Chapter 5

Miscellaneous

In this chapter, we study variations of guarding or illumination problems studied by several authors.

5.1 Guarding triangulations of point sets

Let P_n be a set of n points on the plane. A *triangulation* T of P_n is a partitioning of the convex hull of P_n into a set of triangles with disjoint interiors. The vertices of these triangles are elements of P_n, and each element of P_n is a vertex of at least one triangle in T. The edges of T are the edges of its triangles.

A set S of edges of T guards it if every triangle of T has a vertex that is the end-vertex of an edge in S. A set H of vertices of T guards it if every face of T has a vertex in H. We will prove the following results, proved by Everett and Rivera-Campo [53], and Bose, Shermer, Toussaint and Zhu [16] respectively:

Theorem 5.1.1 *Any triangulation of a set of n points can be guarded with at most $\lfloor \frac{n}{3} \rfloor$ edges.*

Theorem 5.1.2 *Any triangulation of a point set can be guarded with at most $\lfloor \frac{n}{2} \rfloor$ vertices. This bound is tight.*

We notice first that the upper bound for Theorem 5.1.2 follows easily from the Four Color Theorem; simply take a four-coloring of the vertices of our triangulation, and by placing a guard at all the vertices of the two smaller chromatic classes our triangulation is guarded. In fact the first proof of Theorem 5.1.1 also used the Four Color Theorem. We now present a proof of both of these results that does not use the Four Color Theorem. This proof is due to Bose, Kirkpatrick and Li [15]. Some preliminary results will be needed. We first observe that our problems of guarding triangulations of point sets can be reduced to that of guarding maximal planar graphs. If the convex hull of our point set is a triangle, then any triangulation of our point set yields a maximal planar graph, otherwise by adding extra edges to the external face of the graph defined by our triangulation, we obtain a maximal planar graph.

A graph is called k-regular if all of its vertices have degree k. We start by proving the following well known result in Graph Theory:

Theorem 5.1.3 *Every 2-connected planar 3-regular graph has a perfect matching.*

Proof: To prove our result, we will show that any 3-regular 2-connected graph G satisfies Tutte's Theorem 3.1.1. Let S be a set of vertices of G and H an odd component of $G - S$. Since G is 3-regular and H has an odd number of vertices, the number of edges joining vertices of H to vertices in S has to be odd. However since G is 2-connected, there are at least 3 edges connecting S to H. It now follows that $Odd(G-S) \leq \mid S \mid$. ■

A 2-coloring of a graph is called K_3-free if it contains no triangle that is monochromatic. We now prove:

Theorem 5.1.4 *Every planar graph admits a K_3-free 2-coloring.*

Proof: We can assume that G is a maximal planar graph, otherwise insert extra edges until it becomes a maximal planar graph. Let G^* be the dual graph of G, i.e. for each face of G, insert a vertex in G^*. Two vertices in G^* are adjacent if their corresponding triangular faces share an edge of G. Notice that there is a one-to-one correspondence between the edges of G and those of G^*. Since G is maximal, G^* is a 3-regular and 2-connected graph, and thus by Theorem 5.1.3 it has a perfect matching M. Delete from G all the edges corresponding the the edges of G^* in M. It is now easy to see that all the faces of the remaining graph, call it H, are quadrilaterals, and H is therefore bipartite, i.e. the vertices of H can be 2-colored in such a way that no adjacent vertices receive the same color. It now follows that this 2-coloring of H induces a K_3-free coloring of G; see Figure 5.1. ■

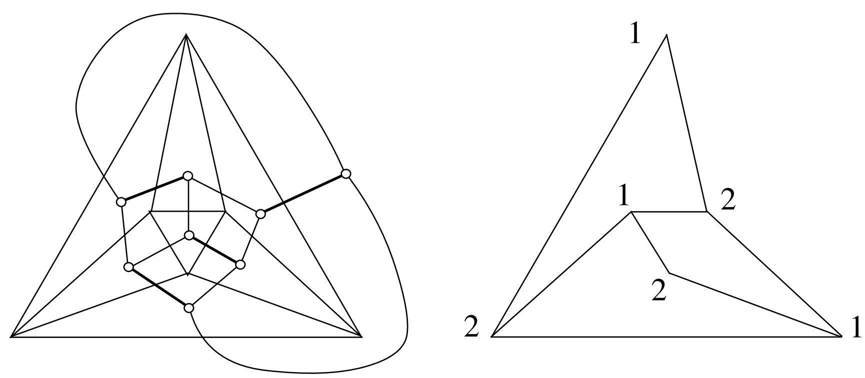

Figure 5.1: Obtaining a K_3-free coloring of a maximal planar graph.

