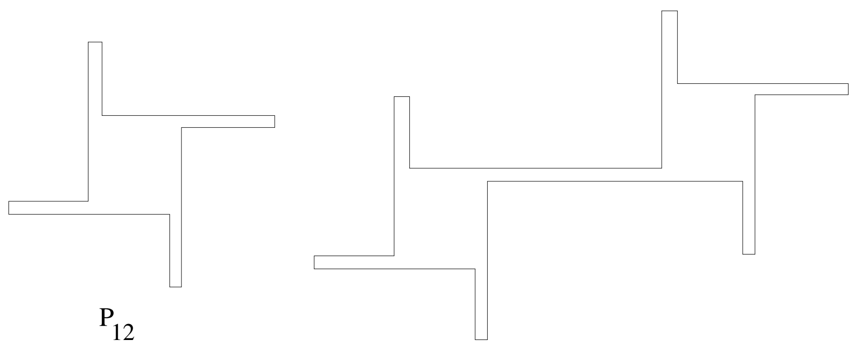

Figure 4.12: A family of polygons that requires $\lfloor\frac{3n-4}{8}\rfloor$ orthogonal vertex floodlights.

We now show that $\frac{\pi}{2}$ is essential in our previous result, i.e. we prove:

Theorem 4.4.2 *For any $\epsilon > 0$ there is an orthogonal polygon that cannot be illuminated with $\frac{\pi}{2}-\epsilon$ vertex floodlights.*

Proof: For any $\epsilon > 0$ construct an orthogonal polygon with 12 vertices consisting of a square S with four long rectangles $R_1, \ldots, R_4$ attached to it as shown in Figure 4.13. Label the vertices of P as shown in the same figure. It is easy to see that P can be constructed in such a way that the following conditions hold:

1. The angle $u_i, v-i, r_i$ is greater than $\frac{\pi}{2}-\epsilon$, $i = 0, \ldots, 3$.
2. The angle p, r_i, r_{i+1} is greater than $\frac{\pi}{2}-\epsilon$, $i = 0, \ldots, 3$ addition taken mod4.

Under these conditions, we can verify that the $\frac{\pi}{2}-\epsilon$ floodlights at u_i and v_i cannot illuminate all of $R_i, i = 0, \ldots, 3$ To complete the illumination of each R_i, we must then use a floodlight placed at a reflex vertex of P, $i = 0, \ldots, 3$. It now follows that the point p at the center of S is not illuminated. ∎

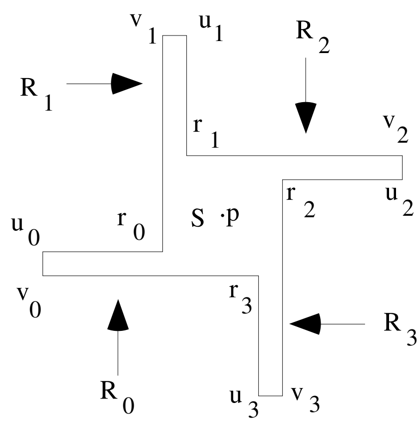

Figure 4.13: An orthogonal polygon that cannot be illuminated with orthogonal vertex floodlights.

4.5 The two floodlight illumination problem

In this section we study the following problem: Given a convex polygon P with n vertices, find two floodlights of sizes α_1 and α_2 that illuminate P and their locations such that $\alpha_1 + \alpha_2$ is minimized.

Consider two points a and b on the boundary of a polygon P, not necessarily vertices of P. The vertex interval (a, b) is defined to be the set of vertices of P that we meet when we move in the clockwise direction on the boundary of P from a to b. For short, we will refer to the interval (a, b) rather than to the vertex interval (a, b). Notice that (a, b)
$neq(b, a)$. In particular, when a and b are interior points of edges of p, $(a, b) \cup (b, a)$ is the set of vertices of P. A pair of floodlights F_1 and F_2 that illuminates a polygon P will be called a floodlight illuminating pair and will be called a FLIP of P. If F_1 and F_2 are such that the sum of their apertures is minimized, we call them an optimal FLIP.

We call F_1, F_2 an opposite FLIP if the intersection of the regions illuminated by F_1 and F_2 is a quadrilateral with all of its vertices on the boundary of P, see Figure 4.14(a). If the interior of the regions illuminated by a FLIP F_1 andF_2 are disjoint, we call F_1, F_2 a dividing FLIP, see Figure 4.14(b). Observe that an optimal FLIP must be either an opposite FLIP or a dividing FLIP.