We can now prove Theorem 5.1.2:

Proof: Take any K_3 2-coloring of T. Clearly each chromatic class in any such coloring guards T. Place a guard at each vertex in the smallest chromatic class, and our result follows. To show that our bound is tight, we generate a family of triangulations that require $\lfloor \frac{n}{2} \rfloor$ vertex guards as follows: Let S_1 be the seven vertex triangulation shown in Figure 5.2. $S_2, \ldots, S_n$ are now obtained by recursively replacing triangle x, y, z by a copy of S_1. We leave it to the reader to verify that S_k has $4k+3$ vertices, and that it requires $2k+1$ guards; see Excercise 1. ∎

The following will be needed to prove Theorem 5.1.1. Let G_1 and G_2 be the subgraphs of G induced by the chromatic classes of our 2-coloring of G, and let M_1 and M_2 be maximal matchings of G_1 and G_2 respectively.

Let us now choose two different sets S_1 and S_2 of edges of G that guard G as follows:

S_1 contains all the edges of M_1 as well as an extra edge incident to each unmatched vertex of G_1. S_2 is chosen in a similar way.

Next we show:

Lemma 5.1.1 *$M_1 \cup M_2$ guard G.*

Proof: Suppose that there is a triangle F of G that is not guarded by $M_1 \cup M_2$. Then none of the vertices of this face is incident to any edge in M_1

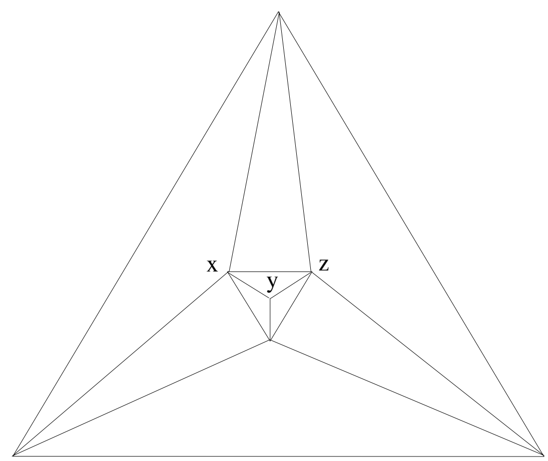

Figure 5.2: S_1 needs 3 guards to be guarded.

or M_2. Since we have a 2-coloring of G, two of the vertices of F receive the same color, say color 1. This contradicts the assumption that M_1 is maximal. ∎

We are now ready to prove our main result:

Proof of Theorem 5.1.1: Notice that S_1, S_2 and $M_1 \cup M_2$ guard G. If $M_1 \cup M_2$ has at most $\lfloor \frac{n}{3} \rfloor$ edges, we are done. Suppose then that $\mid M_1 \cup M_2 \mid$ is greater that $\lfloor \frac{n}{3} \rfloor$. Notice that:

$\mid S_1 \mid = \mid V(G_1) \mid - \mid M_1 \mid$

and

$\mid S_2 \mid = \mid V(G_2) \mid - \mid M_2 \mid$

Moreover $\mid V(G_1) \mid + \mid V(G_2) \mid = \mid V(G) \mid = n$.

Then $\mid S_1 \mid + \mid S_2 \mid = \mid V(G_1) \mid + \mid V(G_2) \mid - (\mid M_1 \mid + \mid M_2 \mid)$

$\leq n - \lfloor \frac{n}{3} \rfloor \leq \lceil \frac{2n}{3} \rceil$.

It now follows that one of S_1 or S_2 has at most $\lfloor \frac{n}{3} \rfloor$elements. ∎

An example of a triangulation with $13n - 12$ vertices that needs $4n - 8$ edge guards can be obtained as follows: Consider the graph H shown in Figure 5.3. It is clear that for this triangulation we need two edges. Let T be any maximal planar graph with n vertices. By Euler's formula, T has exactly $2n - 4$ triangular faces. Place a copy of H in the middle of each face of T and retriangulate the resulting graph. The total number of vertices of the resulting graph is $n + 6(2n - 4)$. Since each copy of H needs two edge guards, and no two copies of H can share an edge guard, we need at least $2(2n - 4)$ edge guards.

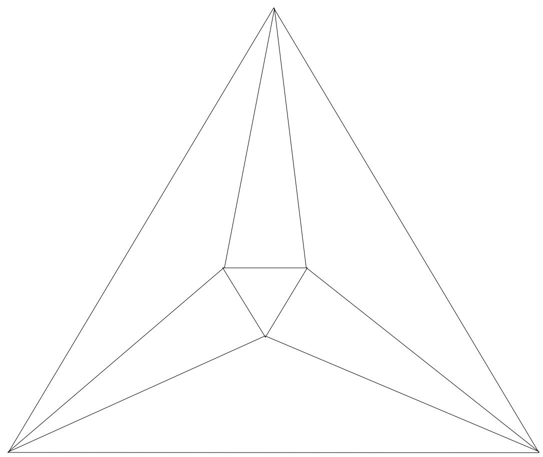

Figure 5.3: To guard H we need 2 edge guards.

A plane graph H is a planar graph G together with an embedding of it on the plane. An edge e and a face F of H are called incident if at least one vertex of e lies on the boundary of F. A set S of edges of H guards it if each face of H is incident to an edge of S. Notice that the proof of the previous theorem does not apply to plane graphs; see exercise.

In a similar way, but now using the Four Color Theorem, we have [53]:

Theorem 5.1.5 *Every plane graph H on n vertices can be guarded with $\lfloor \frac{2n}{5} \rfloor$ edges.*

Proof: Triangulate H and four-color the vertices of H. Let V_1, V_2, V_3 and V_4 be the chromatic clases of our four coloring of H. For $1 \le i < j \le 4$ let $M_{i,j}$ be a maximal matching of the subgraph of H induced by $V_i \cup V_j$. Let $A_{i,j}$ be a set of edges defined as follows: $A_{i,j}$ contains all the edges of $M_{i,j}$ plus choose an extra edge for each vertex of $V_i \cup V_j$ not covered by $M_{i,j}$. Since each face of H must contain three vertices of different colors, it follows that each $A_{i,j}$ guards H. Moreover notice that $M_{1,2} \cup M_{1,3} \cup M_{2,4} \cup M_{3,4}$ is also a guarding set of H. However:

$$| A_{1,2} | + | A_{1,3} | + | A_{2,4} | + | A_{3,4} | + | M_{1,2} \cup M_{1,3} \cup M_{2,4} \cup M_{3,4} | = 2n.$$

It follows that at least one of these sets has at most $\lfloor \frac{2n}{5} \rfloor$ edges. ∎

From an algorithmic point of view, the proofs presented here for Theorems 5.1.1 and 5.1.2 yield algorithms with complexity $O(n^{\frac{3n}{2}})$. To prove this, we notice that calculating the dual graph of a triangulation can be done in linear time, finding its matching in $O(n^{\frac{3n}{2}})$; see [93]. Finding M_1, M_2 again takes $O(n^{\frac{3n}{2}})$ time, and calculating and S_1 and S_2 can be done in linear time. Thus we have:

Theorem 5.1.6 *Finding an edge and a vertex guard sets of sises $\lfloor \frac{n}{3} \rfloor$ and $\lfloor \frac{n}{2} \rfloor$ respectively can be done in $O(n^{\frac{3n}{2}})$*

An interesting open problem here is to find a proof of Theorem 5.1.5 that does not depend on the Four Color Theorem. It is easy to see that a proof based on K_3-free 2-colorings fails since $M_1 \cup M_2$ does not necessarily guard a plane graph. On the other hand if all the faces of a plane graph are odd, i.e. have an odd number of edges on their boundaries, the elements of $M_1 \cup M_2$ guard it. The proof of this is left as an excercise.