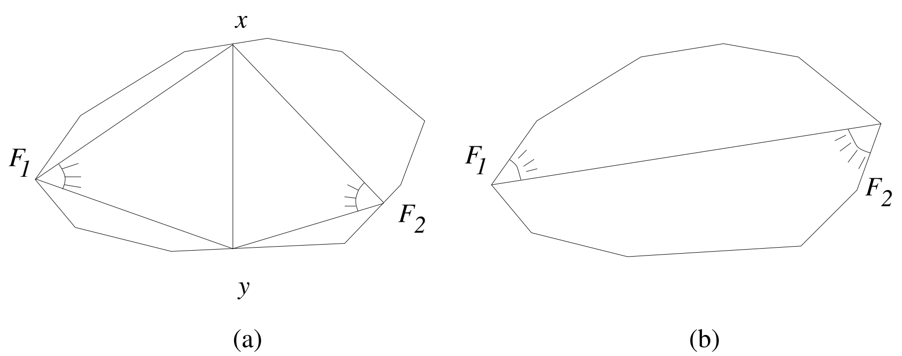

Figure 4.14: An opposite and a dividing FLIP.

We now prove:

Lemma 4.5.1 *Let F_1, F_2 be an optimal FLIP of a polygon P. Then the apexes of F_1 and F_2 are located at vertices of P.*

Proof: Suppose that F_1, F_2 form an optimal FLIP. Two cases arise:

(a) F_1 and F_2 form an opposite FLIP with apexes at q and r such that r is not a vertex of P. Let x, r, y, q be the vertices of the intersection of the areas illuminated by F_1 and F_2. Consider the circle $C(x, y, p)$ that passes through x, y and a vertex p of P such that $C(x, y, p)$ contains all the vertices of the interval (x, y) of P. Notice that the angle spanned by p and the line segment joining x to y is smaller than that spanned by r and the same segment. Therefore if we replace F_2 by a floodlight F_3 with apex in p and illuminating the angular sector determined by x, p, and y, F_1 and F_3 also illuminate P and the sum of their sizes is smaller than that of F_1 and F_2, see Figure 4.15(a).

b) $F_1$and F_2 form a partitioning pair Figure 4.15(b). We can easily verify that in this case, we can slide the apex of F_1 and F_2 towards two vertices of P such that the sum of the sizes of F_1 andF_2 decrease. ∎

Suppose now that we want to find a FLIP F_1 and F_2 of a polygon P in such a way that their apexes are located on two fixed vertices p and q of P, and the sum of the sizes of F_1 and F_2 is minimized. Call such a FLIP an optimal $(p, q) - FLIP$. We now show:

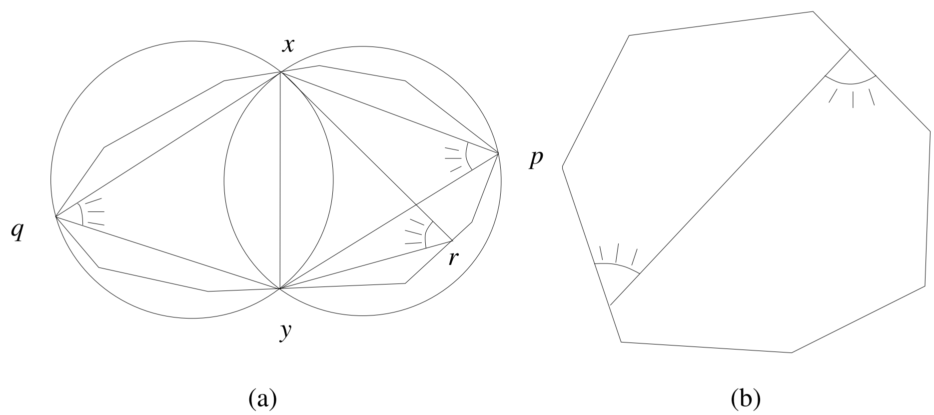

Figure 4.15: The vertices of an optimal FLIP must lie on vertices of P.

Lemma 4.5.2 *Given two vertices p and q of P, we can find an opposite optimal $(p, q) - FLIP$ in linear time.*

Proof: Our objective here is to find two points, x and y, on the boundary of p such that the sum of the angles $\alpha = ypx$ and $\beta = xqy$ is minimized. Consider the angles γ and δ formed by pxq and qyp as in Figure 4.16(a).