5.2 Moving guards

In this section we study a problem introduced by Suzuki and Yamashita, called by T. Shermer *The Hunter's Problem.* In this problem, we have a set of hunters and a prey that are allowed to move freely within a polygon. The hunters move with a bounded velocity, and the prey moves with unbounded velocity.

In his formulation of the problem, the hunters try to catch a prey that is trapped within a polygonal region. The prey is considered caught if it comes within sight of a hunter. Suzuki and Yamashita say that a polygon P is k-searchable if k hunters are always sufficient to catch any prey in P. Three points x_1, x_2, and x_3 of P are called an asteroidal triple if they are such that the shortest path between x_i and x_j is not visible from any point visible from x_k; $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k$. For example in the polygon in Figure 5.4, x_1, x_2, and x_3 form an asteroidal triple. Moreover, it is easy to verify that this polygon is not 1-searchable. Suzuki and Yamashita proved:

Theorem 5.2.1 *A polygon P is 1-searchable if and only if it contains no asteroidal triple.*

The proof of this result is left as an exercise.

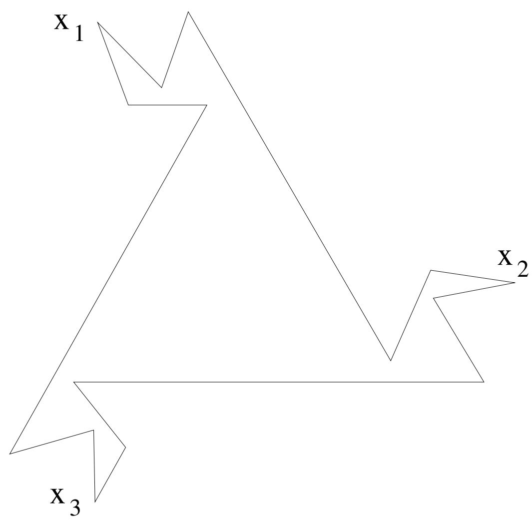

Figure 5.4: A 2-searchable polygon.

A general upper bound on the number of hunters needed to guard a simple polygon was obtained by Urrutia. He proved:

Theorem 5.2.2 *$O(\ln n)$ hunters are always sufficient, and occasionally necessary to catch a prey in any polygon with n vertices.*

Proof: Let $f(n)$ be the minimum number of hunters needed to catch a prey in an n vertex polygon P. We now show that $f(n) \leq f(\frac{2n}{3}) + 1$. This will prove our result. We start by first finding a diagonal joining two vertices of P that cuts it into two subpolygons P_1 and P_2, each of size at most $\frac{2n}{3}$. It is well known that such a diagonal always exists. Station a hunter at an endpoint of this diagonal. This will ensure that no prey can go from P_1 to P_2. Next scan P_1 first, and then P_2. This can be accomplished with at most $f(\frac{2n}{3})$ hunters. Our result now follows. A family of polygons that require $O(\ln n)$ hunters can be obtained from a binary search tree T as follows: first draw T on the plane without crossing edges, and then substitute a sequence of three narrow corridors for each edge of T as in Figure 5.5. ∎

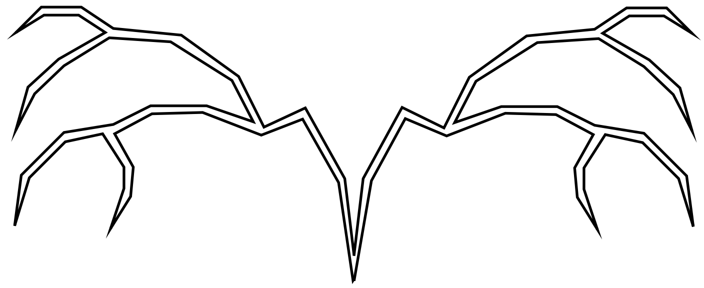

Figure 5.5: A binary polygon.

5.2.1 Lazy guards

The following problem involving moving guards was introduced by Colley, Meijer and Rappaport:

Problem 5.2.1 The lazy guard problem: *Given a simple polygon P, choose the minimum number of stations (points) in P such that a moving guard that visits all the stations guards P, i.e. while traveling to visit all the stations, every point in P will, at some point in time, be visible to our guard.*

Since our guard wants to minimize the distance walked, we will assume that while traveling from one station to the next, he will choose the shortest path connecting them. The motivation for introducing stations is that if we want to make sure a guard patrols a polygon, the easiest way to do so is to place a few *check-in* stations such that if the guard visits them, P is guarded. In Figure 5.6 we show a polygon that needs three stations.

We now prove:

Theorem 5.2.3 *The lazy guard problem can be solved in linear time.*

We begin by proving the following result:

Lemma 5.2.1 *Let S be a set of k stations such that if a guard visits them in a given order, it guards P. Then if we visit the elements of S in any order, P is also guarded.*

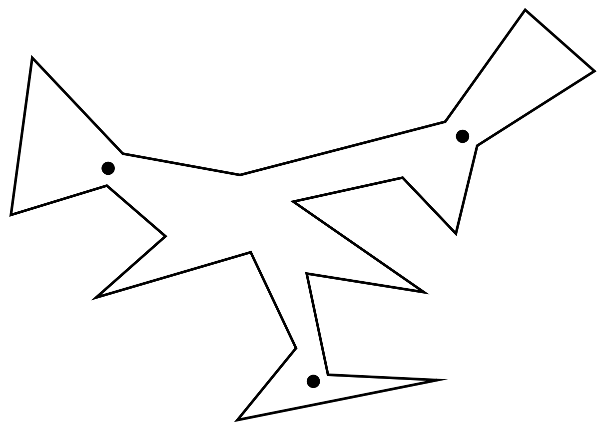

Figure 5.6: Three stations suffice to guard this polygon.

Proof: Suppose that P is guarded while visiting the elements of S in a specific order. Let p and q be two elements of S that are visited in consecutive order, and r any point in P visible from some point x in the shortest path from p to q. Take the longest line segment L containing r and x contained in P. This will cut P into two subpolygons, one containing p and the other containing q. Then if we visit the elements of S in any other order, say first p and then q, at some point we must cross L, and thus r is guarded. ∎

It is easy to see now that we can without loss of generality assume that the stations are visited starting and ending at the same station along the shortest circuit connecting them. This circuit defines a perhaps degenerate polygon P_S that may consist of the union of several simple polygons and line segments connecting them. We refer to this as *the guards polygon.*

We now prove:

Lemma 5.2.2 *If a polygon is guardable using k stations, it can also be guarded with k stations located on the boundary of P.*

Proof: The case $k = 2$ is easy. Let us assume then that $k \geq 3$, and suppose that we have a set S of k stations such that when we visit them, we guard P. Let s be a station in S not located on the boundary of P. Suppose that s is adjacent to two vertices s_1 and s_2 in P_S. Consider the bisector of the external angle of P_S defined by s_1, s, and s_2, and extend it until it hits the boundary of P at a point s'. Let $S' = S - \{s\} \cup \{s'\}$. We now prove that if we visit all the stations in S' we guard P. We notice first that $P_{S'}$ contains P_S. It now follows that if a point is visible from some point in P_S it is also visible from a point in $P_{S'}$. To finish our proof, we continue replacing the elements of S until all stations are on the boundary of P. ∎

Consider a reflex vertex v of P, and let v^- and v^+ be the clockwise and counter-clockwise vertices of P adjacent ot v. Let $r(v^+)$ and $r(v^-)$ be the points at which the rays starting at v^+ and v^- which pass through v hit the boundary of P; see Figure 5.7. Let $C(v, r^-)$ be the segment of the boundary of P traversed when moving from v to $r(v^+)$ in the clockwise direction along the boundary of P. Similarly we define $C(v^-, v)$.

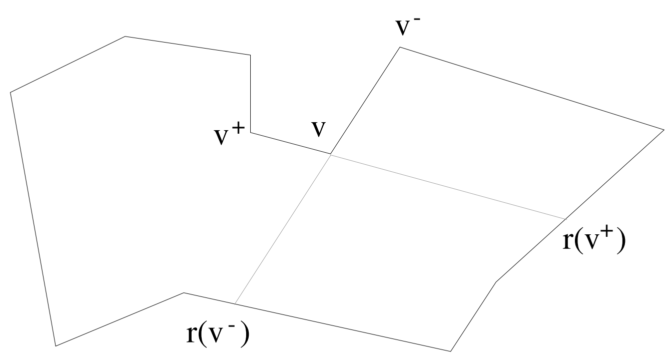

Figure 5.7: Finding $C(v, v^+)$ and $C(v^-, v)$.