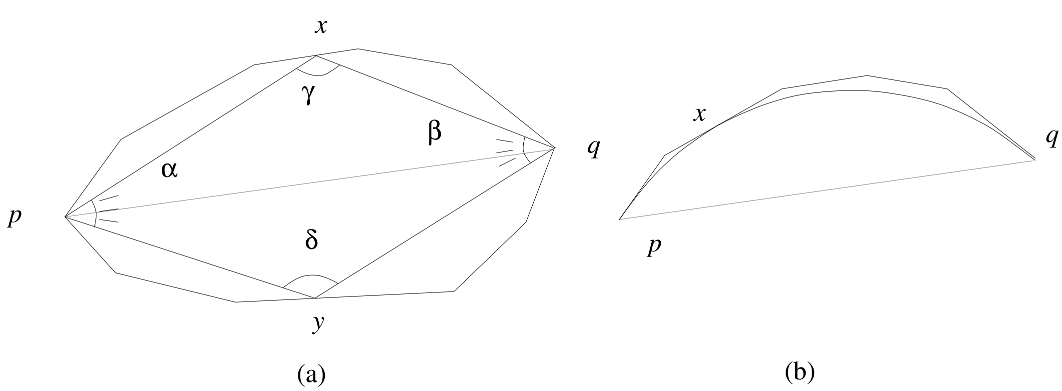

Figure 4.16: Finding an optimal $(p, q) - FLIP$.

We now observe that minimizing $\alpha + \beta$ is equivalent to maximizing $\gamma + \delta$. However, the maximization of $\gamma + \delta$ can be achieved by maximizing independently γ and δ! Thus all we need to do is to locate for each edge e of the chain of edges of P from p to q the point r_e on e that maximizes the angle pr_eq, and keep the point r_e wich maximizes γ. Clearly this can be done in constant time per edge, and x can be found in linear time. The same procedure is applied to find y. ∎

The previous Lemma, provides an $O(n^3)$ algorithm to find the optimal FLIP. To reduce the complexity of our algorithm to $O(n^2)$, we need to provide a criterion to reduce the number of pairs of vertices of P that can be apexes of an optimal opposite FLIP. Consider the set D of all diagonals of P joining all pairs of vertices of P with the property that there is a circle through their endpoints that contains P. Under our general circular position on the vertices of P, it is easy to see that the elements of D induce a triangulation $T(P)$ of P, see Figure 4.17(a). Using standard techniques for calculating Voronoi Diagrams of convex polygons, D and $T(P)$ can be found in linear time [4].

A subset of three vertices $\{u, v, w\}$ of the vertex set of p is called a $c\!-\!triple$ if $\{u, v, w\}$ is the set of vertices of a triangle of $T(P)$. By the definition of $T(P)$, it follows that the circle $C(u, v, w)$ determined by $\{u, v, w\}$ contains P. Under our general position assumption, the number of $c - triples$ of P is exactly $n - 2$. We say that two $c - triples$ $\{u, v, w\}$ and $\{x, y, z\}$ are adjacent if they share two common elements. If $\{u, v, w\}$ and $\{u, v, x\}$ are two adjacent $c - triples$, The vertices w and x will be called the antipodal vertices of $\{u, v, w\}$ and $\{u, v, x\}$.

Given two vertices p and q of P, we say that a $c\!-\!triple$ $\{r, s, t\}$ *separates* them if p and q are both different from r, s, and t and when we traverse the boundary of p from p to q (and from q to p) in the clockwise order, we meet either one of r, s or t before we meet q (resp. p), see Figure 4.17(b).

Lemma 4.5.3 *If two vertices p and q of a polygon P are apexes of an optimal FLIP, then there is no $c - triple$ that separates them.*

Proof: Suppose there is a $c - triple$ that separates p and q, and that there is an optimal opposing FLIP F_1, F_2 of P with apexes at p and q, see Figure 4.17(b). Suppose further that F_1 and F_2 meet at two points x and y in the interior of two edges of P. As in the proof of Lemma 4.5.1, the circle through x, y and q contains in its interior all of the vertices of P between x and y in the clockwise direction, except for q itself, see Figure 4.17(b).

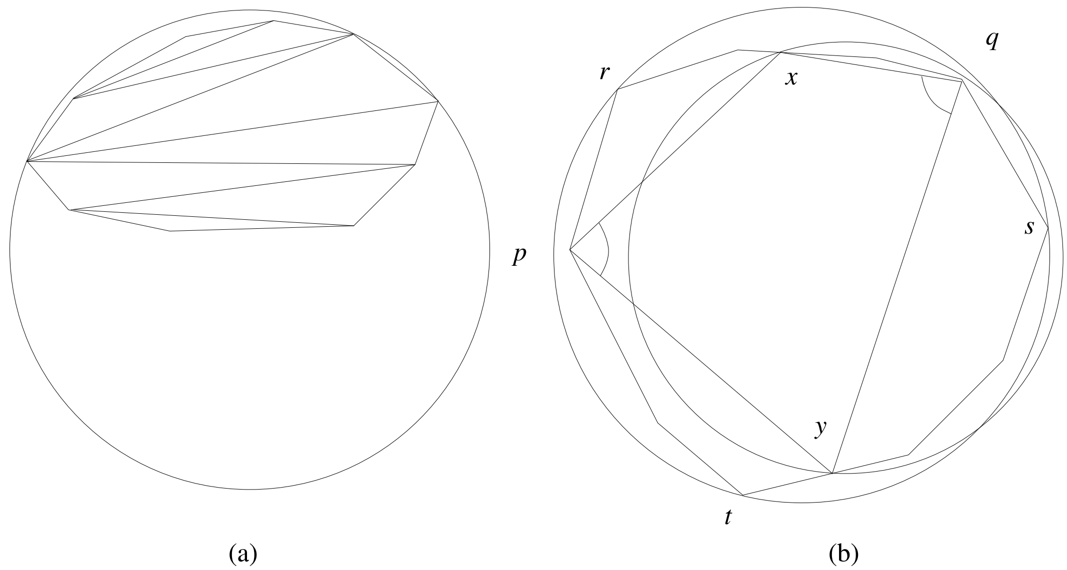

Figure 4.17: A polygon, and its corresponding $T(P)$. The $c-triple$ $\{r,s,t\}$ separates $p$and q .

Similarly for the circle trough y, p and x. However, this will force one of these circles has to intersect the circle through r, s and t at least four times, which is impossible. ■

Corollary 4.5.1 *Let p and q, be the apexes of an optimal FLIP. Then p and q form an antipodal pair of vertices of two adjacent $c-triples$, or there is a circle through p and q containing$P$.*

We are now ready to give our algorithm O-FLIP to obtain an optimal floodlight illumination pair of a polygon.

Algorithm 4.5.1 *O-FLIP Input a polygon P with n vertices.*

1. *Find $T(P)$, and all $c-triples$ of P.*

2. *Find the set S of all pairs of vertices of P that are antipodal pairs, or adjecent in $T(P)$.*

3. *For each pair of p and q in S, find an optimal opposite $(p,q)-FLIP$.*

4. *For each pair of vertices p and q of P find in constant time the partitioning pair of floodlights F_1, F_2 with apexes at p and q that illuminate P and minimizes the sum of the sizes of F_1 and F_2.*

Output the FLIP of minimum weight identified in Steps 3 and 4.

We now have:

Theorem 4.5.1 *Given a convex polygon P with n vertices in general position, O-FLIP finds an optimal FLIP in $O(n^2)$ time.*

Proof: Steps 1 and 2 can be carried out in linear time. Step 3 requires $O(n^2)$ time, since there are at most $n-2$ antipodal pairs and $2n-3$ edges in $T(P)$. Solving each of them requires $O(n)$ time. In Step 4, we need to test all pairs of vertices p and q of P. However, for every of these pairs, there are exactly two partitioning FLIP's, and the complexity of Step 4 is $O(n^2)$. ■

4.6 Exercises

1. Suppose that we associate to each vertex v of P a weight equal to the size of the internal angle of P at v. Show that we can always choose a set of vertices of P such that they illuminate P and the sum of their weights is at most $\frac{(n-2)\pi}{3}$. Show that this bound is tight.

2. Show that Theorem 4.1.2 still holds if we remove the condition that α_1, α_2 and α_3 are less than or equal to π.

3. Prove that Theorem 4.1.1 is not true if we remove the condition that the floodlights are of size at most π!

4. A polygon P is called co-circular if all the vertices of P lie on a circle. Show that the two-floodlight illumination problem for cocircular polygons can be solved in linear time.