We now prove:

Theorem 5.2.4 *A set of stations on the boundary of P suffices to lazy guard it if for every reflex vertex v of P, $C(v, v^+)$ and $C(v^-, v)$ contain an element of S.*

Proof: It is clear that if for some reflex vertex v of P no element of S is located on $C(v, v^+)$ then v^+ is not visible from P_S, which is a contradiction. Similarly for $C(v^-, v)$. The converse is obvious. ∎

From Theorem 5.2.4 we can obtain the following algorithm to solve the lazy guard problem:

Algorithm 5.2.1

1. *For each reflex vertex of* P *compute* $C(v, v^+)$ *and* $C(v^-, v)$.

2. *Determine the smallest set of points* S *on the boundary of* P *such that each* $C(v, v^+)$ *and* $C(v^-, v)$ *contains at least a point in* S.

Step 1 of our algorithm can be done in linear time; see [71]. Step 2 can also be solved in linear time by reducing it to the problem of finding a minimum vertex cover of a circular arc graph, which is known to be solvable in linear time [85], as follows:

Map the boundary of P to a circle C. Each $C(v, v^+)$ and $C(v^-, v)$ is in turn mapped to an arc of our circle. Our problem is now reduced to that of finding a minimum number of points on C that covers the arcs defined by $C(v, v^+)$ and $C(v^-, v)$; v a reflex vertex of P.

Thus we have proved:

Theorem 5.2.5 *An optimal placing of stations to lazy guard a polygon can be found in linear time.*

For polygons with holes, the situation changes drastically. The order in which a set of stations is visited becomes relevant, and finding a minimum set of stations to lazy guard a polygon becomes NP-complete.

We now show that lazy guarding a polygon with holes is NP-complete. We use the following problem that is known to be NP-complete:

Problem 5.2.2 k-vertex cover of planar 3-regular graphs

INSTANCE: *A planar graph* G *with maximum degree* 3, *and an integer* k.

QUESTION: *Is there a set of vertices with* k *elements such that every edge of* G *has an endpoint in* S?

We now prove that the following problem is also NP-complete:

Problem 5.2.3 k-guarding of polygons with holes

INSTANCE: *A polygon* P *possible with holes.*

QUESTION: *Is there a set of points in* p *of cardinality* k *that guards* P?

Lemma 5.2.3 **k-guarding of polygons with holes** *is NP-complete.*

Proof: Take a planar graph G with maximum degree 3, and a plane embedding of it in which its edges are represented by straight lines. We can assume that no two edges of G are collinear. Slightly extend the line segment representing each edge in both sides by a small amount, and substitute this line segment by a thin rectangle $r(e)$ with its longer side parallel to e. Finally add two small spikes aligned with e and of width ϵ. The objective of these spikes is to ensure that any point that sees all of them belongs to $r(e)$ or to any of its the two spikes; see Figure 5.8. Let P be the union of all $r(e)$ and their spikes.

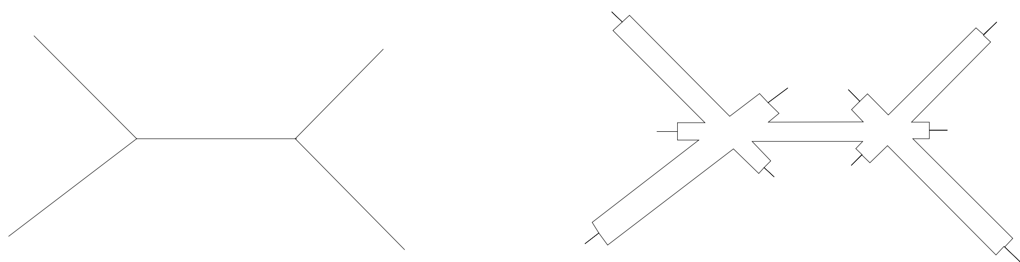

Figure 5.8: Substituing edges in a planar graph.

It is now easy to verify that a k-vertex cover of G generates a set of points that guard the resulting polygon. On the other hand, let S be a set of k points that guards P. If a point s in S guards points in more than one rectangle $r(e)$, these rectangles correspond to edges in G with a common end-vertex v. Move s to the location of v. It now follows that the vertices in which we place a guard correspond to a vertex cover of G. ∎

We now show:

Theorem 5.2.6 *Lazy guarding a polygon with k stations is NP-complete.*

Proof: Take a planar graph with maximum degree 3, and an embedding of it in which no edge is horizontal or vertical. Consider a set of $n + 1$ horizontal line segments and two vertical ones such that the endpoints of each horizontal line segment are on the vertical segments, and each horizontal line segment, except for the bottom one, contain a vertex of G. Obtain from G

the polygon P as in the previous lemma, and substitute each horizontal line segment by a narrow rectangle with a *hook* at each end, and the vertical lines by two narrow rectangles as shown in Figure 5.9. Let Q be the union of these polygons. We now prove that Q can be lazy guarded with $2n+k+2$ stations iff G has a vertex covering of size k.

To cover the spikes of the horizontal rectangles, we need a station in each of them, i.e. we need $2n+2$ stations. Suppose then that G has a vertex cover of size k. Then by placing k stations at the positions corresponding to the vertices of the vertex cover of G, we get a set of $2n+2+k$ stations that lazy guards Q.

Suppose now that Q is $2n+2+k$ lazy guardable, and that S is a set of $2n+2+k$ stations that lazy guard P. To guard the spikes of the horizontal rectangles of Q, we need $2n+2$ stations. Thus the remaining k stations must be used to ensure that P is lazy guarded. Notice first that if a corridor $r(e)$ of Q is such that no station is located in $r(e)$ or its spikes, then due to the way we constructed Q, we can find a path connecting all the stations of S that does not cover $r(e)$, which is a contradiction. As in the previous lemma, we can now move the k stations in S not used to guard hooks to locations corresponding to vertices of G to obtain a k vertex cover of G. It now follows that deciding if Q is lazy guardable with $2n+2+k$ stations is equivalent to deciding if P is k guardable, which is NP-complete. ∎

5.3 Exercises

1. Verify that $S_1, \ldots S_k, \ldots$ as defined in Theorem 5.1.2 require $2K+1$ guards.

2. Prove that there are 2-connected plane graphs for which $\lfloor \frac{n}{3} \rfloor$ edges are necessary to guard them.

3. Prove that any plane graph such that all its faces are odd can be guarded with $\lceil \frac{n}{3} \rfloor$ edges.

4. Prove Theorem 5.2.1.

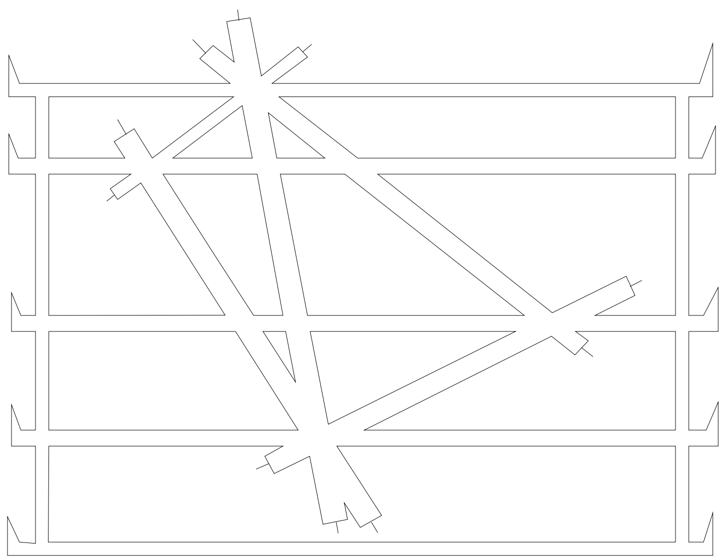

Figure 5.9: Constructing Q.